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ON THE INTERACTION OF STATIONARY CROSSFLOW VORTICES AND TOLLMIEN-SCHLICHTING WAVES IN THE BOUNDARY LAYER ON A ROTATING DISC

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ABSTRACT

There are many fluid flows where the onset of transition can be caused by different instability mechanisms which compete among themselves. Here we consider the interaction of two types of instability mode (at an asymptotically large Reynolds number) which can occur in the flow above a rotating disc. In particular, we examine the interaction between lower-branch Tollmien-Schlichting (TS) waves and the upper-branch, stationary, inviscid crossflow vortex whose asymptotic structure has been described by Hall (1986). This problem is studied in the context of investigating the effect of the vortex on the stability characteristics of a small TS wave. Essentially, it is found that the primary effect is felt through the modification to the mean flow induced by the presence of the vortex. Initially, the TS wave is taken to be linear in character and we show (for the cases of both a linear and a nonlinear stationary vortex) that the vortex can exhibit both stabilizing and destabilizing effects on the TS wave and the nature of this influence is wholly dependent upon the orientation of this latter instability. Further, we examine the problem with a larger TS wave, whose size is chosen so as to ensure that this mode is nonlinear in its own right. An amplitude equation for the evolution of the TS wave is derived which admits solutions corresponding to finite amplitude, stable, traveling waves.

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1. Introduction

Many studies concerned with the instability of three-dimensional boundary layers have been motivated by a desire to understand the phenomenon of transition to turbulence in fluid flows. Here we are concerned with a self-consistent asymptotic description of the interaction of stationary cross-flow vortices and lower-branch Tollmien–Schlichting waves (hereafter referred to as TS waves) in the boundary layer of the flow induced by a rotating disc. This particular flow is susceptible to an instability similar to that which occurs in the boundary layer of flows over a swept wing; a situation which has practical relevance to the development of Laminar Flow Control wings. Further, there is an exact solution of the Navier-Stokes equations which describes the rotating disc flow. This makes the study of this rotating disc problem particularly suitable for a theoretical analysis of the interaction of the vortices and the TS waves.

We will concentrate on the description of the interaction which occurs in flows at an asymptotically large Reynolds number (which is based upon the angular velocity of the disc, a typical lengthscale of the problem and the kinematic viscosity of the flow). The lower-branch TS waves are then described by a classical interactive triple deck structure, the key elements of which are reviewed by Messiter (1979), Stewartson (1981) and Smith (1982). The TS waves, which are travelling modes, have a wavelength much greater than the boundary layer thickness and a small wavespeed. The structure of these disturbances is outlined in Section 3.

The crossflow vortex instability structure can occur only in three-dimensional boundary layers and was first examined both theoretically and experimentally by Gregory et al (1955). The stationary vortex mechanism appears when a direction for the disturbance is chosen such that the effective basic velocity profile contains an inflexion point at the same location at which it vanishes. Gregory et al used the china clay technique to show that for the rotating disc flow the vortex instability takes the form of a regularly spaced pattern of equiangular spiral vortices which is stationary relative to the disc. Stuart (in Gregory et al), using inviscid theory, predicted the number of vortices which would be observed in experiment as approximately four times greater than that actually seen, although his calculation of an angle of 13^{o} between the axes of the vortices and the radius vector on the disc was in excellent agreement with the experiments. The difference between the experimental observations and the results of the inviscid analysis has been shown to be due to viscous effects. Malik (1986*a*) calculated the neutral curve for stationary disturbances and found a second mode. Earlier, Federov et al (1976) had observed experimentally this second mode, which, as in the inviscid case, appeared as a pattern of spiral vortices. The number of these vortices was seen to be between 14 and 16 and these had axes inclined at angles of approximately 20^o to the radius vector.

Hall (1986) has given a linear, asymptotic account of the inviscid mode found by Gregory et al (1955) for large Reynolds numbers. This work has been extended by Gajjar (1989) to examine nonlinear effects. Hall (1986) also elucidated a triple-deck type structure for the second type of stationary vortex which corresponds to an effective velocity profile with zero shear stress at the wall. This problem has also been studied using a weakly nonlinear approach by MacKerrell (1987, 1988). Further, Bassom & Gajjar (1988) have examined the properties of a non-stationary version of the crossflow instability.

Much work has been performed in relation to the important problem of the interaction of the interaction of TS waves with Görtler vortices (an instability associated with flows over curved surfaces). See, for example, Nayfeh (1981), Malik (1986*b*), Bennett & Hall (1988), Bennett et. al. (1988), Hall & Smith (1988, 1989*a*, *b*), Daudpota et al (1988), Bassom & Hall (1988) and the references therein. However, relatively little attention has been paid to the type of crossflow-TS interaction examined in this paper, although, in particular, we refer to the work of Reed (1984, 1985) who investigated this interaction and found that the crossflow vortices lead to a 'double exponential' growth in the TS waves. A linear analysis of the problem was used and the vortex was permitted to grow exponentially although no account was made for nonlinear effects. The vortex was allowed to force a TS wave of appropriate wavelength such that the growth rate of the latter mode was proportional to the amplitude of the vortex. The amplitude of the TS wave then grows like the exponential of an exponential. In the aforementioned papers Reed also examines the question of interaction between crossflow vortices of particular wavelengths. However, in both of these problems, the approach adopted ignores the crucial fact that for a completely rational description of the importance of the interaction process, each of the instability modes involved should be neutrally stable at leading order in their own rights. Otherwise, the vortex growth due to any interaction process is no larger than the growth experienced in its' absence and the importance of the role of the interaction is impossible to assess. For this reason, together with the neglect of nonlinear terms in her subsequent analysis, it must be concluded that the relevance of Reed's work to practical situations is at best doubtful.

The primary motivation for the present investigation stems from the question of how the presence of an inviscid stationary vortex affects the stability characteristics of the TS wave. We emphasize that at this stage we do not concern ourselves with calculations for the growth rates of the respective instabilities. Firstly, we consider the problem of a small crossflow vortex which has an asymptotic structure as described in Hall (1986). The vortex has a critical layer structure in the vicinity of the inflexion point of the basic flow and this critical layer is linear in character. The vortex then has a very small TS wave superimposed upon it and the effect of the vortex on the neutral stability properties of the TS wave is obtained. We find that the vortex can stabilise or destabilise the TS wave depending upon the orientation of the latter mode. The amplitude of the TS wave is then allowed to increase to the point at which it becomes nonlinear in its own right and amplitude equations for the evolution of the TS wave are obtained.

In addition, the problem of the interaction between a stronger, nonlinear vortex and a small TS wave is considered. It is concluded that for both the linear and the nonlinear vortex problems the effect of the vortex on the stability of the TS wave is felt primarily through the correction to the effective mean flow induced by the presence of the vortex.

The procedure in the remainder of the paper is follows. In Section 2 we formulate the problem and indicate the basic disturbance structure for the linear crossflow vortex. We impose a small TS disturbance on this configuration in Section 3 and we examine the neutral stability characteristics of this wave in the presence of the vortex. The TS wave is increased in size to become nonlinear in Section 4 and the problem with a larger, now nonlinear, vortex is considered in Section 5. Finally, we draw some conclusions in Section 6.

2. Formulation of the problem and the linear vortex structure

We consider the case in which the disc rotates about the z-axis with angular velocity Ω. Relative to cylindrical polar axes (r, θ, z) which rotate with the disc and in which r and z have been made dimensionless with respect to some reference lengthscale l, the continuity and Navier-Stokes equations for an incompressible fluid in the region $z \geq 0$ are

$$\nabla . \mathbf{u} = 0, \tag{2.1a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\,\mathbf{u} + 2\,(\boldsymbol{\Omega} \wedge \mathbf{u}) + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}. \tag{2.1b}$$

Here $\mathbf{u}$ is the velocity vector, $\mathbf{r}$ is the coordinate vector, ρ is the fluid density, p is the fluid pressure and ν the kinematic viscosity. The Reynolds number R for the flow is given by $R = \Omega l^2/\nu$ and is taken to be large throughout the following.

Anticipating the structures which govern the interactions described in the following sections it is found convenient to define the small parameter $\epsilon = R^{-\frac{1}{24}}$. As the axes rotate with the disc, the basic steady flow is given by the Von-Kármán solution

$$\mathbf{u} = \mathbf{u_B} = l\Omega\left(r\bar{u}(\xi), r\bar{v}(\xi), \epsilon^{12}\bar{w}(\xi)\right), \qquad p = \rho l^2 \Omega^2 \epsilon^{24} \bar{p}(\xi), \tag{2.2}$$

where $z = \epsilon^{12}\xi$ and $\bar{u}, \bar{v}, \bar{w}$ satisfy the equations

$$\bar{u}^2 - (1+\bar{v})^2 + \bar{u}'\bar{w} = \bar{u}'', \qquad 2\bar{u}\,(1+\bar{v}) + \bar{v}'\bar{w} = \bar{v}'', \tag{2.3a, b}$$

$$\bar{w}' + 2\bar{u} = 0, \qquad \bar{p}' + \bar{w}\bar{w}' - \bar{w}'' = 0. \tag{2.3c, d}$$

Here primes denote differentiation with respect to ξ and the appropriate boundary conditions are

$$\begin{aligned} &\bar{u} = \bar{v} = \bar{w} = 0 \qquad \text{on} \quad \xi = 0, \\ &\bar{u} \longrightarrow 0, \quad \bar{v} \longrightarrow -1 \qquad \text{as} \quad \xi \longrightarrow \infty. \end{aligned} \tag{2.3e}$$

We shall be concerned with perturbations to the basic flow and it is convenient to now derive the equations governing these disturbances. If the steady solution is perturbed by writing

$$\mathbf{u} = \mathbf{u_B} + l\Omega\,(U, V, W)\,, \qquad p = \rho l^2 \Omega^2\,(\bar{p} + P)\,, \tag{2.4}$$

where U, V, W, P are small three-dimensional disturbances and if expressions (2.4) are substituted into the governing equations (2.1) for the flow in the rotating frame, we then obtain the following perturbation equations:

$$\frac{U}{r} + \frac{\partial U}{\partial r} + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial z} = 0, \tag{2.5a}$$

$$\begin{gathered}\left(\frac{\partial}{\partial t} + r\bar{u}\frac{\partial}{\partial r} + \bar{v}\frac{\partial}{\partial \theta} + \epsilon^{12}\bar{w}\frac{\partial}{\partial z}\right) U + \bar{u}U + rW\frac{d\bar{u}}{dz} - 2(1+\bar{v})V - \frac{V^2}{r} \\ + \left(U\frac{\partial}{\partial r} + \frac{V}{r}\frac{\partial}{\partial \theta} + W\frac{\partial}{\partial z}\right) U = -\frac{\partial P}{\partial r} + \frac{1}{R}\left(L(U) - \frac{U}{r^2} - \frac{2}{r^2}\frac{\partial V}{\partial \theta}\right),\end{gathered} \tag{2.5b}$$

$$\begin{gathered}\left(\frac{\partial}{\partial t} + r\bar{u}\frac{\partial}{\partial r} + \bar{v}\frac{\partial}{\partial \theta} + \epsilon^{12}\bar{w}\frac{\partial}{\partial z}\right) V + \bar{u}V + rW\frac{d\bar{v}}{dz} + 2(1+\bar{v})U + \frac{UV}{r} \\ + \left(U\frac{\partial}{\partial r} + \frac{V}{r}\frac{\partial}{\partial \theta} + W\frac{\partial}{\partial z}\right) V = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{1}{R}\left(L(V) - \frac{V}{r^2} + \frac{2}{r^2}\frac{\partial U}{\partial \theta}\right),\end{gathered} \tag{2.5c}$$

$$\begin{gathered}\left(\frac{\partial}{\partial t} + r\bar{u}\frac{\partial}{\partial r} + \bar{v}\frac{\partial}{\partial \theta} + \epsilon^{12}\bar{w}\frac{\partial}{\partial z}\right) W + \epsilon^{12}W\frac{d\bar{w}}{dz} \\ + \left(U\frac{\partial}{\partial r} + \frac{V}{r}\frac{\partial}{\partial \theta} + W\frac{\partial}{\partial z}\right) W = -\frac{\partial P}{\partial z} + \frac{1}{R}L(W).\end{gathered} \tag{2.5d}$$

Here,

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

2.1 The linear vortex structure

Since we will be concerned with the effect of the vortex on a small TS wave which, in the first instance, will be taken to be of such size so as to have an infinitesimal influence on the vortex, we first need to consider the structure of the vortex when the TS wave is absent. Following the work of Gregory et al (1955), the inviscid vortex has wavelengths scaled on the boundary layer thickness and so we consider disturbance quantities with r and θ dependence given by E, defined by

$$E \equiv \exp\left[\frac{i}{\epsilon^{12}}\left\{\int^{r} \alpha(r,\epsilon)dr + \theta\beta(\epsilon)\right\}\right]. \tag{2.6a}$$

We shall restrict our attention to examining disturbances in a neighbourhood of some point $r = r_n$ and expand α and β in the forms

$$\alpha = \alpha_0 + \epsilon^4\alpha_1 + \dots, \qquad \beta = \beta_0 + \epsilon^4\beta_1 + \dots. \tag{2.6b}$$

where $\gamma_0^2 \equiv \alpha_0^2 + \frac{\beta_0^2}{r_n^2}$ is the effective wavenumber of the disturbance. Thus W_{10} satisfies a Rayleigh equation and we obtain stationary modes by demanding that the point of inflexion of the effective basic flow coincides with the point at which this flow vanishes, say at $\xi = \bar{\xi}$. This criterion, together with (2.9) and its appropriate boundary conditions that $W_{10} \longrightarrow 0$ as $\xi \longrightarrow 0$ and as $\xi \longrightarrow \infty$ yields the values for α_0 and β_0. The solution of this eigenproblem gives

$$\bar{\xi} = 1.46, \qquad \gamma_0 = 1.16, \qquad \frac{\alpha_0}{\beta_0} = \frac{4.26}{r_n}. \tag{2.10}$$

In this inviscid zone it is also found that the mean flow quantities U_{30}, V_{30}, W_{30}, P_{30} satisfy the equations

$$2U_{30} + W_{30}' = 0, \tag{2.11a}$$

$$U_{30}'' - \bar{w}U_{30}' - 2\bar{u}U_{30} - \bar{u}'W_{30} + 2(1+\bar{v})V_{30} = 0, \tag{2.11b}$$

$$V_{30}'' - \bar{w}V_{30}' - 2\bar{u}V_{30} - \bar{v}'W_{30} - 2(1+\bar{v})U_{30} = 0, \tag{2.11c}$$

$$P_{30}' = -2\left(|W_{30}|^2\right)'. \tag{2.11d}$$

To determine these correction terms we need to consider explicitly both the wall layer and the critical layer structure (at $\xi = \bar{\xi}$). In the wall layer where we define the O(1) co-ordinate $\bar{Z}$ by $\xi = \epsilon^4\bar{Z}$, the total velocity and pressure fields assume the forms

$$u = r_n(\bar{u}')_{\xi=0}\epsilon^4\bar{Z} + \ldots + \delta\left[(U_{10}^{\dagger} + \ldots)E + c.c.\right]$$
$$+\delta^2\left[(U_{20}^{\dagger} + \ldots)E^2 + c.c. + r_n\epsilon^{-8}(U_{30}^{\dagger} + \ldots)\right] + \ldots, \tag{2.12a}$$

$$v = r_n(\bar{v}')_{\xi=0}\epsilon^4\bar{Z} + \ldots + \delta\left[(V_{10}^{\dagger} + \ldots)E + c.c.\right]$$
$$+\delta^2\left[(V_{20}^{\dagger} + \ldots)E^2 + c.c. + r_n\epsilon^{-8}(V_{30}^{\dagger} + \ldots)\right] + \ldots, \tag{2.12b}$$

$$w = \frac{1}{2}(\bar{w}'')_{\xi=0}\epsilon^{20}\bar{Z}^2 + \ldots$$
$$+\delta\left[\epsilon^4(W_{10}^{\dagger} + \ldots)E + c.c.\right] \tag{2.12c}$$
$$+\delta^2\left[\epsilon^4(W_{20}^{\dagger} + \ldots)E^2 + c.c. + \epsilon^8(W_{30}^{\dagger} + \ldots)\right] + \ldots,$$

$$p = \epsilon^{24}(\bar{p})_{\xi=0} + \ldots + \delta\left[\epsilon^4(P_{10}^{\dagger} + \ldots)E + c.c.\right]$$
$$+\delta^2\left[\epsilon^4(P_{20}^{\dagger} + \ldots)E^2 + c.c. + (P_{30}^{\dagger} + \ldots)\right] + \ldots. \tag{2.12d}$$

In general, the wavenumbers will be complex quantities but we shall only consider neutral disturbances and so take $\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1 \ldots$ to be real. The disturbance structure in the z-direction is as described in Hall (1986), Gajjar (1989). There is an inviscid layer of $O(\epsilon^{12})$, (the same depth as the boundary layer), and to satisfy no-slip conditions at the wall a viscous layer of thickness $O(\epsilon^{16})$ must be present. In contrast to the earlier work we need to explicitly consider the mean flow correction induced by the vortex because this correction plays a crucial role in the interaction problems to be described in subsequent sections. We find that in the inviscid zone the velocity and pressure fields assume the forms

$$\begin{aligned} u = r_n\bar{u}(\xi) &+ \delta\left[\left(U_{10} + \epsilon^4 U_{11} + \ldots\right)E + c.c.\right] \\ &+ \delta^2\left[\left(U_{20} + \ldots\right)E^2 + c.c. + r_n\epsilon^{-12}\left(U_{30} + \ldots\right)\right] + O(\delta^3), \end{aligned} \tag{2.7a}$$

$$\begin{aligned} v = r_n\bar{v}(\xi) &+ \delta\left[\left(V_{10} + \epsilon^4 V_{11} + \ldots\right)E + c.c.\right] \\ &+ \delta^2\left[\left(V_{20} + \ldots\right)E^2 + c.c. + r_n\epsilon^{-12}\left(V_{30} + \ldots\right)\right] + O(\delta^3), \end{aligned} \tag{2.7b}$$

$$\begin{aligned} w = \epsilon^{12}\bar{w}(\xi) &+ \delta\left[\left(W_{10} + \epsilon^4 W_{11} + \ldots\right)E + c.c.\right] \\ &+ \delta^2\left[\left(W_{20} + \ldots\right)E^2 + c.c. + \left(W_{30} + \ldots\right)\right] + O(\delta^3), \end{aligned} \tag{2.7c}$$

$$\begin{aligned} p = \epsilon^{24}\bar{p}(\xi) &+ \delta\left[\left(P_{10} + \epsilon^4 P_{11} + \ldots\right)E + c.c.\right] \\ &+ \delta^2\left[\left(P_{20} + \ldots\right)E^2 + c.c. + \left(P_{30} + \ldots\right)\right] + O(\delta^3). \end{aligned} \tag{2.7d}$$

Here $\delta \ll 1$ is the infinitesimal vortex amplitude, the disturbance quantities $U_{10}, U_{11}, \ldots, U_{20}, U_{30}$ etc. are functions of ξ and r_n and $c.c.$ denotes complex conjugate. We substitute (2.7) into (2.5) and find that the fundamental vortex terms satisfy

$$i\left(\alpha_0 U_{10} + \frac{\beta_0}{r_n}V_{10}\right) + W_{10}' = 0, \tag{2.8a}$$

$$iU_{0B}U_{10} + r_n\bar{u}'W_{10} = -i\alpha_0 P_{10}, \tag{2.8b}$$

$$iU_{0B}V_{10} + r_n\bar{v}'W_{10} = -i\frac{\beta_0}{r_n}P_{10}, \tag{2.8c}$$

$$iU_{0B}W_{10} = -P_{10}', \tag{2.8d}$$

where $U_{0B} \equiv r_n\alpha_0\bar{u} + \beta_0\bar{v}$ is the effective basic velocity profile. Eliminating U_{10}, V_{10} and P_{10} from these equations yields

$$U_{0B}\left(W_{10}'' - \gamma_0^2 W_{10}\right) = U_{0B}''W_{10}, \tag{2.9}$$

Substituting these expansions into (2.5) shows that $U_{10}^{\dagger}, V_{10}^{\dagger}, W_{10}^{\dagger}, P_{10}^{\dagger}$ satisfy the equations given in Hall (1986) for the wall layer quantities. The solution of these equations ultimately leads to the determination of the values of α_1 and β_1 in (2.6). However, this is not of primary interest here. We remark that the quantities $U_{30}^{\dagger}, V_{30}^{\dagger}, W_{30}^{\dagger}$ are proportional to $\bar{Z}, \bar{Z}$ and $\bar{Z}^2$ respectively, and hence the boundary conditions at $\xi = 0$ for the mean-flow disturbance quantities U_{30}, V_{30}, W_{30} (given by (2.11) above) are that

$$U_{30} = V_{30} = W_{30} = 0 \qquad \text{on} \quad \xi = 0. \tag{2.13}$$

Clearly, we also require that the mean flow quantities decay as we leave the boundary layer so

$$U_{30}, V_{30} \quad \longrightarrow 0 \qquad \text{as} \quad \xi \longrightarrow \infty. \tag{2.14}$$

However, we also need to determine the effect of the critical layer at $\xi = \bar{\xi}$ on these mean flow terms. From (2.8) we see that as $\xi \longrightarrow \bar{\xi}$, $U_{0B} = O(\xi - \bar{\xi})$ and singularities exist in the inviscid zone solutions U_{10} and V_{10}. To smooth these singularities we invoke viscous effects by considering a zone of thickness $O(\epsilon^{16})$ surrounding $\xi = \bar{\xi}$. In this region we suppose that $\xi = \bar{\xi} + \epsilon^4 \hat{z}$ where $\hat{z}$ is O(1) and then the velocity and pressure fields are

$$\begin{aligned} u = &\left(r_n \bar{u}_{\bar{\xi}} + \epsilon^4 r_n \bar{u}'_{\bar{\xi}} \hat{z} + \frac{1}{2} \epsilon^8 r_n \bar{u}''_{\bar{\xi}} \hat{z}^2 + \ldots \right) + \\ &\delta \left[\left(\epsilon^{-4} \bar{U}_{10} + \bar{U}_{11} + \epsilon^4 \bar{U}_{12} + \ldots \right) E + c.c \right] \\ &+ \delta^2 \left[r_n \epsilon^{-12} \left(\bar{U}_{30} + \epsilon^4 \bar{U}_{31} + \ldots \right) + \ldots \right], \end{aligned} \tag{2.15a}$$

$$\begin{aligned} v = &\left(r_n \bar{v}_{\bar{\xi}} + \epsilon^4 r_n \bar{v}'_{\bar{\xi}} \hat{z} + \frac{1}{2} \epsilon^8 r_n \bar{v}''_{\bar{\xi}} \hat{z}^2 + \ldots \right) + \\ &\delta \left[\left(\epsilon^{-4} \bar{V}_{10} + \bar{V}_{11} + \epsilon^4 \bar{V}_{12} + \ldots \right) E + c.c. \right] \\ &+ \delta^2 \left[r_n \epsilon^{-12} \left(\bar{V}_{30} + \epsilon^4 \bar{V}_{31} + \ldots \right) + \ldots \right], \end{aligned} \tag{2.15b}$$

$$\begin{aligned} w = &\epsilon^{12} \left(\bar{w}_{\bar{\xi}} + \epsilon^4 \bar{w}'_{\bar{\xi}} \hat{z} + \ldots \right) + \delta \left[\left(\bar{W}_{10} + \epsilon^4 \bar{W}_{11} + \epsilon^8 \bar{W}_{12} + \ldots \right) E + c.c. \right] \\ &+ \delta^2 \left[\left(\bar{W}_{30} + \epsilon^4 \bar{W}_{31} + \ldots \right) + \ldots \right], \end{aligned} \tag{2.15c}$$

$$\begin{aligned} p = &\epsilon^{24} \bar{p}(\bar{\xi}) + \ldots + \delta \left[\left(\bar{P}_{10} + \epsilon^4 \bar{P}_{11} + \epsilon^8 \bar{P}_{12} + \ldots \right) E + c.c. \right] \\ &+ \delta^2 \left[\left(\bar{P}_{30} + \epsilon^4 \bar{P}_{31} + \ldots \right) + \ldots \right]. \end{aligned} \tag{2.15d}$$

The solutions for the fundamental terms are well known. Substituting these expansions into (2.5) and recalling the definitions of U_{10}, V_{10}, W_{10} and P_{10} we find that

$$\bar{W}_{10} = W_{10}(\bar{\xi}), \qquad \bar{P}_{10} = P_{10}(\bar{\xi}) = \frac{iB_{01}}{\gamma_0^2} W_{10}(\bar{\xi}),$$

where we have defined $B_{jk} \equiv r_n \alpha_j \bar{u}^{(k)}(\bar{\xi}) + \beta_j \bar{v}^{(k)}(\bar{\xi})$.

¿From continuity we find that $\bar{V}_{10} = -\frac{r_n \alpha_0}{\beta_0} \bar{U}_{10}$ and equation (2.5b) implies that

$$\frac{d^2 \bar{U}_{10}}{d\hat{z}^2} - i(B_{01}\hat{z} + B_{10})\bar{U}_{10} = \frac{\beta_0}{r_n \gamma_0^2}(\beta_0 \bar{u}'_{\bar{\xi}} - r_n \alpha_0 \bar{v}'_{\bar{\xi}}) W_{10}(\bar{\xi}). \tag{2.16}$$

To match with the solutions in the inviscid zone requires that $\bar{U}_{10} = O(1/\hat{z})$ as $|\hat{z}| \longrightarrow \infty$ and the solution of (2.16) with this property is

$$\bar{U}_{10} = \frac{\beta_0}{r_n \gamma_0^2 B_{01}}(\beta_0 \bar{u}'_{\bar{\xi}} - r_n \alpha_0 \bar{v}'_{\bar{\xi}}) W_{10}(\bar{\xi}) \int_0^\infty \exp\left[i\tau\left(\hat{z} + \frac{B_{10}}{B_{01}}\right)\right] \exp\left(\frac{\tau^3}{3B_{01}}\right) d\tau, \tag{2.17}$$

where we remark that $B_{01} < 0$.

Turning to the equations for the mean flow terms in (2.15) we find that $\bar{W}_{30}$ is constant across the critical layer, but that $\bar{U}_{30}$ satisfies

$$\frac{d\bar{U}_{30}}{d\hat{z}} = \frac{1}{r_n}\left(\bar{W}_{10}^* \bar{U}_{10} + \bar{W}_{10} \bar{U}_{10}^*\right),$$

or,

$$\bar{U}_{30} = \frac{2}{r_n} \int_{-\infty}^{\hat{z}} Re(\bar{U}_{10} \bar{W}_{10}^*) dt + \Lambda, \tag{2.18}$$

where Λ is some constant and an asterisk on a quantity denotes the complex conjugate of that quantity. It is apparent from (2.18) that, unless $\int_{-\infty}^{\infty} Re(\bar{U}_{10}\bar{W}_{10}^*)dt = 0$, the mean flow term U_{30} in the inviscid zone must suffer a discontinuity across the critical layer. We may determine this jump by multiplying (2.17) by $\bar{W}_{10}^*$, integrating with respect to $\hat{z}$ and interchanging the order of integration. The resulting integral assumes a form of the type

$$J(\hat{z}) = \int_0^\infty M(t)\left(\frac{e^{-i\hat{z}t} - 1}{t}\right) dt, \tag{2.19}$$

and the behaviour of $J(\hat{z})$ for large $|\hat{z}|$ is well known, see Haberman (1976). After some algebra, taking real parts and inserting in (2.18), we retrieve the necessary discontinuity across the critical layer,

$$\left[\bar{U}_{30}(\hat{z})\right]_{-\infty}^{\infty} = \frac{2\pi\beta_0(\beta_0\bar{u}'_{\bar{\xi}} - r_n\alpha_0\bar{v}'_{\bar{\xi}})}{\gamma_0^2 r_n^2 B_{01}}|W_{10}(\bar{\xi})|^2. \tag{2.20a}$$

We also find from the governing equations that $\bar{V}_{30} = -r_n\alpha_0\bar{U}_{30}/\beta_0$ so that

$$\left[\bar{V}_{30}(\hat{z})\right]_{-\infty}^{\infty} = -\frac{r_n\alpha_0}{\beta_0}\left[\bar{U}_{30}(\hat{z})\right]_{-\infty}^{\infty}. \tag{2.20b}$$

Moving on to higher order terms in $(2.15a-d)$ reveals that the term $d\bar{U}_{31}/d\hat{z}$ also suffers a non-zero jump across the critical layer which means a discontinuity in the derivative of the mean flow term U_{30} (in the main boundary layer) across $\xi = \bar{\xi}$. Repeating the procedure described above we find that

$$\left[\frac{d\bar{U}_{31}}{d\hat{z}}\right]_{-\infty}^{\infty} = \bar{w}_{\bar{\xi}}\left[\bar{U}_{30}\right]_{-\infty}^{\infty}, \tag{2.20c}$$

and

$$\left[\frac{d\bar{V}_{31}}{d\hat{z}}\right]_{-\infty}^{\infty} = \bar{w}_{\bar{\xi}}\left[\bar{V}_{30}\right]_{-\infty}^{\infty}. \tag{2.20d}$$

To conclude this section, we note that the presence of the vortex has induced a mean flow correction term of size $O(\delta^2\epsilon^{-12})$ across the whole of the boundary layer. This correction is governed by equations (2.11) with boundary conditions (2.13), (2.14), and jump conditions hold across $\xi = \bar{\xi}$, which are given by

$$\begin{aligned} \left[U_{30}(\xi)\right]_{\bar{\xi}-}^{\bar{\xi}+} &= \frac{2\pi\beta_0(\beta_0\bar{u}'_{\bar{\xi}} - r_n\alpha_0\bar{v}'_{\bar{\xi}})}{\gamma_0^2 r_n^2 B_{01}}|W_{10}(\bar{\xi})|^2, \\ \left[V_{30}(\xi)\right]_{\bar{\xi}-}^{\bar{\xi}+} &= -\frac{r_n\alpha_0}{\beta_0}\left[U_{30}(\xi)\right]_{\bar{\xi}-}^{\bar{\xi}+}, \end{aligned} \tag{2.21a,b}$$

$$\left[\frac{dU_{30}}{d\xi}\right]_{\bar{\xi}-}^{\bar{\xi}+} = \bar{w}_{\bar{\xi}}\left[U_{30}\right]_{\bar{\xi}-}^{\bar{\xi}+}, \qquad \left[\frac{dV_{30}}{d\xi}\right]_{\bar{\xi}-}^{\bar{\xi}+} = \bar{w}_{\bar{\xi}}\left[V_{30}\right]_{\bar{\xi}-}^{\bar{\xi}+}. \tag{2.21c,d}$$

We have concentrated on the properties of the mean flow correction terms because when we consider the interaction of the vortex with the TS wave in subsequent sections, it is found that the effect of the vortex on the TS wave is felt primarily through this correction to the mean flow.

We solved the differential system (2.11), (2.13), (2.14), (2.21) using a fourth order Runge-Kutta method combined with a shooting technique. Of importance is the sign of the discontinuity appearing in (2.21a) and it was found that this term is negative. Clearly, the scaled size of the vortex affects the magnitude of the discontinuities in (2.21) and in Figure (1) we illustrate the solutions for the corrections U_{30}, V_{30} and W_{30} when $[U_{30}(\xi)]_{\xi-}^{\xi+} = -1$. We find in this case that $U'_{30}(\xi = 0) = -0.86 \times 10^{-2}$ and $V'_{30}(\xi = 0) = -0.896$. We now consider the effect of adding a small TS wave into the flow.

3. Interaction of a linear vortex with a small TS wave

The lower-branch TS waves we shall concern ourselves with here are the classical type described by a triple deck structure. The waves have $O(\epsilon^{-9})$ wavenumbers (and hence wavelengths much larger than the $O(\epsilon^{12})$ thickness of the boundary layer) and small, $O(\epsilon^3)$, wavespeeds. The details of the triple deck may be found in, for example, Smith (1979a), but we note here that the three decks are of thicknesses $O(\epsilon^9)$, $O(\epsilon^{12})$ and $O(\epsilon^{15})$ in the direction normal to the disc. The first of these decks is a region of potential flow, the middle deck is primarily inviscid but rotational in character and the thin wall layer is viscous.

We shall suppose in the first instance that the size of the TS wave is extremely small, say $\Delta(\ll \delta)$ where we recall that δ is the infinitesimal vortex size. In this case, at leading orders at least, the vortex structure described in the previous section is unaffected by the presence of the TS wave. We have already shown that the vortex induces a correction of size $O(\delta^2 R^{\frac{1}{2}})$ to the mean flow throughout the boundary layer and we anticipate that there are essentially two mechanisms which could be responsible for altering the neutral stability characteristics of the TS wave. Firstly, we have the modification to the mean flow due to the vortex and, secondly, the TS wave will interact with the vortex terms to produce further modes which in turn interact to affect the TS and vortex instabilities. Of crucial importance for the ensuing study is the question as to which of these processes is the more important as far as the TS wave is concerned. We find after analysis that the former mechanism (the mean flow change induced by the vortex) has the dominating effect and this simplifies the following calculations.

As the vortex induces a relative change of $O(\delta^2\epsilon^{-12})$ in the size of the mean flow we expect a similar effect on the neutral wavenumbers and wavespeed for the TS wave. Anticipating this result we seek TS disturbance quantities with r and θ dependence given by F, defined by

$$F \equiv \exp\left[\frac{i}{\epsilon^9}\left\{\int^r \hat{\alpha}(r,\epsilon)dr + \theta\hat{\beta}(\epsilon) - \hat{\Omega}(\epsilon)t\right\}\right], \tag{3.1a}$$

where,

$$\hat{\alpha} = (\hat{\alpha}_0 + \ldots) + \delta^2\epsilon^{-12}(\hat{\alpha}_{00} + \ldots), \qquad \hat{\beta} = (\hat{\beta}_0 + \ldots) + \delta^2\epsilon^{-12}(\hat{\beta}_{00} + \ldots),$$
$$\hat{\Omega} = \epsilon^3(\hat{\Omega}_0 + \ldots) + \delta^2\epsilon^{-9}(\hat{\Omega}_{00} + \ldots). \tag{3.1b}$$

We see that the perturbed wavenumbers $\hat{\alpha}_{00}, \hat{\beta}_{00}$ and the perturbed frequency $\hat{\Omega}_{00}$ measure the effect of the vortex on the neutral stability of the TS wave. Of course there are terms in the wavenumber and frequency expansions not explicitly detailed in (3.1) and larger than the $\hat{\alpha}_{00}, \hat{\beta}_{00}$ and $\hat{\Omega}_{00}$ corrections, but these terms are independent of the vortex properties whereas our aim here is to determine this vortex effect. Hence we shall merely seek expressions for the values of $\hat{\alpha}_{00}, \hat{\beta}_{00}$ and $\hat{\Omega}_{00}$.

The fundamental vortex terms and the TS wave interact to produce instabilities with spatial dependences of the forms EF, $E^{-1}F$ etc. These waves have a structure similar to that for the crossflow vortex but with a more complicated critical layer configuration surrounding $\xi = \bar{\xi}$. Indeed, here these waves have two critical layers separated by an $O(\epsilon^{15})$ distance and each of thickness $O(\epsilon^{16})$. There is the potential for the interaction of these modes with the vortex to drive the TS wave but, as already mentioned, this interaction is of lesser importance than the mean flow modification due to the presence of the vortex. To simplify the analysis below we will hence indicate the sizes of these 'mixed' modes in the various layers but shall not need to give the full details of these modes.

Firstly, we again concentrate on the main boundary layer where $z = \epsilon^{12}\xi$, and we modify expansions $(2.7a-d)$ to consider

$$u = r_n\bar{u}(\xi) + \delta\left[(U_{10} + \ldots)E + c.c.\right] + \delta^2\left[r_n\epsilon^{-12}(U_{30} + \ldots) + \ldots\right] + \ldots$$
$$+ \Delta\left[(\hat{U}_{01} + \ldots)F + O(\delta\epsilon^{-3}E^{\pm 1}F) + \delta^2\epsilon^{-12}\left(\hat{U}_{21}F + \ldots\right) + c.c\right] + \ldots, \tag{3.2a}$$

$$v = r_n\bar{v}(\xi) + \delta\left[(V_{10} + \ldots)E + c.c.\right] + \delta^2\left[r_n\epsilon^{-12}(V_{30} + \ldots) + \ldots\right] + \ldots$$
$$+ \Delta\left[(\hat{V}_{01} + \ldots)F + O(\delta_\epsilon^{-3}E^{\pm}F) + \delta^2\epsilon^{-12}(\hat{V}_{21}F + \ldots) + c.c\right] + \ldots, \tag{3.2b}$$

$$w = \epsilon^{12}\bar{w}(\xi) + \delta\left[(W_{10}+\ldots)E + c.c.\right] + \delta^2\left[(W_{30}+\ldots)+\ldots\right] + \ldots + \Delta \left[\epsilon^3(\hat{W}_{01}+\ldots)F + O(\delta\epsilon^{-3}E^{\pm 1}F) + \delta^2\epsilon^{-9}\left(\hat{W}_{21}F+\ldots\right) + c.c\right] + \ldots, \tag{3.2c}$$

$$p = \epsilon^{24}\bar{p}(\xi) + \delta\left[(P_{10}+\ldots)E + c.c.\right] + \delta^2\left[(P_{30}+\ldots)+\ldots\right] + \ldots + \Delta \left[\epsilon^3(\hat{P}_{01}+\ldots)F + O(\delta\epsilon^{-3}E^{\pm 1}F) + \delta^2\epsilon^{-9}\left(\hat{P}_{21}F+\ldots\right) + c.c\right] + \ldots. \tag{3.2d}$$

Here the vortex terms $(U_{10},\ldots,P_{10})$ and mean flow corrections $(U_{30},\ldots,P_{30})$ satisfy the equations previously given in (2.8), (2.11). Inserting (3.2) in the governing equations (2.5) leads to the usual TS solutions

$$\hat{U}_{01} = \hat{A}_0\bar{u}', \quad \hat{V}_{01} = \hat{A}_0\bar{v}', \quad \hat{W}_{01} = -\frac{i\hat{A}_0}{r_n}\hat{U}_{0B}, \quad \hat{P}_{01} = const., \tag{3.3}$$

where $\hat{A}_0$ denotes an unknown displacement and we have defined $\hat{U}_{jB} \equiv r_n\hat{\alpha}_j\bar{u} + \hat{\beta}_j\bar{v}$. We also find that

$$\hat{U}_{21} = \hat{A}_0U_{30}' + \hat{B}_0\bar{u}', \qquad \hat{V}_{21} = \hat{A}_0V_{30}' + \hat{B}_0\bar{v}',$$

$$\hat{W}_{21} = -\frac{i\hat{A}_0}{r_n}\left(\hat{U}_{00B} + r_n\hat{\alpha}_0U_{30} + \hat{\beta}_0V_{30}\right) - \frac{i\hat{B}_0}{r_n}\hat{U}_{0B}, \qquad \hat{P}_{21} = const., \tag{3.4}$$

where $\hat{B}_0$ is another constant and $\hat{U}_{00B} \equiv r_n\hat{\alpha}_{00}\bar{u} + \hat{\beta}_{00}\bar{v}$.

For the present we assume that these solutions are valid for $\xi > \bar{\xi}$; i.e. above the critical layer. An explicit analysis of the critical layer is required to determine how these solutions need to be modified for $\xi < \bar{\xi}$.

Moving into the upper deck where $z = \epsilon^9\bar{y}$ and $\bar{y}$=O(1) say, the vortex decays exponentially and the expansions are

$$u = \Delta\left[\epsilon^3\left(\hat{\bar{u}}_0+\ldots\right)F + \delta^2\epsilon^{-9}\left(\hat{\bar{u}}_1+\ldots\right)F + c.c.\right], \tag{3.5a}$$

$$v = -r_n + \ldots + \Delta\left[\epsilon^3\left(\hat{\bar{v}}_0+\ldots\right)F + \delta^2\epsilon^{-9}\left(\hat{\bar{v}}_1+\ldots\right)F + c.c.\right], \tag{3.5b}$$

$$w = \epsilon^{12}\bar{w}(\infty) + \delta^2W_{30}(\infty) + \ldots + \Delta\left[\epsilon^3\left(\hat{\bar{w}}_0+\ldots\right)F + \delta^2\epsilon^{-9}\left(\hat{\bar{w}}_1+\ldots\right)F + c.c.\right], \tag{3.5c}$$

$$p = \epsilon^{24}\bar{p}(\infty) + \Delta\left[\epsilon^3\left(\hat{\bar{p}}_0+\ldots\right)F + \delta^2\epsilon^{-9}\left(\hat{\bar{p}}_1+\ldots\right)F + c.c.\right]. \tag{3.5d}$$

Feeding these expansions into (2.5) yields a straightforward upper-deck problem, see Smith (1979*b*) for example. We demand that all disturbance quantities decay as $\bar{y} \longrightarrow \infty$ and obtain matching conditions between the main deck and upper deck solutions. In particular, we find that

$$i(\hat{\alpha}_0\hat{\bar{u}}_0 + \frac{\hat{\beta}_0}{r_n}\hat{\bar{v}}_0) + \frac{\partial \hat{\bar{w}}_0}{\partial \bar{y}} = 0, \quad -i\hat{\beta}_0\hat{\bar{u}}_0 = -i\hat{\alpha}_0\hat{\bar{p}}_0, \quad \hat{\bar{v}}_0 = \frac{\hat{\bar{p}}_0}{r_n}, \quad -i\hat{\beta}_0\hat{\bar{w}}_0 = -\frac{\partial \hat{\bar{p}}_0}{\partial \bar{y}}.$$

We need solutions of this system which match with (3.3) as $\xi \longrightarrow \infty$, and this requires

$$\hat{P}_{01} = \frac{\hat{\beta}_0^2}{r_n\hat{\gamma}_0}\hat{A}_0, \tag{3.6a}$$

where $\hat{\gamma}_0^2 = \hat{\alpha}_0^2 + \frac{\hat{\beta}_0^2}{r_n^2}$. Considering the next order system, matching with (3.4) is achieved if

$$\hat{P}_{21} - \frac{\hat{\beta}_0^2}{r_n\hat{\gamma}_0}\hat{B}_0 = \left[\frac{2\hat{\beta}_{00}\hat{\gamma}_0^2 - \hat{\beta}_0\left(\hat{\alpha}_0\hat{\alpha}_{00} + \frac{\hat{\beta}_0\hat{\beta}_{00}}{r_n^2}\right)}{\hat{\beta}_0\hat{\gamma}_0^2}\right]\hat{P}_{01}. \tag{3.6b}$$

We need next to consider the critical layer where $z = \epsilon^{12}\bar{\xi} + \epsilon^{16}\hat{z}$, with $\hat{z}$=O(1). The main deck solutions (3.3) & (3.4) mean that inside the critical layer our fluid quantities develop as

$$\begin{aligned} u =& r_n(\bar{u}_{\bar{\xi}} + \bar{u}'_{\bar{\xi}}\epsilon^4\hat{z} + \ldots) + \delta\left[(\epsilon^{-4}\bar{U}_{10} + \ldots)E + c.c.\right] + \delta^2\left[r_n\epsilon^{-12}\bar{U}_{30} + \ldots\right] + \ldots \\ &+ \Delta\left[(\hat{\bar{U}}_{01} + \ldots)F + O(\delta^2\epsilon^{-8}E^{\pm1}F) + \delta^2(\epsilon^{-16}\hat{\bar{U}}_{21}F + \ldots) + c.c.\right], \end{aligned} \tag{3.7a}$$

$$\begin{aligned} v =& r_n(\bar{v}_{\bar{\xi}} + \bar{v}'_{\bar{\xi}}\epsilon^4\hat{z} + \ldots) + \delta\left[(\epsilon^{-4}\bar{V}_{10} + \ldots)E + c.c.\right] + \delta^2\left[r_n\epsilon^{-12}\bar{V}_{30} + \ldots\right] + \ldots \\ &+ \Delta\left[(\hat{\bar{V}}_{01} + \ldots)F + O(\delta^2\epsilon^{-8}E^{\pm1}F) + \delta^2(\epsilon^{-16}\hat{\bar{V}}_{21}F + \ldots) + c.c.\right], \end{aligned} \tag{3.7b}$$

$$\begin{aligned} w =& \epsilon^{12}(\bar{w}_{\bar{\xi}} + \ldots) + \delta\left[(\bar{W}_{10} + \ldots)E + c.c.\right] + \delta^2\left[\bar{W}_{30} + \ldots\right] + \ldots \\ &+ \Delta\left[\epsilon^3(\hat{\bar{W}}_{01} + \ldots)F + O(\delta^2\epsilon^{-3}E^{\pm1}F) + \delta^2(\epsilon^{-9}\hat{\bar{W}}_{21}F + \ldots) + c.c.\right], \end{aligned} \tag{3.7c}$$

$$\begin{aligned} p =& \epsilon^{24}\bar{p}(\bar{\xi}) + \ldots + \delta\left[(\bar{P}_{10} + \ldots)E + c.c.\right] + \delta^2\left[\bar{P}_{30} + \ldots\right] + \ldots \\ &+ \Delta\left[\epsilon^3(\hat{\bar{P}}_{01} + \ldots)F + O(\delta^2\epsilon^{-3}E^{\pm1}F) + \delta^2(\epsilon^{-9}\hat{\bar{P}}_{21}F + \ldots) + c.c.\right]. \end{aligned} \tag{3.7d}$$

The vortex terms $\bar{U}_{10}, \bar{V}_{10}, \bar{U}_{30}, \ldots$ are as in expansions $(2.15a-d)$ and follow from the critical layer analysis of the previous section. On using (3.7) in the governing equations (2.5) we find that the TS quantities are given by

$$\hat{\hat{U}}_{01} = \hat{A}_0 \bar{u}'_{\bar{\xi}}, \quad \hat{\hat{V}}_{01} = \hat{A}_0 \bar{v}'_{\bar{\xi}}, \quad \hat{\hat{W}}_{01} = -\frac{i\hat{A}_0}{r_n}\left(\hat{U}_{0B}\right)_{\bar{\xi}}, \quad \hat{\hat{P}}_{01} = \hat{P}_{01}, \tag{3.8a}$$

$$\begin{aligned} \hat{\hat{U}}_{21} &= \hat{A}_0 \frac{\partial \bar{U}_{30}}{\partial \hat{z}}, \qquad \hat{\hat{V}}_{21} = \hat{A}_0 \frac{\partial \bar{V}_{30}}{\partial \hat{z}}, \\ \hat{\hat{W}}_{21} &= -i\hat{A}_0 \left[\hat{\alpha}_0 \bar{U}_{30} + \frac{\hat{\beta}_0}{r_n}\bar{V}_{30}\right] + const., \qquad \hat{\hat{P}}_{21} = \hat{P}_{21}. \end{aligned} \tag{3.8b}$$

Matching these solutions with the main deck quantities (3.3), (3.4) (which we supposed to be valid for $\xi > \bar{\xi}$) shows that, for $\hat{z} \longrightarrow -\infty$, then (3.8) matches with (3.3) & (3.4) if these latter solutions are the main deck solutions for $\xi < \bar{\xi}$ as well. Hence, as far as the TS wave is concerned, the critical layer at $\xi = \bar{\xi}$ is a purely passive affair and (3.3) & (3.4) continue to be the valid main deck solutions below the critical layer. To complete the analysis for the TS wave we need to consider the thin viscous wall layer which is of thickness $O(\epsilon^{15})$. We notice that this TS lower deck is asymptotically much thicker than the $O(\epsilon^{16})$ sized wall layer for the vortex. If inside the lower deck we have $z = \epsilon^{15} Z$ with $Z = O(1)$, then the vortex quantities and the mean flow correction generated by the vortex are easily found by taking appropriate Taylor expansions about $\xi = 0$ for the vortex terms satisfying (2.8) and (2.11). In particular, in this wall layer the fluid quantities take the forms

$$\begin{aligned} u =& r_n \bar{u}'_0 \epsilon^3 Z + \ldots + \delta\left[(\hat{U}^{\ddagger}_{10} + \ldots)E + c.c.\right] + \delta^2\left[r_n \epsilon^{-9} \bar{u}'_m Z + \ldots\right] + \ldots \\ &+ \Delta\left[(U^{\dagger}_{01} + \ldots)F + O(\delta\epsilon^{-3}E^{\pm 1}F) + \delta^2(\epsilon^{-12}U^{\dagger}_{21} + \ldots)F + c.c\right] + \ldots, \end{aligned} \tag{3.9a}$$

$$\begin{aligned} v =& r_n \bar{v}'_0 \epsilon^3 Z + \ldots + \delta\left[(\hat{V}^{\ddagger}_{10} + \ldots)E + c.c.\right] + \delta^2\left[r_n \epsilon^{-9} \bar{v}'_m Z + \ldots\right] + \ldots \\ &+ \Delta\left[(V^{\dagger}_{01} + \ldots)F + O(\delta\epsilon^{-3}E^{\pm 1}F) + \delta^2(\epsilon^{-12}V^{\dagger}_{21} + \ldots)F + c.c\right] + \ldots, \end{aligned} \tag{3.9b}$$

$$\begin{aligned} w =& \bar{w}'_0 \epsilon^{15} Z + \ldots + \delta\left[\epsilon^3(\hat{W}^{\ddagger}_{10} + \ldots)E + c.c.\right] + \delta^2\left[\epsilon^6 \bar{w}''_m \frac{Z^2}{2} + \ldots\right] + \ldots \\ &+ \Delta\left[\epsilon^6(W^{\dagger}_{01} + \ldots)F + O(\delta E^{\pm 1}F) + \delta^2(\epsilon^{-6}W^{\dagger}_{21} + \ldots)F + c.c\right] + \ldots, \end{aligned} \tag{3.9c}$$

$$p = \epsilon^{24}\bar{p}(0) + \ldots + \delta\left[\epsilon^3(\hat{P}_{10}^{\ddagger} + \ldots)E + c.c.\right] + \delta^2\left[\bar{p}_m(0) + \ldots\right] + \ldots$$
$$+ \Delta\left[\epsilon^6(P_{01}^{\dagger} + \ldots)F + O(\delta E^{\pm 1}F) + \delta^2(\epsilon^{-6}P_{21}^{\dagger} + \ldots)F + c.c\right] + \ldots. \tag{3.9d}$$

Here $\bar{u}_0', \bar{v}_0', \bar{w}_0'$ denote the values of $\left(\bar{u}'(\xi)\right)_{\xi=0}, \left(\bar{v}'(\xi)\right)_{\xi=0}, \left(\bar{w}'(\xi)\right)_{\xi=0}$, where $\bar{u}, \bar{v}, \bar{w}$ satisfy (2.3), and similarly $\bar{u}_m', \bar{v}_m', \bar{w}_m''$ denote the values of the derivatives of the mean flow terms U_{30}, V_{30} and W_{30} (defined by (2.11), (2.13), (2.14) & (2.21)) evaluated at $\xi = 0$.

Substituting these expressions into (2.5) and comparing leading order terms in the TS disturbance quantities leads to the usual lower deck problem

$$i(\hat{\alpha}_0 U_{01}^{\dagger} + \frac{\hat{\beta}_0}{r_n} V_{01}^{\dagger}) + \frac{\partial W_{01}^{\dagger}}{\partial Z} = 0, \tag{3.10a}$$

$$i(\hat{\lambda}_0 Z - \hat{\Omega}_0)U_{01}^{\dagger} + r_n \bar{u}_0' W_{01}^{\dagger} = -i\hat{\alpha}_0 P_{01}^{\dagger} + \frac{\partial^2 U_{01}^{\dagger}}{\partial Z^2}, \tag{3.10b}$$

$$i(\hat{\lambda}_0 Z - \hat{\Omega}_0)V_{01}^{\dagger} + r_n \bar{v}_0' W_{01}^{\dagger} = -i\frac{\hat{\beta}_0}{r_n} P_{01}^{\dagger} + \frac{\partial^2 V_{01}^{\dagger}}{\partial Z^2}, \tag{3.10c}$$

$$P_{01}^{\dagger} = \hat{P}_{01}, \tag{3.10d}$$

with the necessary no-slip conditions $U_{01}^{\dagger} = V_{01}^{\dagger} = W_{01}^{\dagger} = 0$ at $Z = 0$ and the requirement that the solutions should match with the main deck solutions (3.3) as $Z \longrightarrow \infty$. Here we have denoted $\hat{\lambda}_0 = \hat{\alpha}_0 r_n \bar{u}_0' + \hat{\beta}_0 \bar{v}_0'$ and we solve (3.10) in the usual way. The unknowns $U_{01}^{\dagger}, V_{01}^{\dagger}$ are eliminated between $(3.10a - c)$ to yield an equation for $W_{01}^{\dagger}$, which may be solved analytically by making the substitutions

$$\Delta_* = i\hat{\lambda}_0, \qquad \eta = \Delta_*^{\frac{1}{3}}\left(Z - \frac{\hat{\Omega}_0}{\hat{\lambda}_0}\right), \qquad \eta_0 = -\frac{\Delta_*^{\frac{1}{3}}\hat{\Omega}_0}{\hat{\lambda}_0}. \tag{3.11}$$

With these definitions it is found that the no-slip conditions at $Z = 0$ together with the matching with (3.3) can only be achieved if

$$r_n \hat{\gamma}_0^2 \hat{P}_{01}\left(\int_{\eta_0}^{\infty} Ai(t)\,dt\right) = -i\hat{\lambda}_0 \Delta_*^{\frac{2}{3}} \hat{A}_0 Ai'(\eta_0), \tag{3.12a}$$

where Ai is the Airy function (see Abramowitz & Stegun 1964). Combining (3.12a) with (3.6a) yields the leading order eigenrelation,

$$\frac{\int_{\eta_0}^{\infty} Ai(t)\,dt}{Ai'(\eta_0)} = -\frac{\Delta_*^{\frac{5}{3}}}{\hat{\gamma}_0 \hat{\beta}_0^2}. \tag{3.12b}$$

It is well known that (3.12b) has solutions for real $\hat{\alpha}_0, \hat{\beta}_0$ and $\hat{\Omega}_0$ with

$$\frac{\hat{\Omega}_0}{\hat{\lambda}_0^{\frac{2}{3}}} = K_1 \equiv 2.297, \qquad \frac{\hat{\beta}_0^2 \hat{\gamma}_0}{\hat{\Omega}_0 \hat{\lambda}_0} = K_2 \equiv 0.4355, \tag{3.13}$$

see Smith (1979a, b). Since our TS waves are three-dimensional, (3.13) admits an infinity of solutions. It is convenient to eliminate the radial distance r_n by writing

$$\hat{\Omega}_d = r_n^{-\frac{1}{2}} \hat{\Omega}_0, \quad \hat{\alpha}_d = r_n^{\frac{1}{4}} \hat{\alpha}_0, \quad \hat{\beta}_d = r_n^{-\frac{3}{4}} \hat{\beta}_0, \quad \hat{\gamma}_d = r_n^{\frac{1}{4}} \hat{\gamma}_0, \quad \text{and} \quad \hat{\lambda}_d = r_n^{-\frac{3}{4}} \hat{\lambda}_0.$$

Then if $\mu = \frac{\hat{\beta}_d}{\hat{\alpha}_d}$ we obtain the solutions

$$\hat{\alpha}_d^{\frac{4}{3}} = \frac{1.001 \left(\bar{u}_0' + \mu \bar{v}_0' \right)^{\frac{5}{3}}}{\mu^2 \sqrt{1 + \mu^2}}, \quad \hat{\beta}_d = \mu \hat{\alpha}_d, \quad \hat{\Omega}_d = 2.297 \hat{\alpha}_d^{\frac{2}{3}} \left(\bar{u}_0' + \mu \bar{v}_0' \right)^{\frac{2}{3}}. \tag{3.14}$$

Here μ measures the angle ϕ which the TS wave makes with the radial direction on the disc, since $\phi = \tan^{-1} \mu$. The solutions $\hat{\alpha}_d, \hat{\beta}_d$ and $\hat{\Omega}_d$ are sketched in Figure (2). The solutions (3.14) are, strictly speaking, valid only for $\mu < \mu_c (= 0.8284)$ or, equivalently, for $\hat{\lambda}_d > 0$. The analysis outlined above has implicitly taken $\hat{\lambda}_d > 0$ although the modifications required for $\hat{\lambda}_d < 0$ are quite straightforward and stem from the consideration of the appropriate branches of the many-valued functions which arise.

We may derive equations satisfied by $U_{21}^{\dagger}, V_{21}^{\dagger}, W_{21}^{\dagger}$ & $P_{21}^{\dagger}$ by equating higher order terms when (3.9) is substituted in (2.5). We obtain an equation for $W_{21}^{\dagger}$ by eliminating $U_{21}^{\dagger}$ & $V_{21}^{\dagger}$ using a suitable combination of the r- and θ- momentum equations and the continuity equation, and we find that

$$\frac{\partial^4 W_{21}^{\dagger}}{\partial \eta^4} - \eta \frac{\partial^2 W_{21}^{\dagger}}{\partial \eta^2} = \frac{i(\hat{\lambda}_{00} + \hat{\lambda}_m) \hat{\gamma}_0^2 \hat{P}_{01}}{\Delta_*^2 Ai'(\eta_0)} \left[\eta + (\Phi - 1) \eta_0 \right] Ai(\eta), \tag{3.15}$$

where $\hat{\lambda}_{00} = \hat{\alpha}_{00} r_n \bar{u}_0' + \hat{\beta}_{00} \bar{v}_0'$, $\hat{\lambda}_m = \hat{\alpha}_0 r_n \bar{u}_m' + \hat{\beta}_0 \bar{v}_m'$, $\Phi = -\frac{\Delta_*^{\frac{1}{3}} \hat{\Omega}_{00}}{(\hat{\lambda}_{00} + \hat{\lambda}_m) \eta_0}$, and the substitutions (3.11) have been made. The solution of (3.15) which satisfies the no-slip conditions at $Z = 0$ and which matches with solutions (3.4) in the main deck provides a second relation between $\hat{P}_{21}$ and $\hat{B}_0$. Combining with the earlier result (3.6b) and using (3.6a), (3.12a) provides the eigenrelation for the correction terms

$\hat{\alpha}_{00}, \hat{\beta}_{00}$ and $\hat{\Omega}_{00}$;-

$$
\begin{aligned}
&-\frac{i\hat{\lambda}_0}{\hat{\gamma}_0\hat{\beta}_0^2}\left(\hat{\alpha}_0\hat{\alpha}_{00}+\frac{\hat{\beta}_0\hat{\beta}_{00}}{r_n^2}\right)-\frac{i(\hat{\lambda}_{00}+\hat{\lambda}_m)}{\hat{\lambda}_0^2}\hat{\gamma}_0^2\left[-\frac{2\hat{\lambda}_0}{3K_2\hat{\Omega}_0}\right.\\
&\left.+\frac{i^{\frac{2}{3}}Ai(\eta_0)}{Ai'(\eta_0)}\left(-\frac{2K_1\hat{\lambda}_0^{\frac{1}{3}}}{3}+\frac{\hat{\lambda}_0^{\frac{2}{3}}\hat{\Omega}_{00}}{(\hat{\lambda}_{00}+\hat{\lambda}_m)}+\frac{2K_1^2\hat{\lambda}_0}{3K_2\hat{\Omega}_0}-\frac{\hat{\lambda}_0^{\frac{4}{3}}K_1\hat{\Omega}_{00}}{(\hat{\lambda}_{00}+\hat{\lambda}_m)}\right)\right] \qquad (3.16)\\
&=-\frac{i\hat{\gamma}_0}{\hat{\beta}_0^2}(\hat{\lambda}_{00}+\hat{\lambda}_m)+\frac{2i\hat{\lambda}_0\hat{\gamma}_0\hat{\beta}_{00}}{\hat{\beta}_0^3}.
\end{aligned}
$$

This equation admits real solutions for $\hat{\alpha}_{00}, \hat{\beta}_{00}$ and $\hat{\Omega}_{00}$ only if the coefficient of $Ai(\eta_0)/Ai'(\eta_0)$ vanishes. Redefining $\hat{\alpha}_{00} = r_n^{-\frac{1}{4}}\hat{\alpha}_{00d}$ and $\hat{\beta}_{00} = r_n^{\frac{3}{4}}\hat{\beta}_{00d}$ implies that

$$
\hat{\alpha}_{00d} = M_1\hat{\beta}_{00d} + M_2, \qquad \hat{\Omega}_{00} = r_n^{\frac{1}{2}}\left[R_1\hat{\beta}_{00d} + R_2\right], \qquad (3.17a,b)
$$

where

$$
M_1 = \frac{\left(\frac{2\hat{\lambda}_d\hat{\gamma}_d}{\hat{\beta}_d^3} - \frac{5\hat{\gamma}_d^2\bar{v}_0'}{3\hat{\beta}_d^2} + \frac{\hat{\lambda}_d}{\hat{\gamma}_d\hat{\beta}_d}\right)}{\left(\frac{5\hat{\gamma}_d\bar{u}_0'}{3\hat{\beta}_d^2} - \frac{\hat{\lambda}_d\hat{\alpha}_d}{\hat{\gamma}_d\hat{\beta}_d^2}\right)}, \qquad M_2 = -\frac{5\hat{\gamma}_d}{3\hat{\beta}_d^2}\frac{(\hat{\alpha}_d\bar{u}_m' + \hat{\beta}_d\bar{v}_m')}{\left(\frac{5\hat{\gamma}_d\bar{u}_0'}{3\hat{\beta}_d^2} - \frac{\hat{\lambda}_d\hat{\alpha}_d}{\hat{\gamma}_d\hat{\beta}_d^2}\right)}, \qquad (3.17c,d)
$$

and

$$
R_1 = \frac{2K_1(M_1\bar{u}_0' + \bar{v}_0')}{3(\hat{\alpha}_d\bar{u}_0' + \hat{\beta}_d\bar{v}_0')^{\frac{1}{3}}}, \qquad R_2 = \frac{2K_1(\hat{\alpha}_d\bar{u}_m' + \hat{\beta}_d\bar{v}_m' + M_2\bar{u}_0')}{3(\hat{\alpha}_d\bar{u}_0' + \hat{\beta}_d\bar{v}_0')^{\frac{1}{3}}}. \qquad (3.17e,f)
$$

The dependences of M_1, M_2, R_1 & R_2 upon the parameter μ are illustrated in Figure (3). We see that depending on the chosen value of $\hat{\beta}_{00d}$, the neutral values for $\hat{\alpha}_{00d}$ and $\hat{\Omega}_{00d}$ can be of either sign. The results (3.17) indicate how the neutral curve for the TS wave is affected by the presence of the vortex and in the following section we demonstrate that the effect of the vortex is largely determined by the inclination of the TS wave ($\tan^{-1}\mu$) to the radial direction. In addition, we now consider the more realistic problem of allowing for a larger TS wave and indeed it is taken to be sufficiently large so as to be nonlinear itself.

4. The interaction of a linear vortex with a weakly nonlinear TS wave

Our aim is to increase the size of the TS wave until the mean flow correction produced by the self-interaction of the TS wave is as large as the mean flow correction produced by the vortex. This will ensure that at this stage the TS wave becomes nonlinear and this approach can be used to determine the stability of the TS wave to the vortex. We still assume that the vortex is small so that $\delta \ll 1$ and then analysis of the system described in Section 3 suggests that the crucial size of the TS wave when nonlinear effects become important is when

$$\Delta = \delta\epsilon^{-3}, \tag{4.1}$$

where Δ is the unscaled amplitude of the TS wave defined in (3.2). It is straightforward to check that at this TS size the analysis presented in Section 2 to determine the structure of the stationary vortex is not affected by interactions of the TS wave with either itself or the vortex. Consequently, to the orders required here the results of Section 2 & 3 for the vortex quantities continue to hold.

As is usual in the weakly nonlinear problem for a TS wave the nonlinearity manifests itself in the lower deck, whereas the main and upper deck solutions remain essentially linear in character. We slightly modify the method used in the previous section by obtaining an evolution equation for the TS amplitude in the vicinity of a chosen point rather than obtaining an equation for the neutral curve and we base our approach on that given by Hall & Smith (1982). If we consider TS waves close to the point (r_n, θ_n) we define the O(1) coordinate r_1 by

$$r = r_n + \delta^2\epsilon^{-12}r_1, \tag{4.2}$$

and allow the TS wave to evolve on the r_1 lengthscale. Consequently, we redefine F given by (3.1) by

$$F \equiv \exp\left[\frac{i}{\epsilon^9}\left\{\int^r \hat{\alpha}(r,\epsilon)dr + \theta\hat{\beta}(\epsilon) - \hat{\Omega}(\epsilon)t\right\}\right], \tag{4.3a}$$

where,

$$\begin{gathered}\hat{\alpha} = \hat{\alpha}_0, \qquad \hat{\beta} = (\hat{\beta}_0 + \ldots) + \delta^2\epsilon^{-12}(\hat{\beta}_{00} + \ldots), \\ \hat{\Omega} = \epsilon^3(\hat{\Omega}_0 + \ldots) + \delta^2\epsilon^{-9}(\hat{\Omega}_{00} + \ldots).\end{gathered} \tag{4.3b}$$

Here the vortex terms $\hat{U}_{10}^{\ddagger}, \ldots, \hat{P}_{10}^{\ddagger}$ again follow from suitable Taylor expansions of the solutions to (2.8) around $\xi = 0$. A significant difference between the expansions (4.5) and the corresponding (3.9) for the linear TS problem is that now the mean flow correction induced by the vortex in the lower deck is of the same size as that produced by the interaction of the TS wave with itself. Hence the terms $U_{mL}, \ldots, P_{mL}$ are no longer the simple Taylor expansions of the mean flows $(U_{30}, \ldots, P_{30})$ generated by the vortex interacting with itself in the main boundary layer and which satisfy (2.11). Since outside the boundary layer the TS wave self-interacts to produce mean flow terms of size only $O(\delta^2\epsilon^{-9})$, (i.e. the same size correction as in the lower deck) we notice that equations (2.11) are still relevant, together with boundary conditions (2.13), (2.14) and jump conditions (2.21). Hence the leading order mean-flow terms across the whole boundary layer are unaffected by the presence of the weakly nonlinear TS wave.

Substituting (4.5) into equations (2.5) and examining the TS quantities yields a system which is very similar in character to that discussed by Smith (1979*b*). At leading orders we find that the linear lower deck problem (3.10) applies so that matching with the main deck solutions yields the linear eigenproblem (3.12*b*) as before which has solution given by (3.14).

The mean flow terms U_{mL} and V_{mL} satisfy equations of the forms

$$r_n \frac{\partial^2 U_{mL}}{\partial Z^2} = \frac{i\hat{\beta}_0}{r_n}\left((V_{01}^{\dagger})^* U_{01}^{\dagger} - V_{01}^{\dagger}(U_{01}^{\dagger})^*\right) + (W_{01}^{\dagger})^* \frac{\partial U_{01}^{\dagger}}{\partial Z} + W_{01}^{\dagger}\frac{\partial (U_{01}^{\dagger})^*}{\partial Z}, \quad (4.6a)$$

$$r_n \frac{\partial^2 V_{mL}}{\partial Z^2} = i\hat{\alpha}_0\left((U_{01}^{\dagger})^* V_{01}^{\dagger} - U_{01}^{\dagger}(V_{01}^{\dagger})^*\right) + (W_{01}^{\dagger})^* \frac{\partial V_{01}^{\dagger}}{\partial Z} + W_{01}^{\dagger}\frac{\partial (V_{01}^{\dagger})^*}{\partial Z}. \quad (4.6b)$$

To match with the mean flow produced by the self-interaction of the vortex in the main deck requires

$$U_{mL} \longrightarrow \bar{u}_m' Z, \qquad V_{mL} \longrightarrow \bar{v}_m' Z, \qquad \text{as} \quad Z \longrightarrow \infty,$$

where $\bar{u}_m'$ and $\bar{v}_m'$ are defined below (3.9). Substituting the solutions of (3.10) together with definitions (3.11) reveals that U_{mL} and V_{mL} are given by

$$U_{mL} = \frac{\hat{\beta}_d^4}{i^{\frac{1}{3}} r_n^{\frac{7}{4}} \hat{\lambda}_d^3}\left(\int_{\eta_0}^{\eta} G_1(t)dt\right)|\hat{A}_0|^2 + \frac{\bar{u}_m'}{i^{\frac{1}{3}} r_n^{\frac{1}{4}} \hat{\lambda}_d^{\frac{1}{3}}}(\eta - \eta_0), \quad (4.7a)$$

$$V_{mL}=\frac{\hat{\beta}_d^4}{i^{\frac{1}{3}}r_n^{\frac{7}{4}}\hat{\lambda}_d^3}\left(\int_{\eta_0}^{\eta}G_2(t)dt\right)|\hat{A}_0|^2+\frac{\bar{v}_m'}{i^{\frac{1}{3}}r_n^{\frac{1}{4}}\hat{\lambda}_d^{\frac{1}{3}}}(\eta-\eta_0), \tag{4.7b}$$

where

$$G_1(t)=i^{-\frac{2}{3}}\left[\hat{\beta}_d(\hat{\beta}_d\bar{u}_0'-\hat{\alpha}_d\bar{v}_0')\mathcal{L}(t)-\frac{\bar{u}_0'\gamma_d^2}{Ai'(\eta_0)}\int_{\eta_0}^{t}Ai(\tau)d\tau\right]\Gamma^*+c.c.,$$

$$G_2(t)=i^{-\frac{2}{3}}\left[\hat{\alpha}_d(\hat{\alpha}_d\bar{v}_0'-\hat{\beta}_d\bar{u}_0')\mathcal{L}(t)-\frac{\bar{v}_0'\gamma_d^2}{Ai'(\eta_0)}\int_{\eta_0}^{t}Ai(\tau)d\tau\right]\Gamma^*+c.c.,$$

$$\Gamma\equiv\left[\frac{\int_{\eta_0}^{t}\left(\int_{\eta_0}^{t_1}Ai(\tau)d\tau\right)dt_1}{Ai'(\eta_0)}\right],$$

and where $\mathcal{L}(t)$ satisfies $\frac{d^2\mathcal{L}}{dt^2}-t\mathcal{L}=1$ with $\mathcal{L}(\eta_0)=0,\mathcal{L}\longrightarrow 0$ as $t\longrightarrow\infty$, so that

$$\mathcal{L}(t)\equiv Ai(t)\int_{\eta_0}^{t}\frac{\left(\int_{\infty}^{t_1}Ai(\tau)d\tau\right)}{\left(Ai(t_1)\right)^2}dt_1.$$

The second harmonic terms in (4.5) satisfy the equations

$$2i(\hat{\alpha}_0U_{21}^{\dagger}+\frac{\hat{\beta}_0}{r_n}V_{21}^{\dagger})+\frac{\partial W_{21}^{\dagger}}{\partial Z}=0, \tag{4.8a}$$

$$2i(\hat{\lambda}_0Z-\hat{\Omega}_0)U_{21}^{\dagger}+r_n\bar{u}_0'W_{21}^{\dagger}=-2i\hat{\alpha}_0P_{21}^{\dagger}+U_{01}^{\dagger}\frac{\partial W_{01}^{\dagger}}{\partial Z}-W_{01}^{\dagger}\frac{\partial U_{01}^{\dagger}}{\partial Z}+\frac{\partial^2U_{21}^{\dagger}}{\partial Z^2}, \tag{4.8b}$$

$$2i(\hat{\lambda}_0Z-\hat{\Omega}_0)V_{21}^{\dagger}+r_n\bar{v}_0'W_{21}^{\dagger}=-\frac{2i\hat{\beta}_0}{r_n}P_{21}^{\dagger}+V_{01}^{\dagger}\frac{\partial W_{01}^{\dagger}}{\partial Z}-W_{01}^{\dagger}\frac{\partial V_{01}^{\dagger}}{\partial Z}+\frac{\partial^2V_{21}^{\dagger}}{\partial Z^2}, \tag{4.8c}$$

$$P_{21}^{\dagger}=const. \tag{4.8d}$$

We may obtain an equation for $W_{21}^{\dagger}$ by eliminating $U_{21}^{\dagger}$ and $V_{21}^{\dagger}$ from $(4.8a,b,c)$. We require the boundary conditions of $U_{21}^{\dagger},V_{21}^{\dagger}$ and $W_{21}^{\dagger}$ vanishing on $Z=0$ and matching conditions as $Z\longrightarrow\infty$ are obtained by considering the main and upper deck equations to be satisfied by the harmonic terms. This is all straightforward and follows from the work of Smith (1979*b*) so we merely state the solution for $W_{21}^{\dagger}$ here. We obtain

$$\frac{\partial^2W_{21}^{\dagger}}{\partial\eta^2}=-\frac{2i\hat{\lambda}_0(\hat{A}_0)^2}{r_n^2\left(\int_{\eta_0}^{\infty}Ai(t)dt\right)^2}\left[F_*+Ai'(\eta)\int_{\eta_0}^{\eta}Ai(t)dt\right]+B_{\dagger}Ai(2^{\frac{1}{3}}\eta),$$

$$V_{mL} = \frac{\hat{\beta}_d^4}{i^{\frac{1}{3}} r_n^{\frac{7}{4}} \hat{\lambda}_d^{\frac{3}{3}}} \left(\int_{\eta_0}^{\eta} G_2(t)\,dt \right) |\hat{A}_0|^2 + \frac{\bar{v}_m'}{i^{\frac{1}{3}} r_n^{\frac{1}{4}} \hat{\lambda}_d^{\frac{1}{3}}} (\eta - \eta_0), \tag{4.7b}$$

where

$$G_1(t) = i^{-\frac{2}{3}} \left[\hat{\beta}_d (\hat{\beta}_d \bar{u}_0' - \hat{\alpha}_d \bar{v}_0') \mathcal{L}(t) - \frac{\bar{u}_0' \gamma_d^2}{Ai'(\eta_0)} \int_{\eta_0}^{t} Ai(\tau)\,d\tau \right] \Gamma^* + c.c.,$$

$$G_2(t) = i^{-\frac{2}{3}} \left[\hat{\alpha}_d (\hat{\alpha}_d \bar{v}_0' - \hat{\beta}_d \bar{u}_0') \mathcal{L}(t) - \frac{\bar{v}_0' \gamma_d^2}{Ai'(\eta_0)} \int_{\eta_0}^{t} Ai(\tau)\,d\tau \right] \Gamma^* + c.c.,$$

$$\Gamma \equiv \left[\frac{\int_{\eta_0}^{t} \left(\int_{\eta_0}^{t_1} Ai(\tau)\,d\tau \right) dt_1}{Ai'(\eta_0)} \right],$$

and where $\mathcal{L}(t)$ satisfies $\frac{d^2\mathcal{L}}{dt^2} - t\mathcal{L} = 1$ with $\mathcal{L}(\eta_0) = 0, \mathcal{L} \longrightarrow 0$ as $t \longrightarrow \infty$, so that

$$\mathcal{L}(t) \equiv Ai(t) \int_{\eta_0}^{t} \frac{\left(\int_{\infty}^{t_1} Ai(\tau)\,d\tau \right)}{(Ai(t_1))^2} dt_1.$$

The second harmonic terms in (4.5) satisfy the equations

$$2i(\hat{\alpha}_0 U_{21}^{\dagger} + \frac{\hat{\beta}_0}{r_n} V_{21}^{\dagger}) + \frac{\partial W_{21}^{\dagger}}{\partial Z} = 0, \tag{4.8a}$$

$$2i(\hat{\lambda}_0 Z - \hat{\Omega}_0) U_{21}^{\dagger} + r_n \bar{u}_0' W_{21}^{\dagger} = -2i\hat{\alpha}_0 P_{21}^{\dagger} + U_{01}^{\dagger} \frac{\partial W_{01}^{\dagger}}{\partial Z} - W_{01}^{\dagger} \frac{\partial U_{01}^{\dagger}}{\partial Z} + \frac{\partial^2 U_{21}^{\dagger}}{\partial Z^2}, \tag{4.8b}$$

$$2i(\hat{\lambda}_0 Z - \hat{\Omega}_0) V_{21}^{\dagger} + r_n \bar{v}_0' W_{21}^{\dagger} = -\frac{2i\hat{\beta}_0}{r_n} P_{21}^{\dagger} + V_{01}^{\dagger} \frac{\partial W_{01}^{\dagger}}{\partial Z} - W_{01}^{\dagger} \frac{\partial V_{01}^{\dagger}}{\partial Z} + \frac{\partial^2 V_{21}^{\dagger}}{\partial Z^2}, \tag{4.8c}$$

$$P_{21}^{\dagger} = const. \tag{4.8d}$$

We may obtain an equation for $W_{21}^{\dagger}$ by eliminating $U_{21}^{\dagger}$ and $V_{21}^{\dagger}$ from (4.8*a*, *b*, *c*). We require the boundary conditions of $U_{21}^{\dagger}, V_{21}^{\dagger}$ and $W_{21}^{\dagger}$ vanishing on $Z = 0$ and matching conditions as $Z \longrightarrow \infty$ are obtained by considering the main and upper deck equations to be satisfied by the harmonic terms. This is all straightforward and follows from the work of Smith (1979*b*) so we merely state the solution for $W_{21}^{\dagger}$ here. We obtain

$$\frac{\partial^2 W_{21}^{\dagger}}{\partial \eta^2} = -\frac{2i\hat{\lambda}_0 (\hat{A}_0)^2}{r_n^2 \left(\int_{\eta_0}^{\infty} Ai(t)\,dt \right)^2} \left[F_* + Ai'(\eta) \int_{\eta_0}^{\eta} Ai(t)\,dt \right] + B_{\dagger} Ai(2^{\frac{1}{3}} \eta),$$

where

$$F_* = Ai(\hat{\eta})\int_{\hat{\eta}_0}^{\hat{\eta}} \frac{\left[\int_{\infty}^{q_1} Ai(q_2)\hat{R}(q_2)dq_2\right]}{(Ai(q_1))^2}dq_1,$$

and

$$\hat{R} = -2^{-\frac{2}{3}}\left[2Ai''(\eta)Ai(\eta) + Ai'(\eta_0)Ai'(\eta)\right], \qquad \hat{\eta} = 2^{\frac{1}{3}}\eta,$$

and the constant $B_\dagger$ is chosen so that $\partial W_{21}^\dagger/\partial\eta = 0$ at $\eta = \eta_0$ and

$$\frac{\partial W_{21}^\dagger}{\partial\eta} \longrightarrow \left[\frac{\hat{\lambda}_0^2}{4\hat{\gamma}_0\hat{\beta}_0^2\Delta_*^{\frac{1}{3}}}\right]\left.\frac{\partial^3 W_{21}^\dagger}{\partial\eta^3}\right|_{\eta=\eta_0} \qquad \text{as} \quad \eta \longrightarrow \infty. \tag{4.9}$$

Finally, to recover the evolution equation for the amplitude parameter $\hat{A}_0$ for the TS solutions we inspect the governing system for the terms $U_{31}^\dagger, V_{31}^\dagger, W_{31}^\dagger, P_{31}^\dagger$. This set of equations take the form

$$i(\hat{\alpha}_0 U_{31}^\dagger + \frac{\hat{\beta}_0}{r_n}V_{31}^\dagger) + \frac{\partial W_{31}^\dagger}{\partial Z} = F_1,$$

$$i(\hat{\lambda}_0 Z - \hat{\Omega}_0)U_{31}^\dagger + r_n\bar{u}_0' W_{31}^\dagger + i\hat{\alpha}_0 P_{31}^\dagger - \frac{\partial^2 U_{31}^\dagger}{\partial Z^2} = F_2,$$

$$i(\hat{\lambda}_0 Z - \hat{\Omega}_0)V_{31}^\dagger + r_n\bar{v}_0' W_{31}^\dagger + \frac{i\hat{\beta}_0}{r_n}P_{31}^\dagger - \frac{\partial^2 V_{31}^\dagger}{\partial Z^2} = F_3,$$

$$P_{31}^\dagger = const.,$$

where F_1, F_2 and F_3 are combinations of lower order TS and mean flow terms. We can use the standard method to obtain a governing equation for $W_{31}^\dagger$ which assumes the form

$$\frac{\partial^3 W_{31}^\dagger}{\partial\eta^3} - \eta\frac{\partial W_{31}^\dagger}{\partial\eta} + W_{31}^\dagger = \frac{\hat{\beta}_0^2\hat{\gamma}_0}{r_n\Delta_*}\hat{B}_0 + \frac{\hat{\beta}_0^2\hat{\gamma}_0}{r_n\Delta_*}\left[\frac{2\hat{\beta}_{00}}{\hat{\beta}_0} + \frac{1}{\hat{\gamma}_0^2}\left(-i\hat{\alpha}_0\frac{\partial}{\partial r_1} + \frac{\hat{\beta}_0\hat{\beta}_{00}}{r_n^2}\right)\right]$$

$$\hat{A}_0 - \frac{1}{\Delta_*}\left[W_{21}^\dagger\frac{\partial^2(W_{01}^\dagger)^*}{\partial Z^2} - \frac{1}{2}\frac{\partial W_{21}^\dagger}{\partial Z}\frac{\partial(W_{01}^\dagger)^*}{\partial Z} - \frac{1}{2}(W_{01}^\dagger)^*\frac{\partial^2 W_{21}^\dagger}{\partial Z^2}\right.$$

$$\left. -i\frac{\partial W_{01}^\dagger}{\partial Z}(\hat{\alpha}_0 r_n U_{mL} + \hat{\beta}_0 V_{mL}) + iW_{01}^\dagger\frac{\partial}{\partial Z}(\hat{\alpha}_0 r_n U_{mL} + \hat{\beta}_0 V_{mL})\right]$$

$$-\frac{(\hat{\lambda}_{00} + \hat{\lambda}_m)}{i\Delta_*}\left[((\eta - \eta_0) + \Phi\eta_0)\frac{\partial W_{01}^\dagger}{\partial\eta} - W_{01}^\dagger\right], \tag{4.11a}$$

where we have used definitions (3.11) and those given below (3.15). The relevant boundary conditions are that

$$W_{31}^{\dagger} = 0 = \frac{\partial W_{31}^{\dagger}}{\partial \eta} \qquad \text{on} \quad \eta = \eta_0,$$

and

$$\frac{\partial W_{31}^{\dagger}}{\partial \eta} \longrightarrow \frac{-i\hat{A}_0}{r_n \Delta_*^{\frac{1}{3}}}(\hat{\lambda}_{00} + \hat{\lambda}_m) - \frac{i\hat{B}_0\hat{\lambda}_0}{r_n \Delta_*^{\frac{1}{3}}}, \tag{4.11b}$$

as $\eta \longrightarrow \infty$.

The system (4.11) only admits a satisfactory solution if a certain solvability criterion is met, and it is this criterion which leads to the desired evolution equation. To obtain this condition, we need to consider the adjoint problem to (4.11), and using the results of Ince (1956), Hall & Smith (1982), it is found that the adjoint function for this problem is

$$S(\eta) \equiv Ai'(\eta) - \frac{Ai'(\eta_0)}{\mathcal{L}'(\eta_0)}\mathcal{L}'(\eta), \tag{4.12}$$

where the function $\mathcal{L}$ is as given in the definition for G_1 and G_2 in (4.7). If we write the right hand side of (4.11a) as $R_*(\eta)$, multiply both sides of (4.11a) by $S(\eta)$, integrate by parts and use (4.11b), we obtain the solvability condition

$$-\frac{i\hat{\beta}_0^2\hat{\gamma}_0 Ai(\eta_0)}{r_n \Delta_*^2}\left(-ir_n\bar{u}_0'\frac{\partial}{\partial r_1} + \hat{\beta}_{00}\bar{v}_0'\right)\hat{A}_0 = \int_{\eta_0}^{\infty} SR_* d\eta. \tag{4.13}$$

We now use the definitions of R_* together with the solutions for U_{mL}, V_{mL} and W_{mL} (given in (4.7),(4.9)) and that for $W_{01}^{\dagger}$ to obtain an evolution equation for $\hat{A}_0$. This equation takes the general form

$$\frac{\partial \hat{A}_0}{\partial r_1} = \left[T_1\hat{\beta}_{00} + T_2\hat{\Omega}_{00} + T_3\right]\hat{A}_0 + \frac{T_4}{r_n^{\frac{3}{2}}}\hat{A}_0|\hat{A}_0|^2, \tag{4.14}$$

where $T_1, \ldots, T_4$ are complex functions of $\hat{\alpha}_d, \hat{\beta}_d, \hat{\gamma}_d$ and η_0. All of the integrations necessary in the determination of $T_1, \ldots, T_4$ were evaluated numerically. Initial conditions at a suitably large value of η, say η_∞, were found asymptotically and then the defining equations for the various functions integrated using a fourth order Runge-Kutta scheme. The integrations were all performed using the trapezium rule and the results were checked by varying the value of η_∞ and the step length used.

These checks lead us to believe that the computed values of $T_1 - T_4$ are correct to within 0.1%.

In the above equation, each of the coefficients $T_1 - T_4$ is a function of the orientation of the TS wave and so are functions of the parameter μ. Further, T_1, T_2 & T_4 are all independent of vortex quantities; i.e. are functions only of the TS wavenumbers $\hat{\alpha}_d, \hat{\beta}_d$ and η_0. Since our prime concern at the outset was to determine the effect of the vortex upon the stability of the TS wave, the precise values of T_1, T_2 and T_4 are not of immediate interest, save to remark that we found that for all admissible values of μ ($< \mu_c \sim 0.8284$), $Re(T_4) < 0$, see Figure (4).

If we write (4.14) in the modified form

$$\frac{\partial \hat{A}_0}{\partial r_1} = b_* \hat{A}_0 + c_* \hat{A}_0 |\hat{A}_0|^2, \tag{4.15}$$

then it is clear that on the basis of linear theory the TS wave is unstable if $Re(b_*) > 0$. However, (4.15) admits an 'equilibrium' solution with

$$|\hat{A}_0|^2 = \left(-\frac{real(b_*)}{real(c_*)} \right),$$

so for $Re(c_*) < 0$ a non-zero, finite, steady solution is possible. It may be easily shown that this solution is stable and then the TS wave is said to be supercritically stable.

The only term in (4.14) which contains vortex-induced quantities is the term T_3 which is given by the formula

$$\begin{aligned} &\left[i \left(\frac{5\hat{\gamma}_d \bar{u}_0'}{3\hat{\beta}_d^2} - \frac{\hat{\lambda}_d \hat{\alpha}_d}{\hat{\gamma}_d \hat{\beta}_d^2} \right) - \frac{2K_1}{3\hat{\lambda}_d^{\frac{5}{3}}} \bar{u}_0' i^{\frac{5}{3}} \left(\frac{K_1}{1.001} - 1 \right) \frac{\hat{\gamma}_d^2 Ai(\eta_0)}{Ai'(\eta_0)} \right] T_3 \\ &= -i \left[\frac{5i}{3\hat{\beta}_d^2} - \frac{2K_1 i^{\frac{5}{3}}}{3\hat{\lambda}_d^{\frac{5}{3}}} \hat{\gamma}_d^2 \left(\frac{K_1}{1.001} - 1 \right) \frac{Ai(\eta_0)}{Ai'(\eta_0)} \right] (\hat{\alpha}_d \bar{u}_m' + \hat{\beta}_d \bar{v}_m'). \end{aligned} \tag{4.16}$$

Here, K_1 is defined by (3.13), η_0 by (3.11), the nondimensionalised TS wavenumbers are $\hat{\alpha}_d, \hat{\beta}_d$ and $\bar{u}_m', \bar{v}_m'$ are the derivatives of the mean flow terms U_{30}, V_{30} induced by the vortex when evaluated at $\xi = 0$. We see immediately that, for a given μ, T_3 is proportional to the size of the mean flow correction due to the vortex and on examining equations (2.11), (2.13), (2.14) and the jump conditions (2.21) which determine this mean flow it is clear that $\bar{u}_m'$ and $\bar{v}_m'$ are, in turn, proportional to the square of the amplitude of the crossflow vortex.

Hence the effect of the vortex on the stability of the TS wave can be measured directly through the coefficient T_3. More particularly, we can see from (4.14) that if $Re(T_3) < 0$ then the TS wave is stabilised by the presence of the vortex, whereas the opposite conclusion may be drawn for $Re(T_3) > 0$. This function is illustrated in Figure (5). We notice that $Re(T_3)$ may be of either sign and this is determined solely by the value of μ (or equivalently by the orientation of the TS wave). In particular, we find that for $\mu > 0.297$ or $-0.296 < \mu < 0$ (corresponding to TS waves making angles between -16.5^o and 0^o or greater than about 16.5^o with the outward radial direction) the effect of $Re(T_3)$ is destabilising. Conversely, for TS waves making other angles with the radial direction, the opposite effect is observed.

Crucially then, our analysis in this section suggests that on a weakly nonlinear basis the TS waves are supercritically stable as is the case in the absence of the vortex, but that the presence of the of vortex can be both of a stabilising or of a destabilising effect, with the change between these states occuring for TS waves inclined at roughly 16.5^o with the outward radius on the disc. Some comments and brief conclusions concerning these results will be made in Section 6.

5. The secondary instability of a fully nonlinear vortex to a TS wave

In this section we attempt to generalise the work contained in Sections 2 & 3. Thus far our concern has been only with the problem involving a linear vortex and now we consider the effects of nonlinearity of this instability mode. We can anticipate that the nonlinearity will affect the governing equations for the mean flow induced by the vortex and that this in turn will alter the evolution equation for the TS wave.

For simplicity, we shall revert to considering the problem of the interaction involving an infinitesimally small TS wave (of size Δ as defined in (3.2)). In this case, the nonlinear vortex structure is unaffected by the presence of the TS wave, at least to the orders that we will be concerned with. We can use the results described by Gajjar (1989), who showed that when the vortex size reaches $O(R^{-\frac{1}{3}})$ $(=O(\epsilon^8))$, then the first effects of nonlinearity are encountered in the critical layer although the remainder of the flow structure remains linear. It is this sized vortex which we consider here and we note that the mean flow correction generated by the vortex is now $O(\epsilon^4)$ which is larger than the vortex itself. This large mean flow correction is necessary due to the properties of the nonlinear critical layer, see Stewartson (1981) and Haberman (1972). Since the vortex induces an $O(\epsilon^4)$ correction and following the ideas presented in Section 3, we expect the TS wave to have spatial and temporal dependence as in the function F_1, defined by

$$F_1 \equiv \exp\left[\frac{i}{\epsilon^9}\left\{\int^r \hat{\alpha}(r,\epsilon)dr + \theta\hat{\beta}(\epsilon) - \hat{\Omega}(\epsilon)t\right\}\right], \tag{5.1a}$$

where,

$$\hat{\alpha} = \hat{\alpha}_0, \qquad \hat{\beta} = (\hat{\beta}_0 + \ldots) + \epsilon^4(\hat{\beta}_{00} + \ldots), \\ \hat{\Omega} = \epsilon^3(\hat{\Omega}_0 + \ldots) + \epsilon^7(\hat{\Omega}_{00} + \ldots). \tag{5.1b}$$

Here, as in (4.3), $\hat{\beta}_{00}$ and $\hat{\Omega}_{00}$ denote the changes in the azimuthal wavenumber and frequency of the TS wave due to the presence of the vortex. We permit the scaled amplitude of the TS wave to evolve on an r_1 lengthscale, where

$$r = r_n + \epsilon^4 r_1, \tag{5.2}$$

replaces the definition (4.2). The linear theory for the TS wave will allow us to derive an evolution equation of the type (4.14) with the nonlinear term on the right hand side of the equation absent. Indeed, following the work contained in Sections 3 & 4 it is easily shown that the stability of the TS wave is governed by an analysis identical to that already performed and that the linear evolution equation of the TS disturbance will satisfy an equation of the form

$$\frac{\partial \hat{A}}{\partial r_1} = \left[Q_1\hat{\beta}_{00d} + Q_2\hat{\Omega}_{00d} + Q_3\right]\hat{A}, \tag{5.3}$$

where $\hat{A}$ is the scaled amplitude of the TS wave, $\hat{\beta}_{00d} = r_n^{-\frac{3}{4}}\hat{\beta}_{00}, \hat{\Omega}_{00d} = r_n^{-\frac{1}{2}}\hat{\Omega}_{00}$ and Q_1, Q_2 & Q_3 are functions of the orientation of the TS and certain mean flow correction parameters which follow from the analysis of the nonlinear critical layer. Further, as in the discussion following (4.14), it is only the function Q_3 which contains the mean flow velocities induced by the vortex and hence is the only term to give information concerning the effect of the vortex on the stability of the TS perturbation. Finally, Q_3 is given by the same formula as determined T_3 save that the mean flow derivative terms present on the right hand side of (4.16) should be replaced by the equivalent terms for the (larger) vortex-induced mean flow.

To summarise the above, we conclude that the evolution equation for the TS wave follows immediately once we have determined the mean flow correction due to the nonlinear vortex. We consequently consider this problem, in a manner similar to that used in Section 2 and below we determine the governing equations for the mean flow.

5.1 The nonlinear vortex

We follow Gajjar (1989) and use scalings for the vortex appropriate to the nonlinear critical layer calculation to be addressed. In the inviscid zone, where $z = \epsilon^{12}\xi$, the implied velocity and pressure fields take the forms

$$u = r_n \bar{u}(\xi) + \epsilon^4 r_n \bar{u}_m + \epsilon^8 \left(U_{10} + \bar{u}_{m1}\right) + \ldots, \tag{5.4a}$$

$$v = r_n \bar{v}(\xi) + \epsilon^4 r_n \bar{v}_m + \epsilon^8 \left(V_{10} + \bar{v}_{m1}\right) + \ldots, \tag{5.4b}$$

$$w = \epsilon^8 W_{10} + \epsilon^{12} \bar{w}(\xi) + \epsilon^{16} \bar{w}_m + \ldots, \tag{5.4c}$$

$$p = \epsilon^8 P_{10} + \ldots, \tag{5.4d}$$

where $\bar{u}_m, \bar{v}_m$ and $\bar{w}_m$ are the leading order mean flow terms induced by the vortex. We reiterate that all the mean flow terms in these expansions (terms with subscript m) are functions of ξ and r_1 and that the mean flow corrections in the radial and azimuthal directions are larger than the fundamental vortex terms. We seek solutions for these latter functions in terms of the crossflow variable x, where x is defined so that (2.6) may be rewritten as

$$E \equiv \exp(ix). \tag{2.6$'$}$$

Substituting expansions (5.4) into (2.5) and comparing suitable terms yields that there is a solution for the vortex terms of the form

$$\begin{aligned}(U_{10}, V_{10}, W_{10}, P_{10}) &= \\ &A(u_{01}(r_n,\xi)\cos x, v_{01}(r_n,\xi)\cos x, w_{01}(r_n,\xi)\sin x, p_{01}(r_n,\xi)\cos x),\end{aligned} \tag{5.5}$$

where A is the scaled size of the vortex and $(u_{01}, v_{01}, w_{01}, p_{01})$ satisfies

$$\alpha_0 u_{01} + \frac{\beta_0}{r_n} v_{01} - \frac{\partial w_{01}}{\partial \xi} = 0, \qquad -U_{0B} u_{01} + r_n \bar{u}' w_{01} = \alpha_0 p_{01},$$
$$-U_{0B} v_{01} + r_n \bar{v}' w_{01} = \frac{\beta_0}{r_n} p_{01}, \qquad U_{0B} w_{01} = -p_{01}'. \tag{5.6}$$

We recall that $U_{0B} \equiv r_n \alpha_0 \bar{u} + \beta_0 \bar{v}$ and note that w_{01} satisfies the Rayleigh equation (2.9).

Additionally, we find that the mean flow terms satisfy equations very similar to (2.11), and in particular

$$2\bar{u}_m + \frac{d\bar{w}_m}{d\xi} = 0, \tag{5.7a}$$

$$\frac{d^2 \bar{u}_m}{d\xi^2} - \bar{w}\frac{d\bar{u}_m}{d\xi} - 2\bar{u}\bar{u}_m - \bar{u}' \bar{w}_m + 2(1+\bar{v})\bar{v}_m = 0, \tag{5.7b}$$

$$\frac{d^2 \bar{v}_m}{d\xi^2} - \bar{w}\frac{d\bar{v}_m}{d\xi} - 2\bar{u}\bar{v}_m - \bar{v}' \bar{w}_m - 2(1+\bar{v})\bar{u}_m = 0. \tag{5.7c}$$

The wall layer structure for this nonlinear vortex is analogous to that in Section 2 for the linear vortex and so we conclude that we need boundary conditions for the induced mean flow of the form

$$\bar{u}_m = \bar{v}_m = \bar{w}_m = 0 \quad \text{on} \quad \xi = 0, \qquad \bar{u}_m, \bar{v}_m \longrightarrow 0 \quad \text{as} \quad \xi \longrightarrow \infty. \tag{5.8}$$

The most dramatic change between the work here and that given earlier is the treatment of the critical layer zone. For now we expect a nonlinear analysis to become appropriate in order to compute the jumps in the mean flow and its' derivative across $\xi = \bar{\xi}$. As before, we suppose that the critical layer is described by $\xi = \bar{\xi} + \epsilon^4 \hat{z}, \hat{z} = O(1)$, and then the analysis of system (5.6) as $\xi \longrightarrow \bar{\xi}$ implies that in the critical layer the relevant expansions are

$$u = r_n \bar{u}_{\bar{\xi}} + \epsilon^4 \left(\hat{u}_0 + r_n \bar{u}'_{\bar{\xi}} \hat{z}\right) + \epsilon^8 \left(\hat{u}_1 + \frac{1}{2} r_n \bar{u}''_{\bar{\xi}} \hat{z}^2\right) + \ldots, \tag{5.9a}$$

$$v = r_n \bar{v}_{\bar{\xi}} + \epsilon^4 \left(\hat{v}_0 + r_n \bar{v}'_{\bar{\xi}} \hat{z}\right) + \epsilon^8 \left(\hat{v}_1 + \frac{1}{2} r_n \bar{v}''_{\bar{\xi}} \hat{z}^2\right) + \ldots, \tag{5.9b}$$

$$w = \epsilon^8 \hat{w}_0 + \epsilon^{12} \left(\hat{w}_1 + \bar{w}_{\bar{\xi}}\right) + \ldots, \tag{5.9c}$$

$$p = \epsilon^8 \hat{p}_0 + \epsilon^{12} \hat{p}_1 + \ldots. \tag{5.9d}$$

Using these expansions in (2.5) reveals that the first order unknowns $\hat{w}_0, \hat{p}_0$ in (5.9) are given by

$$\hat{w}_0 = \hat{w}_* \sin x = \frac{A\gamma_0^2 p_{00}}{B_{01}} \sin x, \qquad \hat{p}_0 = A p_{00} \cos x, \tag{5.10}$$

where $p_{00} = p_{01}(\bar{\xi})$ as defined by (5.6) and $B_{jk} \equiv r_n \alpha_j \bar{u}^{(k)}(\bar{\xi}) + \beta_j \bar{v}^{(k)}(\bar{\xi})$. It is clear that these solutions match on to (5.5) and (5.6) outside the critical layer.

If we define the unknowns $\hat{u}_{jk} = r_n \alpha_j \hat{u}_k + \beta_j \hat{v}_k$, then we find upon substitution in the continuity equation (2.5a) and on using (5.10), that $\hat{u}_{00} = C_{00}$ (a constant). This, when matching with the inviscid layer solutions surrounding the critical layer, gives that

$$\lim_{\xi \longrightarrow \bar{\xi}+} \left[\alpha_0 \bar{u}_m + \frac{\beta_0}{r_n} \bar{v}_m\right] = \lim_{\xi \longrightarrow \bar{\xi}-} \left[\alpha_0 \bar{u}_m + \frac{\beta_0}{r_n} \bar{v}_m\right],$$

or, equivalently, that

$$[\bar{v}_m(\xi)]_{\bar{\xi}-}^{\bar{\xi}+} = -\frac{r_n \alpha_0}{\beta_0} [\bar{u}_m(\xi)]_{\bar{\xi}-}^{\bar{\xi}+},$$

i.e. the jump in $\bar{v}_m$ across the critical layer is proportional to the jump in $\bar{u}_m$ across that layer.

After straightforward analysis (see Gajjar (1989) for further details) it is found that $\hat{v}_0$ satisfies the equation

$$(B_{01}\hat{z} + \bar{B}_{10}) \frac{\partial \hat{v}_0}{\partial x} + \frac{A\gamma_0^2 p_{00}}{B_{01}} \sin x \frac{\partial \hat{v}_0}{\partial \hat{z}} = A p_{00} \left[\frac{\beta_0}{r_n} - \frac{r_n \gamma_0^2 \bar{v}'_{\bar{\xi}}}{B_{01}}\right] \sin x + \frac{\partial^2 \hat{v}_0}{\partial \hat{z}^2}, \tag{5.11}$$

where $\bar{B}_{10} \equiv B_{10} + C_{00}$.

We recall that $B_{01} < 0$ and this enables us to write (5.11) in a canonical form. If we define

$$\hat{v}_0 = K_0(Y - V^*), \qquad C_* Y = \frac{\bar{B}_{10}}{B_{01}} + \hat{z}, \qquad K_0 = \frac{B_{01} C_*}{\gamma_0^2} \left(\frac{\beta_0}{r_n} - \frac{r_n \gamma_0^2 \bar{v}'_{\bar{\xi}}}{B_{01}}\right),$$
$$B_{01} C_*^3 = \lambda_c^{-1}, \quad (> 0) \qquad C_*^2 = \frac{A\gamma_0^2 p_{00}}{B_{01}^2}, \tag{5.12}$$

we can reduce (5.11) to the form

$$Y \frac{\partial V^*}{\partial x} + \sin x \frac{\partial V^*}{\partial Y} - \lambda_c \frac{\partial^2 V^*}{\partial Y^2} = 0. \tag{5.13}$$

We note that $C_* < 0$ so $\hat{z} \longrightarrow \pm\infty$ corresponds to $Y \longrightarrow \mp\infty$ and the boundary conditions to ensure a match with the inviscid zone solutions now take the forms

$$V^* \longrightarrow Y - \frac{r_n(\bar{v}_m)^{\mp}}{K_0} + \frac{\cos x}{Y} + O\left(\frac{1}{Y^2}\right), \qquad (5.14)$$

as $Y \longrightarrow \pm\infty$ where $(\bar{v}_m)^{\mp}$ denotes $\lim_{\xi \longrightarrow \bar{\xi}^{\mp}} (\bar{v}_m)$ respectively.

The system (5.13) & (5.14) is a classical one and the solution characteristics are well known, see Haberman (1972), Stewartson (1981). If we define $V^* = \partial^2\psi/\partial Y^2$ then we obtain the system

$$Y \frac{\partial^3 \psi}{\partial x \partial Y^2} + \sin x \frac{\partial^3 \psi}{\partial Y^3} = \lambda_c \frac{\partial^4 \psi}{\partial Y^4}, \qquad (5.15a)$$

$$\frac{\partial \psi}{\partial Y} \longrightarrow \frac{1}{2}Y^2 - \frac{r_n(\bar{v}_m)^{\mp}}{K_0} Y + \ln|Y| \cos x + B_0^{\pm}(x) + \ldots, \quad \text{as} \quad Y \longrightarrow \pm\infty. \quad (5.15b)$$

We may apply the standard technique of integrating (5.15a) with respect to Y, applying the boundary conditions (5.15b) and then integrating over a period in x. After appealing to periodicity and a further integration with respect to Y we find (see Haberman 1972) that, on using (5.12),

$$(\bar{v}_m^- - \bar{v}_m^+) = -\frac{K_0}{2\pi\lambda_c r_n} \int_0^{2\pi} \left(B_0^+ - B_0^-\right) \sin x dx. \qquad (5.16)$$

The numerical solution of (5.15) is well documented (Haberman (1972), Smith & Bodonyi (1982)) and if we define the phase shift $(-\hat{\phi})$ by

$$-\hat{\phi} = \frac{1}{\pi} \int_0^{2\pi} \left(B_0^+ - B_0^-\right) \sin x dx, \qquad (5.17)$$

then the dependence of $\hat{\phi}$ on the parameter λ_c is shown in Figure (6). (This figure was plotted using data kindly supplied by Dr. J. Gajjar.) It is known that as λ_c becomes very large $\hat{\phi} \longrightarrow \pi$ which corresponds to the result for a linear vortex, whereas for small λ_c, $\hat{\phi} \longrightarrow 0$. Using the transformations (5.10, 5.12) we find that (5.16) becomes

$$[\bar{u}_m(\xi)]_{\bar{\xi}-}^{\bar{\xi}+} = \frac{\beta_0(\beta_0 \bar{u}'_{\bar{\xi}} - r_n \alpha_0 \bar{v}'_{\bar{\xi}})}{2 r_n^2 \gamma_0^2 B_{01}} \left(\hat{\phi}\right) (\hat{w}_*)^2, \qquad (5.18a)$$

$$[\bar{v}_m(\xi)]_{\bar{\xi}-}^{\bar{\xi}+} = -\frac{r_n \alpha_0}{\beta_0} [\bar{u}_m(\xi)]_{\bar{\xi}-}^{\bar{\xi}+}. \qquad (5.18b)$$

These are the mean flow jump conditions which are the counterparts of (2.21). We can easily show that for $\lambda_c \gg 1$ (and hence $\hat{\phi} \longrightarrow \pi$), the size of $\hat{w}_*$ diminishes (from (5.10), (5.12)) and the jump criteria approach those of (2.21) for the linear vortex.

We need to extend the above to consider the shift in the derivative of the mean flow across the critical layer and this necessitates consideration of the next order terms in expansions (5.9).

To derive the necessary information we can proceed to obtain a differential equation to determine $\hat{v}_1$. The governing equations for these unknowns are given by Gajjar (1989) so we do not repeat them here. Instead, we merely state that we can reduce the problem for $\hat{v}_1$ to the canonical form

$$Y\frac{\partial^3\Psi}{\partial x\partial Y^2} + \sin x\frac{\partial^3\Psi}{\partial Y^3} - \lambda_c\frac{\partial^4\Psi}{\partial Y^4} = \chi(x,Y) - \frac{K_0\bar{w}_{\bar{\xi}}}{C_*}, \tag{5.19a}$$

with boundary conditions

$$\Psi \longrightarrow \frac{1}{6}\left[1 + r_n B_{01}C_*^2(\bar{v}_m')^{\mp}\right]Y^3 + \frac{r_n C_*}{2}\left[B_{01}(\bar{v}_{m1})^{\mp} - \bar{B}_{10}(\bar{v}_m')^{\mp}\right]Y^2 + O(Y\ln|Y|), \tag{5.19b}$$

as $|Y| \longrightarrow \infty$. Here $(\bar{v}_m')^{\mp}, (\bar{v}_{m1})^{\mp}$ denote $\left[\bar{v}_m'(\xi)\right]_{\xi\longrightarrow\bar{\xi}\mp}, [\bar{v}_{m1}(\xi)]_{\xi\longrightarrow\bar{\xi}\mp}$ respectively, where these mean flow terms are defined in the expansions (5.4). Also $\chi(x,Y)$ is a very complicated expression involving ψ (the solution of (5.15)). However, we find upon integrating (5.19a) with respect to Y and integrating over a period in x, that the function $\chi(x,Y)$ plays no part in determining the jump in $\bar{v}_m'$ across the critical layer and, on reapplying transformations (5.12), we obtain the result

$$\left[\bar{v}_m'(\xi)\right]_{\bar{\xi}-}^{\bar{\xi}+} = \bar{w}_{\bar{\xi}}\left[\bar{v}_m(\xi)\right]_{\bar{\xi}-}^{\bar{\xi}+}. \tag{5.20a}$$

Similarly, we also find that

$$\left[\bar{u}_m'(\xi)\right]_{\bar{\xi}-}^{\bar{\xi}+} = \bar{w}_{\bar{\xi}}\left[\bar{u}_m(\xi)\right]_{\bar{\xi}-}^{\bar{\xi}+}. \tag{5.20b}$$

Consequently, we now have a complete determination of the mean flow induced by the nonlinear vortex; namely the defining equations (5.7), boundary conditions (5.8) and jump conditions (5.18) & (5.20). This system is almost identical to that studied in Section 2 and in practice $(\bar{u}_m, \bar{v}_m, \bar{w}_m)$ is merely a multiple of the solutions (U_{30}, V_{30}, W_{30}) found earlier. This multiplication factor arises due to the difference in the jump conditions (5.18a) and (2.21a). This observation leads us to some immediate conclusions concerning the effect of the vortex on the stability of an infinitesimal TS wave. For a given orientation of the TS wave, the function Q_3 appearing in (5.3) is a real multiple of T_3 in the evolution equation (4.14). Consequently, the nonlinear vortex has a stabilising influence on the TS wave if the latter mode makes an angle ϕ with the outward radial direction if $0^o < \phi < 16.5^o$ or if $\phi < -16.5^o$, and is destabilising otherwise.

To conclude this investigation of the weakly nonlinear vortex problem we consider the size of the mean flow jump across the critical layer. We know from the forms of (5.18a) and (4.16) that the stabilising or destabilising effect on the TS wave is proportional to the jump across the critical layer. From the transformations (5.12), solutions (5.10) and (5.18a), it is seen that the jump in $\bar{u}_m$ across the critical layer is proportional to $\hat{\phi}A^2$, which is a function of λ_c. Figure (7) illustrates this dependence, and two asymptotic cases naturally arise. Firstly, as $A \longrightarrow 0$ we know that $\lambda_c \longrightarrow \infty$ and $\hat{\phi} \longrightarrow \pi$ so the jump across the critical layer becomes small. This corresponds to returning to the linear theory of Section 2. Additionally, for small λ_c (which corresponds to a large vortex), we have from (5.12) that the scaled amplitude of the vortex $A \sim O(\lambda_c^{-\frac{2}{3}})$. It is also known that in this limit $\hat{\phi} \longrightarrow \lambda_c C^{(1)}$, where $\frac{1}{4}C^{(1)} = 1.379$ (Smith & Bodonyi 1982). Hence the mean flow jump is $O(\lambda_c^{-\frac{1}{3}})$ and so becomes large. Following the work of Gajjar (1989) it can be shown that when the scaled amplitude $A \sim O(\epsilon^{-\frac{16}{3}})$, the nature of the critical layer changes to a structure similar to that given by Bodonyi et al (1983) for a strongly nonlinear critical layer and thus a modified analysis of the flow is required.

We have thus shown in this section that the effect of a nonlinear vortex on the stability of the TS wave is very similar to that of the linear vortex considered earlier with the main difference arising from the changed mean flow jump across the critical layer. As previously, the nature of the influence of the vortex is largely determined by the orientation of the TS wave. Of interest would be the extension of our work to study the case of a strongly nonlinear vortex; the flow structure for this problem following from the discussion of the limit $\lambda_c \longrightarrow 0$ given above.

6. Conclusions

In this paper we have attempted to provide a rational asymptotic analysis of the problem of interaction between a stationary crossflow vortex in the flow induced by a rotating disc and a classical lower branch TS wave. The interaction has been studied in the context of this particular basic flow because this flow is susceptible to instabilities which occur in the boundary layer of a swept wing and this is of relevance to the development of Laminar Flow Control wings.

At first, the asymptotic structure of a linear crossflow vortex was obtained and we have investigated the effect of this vortex on both linear and on weakly nonlinear TS waves. Of particular interest in our work has been the problem of the behaviour of the TS wave in the neighbourhood of the critical layer of the vortex which is situated inside the main part of the boundary layer of the flow. As far as the TS wave is concerned, the critical layer plays a passive role and the leading order TS solutions pass through the critical layer region unaffected. In addition, we have demonstrated that the effect of the vortex on the stability of the travelling TS wave is felt entirely through the mean flow generated by the presence of the vortex. On analysing this phenomenon, we have found that depending on the orientation of the TS wave, the vortex can have either a stabilising or a destabilising effect- in particular the vortex stabilises the TS wave if the latter makes an angle between 0^o and 16.5^o or less than -16.5^o with the outward radial direction. This work has an obvious practical implication, namely that it is possible that when these two types of instability are present in a three-dimensional flow that the crossflow vortex can destabilise the TS wave.

The above interaction structure has also been extended to the study of a non-linear vortex. The findings described in the last paragraph are largely unaltered by this change, at least as far as the effect of the vortex on the stability of an infinitesimal TS wave is concerned. The analysis for the linear vortex was easily adapted to study the implications for the stability of a weakly nonlinear TS wave in the presence of a linear vortex, although the work for the case of a nonlinear vortex is not so readily extendable for the problem with this larger TS amplitude. This extension is currently being considered by the authors.

Further, the study of Section 5 and that of Gajjar (1989) points to an even larger crossflow vortex which may be described by asymptotic means. This again provides scope for future work.

Finally, we should remark that here we have considered the problem of interaction between a pair of specific crossflow and TS instabilities. In realistic flows it is likely that other modes could be present and it is desirable to be able to classify the relative importance of other possible interaction mechanisms. In particular, we are looking at the problem of interaction between TS waves and the stationary vortex mode described by Hall (1986) which is characterised by having zero wall shear stress for the effective crossflow velocity profile. In addition, we note that our approach developed here can only deal with interactions at asymptotically large Reynolds numbers and the importance of the crossflow vortex on the stability of the TS wave at lower Reynolds numbers can only be resolved by pursuing extensive numerical calculations.

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Figure Captions

Fig (1). The dependence of the mean flow terms U_{30}, V_{30} and W_{30} (defined by (2.7) and satisfying (2.11), (2.13), (2.14), (2.21)) upon the boundary layer coordinate ξ.

Fig (2). The neutral non-dimensional wavenumbers $\hat{\alpha}_d, \hat{\beta}_d$ and frequency $\hat{\Omega}_d$ for the TS wave defined by (3.1) expressed as functions of the waveangle $\phi = \tan^{-1} \mu$.

Fig (3). The functions (i) M_1, M_2 and (ii) R_1, R_2 which determine the corrections to the neutral wavenumbers and frequency of the TS wave due to the presence of the vortex.

Fig (4). The real part of the coefficient T_4 of the nonlinear term in the TS evolution equation (4.14) expressed as a function of μ.

Fig (5). The real part of the coefficient T_3 in the evolution equation (4.14) which determines the effect of the crossflow vortex on the stability of the TS wave.

Fig (6). The function $\hat{\phi} = \hat{\phi}(\lambda_c)$ given by (5.15) and (5.17).

Fig (7). Dependence of the vortex quantity $\hat{\phi}A^2$ upon the parameter λ_c. Here $\hat{\phi}$ is defined by (5.15), (5.17) and A is the scaled vortex amplitude size.

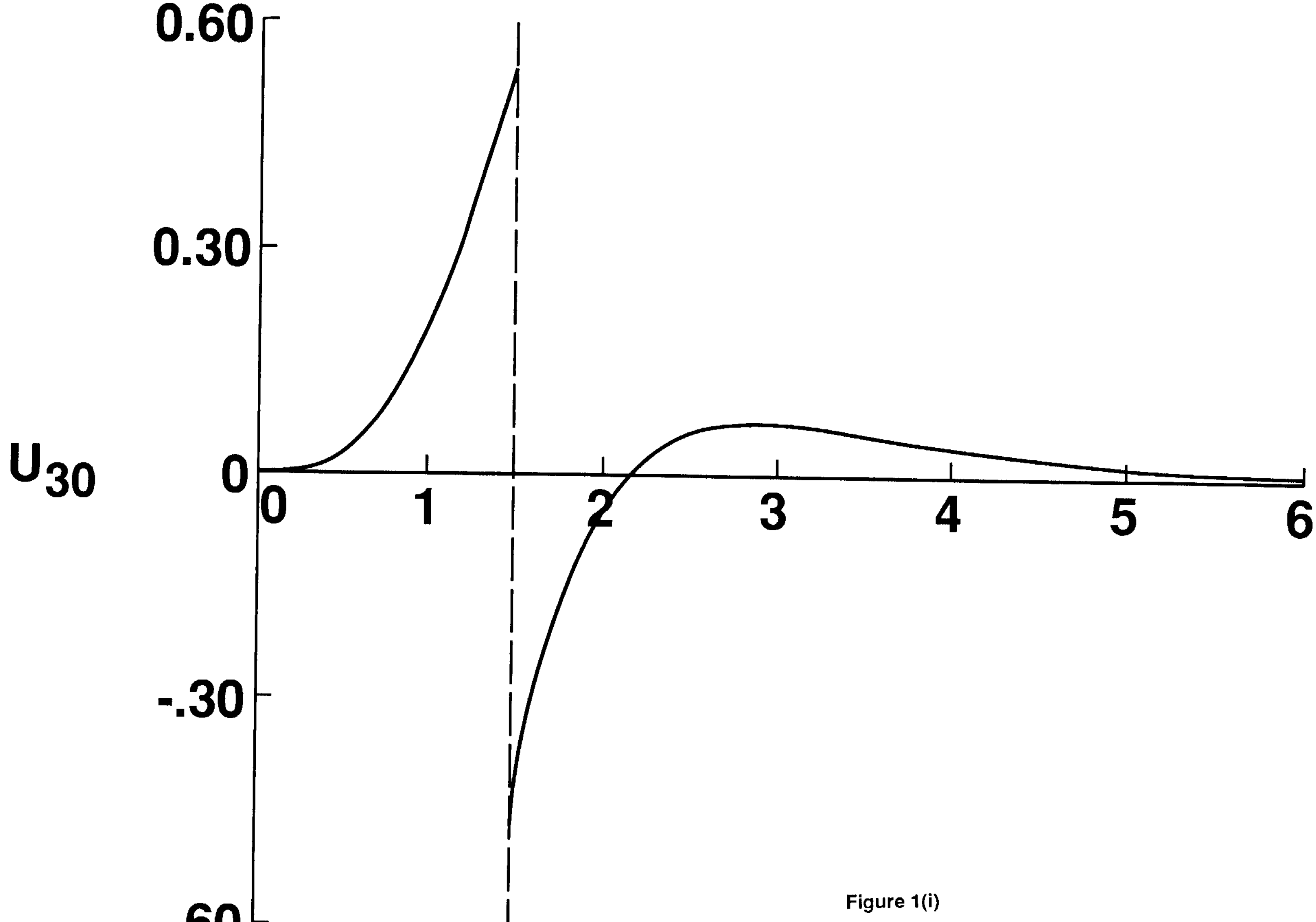
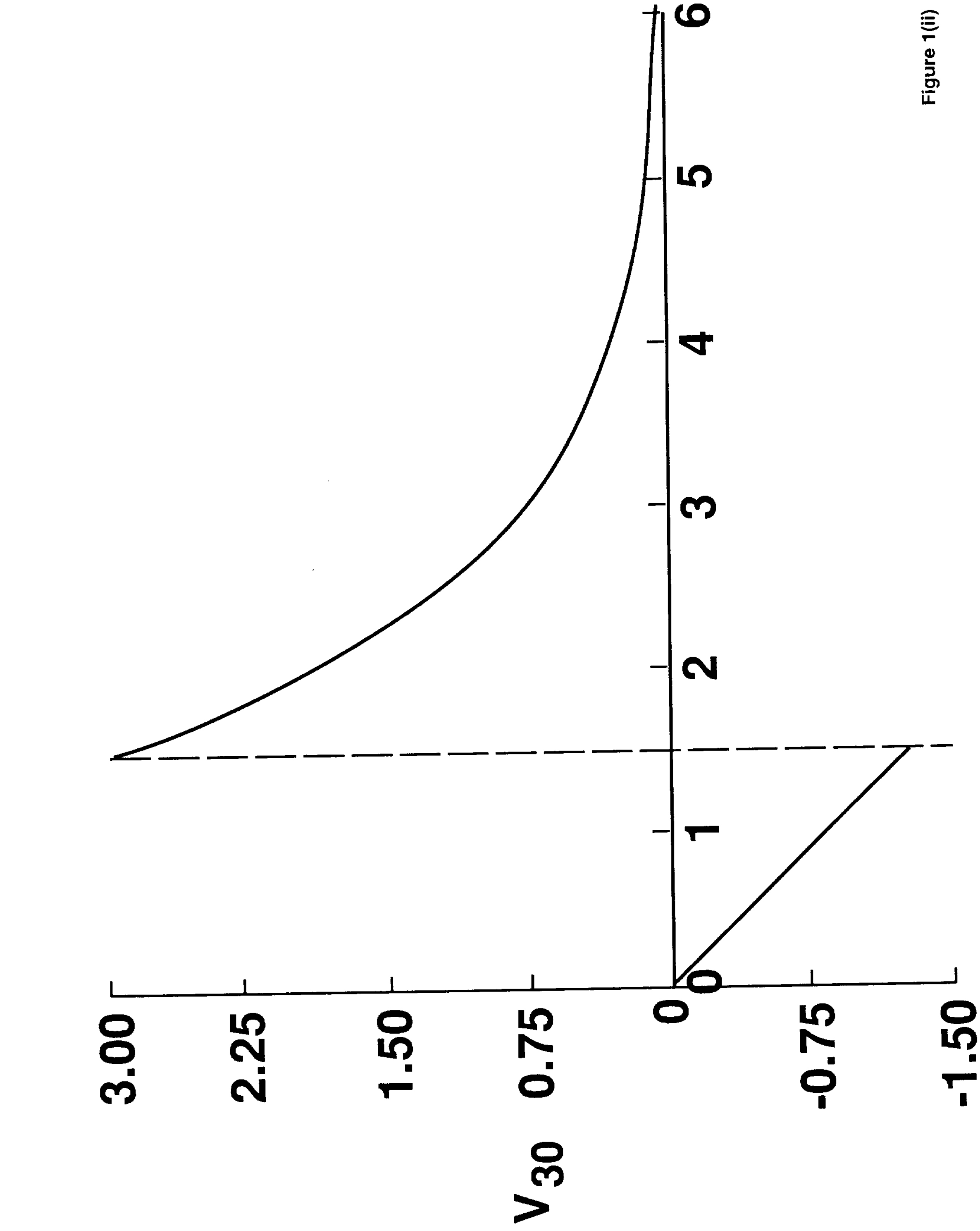


Figure 1(i)



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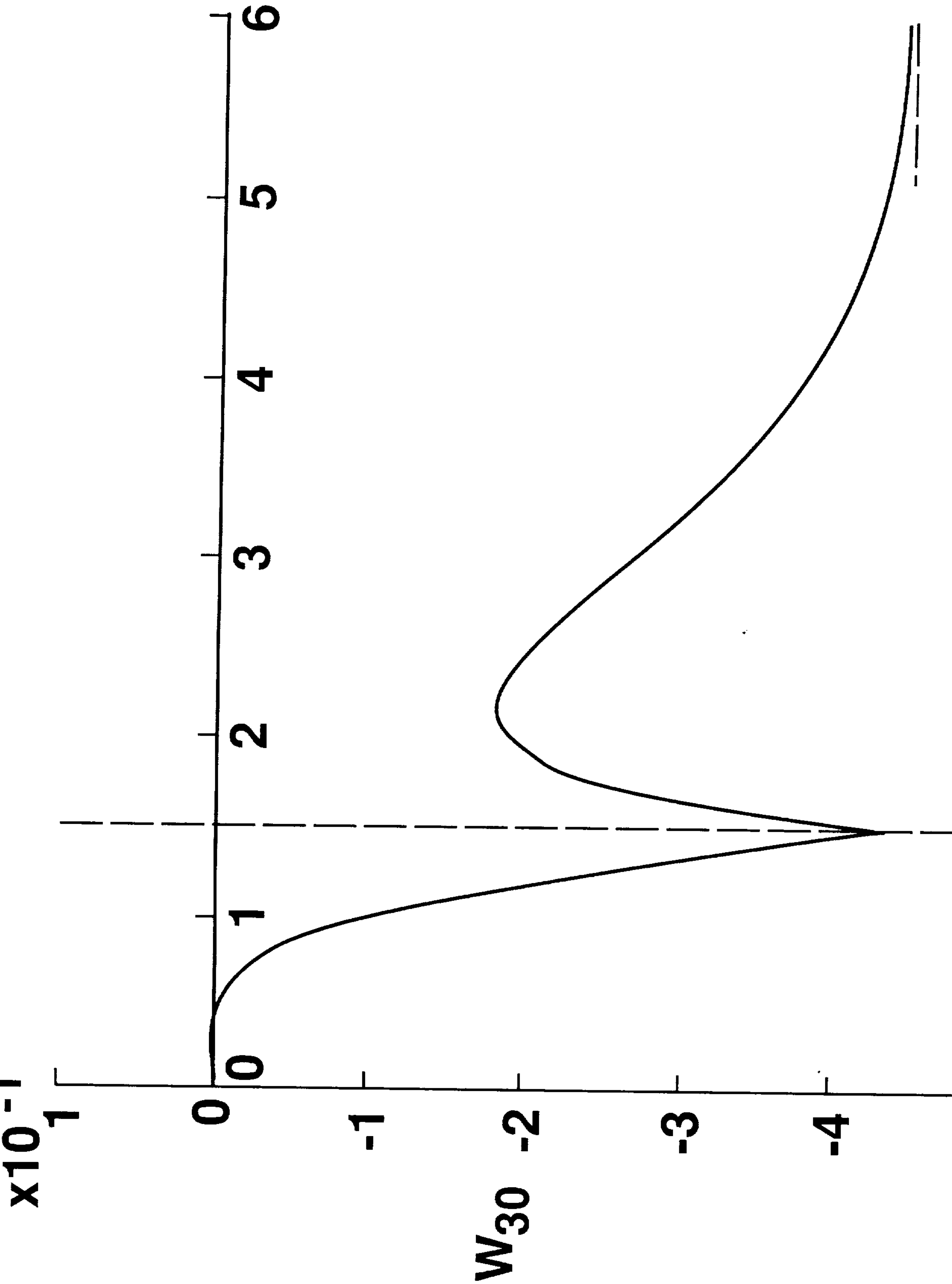

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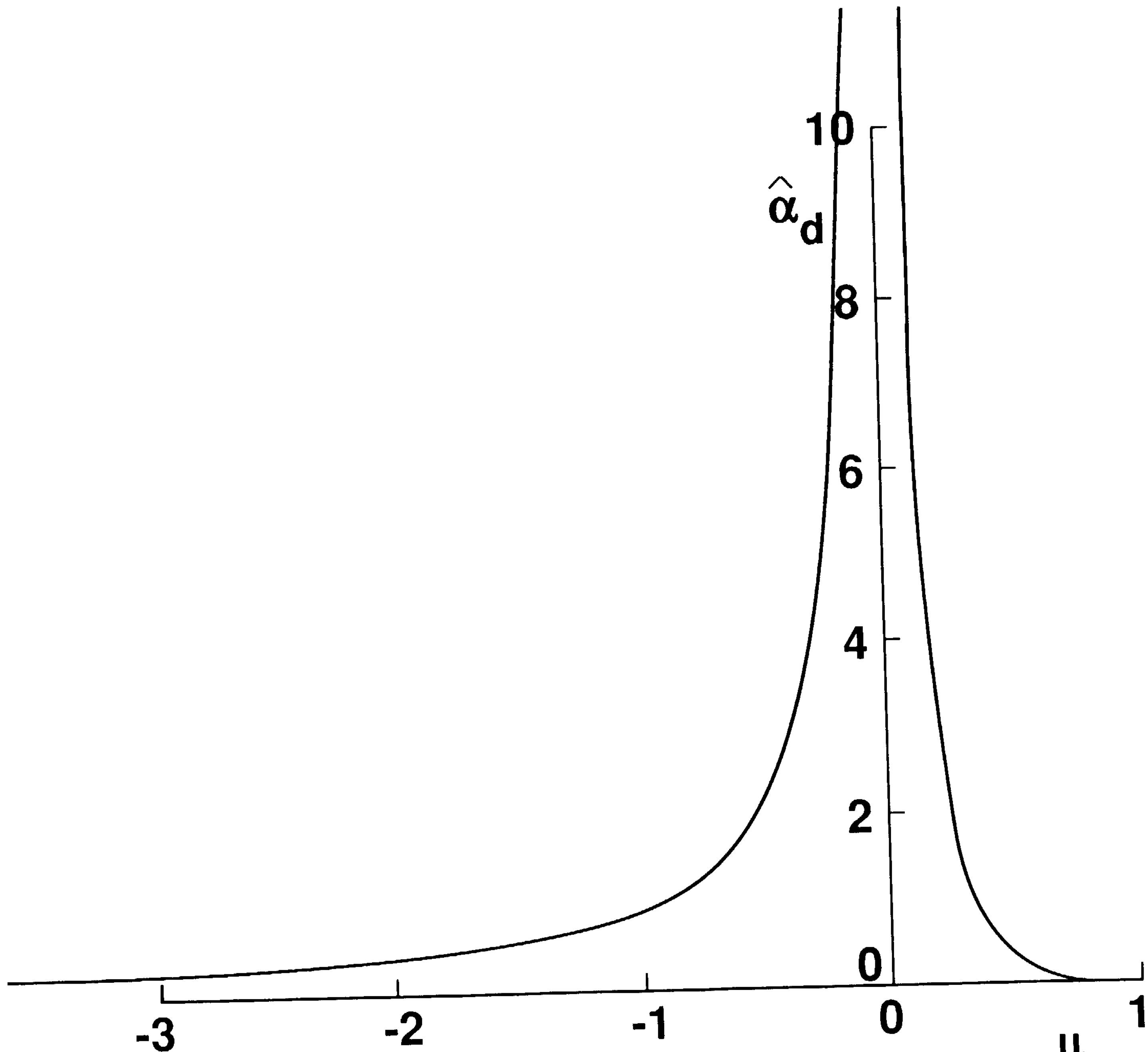

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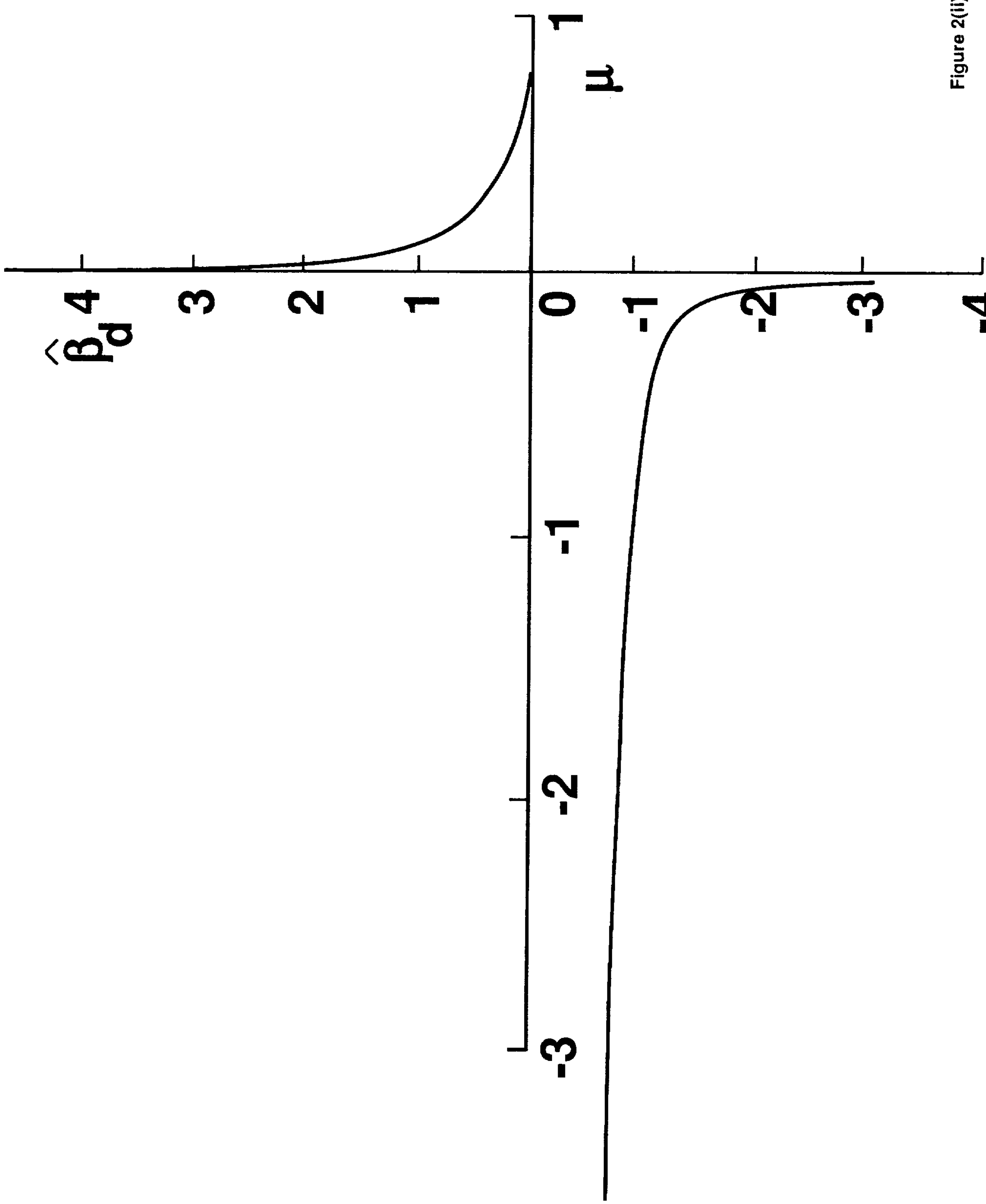

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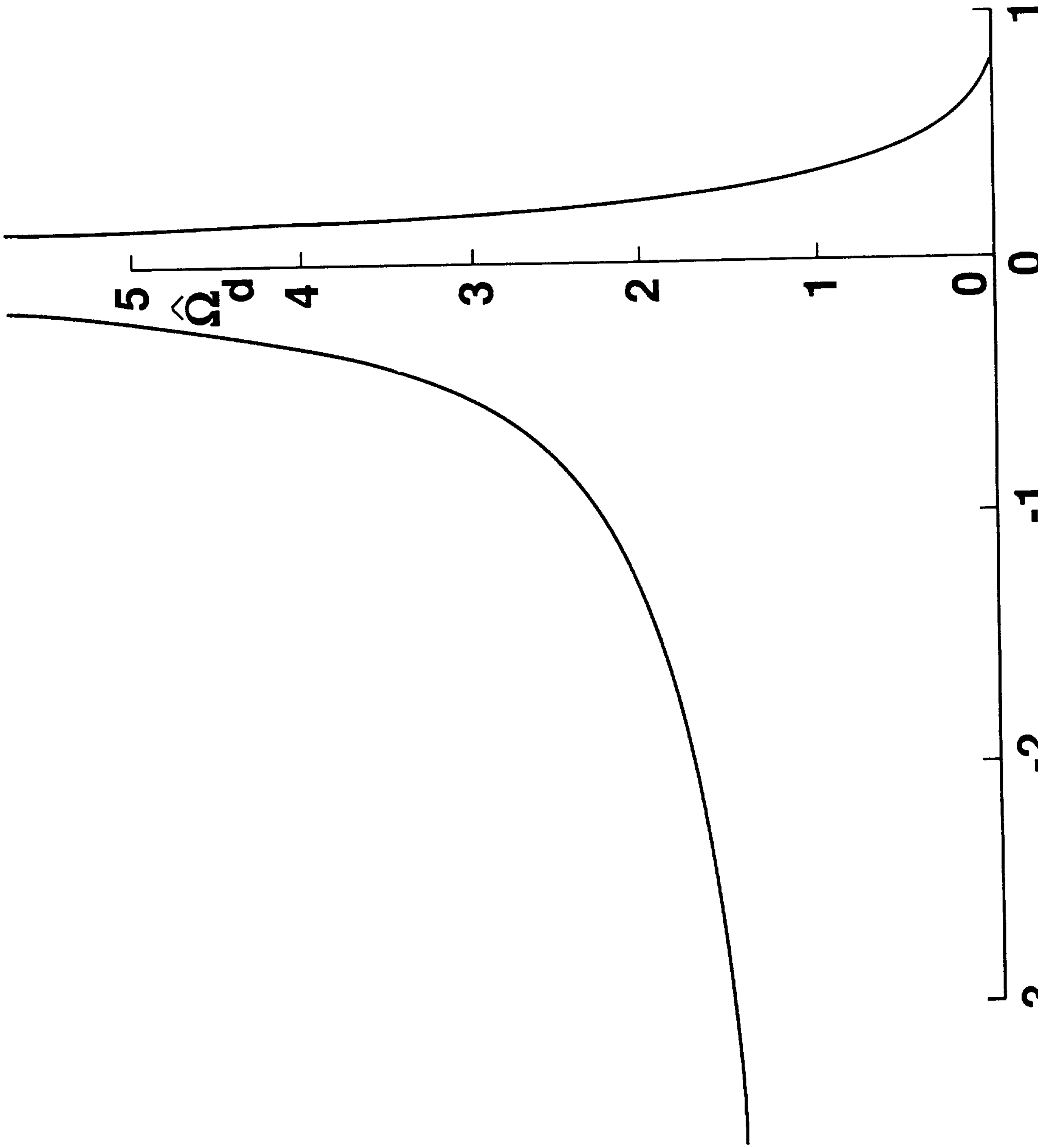

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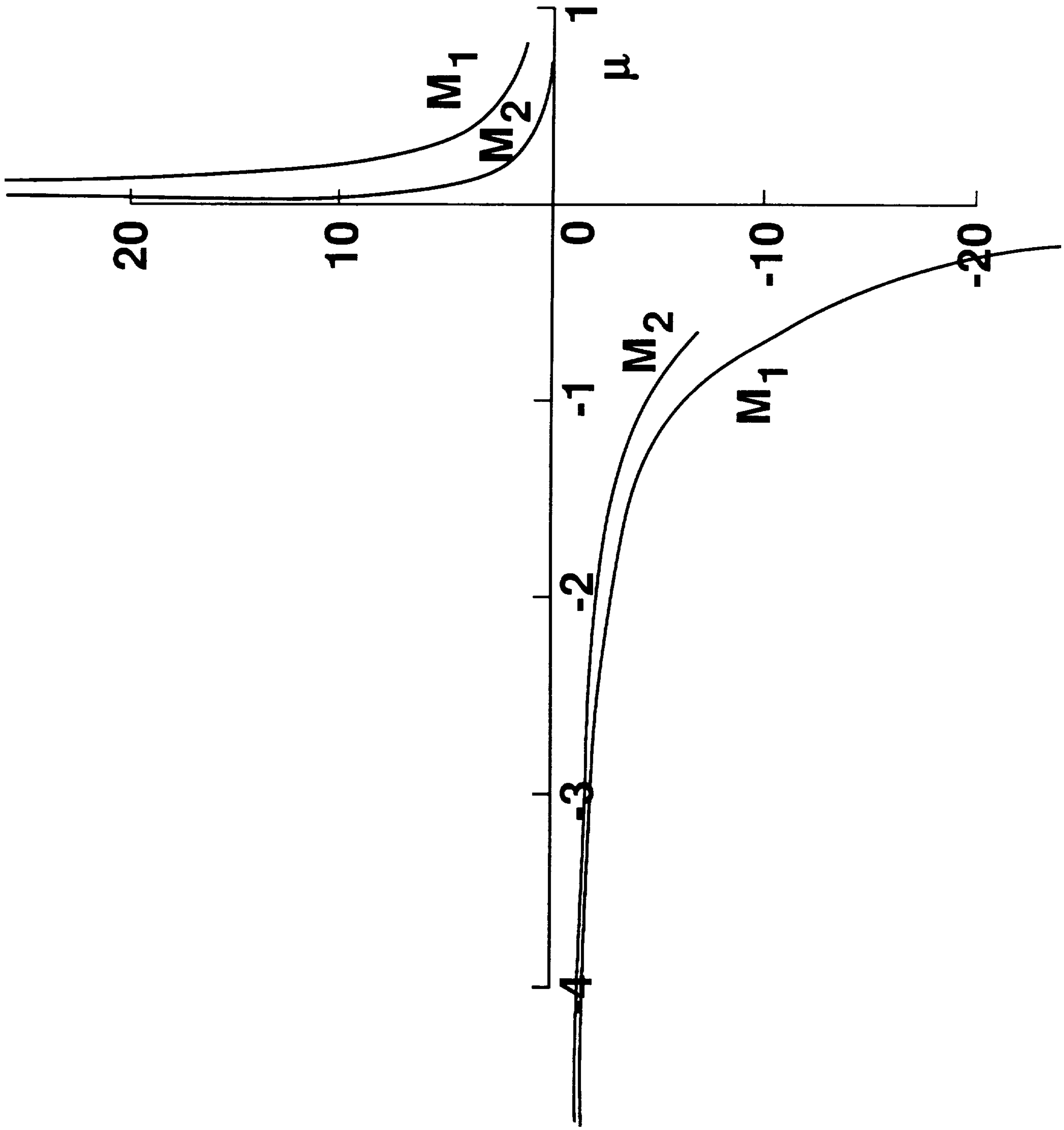

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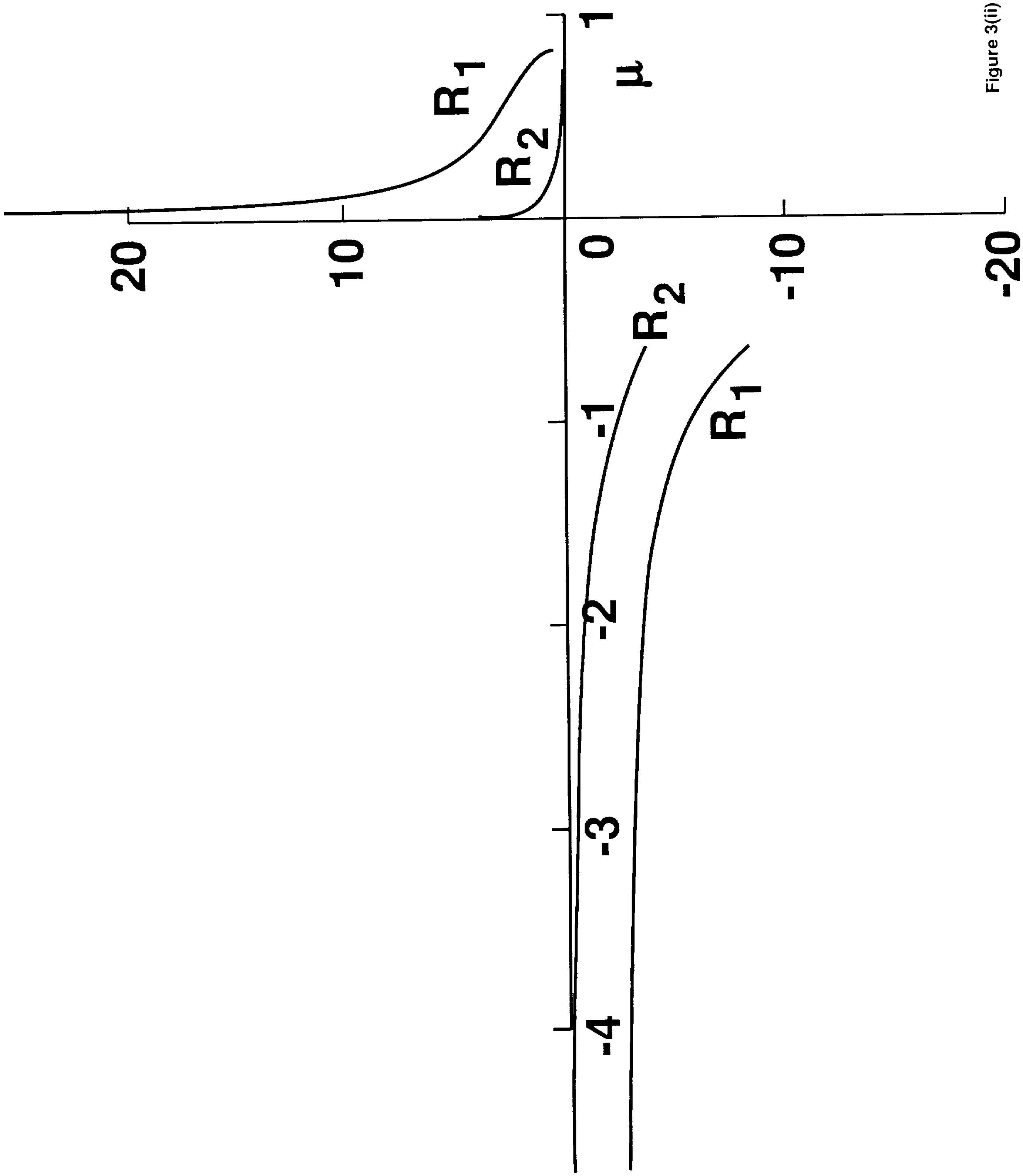

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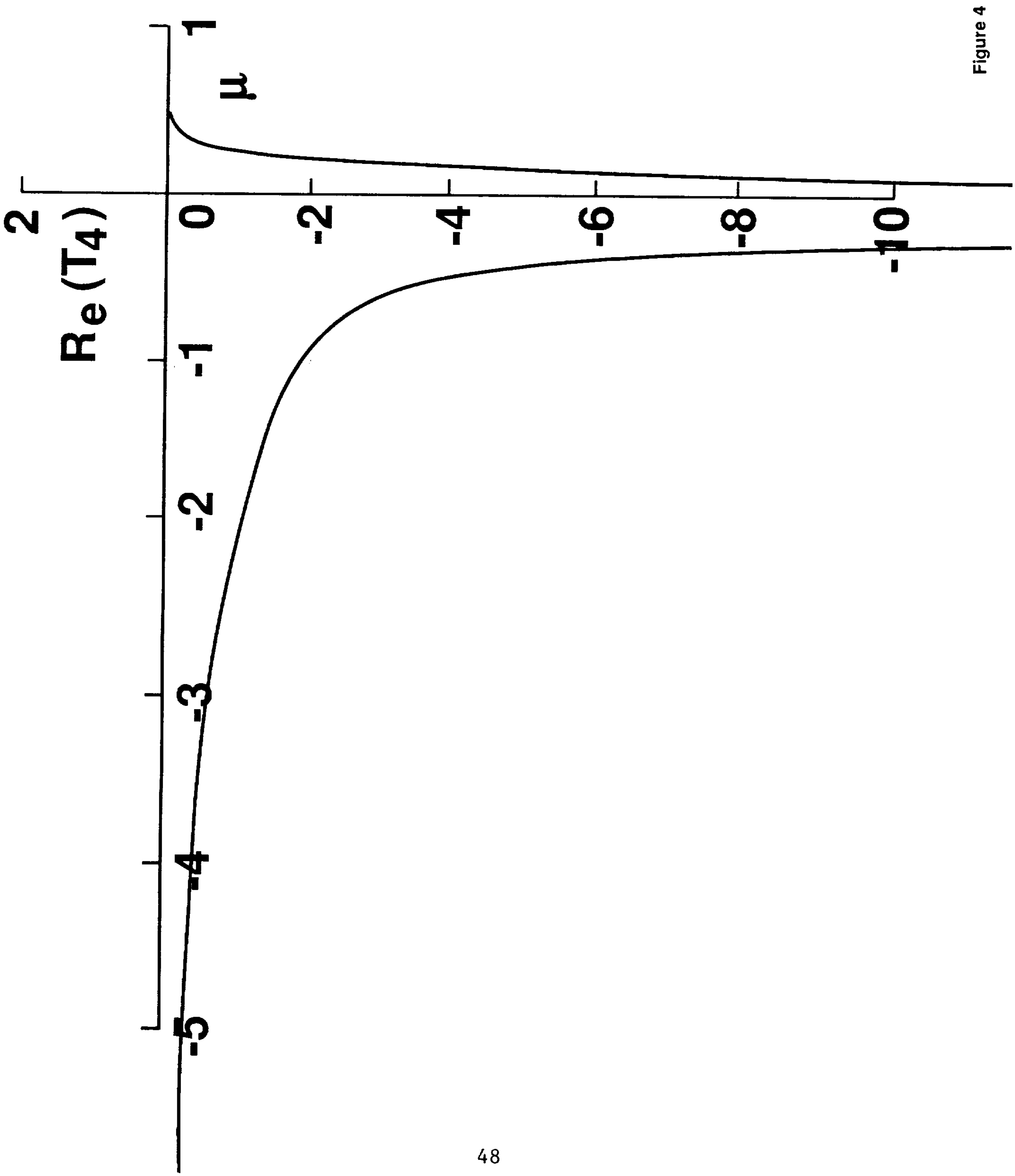

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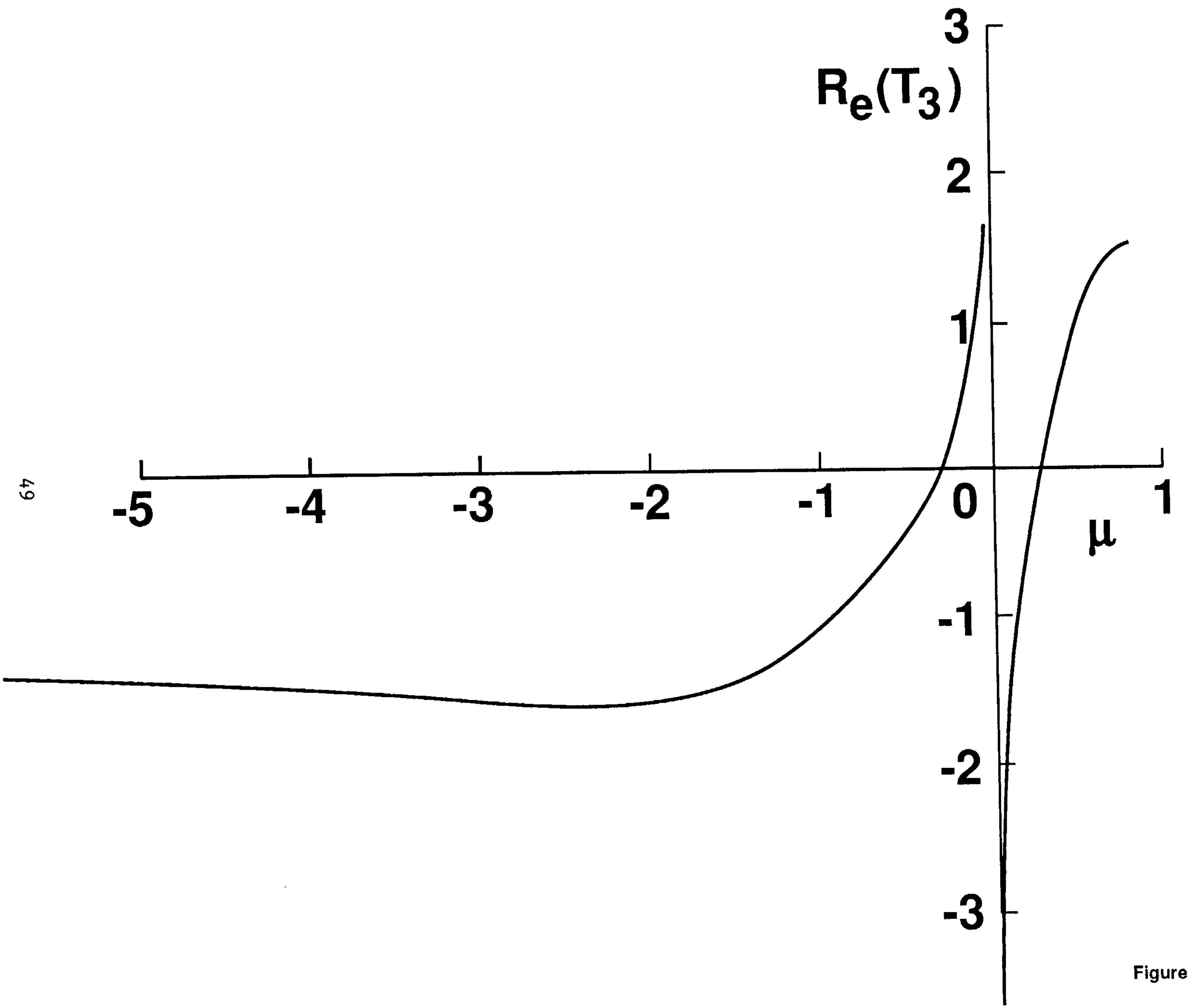

Figure 5

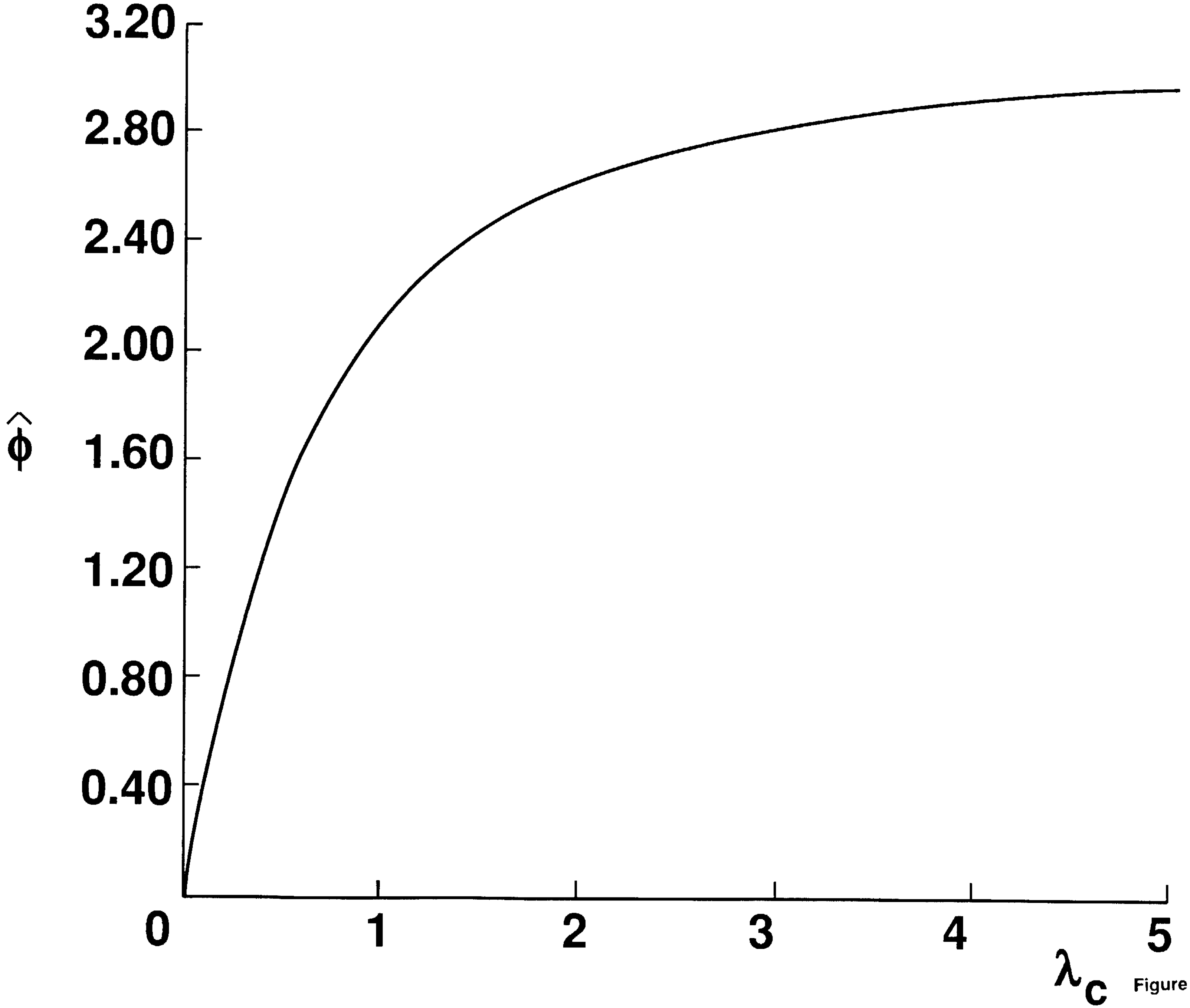

Figure 6

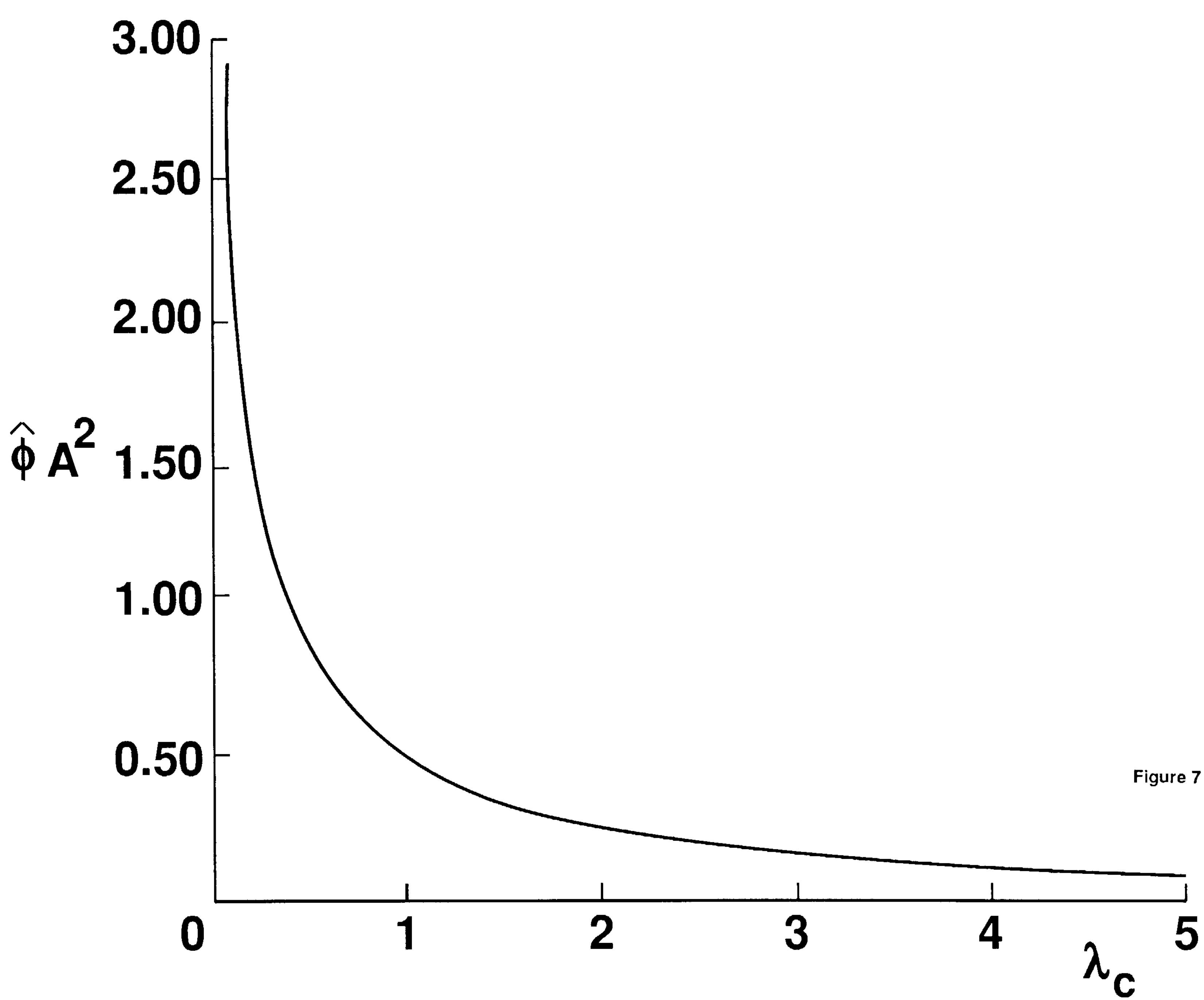

Figure 7

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16. Abstract

There are many fluid flows where the onset of transition can be caused by different instability mechanisms which compete among themselves. Here we consider the interaction of two types of instability mode (at an asymptotically large Reynolds number) which can occur in the flow above a rotating disc. In particular, we examine the interaction between lower-branch Tollmien-Schlichting(TS) waves and the upper-branch, stationary, inviscid crossflow vortex whose asymptotic structure has been described by Hall (1986). This problem is studied in the context of investigating the effect of the vortex on the stability characteristics of a small TS wave. Essentially, it is found that the primary effect is felt through the modification to the mean flow induced by the presence of the vortex. Initially, the TS wave is taken to be linear in character and we show (for the cases of both a linear and a nonlinear stationary vortex) that the vortex can exhibit both stabilizing and destabilizing effectson tha TS wave and the nature of influence is wholly dependent upon theorientation of this latter instability. Further, we examine the problem with a larger TS wave, whose size is chosen so as to ensure that this mode is nonlinear in its own right.

17. Key Words (Suggested by Author(s)) crossflow vortex, Tollmein-Schlichting Wave	18. Distribution Statement Unclassified - Unlimited; 02 - Aerodynamics; 34 - Fluid Mechanics and Heat Transfer

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