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FLEXIBLE AIRCRAFT DYNAMIC MODELING FOR DYNAMIC ANALYSIS AND CONTROL SYNTHESIS

By

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ABSTRACT

The linearization and simplification of a nonlinear, literal model for flexible aircraft is highlighted. Areas of model fidelity that are critical if the model is to be used for control system synthesis are developed and several simplification techniques that can deliver the necessary model fidelity are discussed. These techniques include both numerical and analytical approaches. An analytical approach, based on first-order sensitivity theory is shown to lead not only to excellent numerical results, but also to closed-form analytical expressions for key system dynamic properties such as the pole/zero factors of the vehicle transfer-function matrix. The analytical results are expressed in terms of vehicle mass properties, vibrational characteristics, and rigid-body and aeroelastic stability derivatives, thus leading to the underlying causes for critical dynamic characteristics.

MODELING FLEXIBLE AIRCRAFT FOR DYNAMIC ANALYSIS AND CONTROL SYNTHESIS

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TOPICAL OUTLINE

- What Constitutes a Valid Model?
  - How Validity Will Be Measured
  - What's Important in a Feedback System
- Some Approaches to Obtain (Simple) Valid Models
  - Numerical
  - Literal
- Physical Causes of Critical Dynamic Characteristics

WHAT VEHICLE DYNAMIC CHARACTERISTICS ARE CRITICAL IN A FEEDBACK SYSTEM?

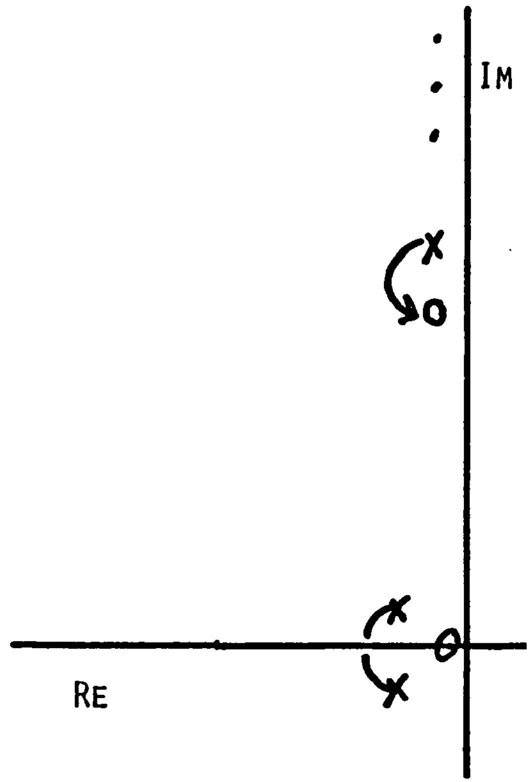
They Are Important If They Can:

— Induce Critical Pole/Zero Interactions

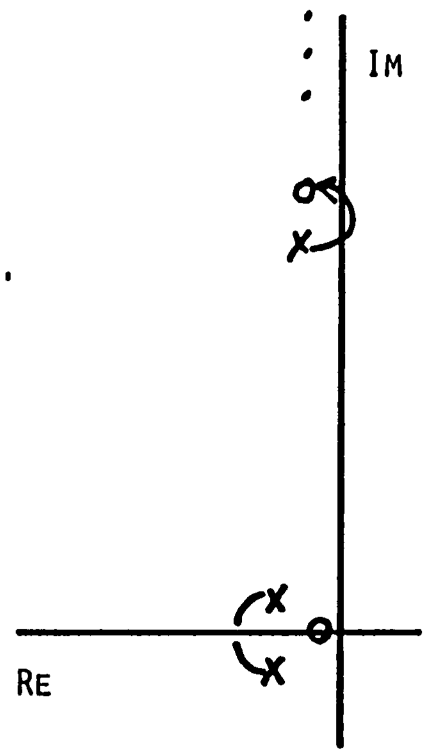
— Significantly Affect Frequency Response Near Crossover

— Significantly Affect Time Response

CRITICAL DI-POLE EFFECT

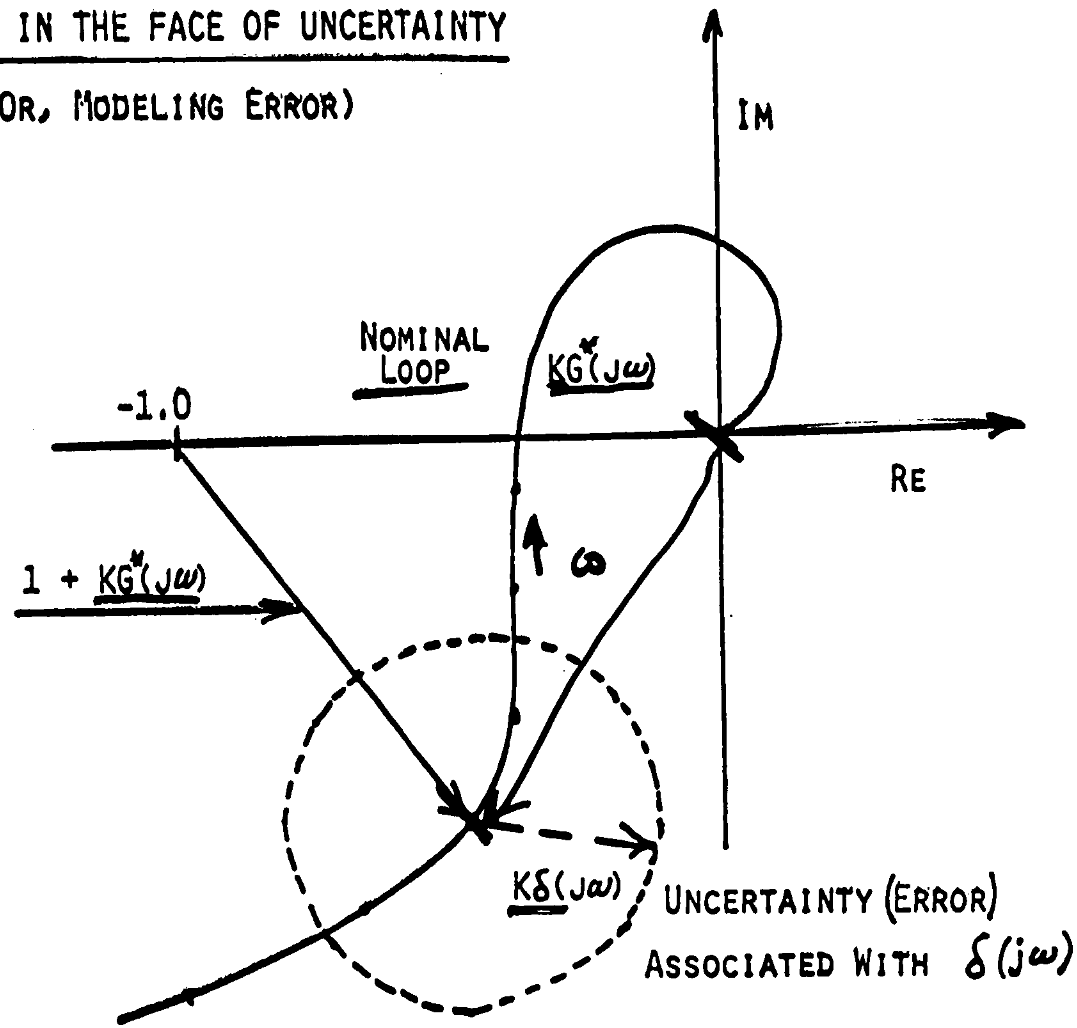


VS.



STABILITY IN THE FACE OF UNCERTAINTY

(OR, MODELING ERROR)



→ DESIGN MODEL = $G^*(j\omega)$

→ "TRUE" MODEL = $G^* + \delta$

→ STABILITY GUARANTEED IF

$$|K\delta(j\omega)| < |1 + KG^*(j\omega)|$$

or if

$$|\varepsilon| < |1 + (KG^*)^{-1}|$$

with $\varepsilon = (G^*)^{-1}\delta G$

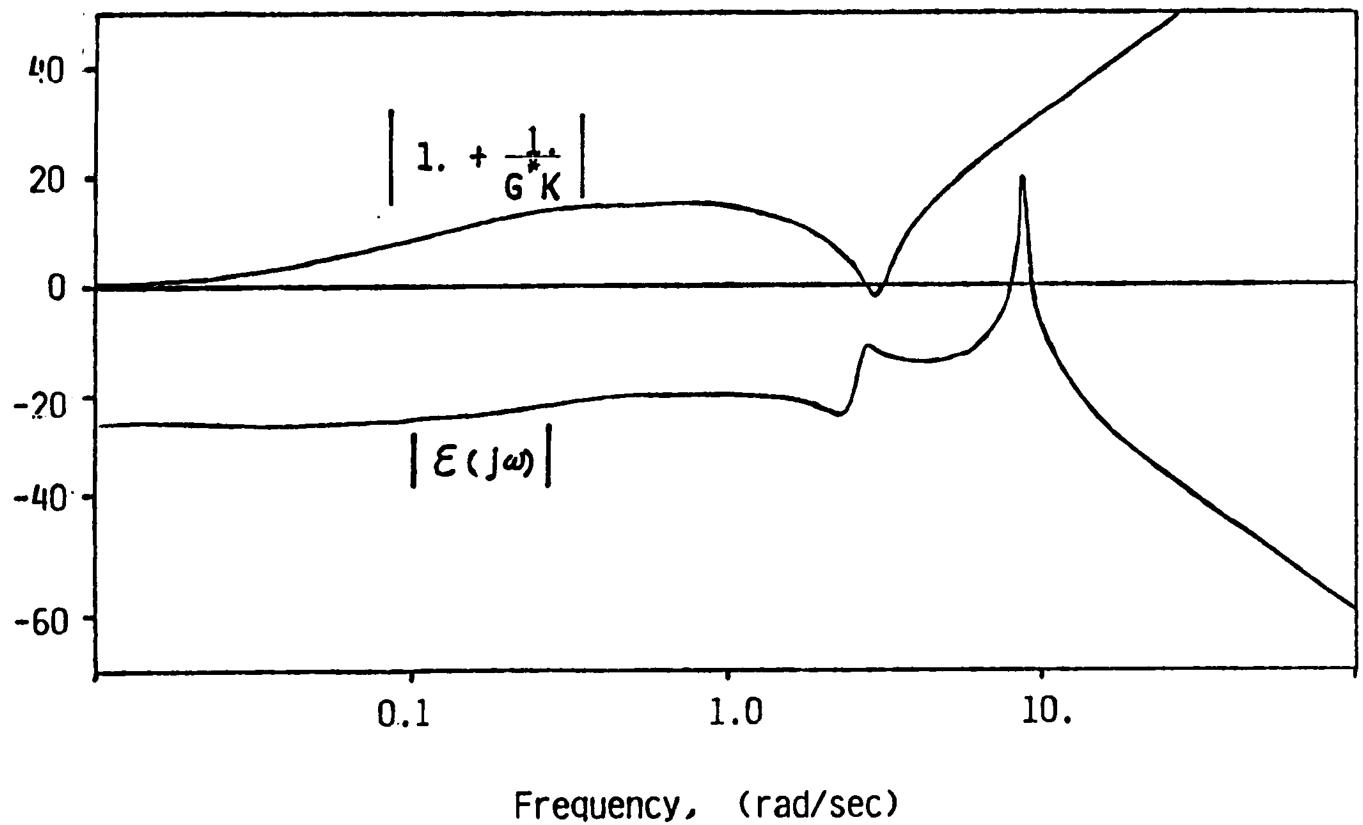
ROBUSTNESS EVALUATION

Slender-Body Configuration

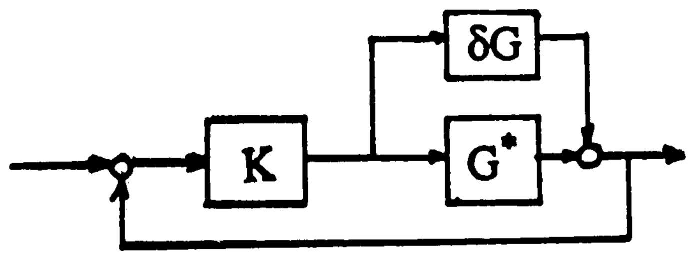
$G^* =$ Rigid Model (M=1.8, S.L.)

$\omega_1 = 9$ rad/sec

$\omega_{sp}^* = 7.2$ rad/sec



CHARACTERIZING UNCERTAINTY

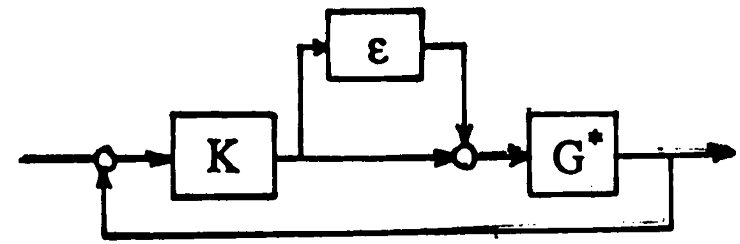


$G = G^* + \delta G$

— If N.U.P.(G) = N.U.P(G*)

Stability of the Loop
Guaranteed If

$\bar{\sigma}[\delta G K] < \underline{\sigma}[I + GK(j\omega)], \omega > 0$



$G = G^*(1 - \varepsilon)$

— If N.U.P.(G) = N.U.P.(G*)

Stability of the Loop
Guaranteed If

$\bar{\sigma}[\varepsilon] < \underline{\sigma}[I + [GK(j\omega)]^{-1}]$

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Most Critical When $\|GK\| \approx 1$ (crossover)

→ Let's Look At Potential Sources For δG or $\varepsilon(j\omega)$ ←

A MEANINGFUL METRIC FOR MODEL VALIDITY

$$\delta G = E(j\omega) = G(j\omega) - G_s(j\omega)$$
$$(\text{or } E(j\omega) = I - G^{-1}G_s)$$

In Crossover Region

Can Use Singular Values If Desired

$$\bar{\sigma}(E) = \bar{\lambda}^{1/2}(EE^*),$$

A Possible Metric

$$\|E(j\omega)\|_{CF} = \sup_{\omega_1 < \omega < \omega_2} \bar{\sigma}(E(j\omega))$$

A Conservative Metric

$$\|E(j\omega)\|_\infty = \sup_{0 < \omega < \infty} \bar{\sigma}(E(j\omega))$$

Or Just look at Bode Plots

CONSIDER THE SYSTEM IN POLYNOMIAL-MATRIX FORM

$$\begin{bmatrix} A(s) & c(s) \\ r(s) & m(s) \end{bmatrix} \begin{bmatrix} Z(s) \\ z_r(s) \end{bmatrix} = \begin{bmatrix} B(s) \\ b_r(s) \end{bmatrix} U(s)$$

$$Y(s) = M(s)Z(s) + m_r(s)z_r(s) \Rightarrow Y(s) = [G(s)]U(s)$$

$G_{ij}(s)$ Determined From

$$\frac{Z_i(s)}{U_j(s)} = \frac{m \det [A_i \mid B_j - cm^{-1}(r_i \mid b_{r_j})]}{m \det[A - cm^{-1}r]}$$

$$\frac{z_r(s)}{U_j(s)} = \frac{b_{r_j} \det[A - B_j b_{r_j}^{-1} r]}{m \det[A - cm^{-1}r]}$$

(Assumes $z_r(s)$ Scalar)

NOW ASSUME

$$c_k r_k << m \quad ; \quad k = 1, \cdots, n$$

$$B_{j_k} r_k << b_{r_j} \quad ; \quad k = 1, \cdots, n$$

$$c_k (r_i \mid b_{r_j})_k << m \quad ; \quad k = 1, \cdots, n$$

Over The Freq. Range Of Interest

In This Case

$$\left\{ \begin{array}{l} A_i \mid B_j - c m^{-1} (r_i \mid b_{r_j}) \approx A_i \mid B_j \\ A - B_j b_{r_j}^{-1} r \approx A \\ A - c m^{-1} r \approx A \end{array} \right.$$

<u>And</u>

$$\left\{ \begin{array}{l} \dfrac{Z_i(s)}{U_j(s)} \approx \dfrac{\hat{Z}_i(s)}{U_j(s)} = \dfrac{\det [A_i \mid B_j]}{\det [A]} \\ \\ \dfrac{z_r(s)}{U_j(s)} \approx b_{r_j}/m \end{array} \right.$$

CASE I

If $\frac{b_{rj}}{m} \approx 0$ In Freq. of Interest

Truncated Model

$$Y(s) \approx M(s)\hat{Z}(s)$$

$$\frac{\hat{Z}_i(s)}{U_j(s)} = \det[A_i \mid B_j]/\det[A]$$

Adequate Model When z_r Corresponds To A Low-Frequency, Or "Slow" Degree Of Freedom

Special Case – Modal Form

$$\begin{bmatrix} (sI - \Lambda) & 0 \\ 0 & (sI - \Lambda_r) \end{bmatrix} \begin{bmatrix} N(s) \\ N_r(s) \end{bmatrix} = \begin{bmatrix} B \\ B_r \end{bmatrix} U(s)$$

$$Y = M\,N(s) + M_r N_r(s)$$

Here

$$E(s) = G(s) - G_r(s) = M_r(sI - \Lambda_r)^{-1} B_r$$

$$E_{ij}(j\omega) \to 0 \text{ as } |(j\omega - \lambda_r)| \to \infty$$

CASE II

If $\frac{b_{rj}}{m} \to$ Constant ($\neq 0$) In Freq. of Interest

Residualized Model

$$Y(s) \approx M(s)\hat{Z}(s) + m_r(s)\frac{b_r(0)}{k_r}U(s)$$

$$\frac{\hat{Z}_i(s)}{U_j(s)} = \frac{\det[A_i \mid B_j]}{\det[A]}$$

Adequate Model When Zr Corresponds to A (stable) High-Frequency ("Fast") Degree Of Freedom

Special Case

Modal Resid.

$$E(s) = M_r(sI - \Lambda_r)^{-1}B_r + M_r(\Lambda_r)^{-1}B_r$$

$$= M_r[s(sI - \Lambda_r)^{-1}\Lambda_r^{-1}]B_r$$

$|E_{ij}(j\omega)| \to 0$ as $|j\omega| \to 0$

c.f. Bacon & Schmidt '88 JGCD.

Table 6. Frequency Weighted Internally Balanced Reduction

Good Approx when $|a_{ij}|$ large

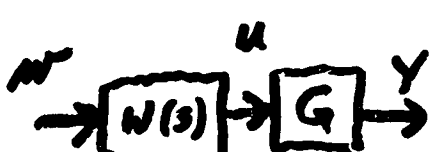
Given: System state space description A, B, C and weighting filter state space description A_w, B_w, C_w.

Find: r^{th} order system

Step 1: Solve for X and Y

$$\begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} X & X_{12} \\ X_{21} & X_{22} \end{bmatrix} + \begin{bmatrix} X & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} A^T & 0 \\ C_w^T B^T & A_w^T \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & B_w B_w^T \end{bmatrix} = 0$$

$$\begin{bmatrix} A^T & 0 \\ C_w^T B^T & A_w^T \end{bmatrix} \begin{bmatrix} Y & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} + \begin{bmatrix} Y & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix} + \begin{bmatrix} CC^T & 0 \\ 0 & 0 \end{bmatrix} = 0$$



Step 2: Find T and Σ where $XY = T\Sigma^2T^{-1}$, $T = [T_r\ T_{n-r}]$, $T^{-T} = [U_r, U_{n-r}]$

$$\Sigma^2 = \begin{bmatrix} \Sigma_r^2 & 0 \\ 0 & \Sigma_{n-r}^2 \end{bmatrix} \quad \text{where}$$

$$\Sigma_r = \mathrm{diag}(v_{c_i} v_{o_i}) \quad i = 1, \ldots, r$$

$$\Sigma_{n-r} = \mathrm{diag}(v_{c_i} v_{o_i}) \quad i = r+1, \cdots, n$$

$$v_{c_1} v_{o_1} \geq \cdots \geq v_{c_n} v_{o_n} \geq 0$$

Step 3: r^{th} order system is

$$A_r = U_r^T A T_r$$

$$B_r = U_r^T B$$

$$C_r = C T_r$$

THE SYSTEM IS DESCRIBED BY
(After Linearization)

$$\underbrace{\left[\begin{array}{c|ccc} [F_{RB}(s)] & 0 & 0 & \cdots \\ \hline [0] & s^2+2\underline{\xi_1\omega_1}s+\underline{\omega_1^2} & 0 & \cdots \\ 0 & & \ddots & \\ & & & \cdot \end{array}\right]}_{[F(s)]} \begin{bmatrix} \bar{y}_R \\ \hline \eta_1 \\ \vdots \\ \eta_m \end{bmatrix} = \underbrace{\begin{bmatrix} M_R \\ \hline F_1 \\ \vdots \\ F_m \end{bmatrix}}_{[M(s)]} \bar{\delta}_c(s)$$

(Elastically Decoupled)

$\bar{y}_R$ { Rigid-Body D.O.F.

$$\text{Aero Coupling (can include unsteady effects)} \left\{ + \underbrace{\left[\begin{array}{c|cc} [A_{RR}(s)] & [A_{RE_1}(s)] & \cdots \\ \hline [A_{E_1R}(s)] & A_{E_1}(s) & \cdots \\ \vdots & & \end{array}\right]}_{[A(s)]} \right\} \begin{bmatrix} \bar{y}_R \\ \hline \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}$$

e.g. $A_{E_i}(s) = \dfrac{N_{E_i}(s)}{D_{E_i}(s)}$, (Unsteady)

FOR QUASI-STEADY AERODYNAMICS A STANDARD FORM IS

$$\left[\begin{array}{c|c} \overbrace{\begin{matrix} (s-Z_w) & -(Z_q+U_o) \\ -M_w & (s-M_q) \end{matrix}}^{F_{RR}} & \overbrace{\begin{matrix} -(\underline{Z}_{\dot{\eta}_1} s+\underline{Z}_{\eta_1}) & \cdots \\ -(\underline{M}_{\dot{\eta}_1} s+\underline{M}_{\eta_1}) & \cdots \end{matrix}}^{F_{RE}} \\ \hline \underbrace{\begin{matrix} -\underline{F}_{1_w} & -\underline{F}_{1_q} \\ \vdots & \end{matrix}}_{F_{ER}} & \underbrace{\begin{matrix} \{s^2+(2\xi_1\omega_1-\underline{F}_{1_{\dot{\eta}_1}})s+(\omega_1^2-\underline{F}_{1\eta_1})\} & \\ & \ddots \end{matrix}}_{F_{EE}} \end{array}\right] \begin{bmatrix} w \\ \dot{\theta} \\ \hline \eta_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} Z_\delta \\ M_\delta \\ \hline F_{1\delta} \\ \vdots \end{bmatrix} \bar{\delta}$$

And, for example

$$\frac{\dot{\theta}(s)}{\delta_E(s)} = \frac{C_\theta(s+1/T_\theta)\prod^{m}(s^2+2\xi_{\theta_i}\omega_{\theta_i}s+\omega_{\theta_i}^2)}{\underbrace{(s^2+2\xi_{sp}\omega_{sp}s+\omega_{sp}^2)}_{\text{Like Rigid Body But Mod}}\underbrace{\prod^{m}(s^2+2\xi_{E_i}\omega_{E_i}s+\omega_{E_i}^2)}_{\text{Aero-Elastic Dipoles}}}$$

From Cramer's Rule

Table 4. Elastic Aircraft Linear Longitudinal Equations In Polynomial Matrix Form

$$
\begin{bmatrix}
s-X_u & -X_\alpha & -X_q s - X_\theta & -X_{\dot{\eta}_1}s - X_{\eta_1} & -X_{\dot{\eta}_2}s - X_{\eta_2} & \cdots & 0 & 0 \\
\frac{Z_u}{V_{T_1}} & s-\frac{Z_\alpha}{V_{T_1}} & -(1+\frac{Z_q}{V_{T_1}})s-\frac{Z_\theta}{V_{T_1}} & -\frac{Z_{\dot{\eta}_1}}{V_{T_1}}s-\frac{Z_{\eta_1}}{V_{T_1}} & -\frac{Z_{\dot{\eta}_2}}{V_{T_1}}s-\frac{Z_{\eta_2}}{V_{T_1}} & & 0 & 0 \\
-M_u & -M_\alpha & s^2-M_q s & -M_{\dot{\eta}_1}s-M_{\eta_1} & -M_{\dot{\eta}_2}s-M_{\eta_2} & & 0 & 0 \\
-F_{1_u} & -F_{1_\alpha} & -F_{1_q}s & s^2+(2\zeta_1\omega_1-F_{1_{\dot{\eta}_1}})s+(\omega_1^2-F_{1_{\eta_1}}) & -F_{1_{\dot{\eta}_2}}s-F_{1_{\eta_2}} & & 0 & 0 \\
-F_{2_u} & -F_{2_\alpha} & -F_{2_q}s & -F_{2_{\dot{\eta}_1}}s-F_{2_{\eta_1}} & s^2+(2\zeta_2\omega_2-F_{2_{\dot{\eta}_2}})s+(\omega_2^2-F_{2_{\eta_2}}) & & 0 & 0 \\
\vdots & & & & & \ddots & & \vdots \\
-K_u & -K_\alpha & -K_q s & -K_{\dot{\eta}_1}s-K_{\eta_1} & -K_{\dot{\eta}_2}s-K_{\eta_2} & & 1 & 0 \\
0 & 0 & -s & \phi_1' s & \phi_2' s & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u(s) \\ \alpha(s) \\ \theta(s) \\ \eta_1(s) \\ \eta_2(s) \\ \vdots \\ a_z(s) \\ q_p(s)
\end{bmatrix}
=
\begin{bmatrix}
X_{\delta_E} & X_{\delta_C} \\
\frac{Z_{\delta_E}}{V_{T_1}} & \frac{Z_{\delta_C}}{V_{T_1}} \\
M_{\delta_E} & M_{\delta_C} \\
F_{1_{\delta_E}} & F_{1_{\delta_C}} \\
F_{2_{\delta_E}} & F_{2_{\delta_C}} \\
 & \vdots \\
K_{\delta_E} & K_{\delta_C} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_E \\ \delta_C
\end{bmatrix}
$$

where

$$K_u = Z_u - x_p M_u + \sum_{i=1}^{\infty}\phi_i F_{i_u}$$

$$K_\alpha = Z_\alpha - x_p M_\alpha + \sum_{i=1}^{\infty}\phi_i F_{i_\alpha}$$

$$K_q = Z_q - x_p M_q + \sum_{i=1}^{\infty}\phi_i F_{i_q}$$

$$K_{\eta_i} = Z_{\eta_i} - x_p M_{\eta_i} - \phi_i(\omega_i^2 - F_{i_{\eta_i}}) + \sum_{j=1,\, j\neq i}^{\infty}\phi_i F_{i_{\eta_j}}$$

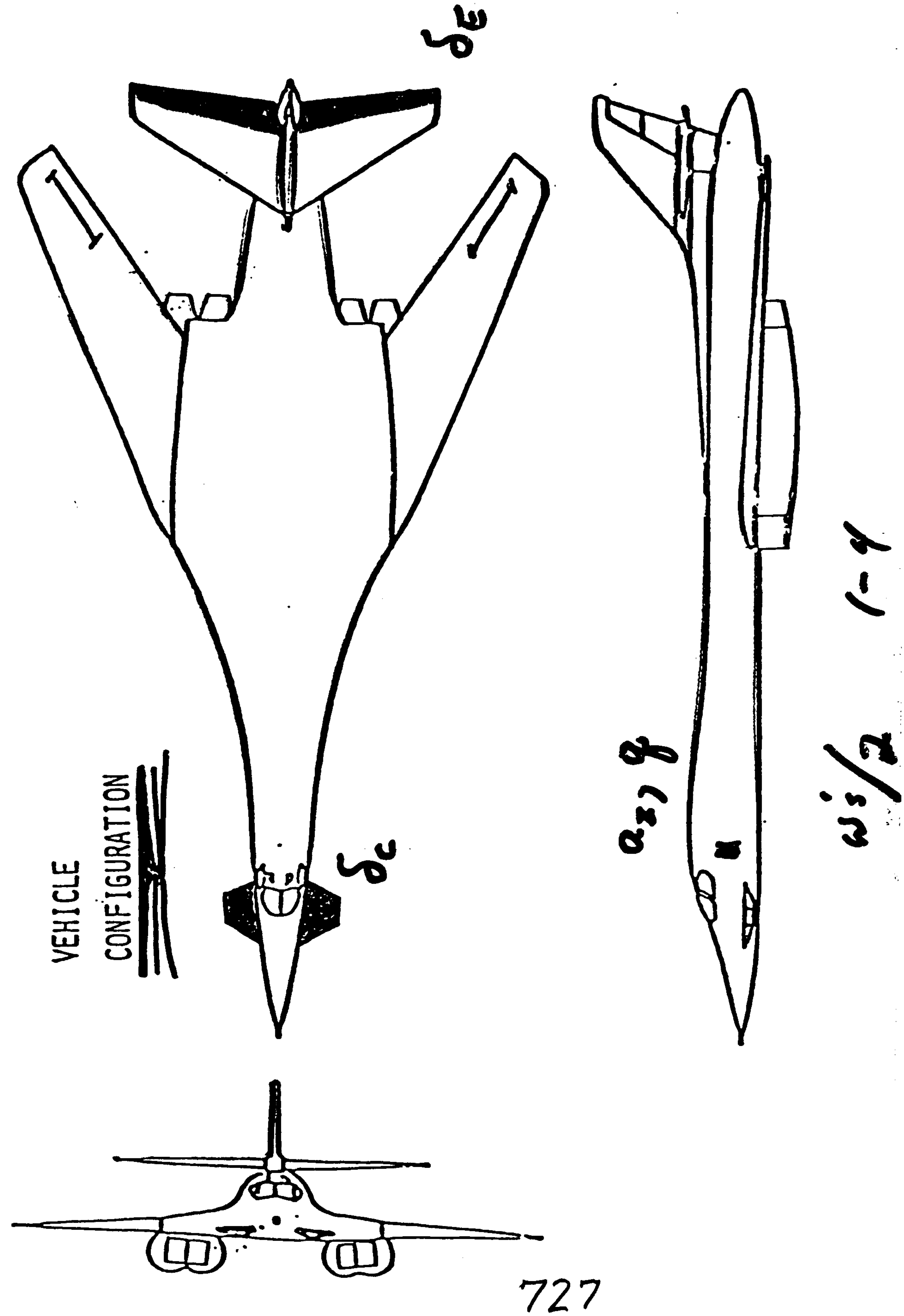
$$K_{\dot{\eta}_i} = Z_{\dot{\eta}_i} - x_p M_{\dot{\eta}_i} - \phi_i(2\zeta_i\omega_i - F_{i_{\dot{\eta}_i}}) + \sum_{j=1,\, j\neq i}^{\infty}\phi_i F_{i_{\dot{\eta}_j}}$$

$$K_{\delta_E} = Z_{\delta_E} - x_p M_{\delta_E} + \sum_{i=1}^{\infty}\phi_i F_{i_{\delta_E}}$$

$$K_{\delta_C} = Z_{\delta_C} - x_p M_{\delta_C} + \sum_{i=1}^{\infty}\phi_i F_{i_{\delta_C}}$$

$$u = \begin{Bmatrix}\delta_E \\ \delta_C\end{Bmatrix}$$

$$Y = [G(s)]\, U(s)$$



VEHICLE CONFIGURATION

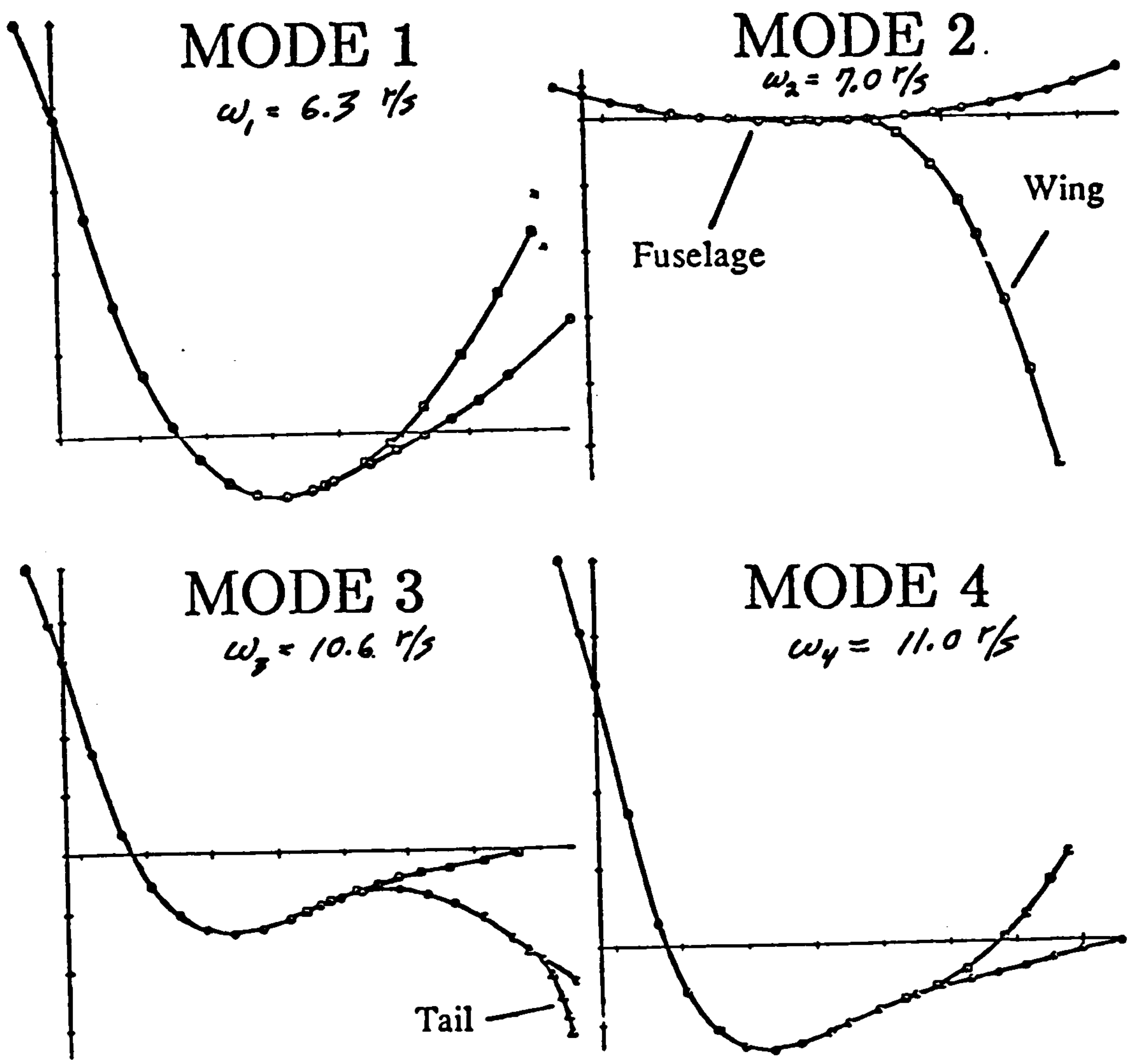
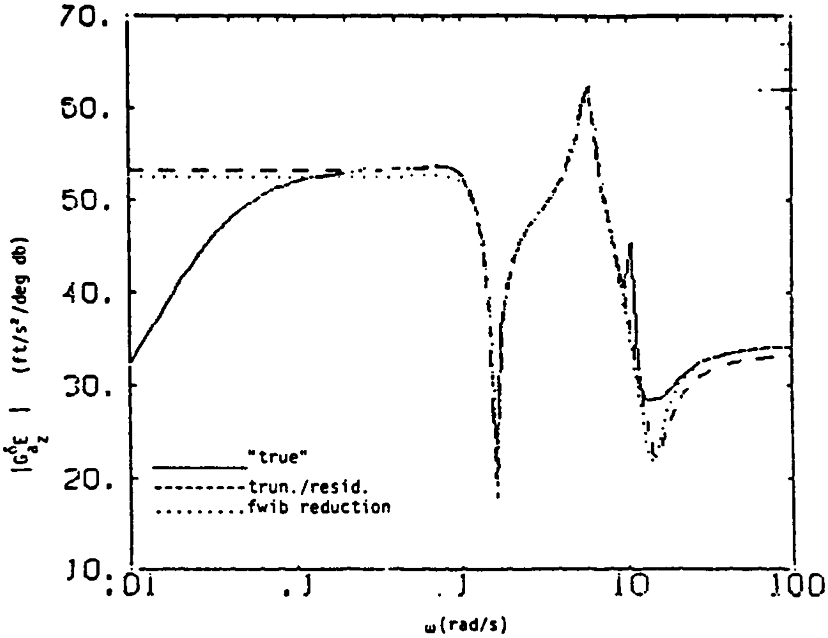

MODE 1
$\omega_1 = 6.3$ r/s
MODE 2
$\omega_2 = 7.0$ r/s
Fuselage
Wing
MODE 3
$\omega_3 = 10.6$ r/s
Tail
MODE 4
$\omega_4 = 11.0$ r/s

Table 10. Transfer Functions For the True Model

	$G_{a_z}^{\delta_E}$ (ft/s^2/deg)	$G_{q_p}^{\delta_E}$ (rad/s/deg)	$G_{a_z}^{\delta_c}$ (ft/s^2/deg)	$G_{q_p}^{\delta_c}$ (rad/s/deg)
gains	52.01	8.001	-244.5	15.65
zeros	6.473E-5	0	1.087E-4	0
	-.008887	-.05103	-.008093	-.05541
	−.01958±j1.661	-.2020	.1703±j1.795	-.1172
	−.3610±j11.00	3.642	−.8996±j4.132	−.5973±j2.912
	−1.003±j11.13	-4.020	−.2252±j10.77	−.2556±j10.68
	1.549±j11.71	−.3610±j11.00	−.3607±j11.00	−.3562±j10.99
	−3.144±j14.34	−2.838±j12.71	−2.601±j13.06	−2.564±j13.12
		.5735±j13.41		
poles	.03324			
	-.04268			
	−.4513±j1.171			
	−.4408±j6.010			
	−.2240±j10.78			
	−.3611±j11.00			
	−2.558±j13.05			

Bode Magnitude

$G_{a_z}^{\delta_E}$



Bode Phase

$G_{a_z}^{\delta_E}$

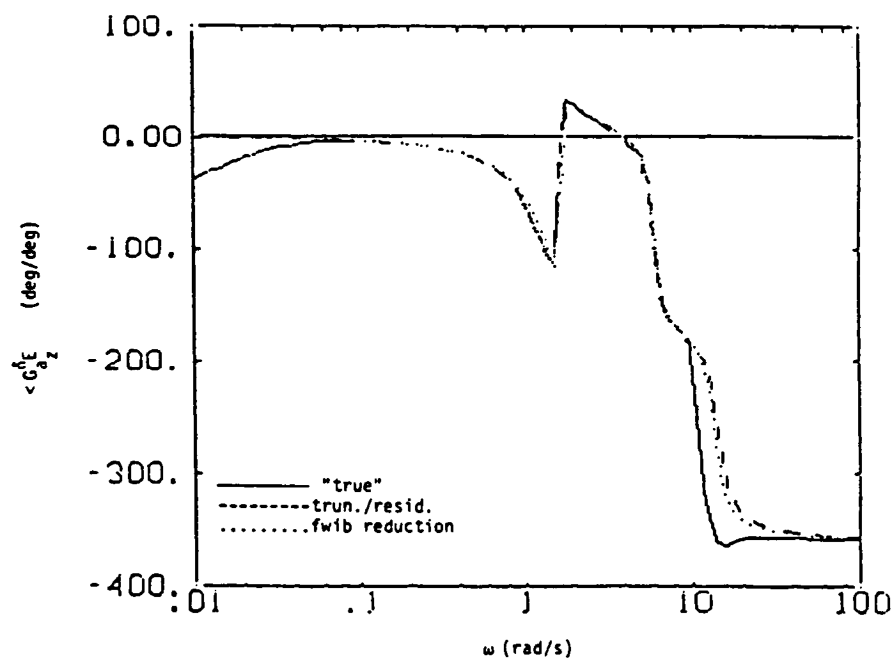
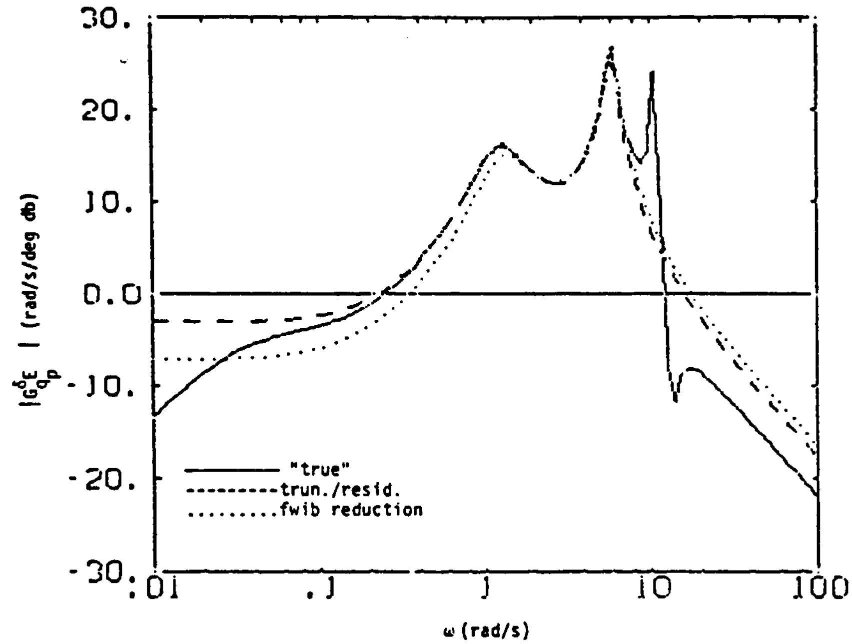


Figure 4. $< G_{a_z}^{\delta_E}$ Frequency Response

Bode Magnitude

$G_q^{\delta_E}$



Bode Phase

$G_q^{\delta_E}$

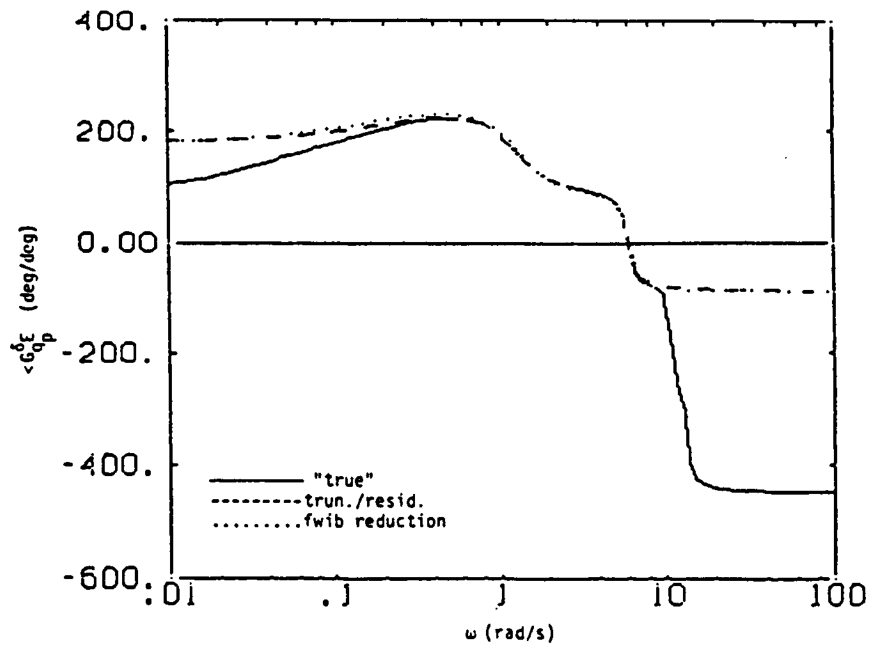
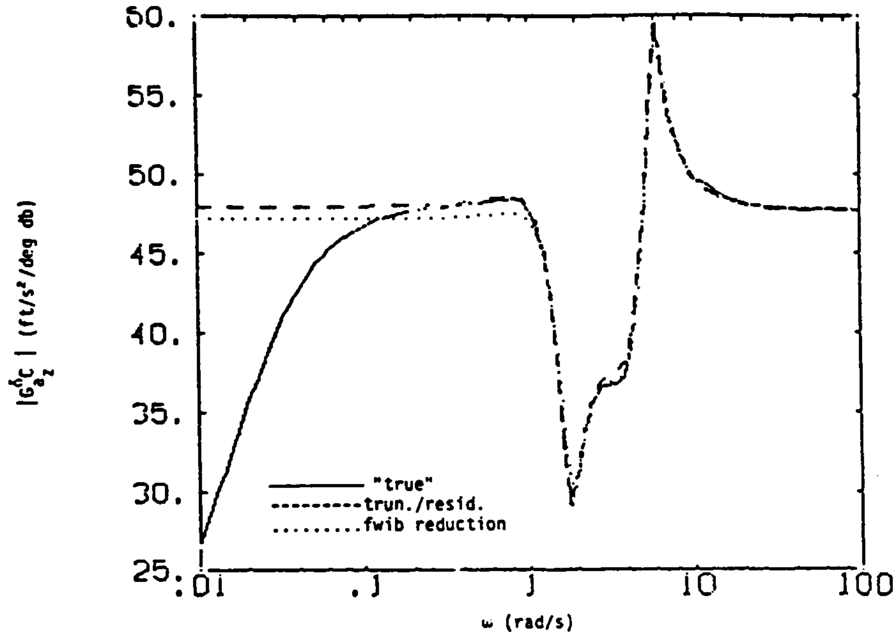


Figure 6. $<G_{q_p}^{\delta_E}$ Frequency Response

Bode Magnitude

$G_{a_z}^{\delta_c}$



Bode Phase

$G_{a_z}^{\delta_c}$

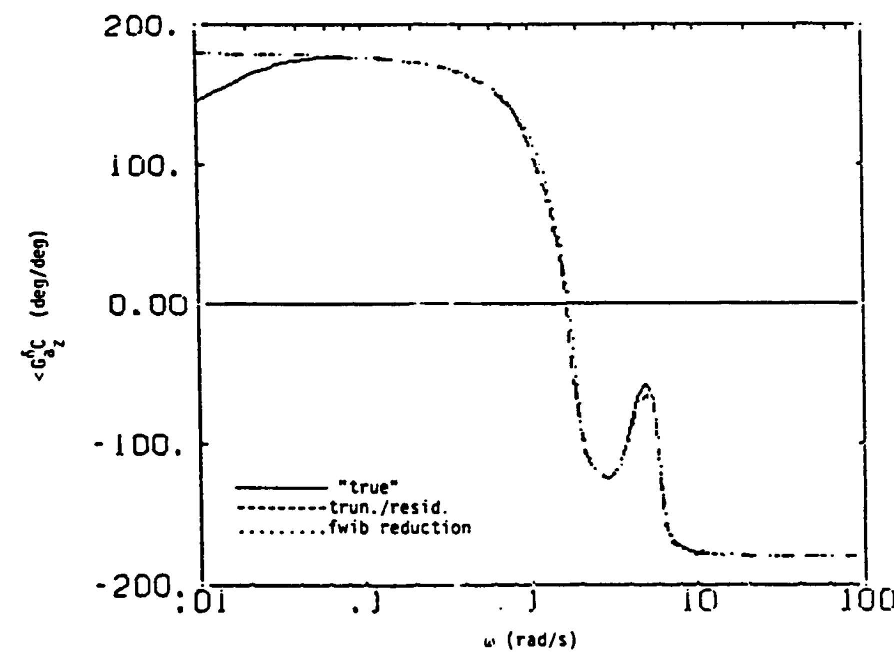
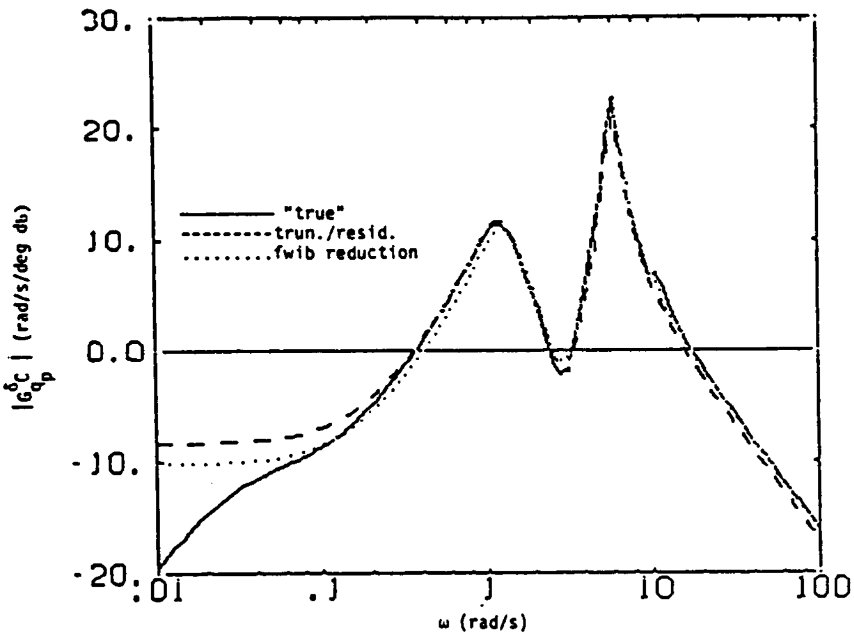


Figure 8. $< G_{a_z}^{\delta_c}$ Frequency Response

Bode Magnitude

$G_q^{\delta_c}$



Bode Phase

$G_q^{\delta_c}$

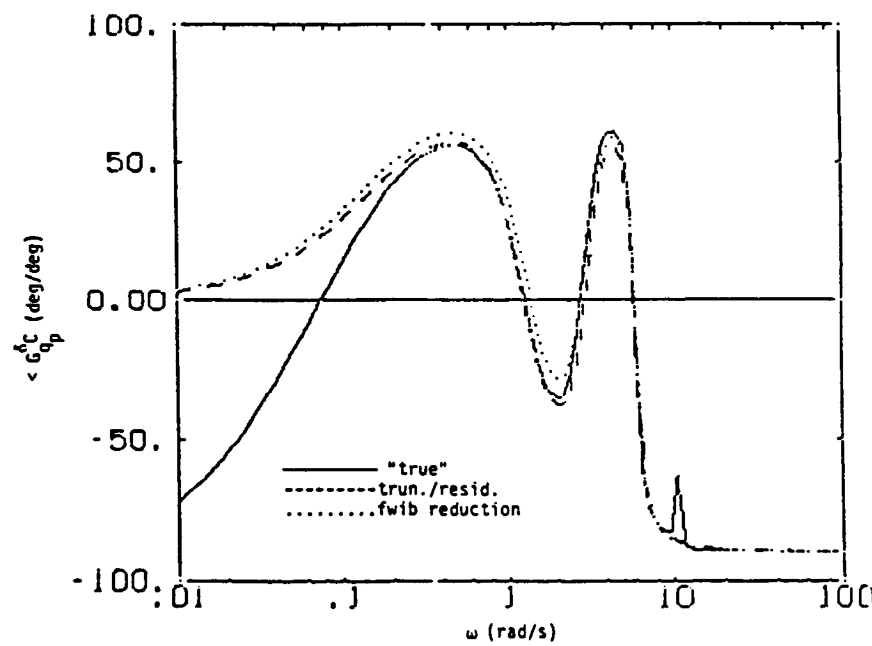


Figure 10. $< G_{q_p}^{\delta_c}$ Frequency Response

Table 12. Transfer Functions For FWIB Reduction Model

	$G_{a_z}^{\delta_B}$ (ft/s^2/deg)	$G_{q_p}^{\delta_B}$ (rad/s/deg)	$G_{a_z}^{\delta_c}$ (ft/s^2/deg)	$G_{q_p}^{\delta_c}$ (rad/s/deg)
gains	52.01	14.96	-244.5	15.29
zeros	$-.02102 \pm j1.670$	-.1789	$.1725 \pm j1.806$	-.1437
	$1.244 \pm j13.51$	2.781	$-.9177 \pm j4.143$	$-.6806 \pm j2.900$
		-3.732		
poles	$-.4679 \pm j1.234$			
	$-.4413 \pm j6.015$			

Table 14. Lower–Order Transfer Function Forms

$$G^{\delta_E}_{a_z}(s) = \frac{K^{\delta_E}_{a_z} s[s^2 + {}_{sp}(2\zeta\omega)^{\delta_E}_{a_z} s + {}_{sp}(\omega^2)^{\delta_E}_{a_z}][s^2 + {}_{f1}(2\zeta\omega)^{\delta_E}_{a_z} s + {}_{f1}(\omega^2)^{\delta_E}_{a_z}]}{D(s)}$$

$$G^{\delta_E}_{q_p}(s) = \frac{K^{\delta_E}_{q_p} s[s + {}_{sp}(\frac{1}{T})^{\delta_E}_{q_p}][s + {}_{f11}(\frac{1}{T})^{\delta_E}_{q_p}][s + {}_{f12}(\frac{1}{T})^{\delta_E}_{q_p}]}{D(s)}$$

$$G^{\delta_c}_{a_z}(s) = \frac{K^{\delta_c}_{a_z} s[s^2 + {}_{sp}(2\zeta\omega)^{\delta_c}_{a_z} s + {}_{sp}(\omega^2)^{\delta_c}_{a_z}][s^2 + {}_{f1}(2\zeta\omega)^{\delta_c}_{a_z} s + {}_{f1}(\omega^2)^{\delta_c}_{a_z}]}{D(s)}$$

$$G^{\delta_c}_{q_p}(s) = \frac{K^{\delta_c}_{q_p} s[s + {}_{sp}(\frac{1}{T})^{\delta_c}_{q_p}][s^2 + {}_{f1}(2\zeta\omega)^{\delta_c}_{q_p} s + {}_{f1}(\omega^2)^{\delta_c}_{q_p}]}{D(s)}$$

where $D(s)=s[s^2 + (2\zeta\omega)_{sp}s + (\omega^2)_{sp}][s^2 + (2\zeta\omega)_{f1}s + (\omega^2)_{f1}]$

Table 7. Truncated/Residulized Model

$$
\begin{bmatrix}
s-\dfrac{Z_\alpha}{V_{T_1}} & -\left(1+\dfrac{Z_q}{V_{T_1}}\right)s-\dfrac{Z_\theta}{V_{T_1}} & -\dfrac{Z_{\dot{\eta}_1}}{V_{T_1}}s-\dfrac{Z_{\eta_1}}{V_{T_1}} & 0 & 0 \\
-M_\alpha & s^2-M_q s & -M_{\dot{\eta}_1}s-M_{\eta_1} & 0 & 0 \\
-F_{1_\alpha} & -F_{1_q}s & s^2+(2\zeta_1\omega_1-F_{1_{\dot{\eta}_1}})s+(\omega_1^2-F_{1_{\eta_1}}) & 0 & 0 \\
-K_\alpha & -K_q s & -K_{\dot{\eta}_1}s-K_{\eta_1} & 1 & 0 \\
0 & -s & \phi_i' s & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha(s) \\ \theta(s) \\ \eta_1(s) \\ a_z(s) \\ q_p(s)
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{Z_{\delta_E}}{V_{T_1}} & \dfrac{Z_{\delta_C}}{V_{T_1}} \\
M_{\delta_E} & M_{\delta_C} \\
F_{1_{\delta_E}} & F_{1_{\delta_C}} \\
K_{\delta_E} & K_{\delta_C} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_E(s) \\ \delta_C(s)
\end{bmatrix}
$$

$$D(s) = -\left(\frac{Z_{\dot{\eta}_1}}{V_{T_1}}s + \frac{Z_{\eta_1}}{V_{T_1}}\right)[M_\alpha F_{1_q}s + F_{1_\alpha}(s^2 - M_q s)]$$

$$- (M_{\dot{\eta}_1}s + M_{\eta_1})\,[F_{1_q}s(s - \frac{Z_\alpha}{V_{T_1}}) + (1 + \frac{Z_q}{V_{T_1}})F_{1_\alpha}s] \tag{23}$$

$$\underline{+ (s^2 + (2\zeta_1\omega_1 - F_{1_{\dot{\eta}_1}})s + (\omega_1^2 - F_{1_{\eta_1}}))\,[(s - \frac{Z_\alpha}{V_{T_1}})(s^2 - M_q s) - (1 + \frac{Z_q}{V_{T_1}})M_\alpha s]}$$

$$N(s) = \frac{Z_{\delta_B}}{V_{T_1}}\{s[M_\alpha(s^2 + (2\zeta_1\omega_1 - F_{1_{\dot{\eta}_1}})s + (\omega_1^2 - F_{1_{\eta_1}})) + F_{1_\alpha}(M_{\dot{\eta}_1}s + M_{\eta_1})],$$

$$-\phi_1' s[M_\alpha F_{1_q}s + F_{1_\alpha}(s^2 - M_q s)]\}$$

$$+ M_{\delta_B}\{s[\underline{(s - \frac{Z_\alpha}{V_{T_1}})(s^2 + (2\zeta_1\omega_1 - F_{1_{\dot{\eta}_1}})s + (\omega_1^2 - F_{1_{\eta_1}}))} - F_{1_\alpha}(\frac{Z_{\dot{\eta}_1}}{V_{T_1}}s + \frac{Z_{\eta_1}}{V_{T_1}})]$$

$$- \phi_1' s[F_{1_q}s(s - \frac{Z_\alpha}{V_{T_1}}) + (1 + \frac{Z_q}{V_{T_1}})F_{1_\alpha}s]\}$$

$$+ F_{1_{\delta_B}}\{s[(s - \frac{Z_\alpha}{V_{T_1}})(M_{\dot{\eta}_1}s + M_{\eta_1}) + M_\alpha(\frac{Z_{\dot{\eta}_1}}{V_{T_1}}s + \frac{Z_{\eta_1}}{V_{T_1}})]$$

$$- \phi_1' s[\underline{(s - \frac{Z_\alpha}{V_{T_1}})}(s^2 - M_q s) - M_\alpha(1 + \frac{Z_q}{V_{T_1}})s]\}$$

Table 8. Polynomial Coefficient and Factor Relationships

$$\tilde{d}_1 = (\tilde{\omega}^2)_{sp}(\tilde{\omega}^2)_{f1}$$

$$\tilde{d}_2 = (2\tilde{\zeta}\omega)_{sp}(\tilde{\omega}^2)_{f1} + (2\tilde{\zeta}\omega)_{f1}(\tilde{\omega}^2)_{sp}$$

$$\tilde{d}_3 = (\tilde{\omega}^2)_{sp} + (\tilde{\omega}^2)_{f1} + (2\tilde{\zeta}\omega)_{sp}(2\tilde{\zeta}\omega)_{f1}$$

$$\tilde{d}_4 = (2\tilde{\zeta}\omega)_{sp} + (2\tilde{\zeta}\omega)_{f1}$$

$$\tilde{n}_1 = \tilde{K}_{q_p}^{\delta_B}\ {}_{sp}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B}\ {}_{f11}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B}\ {}_{f12}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B}$$

$$\tilde{n}_2 = \tilde{K}_{q_p}^{\delta_B}\left[\ {}_{sp}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B}\ {}_{f11}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B} + {}_{sp}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B}\ {}_{f12}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B} + {}_{f11}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B}\ {}_{f12}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B}\right]$$

$$\tilde{n}_3 = \tilde{K}_{q_p}^{\delta_B}\left[\ {}_{sp}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B} + {}_{f11}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B} + {}_{f12}\left(\frac{1}{\tilde{T}}\right)_{q_p}^{\delta_B}\right]$$

$$\tilde{n}_4 = \tilde{K}_{q_p}^{\delta_B}$$

Table 9. Polynomial Coefficient and Factor Difference Relationships

$$\Delta\tilde{d}_1 = \tilde{\omega}_{sp}^2 \Delta(\tilde{\omega}^2)_{f1} + (\tilde{\omega}^2)_{f1} \Delta(\tilde{\omega}^2)_{sp}$$

$$\Delta\tilde{d}_2 = (2\tilde{\zeta}\omega)_{sp} \Delta(\tilde{\omega}^2)_{f1} + (\tilde{\omega}^2)_{f1} \Delta(2\tilde{\zeta}\omega)_{sp} + (2\tilde{\zeta}\omega)_{f1} \Delta(\tilde{\omega}^2)_{sp} + (\tilde{\omega}^2)_{sp} \Delta(2\tilde{\zeta}\omega)_{f1}$$

$$\Delta\tilde{d}_3 = \Delta(\tilde{\omega}^2)_{sp} + \Delta(\tilde{\omega}^2)_{f1} + (2\tilde{\zeta}\omega)_{sp} \Delta(2\tilde{\zeta}\omega)_{f1} + (2\tilde{\zeta}\omega)_{f1} \Delta(2\tilde{\zeta}\omega)_{sp}$$

$$\Delta\tilde{d}_4 = \Delta(2\zeta\omega)_{sp} + \Delta(2\tilde{\zeta}\omega)_{f1}$$

$$\Delta\tilde{n}_1 = \tilde{K}_{q_p}^{\delta_B}\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}\, {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} \Delta\, {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + \tilde{K}_{q_p}^{\delta_B}\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}\, {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} \Delta\, {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}$$

$$+ \tilde{K}_{q_p}^{\delta_B}\, {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}\, {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} \Delta\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}\, {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}\, {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} \Delta\tilde{K}_{q_p}^{\delta_B}$$

$$\Delta\tilde{n}_2 = \tilde{K}_{q_p}^{\delta_B}[\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}] \Delta\, {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + \tilde{K}_{q_p}^{\delta_B}[\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}] \Delta\, {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}$$

$$+ \tilde{K}_{q_p}^{\delta_B}[\, {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}] \Delta\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + [\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}\, {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}$$

$$+ {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}\, {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}\, {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}] \Delta\tilde{K}_{q_p}^{\delta_B}$$

$$\Delta\tilde{n}_3 = \tilde{K}_{q_p}^{\delta_B} \Delta\, {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + \tilde{K}_{q_p}^{\delta_B} \Delta\, {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + \tilde{K}_{q_p}^{\delta_B} \Delta\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + [\, {}_{sp}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + {}_{f11}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B} + {}_{f12}(\frac{1}{\tilde{T}})_{q_p}^{\delta_B}] \Delta\tilde{K}_{q_p}^{\delta_B}$$

$$\Delta\tilde{n}_4 = \Delta\tilde{K}_{q_p}^{\delta_B}$$

Table 14. Transfer Functions from Literal Approximations

	$G_{a_z}^{\delta_B}$ (ft/s^2/deg)	$G_{q_p}^{\delta_B}$ (rad/s/deg)	$G_{a_z}^{\delta_c}$ (ft/s^2/deg)	$G_{q_p}^{\delta_c}$ (rad/s/deg)
gains	46.43	13.06	-241.5	14.06
zeros	0 −.1301±j2.039 1.854±j14.25	0 -.2038 3.497 -3.946	0 .1092±j1.834 −.7248±j4.006	0 -.1592 −.7430±j2.825
poles	0 −.4497±j1.246 −.4838±j6.050			

CLOSED-FORM EXPRESSIONS FOR THESE TERMS

Specifically, it has been shown that for a slender (low aspect ratio) vehicle

$$\omega_{sp}^{*2} - \omega_{sp}^{2} = \frac{(U_o+Z_q)M_{\eta_1}F_{1_w}}{(\omega_1^2-F_{1_{\eta_1}})+(U_o+Z_q)M_w}$$

$$\omega_{1_E}^{2} - \omega_{\theta_1}^{2} = \frac{(U_o+Z_q)M_{\eta_q}F_{1_w}}{(\omega_1^2-F_{1_{\eta_a}})+(U_o+Z_q)M_w} - M_{\eta_1} F_{1\delta}/M_\delta$$

$$2\xi_{E_1}\omega_{E_1} = (2\xi_1\omega_1-F_{1_{\dot{\eta}_1}}) + \frac{Z_{\eta_1}F_{1_w}+M_{\eta_1}F_{1_q}}{(\omega_1^2-F_{1_{\eta_1}})+(U_o+Z_q)M_w}$$

FURTHERMORE, IT CAN BE SHOWN

For Example, That

$$M_{\eta_i} = \frac{1}{2} \rho V^2 S\bar{c} C_{m_{\eta_i}}$$

$$C_{m_{\eta_i}} = \frac{1}{S\bar{c}} \left\{ \int_{-b/2}^{b/2} c_{l_{\alpha_w}} \left(\frac{\Delta x_w}{c_w}\right)\left(\frac{d\phi_i}{dx}\right)_w c_w^2 dy_w + C_{L_{\alpha_c}} \left(\frac{\Delta x_c}{\bar{c}}\right)\left(\frac{d\phi_i}{dx}\right)_c S_c \bar{c} \right\}$$

where

$c_{l_{\alpha_w}}$ = wing section lift–curve slope

$\left(\frac{d\phi_i}{dx}\right)_{(\cdot)}$ = mode i slope at location $(\cdot)$

SUMMARY

- Key Issues In Feedback Systems Reviewed
- Physics Of Model Uncertainty Addressed
- Mode Frequencies Near Crossover
- Mode Shapes And Dipoles

Purely Elastic

- With Aeroelastic Coupling, Both Affected
- Closed-Form Expressions Developed To Show Sources Of Interactions
- Sensitivity To Uncertainty In These Parameter May Be Further Explored