Effect of Particle Velocity Fluctuations
on the Inertia Coupling in Two-Phase Flow

by

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Introduction

The fluid dynamics of flows of dispersed materials in a fluid is fundamental to suspensions, bubbly liquids, droplet flows, pneumatic transport, fluidization and erosion. Equations of motion to describe these materials must deal with the interactions between them as well as the deformation of the carrier fluid. Models that treat assemblages of solid particles have been proposed and studied (Jenkins and Savage, 1983) that result in the particles behaving like a gas, with a pressure due to the fluctuations in the velocities that is attributed to collisional motions of the indivual particles. Models for fluid-particle mixtures (Drew 1983) do not include this effect (Passman 1989) The purpose of the present paper is to derive constitutive equations to supplement the equations of motion that include the effects of the particle velocity fluctuations. The particle motions are assumed to be at a sufficiently high Reynolds number that the fluid motion is inviscid, but viscous effects such as boundary layer separation are neglected.

Equations of Motion

The appropriate general average is the ensemble average. The ensemble average is the appropriate generalization of adding the values of the variable for each realization, and dividing by the number of observations. We shall refer to a "process" as the set of possible flows that could occur, given that the initial and boundary conditions are those appropriate to the physical situation that we wish to describe. We refer to a "realization" of the flow as a possible motion that could have happened. Generally, we expect an infinite number of realizations of the flow, consisting of variations of position, attitude, and velocities of the discrete units and the fluid between them.

If f is some field (i.e., a function of position $\mathbf{x}$ and time t) for some particular real-

ization, μ, of the process, then the average of f is

$$(1) \qquad \overline{f}(\mathbf{x}, t) = \int_M f(\mathbf{x}, t; \mu) dm(\mu)$$

where $dm(\mu)$ is the measure (probability) of observing process μ and M is the set of all processes. We refer to M as the ensemble. The ensemble average is the fundamental average that allows the interpretation of the phenomena in terms of the repeatability of experiments. Any one exact experiment or realization will not be repeatable; however, any repetition of the experiment will lead to another member of the ensemble.

In order to apply the averaging procedure to the equations of motion, we shall need some results about the averaging procedure. We shall also give these results for time- and volume averaging.

In order to average to the exact equations, we need expressions for $\overline{\partial f / \partial t}$ and $\overline{\nabla f}$. If f is "well behaved", then it is clear from the definition of the ensemble average that

$$(2) \qquad \overline{\frac{\partial f}{\partial t}} = \frac{\partial \overline{f}}{\partial t}$$

and

$$(3) \qquad \overline{\nabla f} = \nabla \overline{f}$$

Functions are generally discontinuous at the interface in most multiphase flow. They are well behaved within each phase, however. Thus, consider $\overline{X_k \nabla f}$, where X_k is the phase indicator function for phase k:

$$(4) \qquad X_k = \begin{cases} 1, & \text{if } \mathbf{x} \in k; \\ 0, & \text{otherwise.} \end{cases}$$

In the averaging process we will require the result

$$(5) \qquad \frac{\partial X_k}{\partial t} + \mathbf{v}_i \cdot \nabla X_k = 0.$$

This relation has a reasonable physical explanation. Note that it is the "material" derivative of X_k following the interface. If we look at a point that is not on the interface, then either $X_k = 1$ or $X_k = 0$. In either case, the partial derivatives are both zero, and hence

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the expression (5) is zero. If we consider a point on the interface, if we move with the interface, we see the function X_k as a constant jump. Thus, its material derivative is zero.

The averaged equations are

Mass

$$(6) \qquad \frac{\partial \overline{X_k \rho}}{\partial t} + \nabla \cdot \overline{X_k \rho \mathbf{v}} = \overline{\rho(\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k}$$

Momentum

$$(7) \qquad \frac{\partial \overline{X_k \rho \mathbf{v}}}{\partial t} + \nabla \cdot \overline{X_k \rho \mathbf{v} \mathbf{v}} = \nabla \cdot \overline{X_k \mathbf{T}} + \overline{X_k \rho \mathbf{g}} + \overline{(\rho \mathbf{v}(\mathbf{v} - \mathbf{v}_i) - \mathbf{T}) \cdot \nabla X_k}.$$

Next, we define the appropriate average variables describing multiphase mechanics.

First, the geometry of the exact, or microscopic situation is defined in terms of the phase function X_k. The average of X_k is the average fraction of the occurrences of phase k at point $\mathbf{x}$ at time t.

$$(8) \qquad \alpha_k = \overline{X_k}$$

The quantity $\partial X_k / \partial n$ is the delta function defining the interface. its average is the interfacial area density.

$$(9) \qquad a_i = \overline{\frac{\partial X_k}{\partial n}}$$

All the remaining variables are defined in terms of weighted averages. The main, or "phasic" variables are either phasic weighted variables (weighted with the phase function X_k) or mass-weighted (or Favré) averaged (weighted by $X_k \rho$).

The "conserved" variables are the density

$$(10) \qquad \overline{\rho}_k^x = \overline{X_k \rho} / \alpha_k,$$

and the velocity

$$(11) \qquad \overline{\mathbf{v}}_k^{x\rho} = \overline{X_k \rho \mathbf{v}} / \alpha_k \overline{\rho}_k^x$$

The averaged stress is defined by

$$(12) \qquad \overline{\mathbf{T}}_k^x = \overline{X_k \mathbf{T}}/\alpha_k$$

The average body force is

$$(13) \qquad \overline{\mathbf{g}}_k^{x\rho} = \overline{X_k \rho \mathbf{g}}/\alpha_k \overline{\rho}_k^x$$

As discussed above, several terms appear representing the actions of the convective and molecular fluxes at the interface. The convective flux terms are the mass generation rate

$$(14) \qquad \Gamma_k = \overline{\rho(\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k}$$

and the interfacial momentum flux

$$(15) \qquad \mathbf{v}_{ki}^m \Gamma_k = \overline{\rho \mathbf{v}(\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k}$$

The interfacial momentum source is defined by

$$(16) \qquad \mathbf{M}_k = -\overline{\mathbf{T} \cdot \nabla X_k}$$

The motion of the interfaces gives rise to velocities that are not "laminar" in general. The velocity fluctuations may be due to turbulence or to the motion in the phases due to the motion of the interfaces. The effect of these velocity fluctuations, whatever their source, on a variable is accounted for by introducing its fluctuating field (denoted by the prime superscript), which is the difference between the complete field and the appropriate mean field. For example,

$$\mathbf{v}_k' = \mathbf{v} - \overline{\mathbf{v}}_k^{x\rho}$$

Then,

$$(17) \qquad \begin{aligned} \overline{X_k \rho \mathbf{v} \mathbf{v}} &= \overline{X_k \rho (\overline{\mathbf{v}}_k^{x\rho} + \mathbf{v}_k')(\overline{\mathbf{v}}_k^{x\rho} + \mathbf{v}_k')} \\ &= \overline{X_k \rho} \, \overline{\mathbf{v}}_k^{x\rho} \overline{\mathbf{v}}_k^{x\rho} + \overline{X_k \rho \mathbf{v}_k' \mathbf{v}_k'} \\ &= \alpha_k \overline{\rho}_k^x \overline{\mathbf{v}}_k^{x\rho} \overline{\mathbf{v}}_k^{x\rho} - \alpha_k \mathbf{T}_k^{Re}. \end{aligned}$$

The Reynolds stress is defined by

$$(18) \qquad \mathbf{T}_k^{Re} = -\overline{X_k \rho \mathbf{v}_k' \mathbf{v}_k'}/\alpha_k,$$

The averaged interfacial pressure p_{ki} and shear stress τ_{ki} are introduced to separate mean field effects from local effects in the interfacial force. The interfacial pressure is defined by

$$(19) \qquad p_{ki} = \overline{p \partial X_k / \partial n_k}/a_i$$

and the interfacial shear stress is

$$(20) \qquad \tau_{ki} = \overline{\tau_k \partial X_k / \partial n_k}/a_i$$

Thus,

$$\mathbf{M}_k = -\,\overline{\mathbf{T} \cdot \nabla X_k}$$
$$= \overline{p \nabla X_k} - \overline{\tau \cdot \nabla X_k}$$
$$= p_{ki} \overline{\nabla X_k} - \tau_{ki} \cdot \overline{\nabla X_k} - \overline{\mathbf{T}_{ki}' \cdot \nabla X_k}$$
$$(21) \qquad = p_{ki} \nabla \alpha_k - \tau_{ki} \nabla \alpha_k + \mathbf{M}_k',$$

where we define the interfacial extra momentum source

$$(22) \qquad \mathbf{M}_k' = \mathbf{M}_k + p_{ki} \nabla \alpha_k - \tau_{ki} \cdot \nabla \alpha_k$$

and introduce

$$\mathbf{T}_{ki}' = -p_{ki}' \mathbf{I} + \tau_{ki}' = -(p - p_{ki})\mathbf{I} + (\tau - \tau_{ki}).$$

The averaged equations governing each phase are

Mass

$$(23) \qquad \frac{\partial \alpha_k \overline{\rho}_k^x}{\partial t} + \nabla \cdot \alpha_k \overline{\rho}_k^x \overline{\mathbf{v}}_k^{x\rho} = \Gamma_k$$

Momentum

$$(24) \qquad \frac{\partial \alpha_k \overline{\rho}_k^x \overline{\mathbf{v}}_k^{x\rho}}{\partial t} + \nabla \cdot \alpha_k \overline{\rho}_k^x \overline{\mathbf{v}}_k^{x\rho} \overline{\mathbf{v}}_k^{x\rho} = \nabla \cdot \alpha_k \left(\overline{\mathbf{T}}_k^x + \mathbf{T}_k^{Re} \right) + \alpha_k \overline{\rho}_k \mathbf{g} + \mathbf{M}_k + \mathbf{v}_{ki}^m \Gamma_k$$

The equation of conservation of mass for phase k (23) can be used in the momentum equation (24) to yield the Lagrangian form of the momentum equation:

$$\alpha_k \overline{\rho}_k^x \frac{D_k \overline{\mathbf{v}}_k^{x\rho}}{Dt} = \alpha_k \overline{\rho}_k^x \left(\frac{\partial \overline{\mathbf{v}}_k^{x\rho}}{\partial t} + \overline{\mathbf{v}}_k^{x\rho} \cdot \nabla \overline{\mathbf{v}}_k^{x\rho} \right)$$

$$= \nabla \cdot \alpha_k (\overline{\mathbf{T}}_k^x + \mathbf{T}_k^{Re}) + \alpha_k \overline{\rho}_k^x \mathbf{g} + \mathbf{M}_k'$$

(25)
$$+ p_{ki} \nabla \alpha_k - \tau_{ki} \cdot \nabla \alpha_k + (\mathbf{v}_{ki}^m - \overline{\mathbf{v}}_k^{x\rho}) \Gamma_k .$$

The jump conditions are derived by multiplying the exact jump condition by $\mathbf{n}_1 \cdot \nabla X_1$ and averaging. This process yields the following conditions:

Mass

(26)
$$\Gamma_1 + \Gamma_2 = 0$$

Momentum

(27)
$$\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{v}_{1i}^m \Gamma_1 + \mathbf{v}_{2i}^m \Gamma_2 = \mathbf{m}$$

Constitutive Equations

The exact equations of motion can be solved for the flow of an inviscid, incompressible irrotational fluid around an isolated sphere. We shall use this solution to derive information about constitutive equations for the force on the dispersed phase, the average stress, the Reynolds stress, and the interfacial pressure when the particle phase is allowed to have a random velocity.

The fluid velocity at $\mathbf{x}$ for the irrotational flow of an incompressible inviscid fluid is expressed in terms of the velocity potential by

(28)
$$\mathbf{v}(\mathbf{x}) = \nabla \phi(\mathbf{x})$$

The continuity equation becomes

(28)
$$0 = \nabla \cdot \mathbf{v} = \nabla \cdot \nabla \phi = \nabla^2 \phi .$$

The pressure at any point $\mathbf{x}$ is given in terms of the velocity by Bernoulli's equation.

(30)
$$p - \rho \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) = p_0 = \text{constant.}$$

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Consider a sphere located at a point $\mathbf{z}$ in a flow field, moving with velocity $\mathbf{v}_p$. The boundary condition at the surface of the sphere is

$$(31) \qquad \mathbf{n} \cdot \mathbf{v}_p = \mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \nabla\phi \text{ at } |\mathbf{x} - \mathbf{z}| = a,$$

where a is the radius of the sphere, $\mathbf{n}$ is the normal to the surface of the sphere and $\mathbf{v}_p$ is the velocity of the sphere. The boundary condition far from the sphere is

$$(32) \qquad \phi \to \phi_\infty \text{ as } |\mathbf{x} - \mathbf{z}| \to \infty,$$

where

$$\phi_\infty = \mathbf{v}_0(t) \cdot \mathbf{x} + \frac{1}{2}\mathbf{x} \cdot \mathbf{e}_f \cdot \mathbf{x}$$

is the velocity potential that would exist in the fluid if the sphere were not present. Here $\mathbf{v}_0(t)$ is the (unsteady) velocity of the fluid at the origin, and $\mathbf{e}_f$ is the rate of strain tensor for the fluid. We shall assume that $\mathbf{e}_f$ is constant.

A convenient form for the solution of this problem is given by Voinov (1973), and is

$$
\begin{aligned}
\phi = {}& \mathbf{v}_0(t) \cdot \mathbf{x} + \frac{1}{2}\mathbf{x} \cdot \mathbf{e}_f \cdot \mathbf{x} \\
& + \frac{1}{2}\left(\mathbf{v}_p - \mathbf{v}_0(t) - \mathbf{z} \cdot \mathbf{e}_f\right) \cdot (\mathbf{x} - \mathbf{z})\left(\frac{a^3}{r^3}\right) \\
& + \frac{1}{3}(\mathbf{x} - \mathbf{z}) \cdot \mathbf{e}_f \cdot (\mathbf{x} - \mathbf{z})\left(\frac{a^5}{r^5}\right)
\end{aligned}
$$

$$(33)$$

If there are many spheres in the flow field, the solution given above (33) will still be a good approximation if the spheres are sufficiently far apart that the flow disturbances due to the individual spheres do not interact. That is, the flow must be sufficiently dilute. Thus, we can think of each sphere as "isolated" in the sense that it only interacts with its neighbors through the averaged fields. We assume that each sphere lies in a "cell," and inside that cell, the velocity is given by eq. (33). We shall approximate the cell to be a sphere of radius R. We choose R so that

$$\frac{4}{3}\pi a^3 \Big/ \frac{4}{3}\pi R^3 = \alpha.$$

Averaging

We now introduce a means of evaluating the averaging process. The first aspect of the ensemble average is that the sphere velocity is random, with a distribution function $f^{(1)}(\mathbf{v}_p, \mathbf{x}, t)$. The second aspect is that the sphere and the surrounding cell can lie with the sphere center anywhere within radius R of the space point $\mathbf{x}$. The average is performed by first integrating over the distribution of the velocities that the sphere can have, followed by an integration over the possible positions that the sphere center can have. Note that if $|\mathbf{x} - \mathbf{z}| < a$, the material making up the sphere occupies the field point $\mathbf{x}$ and if $|\mathbf{x} - \mathbf{z}| > a$ the fluid occupies the field point $\mathbf{x}$.

The average of a quantity $g(\mathbf{x}, t; \mathbf{z}, \mathbf{v}_p)$ is performed in two parts, that is first, we perform a conditional average of g for given sphere position $\mathbf{z}$, integrating over the velocity space $\mathbf{v}_p$, followed by the average over the spatial positions the sphere can have. We assume that the distribution of positions is such that

$$\frac{dV}{\frac{4}{3}\pi R^3} \left[1 - \mathbf{x}' \cdot \frac{\nabla \alpha_d(\mathbf{x}, t)}{\alpha_d(\mathbf{x}, t)} \right]$$

is the probability of finding the sphere in a volume dV surrounding the point $\mathbf{z}$, where $\mathbf{x}' = \mathbf{x} - \mathbf{z}$. Thus, for the average over the fluid of a quantity g, we have

$$(34) \qquad \overline{g}(\mathbf{x}, t | \mathbf{z}) = \int_{R^3} g(\mathbf{x}, t; \mathbf{z}, \mathbf{v}_p) f^{(1)}(\mathbf{v}_p, \mathbf{z}, t) dV_{\mathbf{v}_p}.$$

Here the notation $\overline{g}(\mathbf{x}, t | \mathbf{z})$ is intended to suggest the conditional average assuming the sphere is located at $\mathbf{z}$. The average of g over the fluid phase is then given by

$$(35) \qquad \overline{g}_c^x(\mathbf{x}, t) = \frac{1}{\frac{4}{3}\pi(R^3 - a^3)} \int_a^R \int \int_{\Omega(r)} \overline{g}(\mathbf{x}, t | \mathbf{z}) d\Omega dr,$$

where $\Omega(r)$ is the sphere of radius r centered at $\mathbf{x}$, and the integration is over the $\mathbf{z}$ variable.

It will be convenient to introduce the average particle velocity and the fluctuation particle velocity as

$$(36) \qquad \overline{\mathbf{v}}_p(\mathbf{x}, t) = \int_{R^3} \mathbf{v}_p f^{(1)}(\mathbf{v}_p, \mathbf{x}, t) \, dV_{\mathbf{v}_p}$$

$$(37) \qquad \mathbf{v}_p'(\mathbf{x}, t) = \mathbf{v}_p - \overline{\mathbf{v}}_p(\mathbf{x}, t)$$

Note that

$$(38) \qquad \overline{\mathbf{v}'_p} = 0$$

The particle kinetic energy per unit particle mass is

$$(39) \qquad u_d^{Re}(\mathbf{x},t) = \frac{1}{2} \int_{R^3} |\mathbf{v}'_p|^2 f^{(1)}(\mathbf{v}_p,\mathbf{x},t)\, dV_{\mathbf{v}_p}$$

and the Reynolds stress for the particles is defined by

$$(40) \qquad \mathbf{T}_d^{Re}(\mathbf{x},t) = -\rho_d \int_{R^3} \mathbf{v}'_p \mathbf{v}'_p f^{(1)}(\mathbf{v}_p,\mathbf{x},t)\, dV_{\mathbf{v}_p}$$

In order to evaluate the integrals appearing in the averaging process, we must express the $\mathbf{z}$ dependence of the velocities in terms of $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} - \mathbf{z}$. We have

$$\mathbf{v}_f(\mathbf{z}) = \mathbf{v}_f(\mathbf{x}) - \mathbf{x}' \cdot \mathbf{e}_f$$

and

$$\overline{\mathbf{v}}_p(\mathbf{z}) = \overline{\mathbf{v}}_p(\mathbf{x}) - \mathbf{x}' \cdot \mathbf{e}_p$$

where $\mathbf{e}_p$ is the velocity gradient tensor for the average particle motion. We shall assume that this tensor is constant and symmetric.

We have

$$\mathbf{v}(\mathbf{x},t;\mathbf{z},\mathbf{v}_p) = \nabla\phi(\mathbf{x},t;\mathbf{z},\mathbf{v}_p)$$

$$= \mathbf{v}_f(\mathbf{x}) + \frac{1}{2}(\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{x}) - \mathbf{v}'_p - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p))\left(\frac{a^3}{r^3}\right)$$

$$- \frac{3}{2}(\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{x}) - \mathbf{v}'_p - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \cdot \mathbf{x}' \left(\frac{a^3}{r^5}\right)\mathbf{x}'$$

$$(41) \qquad + \frac{2}{3}\mathbf{x}' \cdot \mathbf{e}_f \left(\frac{a^5}{r^5}\right) - \frac{5}{3}\mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^5}{r^7}\right)\mathbf{x}'$$

Note that $\mathbf{v}_f(\mathbf{x})$ is the fluid velocity that would exist at $\mathbf{x}$ if the sphere were not present, and $\mathbf{v}_f(\mathbf{z}) - \mathbf{v}_p$ is the relative velocity between the sphere and the fluid evaluated at the sphere center. It is convenient to have expressions for the integrals of powers of $\mathbf{x}'$ over $\Omega(r)$. For these integrals, we note that

$$(42\mathrm{a}) \qquad \int_{\Omega(r)} \mathbf{x}' \ldots \mathbf{x}' d\Omega = 0$$

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if the factor $\mathbf{x}'$ appears on odd number of times, and

(42b)
$$\int_{\Omega(r)} d\Omega = 4\pi r^2$$

(42c)
$$\int_{\Omega(r)} \mathbf{x}'\mathbf{x}' d\Omega = \frac{4}{3}\pi r^4 \mathbf{I}$$

(42d)
$$\int_{\Omega(r)} \mathbf{x}'\mathbf{x}'\mathbf{x}'\mathbf{x}' d\Omega = \frac{4}{15}\pi r^6 \mathbf{\Sigma}$$

where $\mathbf{\Sigma}$ is a fourth order isotropic tensor defined in Cartesian coordinates by

$$\Sigma_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}.$$

We further note that if $\mathbf{v}$ is a vector, and $\mathbf{e}$ is a symmetric second order tensor with $e_{ii} = 0$, then

$$\Sigma_{ijkl}v_j e_{kl} = 2v_j e_{ji}.$$

Derivation of Averaged Quantities

In order to average eq. (41), we note that the average over the velocity fluctuations gives no contribution, by eq. (38). Then averaging over $\mathbf{z}$ gives

(43)
$$\overline{\mathbf{v}}_c^{xp}(\mathbf{x}, t) = \mathbf{v}_f(\mathbf{x}, t).$$

Similarly, to obtain the interfacial averaged velocity of the fluid, again the integration over the velocity fluctuations gives no contribution, and we have

$$\overline{\mathbf{v}}_{ci}(\mathbf{x}, t) = \frac{1}{4\pi a^2} \int_{\Omega(a)} \overline{\mathbf{v}}(\mathbf{x}, t|\mathbf{z}) d\Omega.$$

Substituting and performing the integrations lead to the result

(44)
$$\overline{\mathbf{v}}_{ci}(\mathbf{x}, t) = \mathbf{v}_f(\mathbf{x}, t).$$

This result is a little surprising at first. The fluid at the surface of the sphere satisfies the condition $\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{v}_p$, but is allowed to slip in the tangential direction. After the passage

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of the sphere, the fluid that was momentarily in contact with the surface of the sphere is again moving with the fluid. The result says that even during the time that it is in contact with the surface of the sphere, its average velocity is still equal to the average velocity of the fluid, and not of the sphere.

Now let us compute averaged pressures using this formalism. The exact pressure can be computed by Bernoulli's equation (30). In order to evaluate the derivatives in eq. (30), we note that $\mathbf{x}$ is constant during t derivatives, but $\partial \mathbf{z}/\partial t = \mathbf{v}_p = \overline{\mathbf{v}}_p(\mathbf{z}, t) + \mathbf{v}'_p$. Also, when evaluating $\nabla \phi$, both t and $\mathbf{z}$ are held constant. The pressure is given by

$$
\begin{aligned}
(45) \qquad p(\mathbf{x}, t; \mathbf{z}, \mathbf{v}_p) = p_0 - \rho_c & \left(\frac{\partial \mathbf{v}_0}{\partial t} \cdot \mathbf{x} + \frac{1}{2} \left[\frac{\partial \mathbf{v}_0}{\partial t} - \frac{\partial \overline{\mathbf{v}}_p(\mathbf{x}, t)}{\partial t} \right. \right. \\
& \left. + (\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p) \cdot \mathbf{e}_f - (\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p) \cdot \mathbf{e}_p \right] \cdot \mathbf{x}' \left(\frac{a^3}{r^3} \right) \\
& - \frac{1}{2}(\mathbf{v}_f(\mathbf{x}) - (\overline{\mathbf{v}}_p(\mathbf{z}, t) + \mathbf{v}'_p) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \cdot (\overline{\mathbf{v}}_p(\mathbf{z}, t) + \mathbf{v}'_p) \left(\frac{a^3}{r^3} \right) \\
& - \frac{3}{2}(\mathbf{v}_f(\mathbf{x}) - (\overline{\mathbf{v}}_p(\mathbf{z}, t) + \mathbf{v}'_p) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \cdot \mathbf{x}' \left(\frac{a^3}{r^5} \right) \mathbf{x}' \cdot (\overline{\mathbf{v}}_p(\mathbf{z}, t) + \mathbf{v}'_p) \\
& - \frac{2}{3}(\overline{\mathbf{v}}_p(\mathbf{z}, t) + \mathbf{v}'_p) \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^5}{r^5} \right) + \frac{5}{3} \mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}' \mathbf{x}' \cdot (\overline{\mathbf{v}}_p(\mathbf{z}, t) + \mathbf{v}'_p) \left(\frac{a^5}{r^5} \right) + \frac{1}{2} \mathbf{v}_f \cdot \mathbf{v}_f \\
& + \frac{1}{8} |\mathbf{v}_f(\mathbf{x}) - (\overline{\mathbf{v}}_p(\mathbf{z}, t) + \mathbf{v}'_p) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)|^2 \left(\frac{a^6}{r^6} \right) \\
& + \frac{1}{2} \mathbf{v}_f \cdot (\mathbf{v}_f(\mathbf{x}) - (\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \left(\frac{a^3}{r^3} \right) \\
& + \frac{9}{8} ((\mathbf{v}_f(\mathbf{x}) - (\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \cdot \mathbf{x}')^2 \left(\frac{a^6}{r^{10}} \right) \\
& + \frac{2}{3}(\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p) \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^5}{r^5} \right) - \frac{5}{3}(\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p) \cdot \mathbf{x}' \mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^5}{r^7} \right) \\
& \frac{1}{3}(\mathbf{v}_f(\mathbf{x}) - (\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p)) \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^8}{r^8} \right) \\
& - \frac{5}{6}(\mathbf{v}_f(\mathbf{x}) - (\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p)) \cdot \mathbf{x}' \mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^8}{r^{10}} \right) \\
& \left. + \frac{3}{2}(\mathbf{v}_f(\mathbf{x}) - (\overline{\mathbf{v}}_p(\mathbf{x}, t) + \mathbf{v}'_p)) \cdot \mathbf{x}' \mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^8}{r^{10}} \right) \right)
\end{aligned}
$$

where we have ignored terms of order ϵ_f^2, ϵ_p^2, and $\epsilon_f \epsilon_p$. Averaging over the velocity fluctuations gives

$$\bar{p}(\mathbf{x},t|\mathbf{z}) = p_0 - \rho_c \left(\frac{\partial \mathbf{v}_0}{\partial t} \cdot \mathbf{x} + \frac{1}{2} \left[\frac{\partial \mathbf{v}_0}{\partial t} - \frac{\partial \overline{\mathbf{v}}_p(\mathbf{x},t)}{\partial t} \right. \right.$$

$$+ \overline{\mathbf{v}}_p(\mathbf{x},t) \cdot \mathbf{e}_f - \overline{\mathbf{v}}_p(\mathbf{x},t) \cdot \mathbf{e}_p \Big] \cdot \mathbf{x}' \left(\frac{a^3}{r^3} \right)$$

$$- \left[\frac{1}{2} (\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{z},t) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \cdot \overline{\mathbf{v}}_p(\mathbf{z},t) - u_d^{Re}(\mathbf{z},t) \right] \left(\frac{a^3}{r^3} \right)$$

$$- \left[\frac{3}{2} (\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{z},t) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \cdot \mathbf{x}'\mathbf{x}' \cdot \overline{\mathbf{v}}_p(\mathbf{z},t) + \frac{3}{2} \frac{\mathbf{x}' \cdot \mathbf{T}_d^{Re}(\mathbf{z},t) \cdot \mathbf{x}'}{\rho_d} \right] \left(\frac{a^3}{r^5} \right)$$

$$- \frac{2}{3} \overline{\mathbf{v}}_p(\mathbf{z},t) \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^5}{r^5} \right) + \frac{5}{3} \mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}'\mathbf{x}' \cdot \overline{\mathbf{v}}_p(\mathbf{z},t) \left(\frac{a^5}{r^5} \right) + \frac{1}{2} \mathbf{v}_f \cdot \mathbf{v}_f$$

$$+ \left[\frac{1}{8} |\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{z},t) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)| + \frac{1}{4} u_d^{Re}(\mathbf{z},t) \right] \left(\frac{a^6}{r^6} \right)$$

$$+ \frac{1}{2} \mathbf{v}_f \cdot (\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{x},t) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \left(\frac{a^3}{r^3} \right)$$

$$+ \left[\frac{9}{8} ((\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{x},t) - \mathbf{x}' \cdot (\mathbf{e}_f - \mathbf{e}_p)) \cdot \mathbf{x}')^2 - \frac{9}{8} \frac{\mathbf{x}' \cdot \mathbf{T}_d^{Re}(\mathbf{z},t) \cdot \mathbf{x}'}{\rho_d} \right] \left(\frac{a^6}{r^{10}} \right)$$

$$+ \frac{2}{3} \overline{\mathbf{v}}_p(\mathbf{x},t) \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^5}{r^5} \right) - \frac{5}{3} \overline{\mathbf{v}}_p(\mathbf{x},t) \cdot \mathbf{x}'\mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^5}{r^7} \right)$$

$$\frac{1}{3} (\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{x},t)) \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^8}{r^8} \right)$$

$$- \frac{5}{6} (\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{x},t)) \cdot \mathbf{x}'\mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^8}{r^{10}} \right)$$

$$\left. \left. + \frac{3}{2} (\mathbf{v}_f(\mathbf{x}) - \overline{\mathbf{v}}_p(\mathbf{x},t)) \cdot \mathbf{x}'\mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}' \left(\frac{a^8}{r^{10}} \right) \right) \right)$$

The spatial integration is tedious, but results in

$$(46) \qquad \bar{p}_c^x = p_0 - \rho_c \frac{\partial \mathbf{v}_f}{\partial t} \cdot \mathbf{x} - \frac{1}{2} \mathbf{v}_f(\mathbf{x}) \cdot \mathbf{v}_f(\mathbf{x})$$

where we ignore terms of order a/R in addition to those ignored previously. We also obtain

$$(47) \qquad \bar{p}_{ci}(\mathbf{x},t) = \bar{p}_c^x - \frac{1}{4} \rho_c |\overline{\mathbf{V}}_c(\mathbf{x},t) - \overline{\mathbf{v}}_d(\mathbf{x},t)|^2 + \frac{1}{2} \rho_c u_d^{Re}.$$

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The interfacial force density $\mathbf{M}_d$ is given by $\mathbf{M}_d = \overline{p \nabla X_d}$. Thus,

$$(48) \qquad \mathbf{M}_d(\mathbf{x},t) = -\frac{1}{\frac{4}{3}\pi R^3} \int_{\Omega(a)} n p(\mathbf{x},t|\mathbf{z}) \left(1 - \frac{\mathbf{x}' \cdot \alpha_d}{\alpha_d}\right) d\Omega.$$

This can be computed by substituting eq. (45) for the pressure, and recognizing that $\mathbf{n} = \mathbf{x}'/a$. We must also expand the terms $u_d^{Re}(\mathbf{z},t) = u_d^{Re}(\mathbf{x},t) - \mathbf{x}' \cdot \nabla u_d^{Re}(\mathbf{x},t)$ and $\mathbf{T}_d^{Re}(\mathbf{z},t) = \mathbf{T}_d^{Re}(\mathbf{x},t) - \mathbf{x}' \cdot \nabla \mathbf{T}_d^{Re}(\mathbf{x},t)$ The result is that

$$(49) \qquad \mathbf{M}_d(\mathbf{x},t) = \overline{p}_c^x \nabla \alpha_d$$

$$+ \alpha_d \overline{\rho}_c^x \left(\frac{1}{2}\left[\frac{\partial \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t)}{\partial t} - \frac{\partial \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)}{\partial t} + \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) \cdot \nabla \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t) \cdot \nabla \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)\right]\right.$$

$$\left. - \frac{7}{20}(\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \cdot (\nabla \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \nabla \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t))\right)$$

$$+ \overline{\rho}_c^x \left(\frac{2}{5}(\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \cdot (\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \nabla \alpha_d\right.$$

$$\left. + \frac{9}{20}(\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)(\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \cdot \nabla \alpha_d\right)$$

$$- \alpha_d \overline{\rho}_c^x \frac{7}{20} \nabla u_d^{Re} - \alpha_d \frac{\overline{\rho}_c^x}{\overline{\rho}_d^x} \frac{9}{20} \nabla \cdot \mathbf{T}_d^{Re} - \nabla \alpha_d \overline{\rho}_c^x \frac{7}{20} u_d^{Re} - \frac{\overline{\rho}_c^x}{\overline{\rho}_d^x} \frac{9}{20} \nabla \alpha_d \cdot \mathbf{T}_d^{Re}$$

Note that no drag force is present in eq. (49). This is the result of D'Alembert's paradox, that is, there is no net force on a body moving at a constant velocity through an inviscid fluid at rest.

If a distribution of stresses is applied to the surface of an elastic body, there results a distribution of stresses inside the body. These induced stresses are important in computing constitutive equations for solid-fluid mixtures. The average stress inside the sphere is given by

$$\overline{\mathbf{T}}_d^x(\mathbf{x},t) = \frac{1}{\frac{4}{3}\pi a^3} \int_0^a \int \int_{\Omega(r)} \mathbf{T}(\mathbf{x},t|\mathbf{z}) d\Omega$$

where

$$\mathbf{T}(\mathbf{x},t|\mathbf{z}) = \int_{R^3} \mathbf{T}(\mathbf{x},t;\mathbf{z},\mathbf{v}_p) f^{(1)}(\mathbf{v}_p,\mathbf{z},t) \, dV_{\mathbf{v}_p}$$

Here $\mathbf{T}(\mathbf{x},t;\mathbf{z},\mathbf{v}_p)$ is the stress at point $\mathbf{x}$ inside a sphere having its center at $\mathbf{z}$ at time t. We shall assume that the spheres are linearly elastic solids, but we shall assume that the

deformation is sufficiently small that the fluid motion is unaffected by the deformation of the spheres. Then the stress-strain relation is given by

(50)
$$\mathbf{T} = \mu_s[\nabla\mathbf{u} + (\nabla\mathbf{u})^{tr}] + \lambda_s \nabla \cdot \mathbf{uI}$$

This can be written as

(51)
$$\mathbf{T} = \boldsymbol{\sigma} + \Theta\mathbf{I}$$

where

$$\Theta = \left(\lambda_s + \frac{2}{3}\mu_s\right) \nabla \cdot \mathbf{u}$$

$$\boldsymbol{\sigma} = \mu_s \left\{ [\nabla\mathbf{u} + (\nabla\mathbf{u})^{tr}] - \frac{2}{3}\nabla \cdot \mathbf{uI} \right\}$$

The spherical part of the stress satisfies (Love, 1932)

(52a)
$$\nabla^2\Theta(\mathbf{x}, t; \mathbf{z}, \mathbf{v}_p) = 0$$

(52b)
$$\Theta(\mathbf{x}, t; \mathbf{z}, \mathbf{v}_p) = -p(\mathbf{x}, t; \mathbf{z}, \mathbf{v}_p) \text{ on } |\mathbf{x} - \mathbf{z}| = a$$

Averaging over $\mathbf{v}_p$ gives

(53a)
$$\nabla^2\Theta(\mathbf{x}, t|\mathbf{z}) = 0$$

(53b)
$$\Theta(\mathbf{x}, t|\mathbf{z}) = -p(\mathbf{x}, t|\mathbf{z}) \text{ on } |\mathbf{x} - \mathbf{z}| = a$$

Solving and performing the integration over $\mathbf{z}$ gives

(54)
$$\overline{\Theta}_d^x(\mathbf{x}, t) = -\overline{p}_{ci}(\mathbf{x}, t).$$

The solution for $\boldsymbol{\sigma}$ is also given in Love (1932) and can be averaged in $\mathbf{v}_p$ and then integrated in $\mathbf{z}$ to give

(55)
$$\overline{\boldsymbol{\sigma}}_d^x(\mathbf{x}, t) = \overline{\rho}_c^x \left[-\frac{9}{20} \left((\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x}, t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x}, t))(\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x}, t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x}, t)) - \frac{\mathbf{T}_d^{Re}}{\overline{\rho}_d^x} \right) \right.$$
$$\left. + \frac{8}{20} \left(|\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x}, t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x}, t)|^2 + 2u_d^{Re} \right) \mathbf{I} \right].$$

We next turn to computations of the Reynolds stress using the velocity fluctuations due to the inviscid flow around a sphere. Using the expression for the velocity (41) and the average fluid velocity (43), we see that

$$
\mathbf{v}'_c(\mathbf{x}, t; \mathbf{z}, \mathbf{v}_p) = \frac{1}{2}(\mathbf{v}_f(\mathbf{z}) - \mathbf{v}_p)\left(\frac{a^3}{r^3}\right)
$$
$$
- \frac{3}{2}(\mathbf{v}_f(\mathbf{z}) - \mathbf{v}_p) \cdot \mathbf{x}'\left(\frac{a^3}{r^5}\right)\mathbf{x}'
$$
(56)
$$
+ \frac{2}{3}\mathbf{x}' \cdot \mathbf{e}_f\left(\frac{a^5}{r^5}\right) - \frac{5}{3}\mathbf{x}' \cdot \mathbf{e}_f \cdot \mathbf{x}'\left(\frac{a^5}{r^7}\right)\mathbf{x}'
$$

Averaging over the particle velocity fluctuations yields

$$
\mathbf{T}_c^{Re}(\mathbf{x}, t|\mathbf{z}) = -\rho_c \overline{\mathbf{v}'_c}(\mathbf{x}, t|\mathbf{z})\overline{\mathbf{v}'_c}(\mathbf{x}, t|\mathbf{z})
$$
$$
+ \rho_c \left[\frac{1}{4}\left(\frac{a^6}{r^6}\right)\frac{\mathbf{T}_d^{Re}}{\rho_d} - \frac{3}{4}\left(\frac{a^6}{r^8}\right)[\mathbf{x}'(\mathbf{x}' \cdot \frac{\mathbf{T}_d^{Re}}{\rho_d}) + (\mathbf{x}' \cdot \frac{\mathbf{T}_d^{Re}}{\rho_d})\mathbf{x}']\right.
$$
$$
\left. + \frac{9}{4}\left(\frac{a^6}{r^{10}}\right)\mathbf{x}'\mathbf{x}'(\mathbf{x}' \cdot \frac{\mathbf{T}_d^{Re}}{\rho_d} \cdot \mathbf{x}')\right]
$$

The integration over $\mathbf{z}$ can be performed, yielding

$$
\mathbf{T}_c^{Re}(\mathbf{x}, t) = -\frac{1}{20}\alpha_d \overline{\rho}_c^x \left[\left((\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho})(\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho}) - \frac{\mathbf{T}_d^{Re}}{\overline{\rho}_d^x}\right)\right.
$$
(57)
$$
\left. + 3\left((\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho}) \cdot (\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho}) + 2u_d^{Re}\right)\mathbf{I}\right].
$$

The fluid fluctuation kinetic energy is $u_c^{Re} = \frac{1}{2}\overline{\mathbf{v}' \cdot \mathbf{v}'}^x$, and can be computed by taking the trace of eq. (57) for $\mathbf{T}_c^{Re}$. The result is

(58)
$$
u_c^{Re} = \frac{1}{4}\alpha_d|\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho}|^2 + \frac{1}{2}\alpha_d u_d^{Re}.
$$

Conservation of Fluctuation Kinetic Energy

In analogy with statistical mechanics for assemblages of molecules, the theory of averaging as applied to multiphase flows allows the computation of averaged equations for higher moments of velocity and pressure correlations.

We start with the derivation of averaged equations for the fluctuation kinetic energy for each phase. The exact momentum equation is

(59)
$$
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \rho \mathbf{v}\mathbf{v} = \nabla \cdot \mathbf{T} + \rho \mathbf{g}
$$

We shall derive an equation for the evolution of the kinetic energy. The Lagrangian form of the momentum equation is

$$(60) \qquad \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \nabla \cdot \mathbf{T} + \rho \mathbf{g}$$

If we take the dot product of $\mathbf{v}$ with eq. (60), we have

$$(61) \qquad \rho \left(\frac{\partial \frac{1}{2} v^2}{\partial t} + \mathbf{v} \cdot \nabla \frac{1}{2} v^2 \right) = \mathbf{v} \cdot (\nabla \cdot \mathbf{T}) + \rho \mathbf{v} \cdot \mathbf{g}$$

We note that

$$\mathbf{v} \cdot (\nabla \cdot \mathbf{T}) = \nabla \cdot (\mathbf{T} \cdot \mathbf{v}) - \mathbf{T} : \nabla \mathbf{v}$$

If we also return to the Eulerian form, we have

$$(62) \qquad \frac{\partial \frac{1}{2} \rho v^2}{\partial t} + \nabla \cdot \frac{1}{2} \rho v^2 \mathbf{v} = \nabla \cdot (\mathbf{T} \cdot \mathbf{v}) - \mathbf{T} : \nabla \mathbf{v} + \rho \mathbf{v} \cdot \mathbf{g}$$

If we apply the ensemble average to eq. (62), we have

$$
\frac{\partial \overline{X_k \frac{1}{2} \rho v^2}}{\partial t} + \nabla \cdot \overline{X_k \frac{1}{2} \rho v^2 \mathbf{v}} = \nabla \cdot \overline{X_k \mathbf{T} \cdot \mathbf{v}} - \overline{X_k \mathbf{T} : \nabla \mathbf{v}}
$$
$$
(63) \qquad + \overline{X_k \rho \mathbf{v} \cdot \mathbf{g}} - \overline{[\rho \frac{1}{2} v^2 (\mathbf{v} - \mathbf{v}_i) + \mathbf{T} \cdot \mathbf{v}) \cdot \nabla X_k]}
$$

We define the fluctuation velocity of phase k by

$$(64) \qquad \mathbf{v}'_k = \mathbf{v} - \overline{\mathbf{v}}_k^{x\rho}$$

Then

$$(65) \qquad v^2 = (v'_k)^2 + 2\mathbf{v}'_k \cdot \overline{\mathbf{v}}_k^{x\rho} + (\overline{v}_k^{x\rho})^2$$

so that, noting that $\overline{X_k \rho \mathbf{v}'_k} = \overline{X_k \rho \mathbf{v}} - \overline{X_k \rho \mathbf{v}}_k^{x\rho} = 0$, we have

$$
\overline{X_k \rho \frac{1}{2} v^2} = \overline{X_k \rho (v'_k)^2} + \overline{X_k \rho 2 \mathbf{v}'_k \cdot \mathbf{v}_k^{x\rho}} + \overline{X_k \rho (v_k^{x\rho})^2} = \alpha_k \overline{\rho}_k^x u_k^{Re} + \alpha_k \overline{\rho}_k^x \frac{1}{2} v_k^{x\rho}
$$

Furthermore,

$$
v^2 \mathbf{v} = (v'_k)^2 \mathbf{v}'_k + (v'_k)^2 \overline{\mathbf{v}}_k^{x\rho} + 2\mathbf{v}'_k \cdot \overline{\mathbf{v}}_k^{x\rho} \overline{\mathbf{v}}_k^{x\rho} + 2\mathbf{v}'_k \cdot \overline{\mathbf{v}}_k^{x\rho} \mathbf{v}'_k + (\overline{v}_k^{x\rho})^2 \mathbf{v}'_k + (\overline{v}_k^{x\rho})^2 \overline{\mathbf{v}}_k^{x\rho}
$$

so that

$$\overline{X_k \rho \frac{1}{2} v^2 \mathbf{v}} = \overline{X_k \rho \frac{1}{2}(v_k')^2 \mathbf{v}_k'} + \overline{X_k \rho \frac{1}{2}(v_k')^2} \overline{\mathbf{v}}_k^{x\rho} + \overline{\mathbf{v}}_k^{x\rho} \cdot \overline{X_k \rho \mathbf{v}_k' \mathbf{v}_k'} + \overline{X_k \rho \frac{1}{2}(\overline{v}_k^{x\rho})^2 \overline{\mathbf{v}}_k^{x\rho}}$$

$$= \alpha_k \mathbf{q}_k^K + \alpha_k \overline{\rho}_k^x u_k^{Re} \overline{\mathbf{v}}_k^{x\rho} - \alpha_k \overline{\mathbf{v}}_k^{x\rho} \cdot \mathbf{T}_k^{Re} + \alpha_k \overline{\rho}_k^x \frac{1}{2}(\overline{v}_k^{x\rho})^2 \overline{\mathbf{v}}_k^{x\rho}$$

Also, note that

$$\mathbf{T} \cdot \mathbf{v} = \mathbf{T} \cdot \overline{\mathbf{v}}_k^{x\rho} + = \mathbf{T} \cdot \mathbf{v}_k'.$$

Note further that

$$\mathbf{T} : \nabla \mathbf{v} = \mathbf{T} : \nabla \overline{\mathbf{v}}_k^{x\rho} + \mathbf{T} : \nabla \mathbf{v}_k'$$

Then we have

$$\overline{X_k \mathbf{T} \cdot \mathbf{v}} = \overline{X_k \mathbf{T}} \cdot \overline{\mathbf{v}}_k^{x\rho} + \overline{X_k \mathbf{T} \cdot \mathbf{v}_k'} = \alpha_k \mathbf{T}_k^x \cdot \overline{\mathbf{v}}_k^{x\rho} - \alpha_k \mathbf{q}_k^T$$

and

$$\overline{X_k \mathbf{T} : \nabla \mathbf{v}} = \overline{X_k \mathbf{T}} : \nabla \overline{\mathbf{v}}_k^{x\rho} + \overline{X_k \mathbf{T} : \nabla \mathbf{v}_k'} = \alpha_k \overline{\mathbf{T}}_k^x : \nabla \overline{\mathbf{v}}_k^{x\rho} + D_k,$$

where $\mathbf{q}_k^T = \mathbf{q}_k^p + \mathbf{q}_k^\tau$ and $D_k = \overline{X_k \mathbf{T} : \nabla \mathbf{v}_k'}$.

Next, in the interfacial terms, we have

$$\overline{\rho \frac{1}{2} v^2 (\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k} = \frac{1}{2}(\overline{v}_k^{x\rho})^2 \Gamma_k + \overline{\mathbf{v}}_k^{x\rho} \cdot \mathbf{v}_{ki}^m \Gamma_k + \overline{\rho \frac{1}{2}(v_k')^2 (\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k}$$

and

$$\overline{(\mathbf{T} \cdot \mathbf{v}) \cdot \nabla X_k} = \overline{(\mathbf{T} \cdot \overline{\mathbf{v}}_k^{x\rho}) \cdot \nabla X_k} + \overline{(\mathbf{T} \cdot \mathbf{v}_k') \cdot \nabla X_k} = \mathbf{M}_k \cdot \overline{\mathbf{v}}_k^{x\rho} + W_k$$

The equation for the conservation of fluctuation kinetic energy then becomes

$$\frac{\partial}{\partial t} \alpha_k \overline{\rho}_k^x \left(u_k^{Re} + \frac{1}{2}(\overline{v}_k^{x\rho})^2 \right) + \nabla \cdot \alpha_k \overline{\rho}_k^x \left(u_k^{Re} + \frac{1}{2}(\overline{v}_k^{x\rho})^2 \right) \mathbf{v}_k^{x\rho} - \nabla \cdot \left(\alpha_k \overline{\mathbf{v}}_k^{x\rho} \cdot \mathbf{T}_k^{Re} \right) =$$

$$- \nabla \cdot \alpha_k (\mathbf{q}_k^K + \mathbf{q}_k^T) + \nabla \cdot \alpha_k \overline{\mathbf{T}}_k^x \cdot \overline{\mathbf{v}}_k^{x\rho} + \alpha_k \overline{\rho}_k^x \overline{\mathbf{v}}_k^{x\rho} \cdot \mathbf{g}$$

$$- \alpha_k \overline{\mathbf{T}}_k^x : \nabla \overline{\mathbf{v}}_k^{x\rho} - D_k + \frac{1}{2}(\overline{v}_k^{x\rho})^2 \Gamma_k + \overline{\mathbf{v}}_k^{x\rho} \cdot \mathbf{v}_{ki}^m \Gamma_k$$

$$+ \overline{\rho \frac{1}{2}(v_k')^2 (\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k} + \mathbf{M}_k \cdot \overline{\mathbf{v}}_k^{x\rho} + W_k$$

(66)

The equation for the average kinetic energy is

$$\frac{\partial \alpha_k \overline{\rho}_k^x \frac{1}{2}(\overline{v}_k^{x\rho})^2}{\partial t} + \nabla \cdot \alpha_k \rho_k \overline{\mathbf{v}}_k^{x\rho} \frac{1}{2}(\overline{v}_k^{x\rho})^2 =$$

$$\overline{\mathbf{v}}_k^{x\rho} \cdot \nabla \cdot \alpha_k(\overline{\mathbf{T}}_k + \mathbf{T}_k^{Re}) + \mathbf{M}_k \cdot \overline{\mathbf{v}}_k^{x\rho}$$

(67)
$$+ \alpha_k \rho_k \mathbf{g} \cdot \overline{\mathbf{v}}_k^{x\rho} + \overline{\mathbf{v}}_k^{x\rho} \cdot \mathbf{v}_{ki}^m \Gamma_k.$$

Subtracting this from eq. (66), we have

$$\frac{\partial \alpha_k \overline{\rho}_k^x u_k^{Re}}{\partial t} + \nabla \cdot \alpha_k \overline{\rho}_k^x u_k^{Re} \overline{\mathbf{v}}_k^{x\rho} = \alpha_k \mathbf{T}_k^{Re} : \nabla \overline{\mathbf{v}}_k^{x\rho} - \nabla \cdot \alpha_k(\mathbf{q}_k^K + \mathbf{q}_k^T)$$

(68)
$$+ \frac{1}{2}(\overline{v}_k^{x\rho})^2 \Gamma_k - \overline{\rho \frac{1}{2}(v_k')^2(\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k} + W_k - D_k$$

This equation has some interesting interpretations. First, note that the dissipation due to the Reynolds stress $\alpha_k \mathbf{T}_k^{Re} : \nabla \overline{\mathbf{v}}_k^{x\rho}$ acts as a source of fluctuation kinetic energy, while its counterpart for the molecular dissipation $\alpha_k \overline{\mathbf{T}}_k^x : \nabla \overline{\mathbf{v}}_k^{x\rho}$ does not appear in this equation. Dissipation on the macroscopic scale, then, winds up as different things on the microscopic scale. Also, the dissipation due to microscopic velocity fluctuations D_k implies a loss of fluctuation kinetic energy. Thus, loss mechanisms, such as inelastic collisions or viscous dissipation in the velocity fulctuations, cause a loss of fluctuation kinetic energy to heat. Finally, the working of the fluctuations at the interface, W_k appears as a source of fluctuation kinetic energy.

Since this equation is unnecessary for the fluid phase, we shall ignore it for $k = c$. For $k = d$, we note that $D_d = 0$ is consistent with the linear elasticity assumption and the assumption that the particle radius a does not change. Furthermore, if assume no phase change ($\Gamma_k = 0$), and we ignore triple correlations in the particle velocity fluctuations ($\overline{\mathbf{v}_p' \mathbf{v}_p' \mathbf{v}_p'} = 0$), we have

(69)
$$\alpha_d \overline{\rho}_d^x \frac{\partial u_d^{Re}}{\partial t} + \alpha_d \overline{\rho}_d^x \cdot \nabla u_d^{Re} \overline{\mathbf{v}}_d^{x\rho} = \alpha_d \mathbf{T}_d^{Re} : \nabla \overline{\mathbf{v}}_d^{x\rho} + W_d$$

Discussion of the Force on a Sphere

The equations of motion for the mixture are

(70)
$$\frac{\partial \alpha_d}{\partial t} + \nabla \cdot \alpha_d \overline{\mathbf{v}}_d^{x\rho} = 0$$

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$$(71) \qquad \frac{\partial \alpha_c}{\partial t} + \nabla \cdot \alpha_c \overline{\mathbf{v}}_c^{x\rho} = 0$$

$$(72) \quad \alpha_d \overline{\rho}_d^x \left(\frac{\partial \overline{\mathbf{v}}_d^{x\rho}}{\partial t} + \overline{\mathbf{v}}_d^{x\rho} \cdot \nabla \overline{\mathbf{v}}_c^{x\rho} \right) = -\alpha_d \nabla (\overline{p}_c^x + \frac{1}{2} \overline{\rho}_c^x u_d^{Re} - \frac{1}{4} \overline{\rho}_c^x |\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho}|^2) + \nabla \cdot \alpha_d \mathbf{T}_d^{Re}$$

$$+ \nabla \cdot \alpha_d \left\{ \overline{\rho}_c^x \left[-\frac{9}{20} (\overline{\mathbf{v}}_c(\mathbf{x},t) - \overline{\mathbf{v}}_d(\mathbf{x},t))(\overline{\mathbf{v}}_c(\mathbf{x},t) - \overline{\mathbf{v}}_d(\mathbf{x},t)) \right. \right.$$

$$\left. \left. + \frac{3}{20} |\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)|^2 \mathbf{I} \right] \right\}$$

$$+ \alpha_d \overline{\rho}_c^x \left(\frac{1}{2} \left[\frac{\partial \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t)}{\partial t} - \frac{\partial \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)}{\partial t} + \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) \cdot \nabla \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t) \cdot \nabla \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t) \right] \right.$$

$$\left. - \frac{7}{20} [\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)] \cdot [\nabla \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \nabla \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)] \right)$$

$$+ \overline{\rho}_c^x \left(\frac{2}{5} (\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \cdot (\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \nabla \alpha_d \right.$$

$$\left. + \frac{9}{20} (\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t))(\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \cdot \nabla \alpha_d \right)$$

$$+ \overline{\rho}_c^x \frac{9}{20} \nabla (\alpha_d u_d^{Re})$$

$$(73) \qquad \alpha_c \overline{\rho}_c^x \left(\frac{\partial \overline{\mathbf{v}}_c^{x\rho}}{\partial t} + \overline{\mathbf{v}}_c^{x\rho} \cdot \nabla \overline{\mathbf{v}}_c^{x\rho} \right) = -\alpha_c \nabla \overline{p}_c^x$$

$$+ \nabla \cdot \left\{ -\frac{1}{20} \alpha_d \overline{\rho}_c^x \left[\left((\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho})(\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho}) - \frac{\mathbf{T}_d^{Re}}{\overline{\rho}_d^x} \right) \right. \right.$$

$$\left. \left. + 3 \left((\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho}) \cdot (\overline{\mathbf{v}}_c^{x\rho} - \overline{\mathbf{v}}_d^{x\rho}) + 2u_d^{Re} \right) \mathbf{I} \right] \right\}$$

$$+ \left(\frac{1}{4} \overline{\rho}_c^x |\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) + \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)|^2 + \frac{1}{2} \overline{\rho}_c^x u_d^{Re} \right) \nabla \alpha_d$$

$$- \alpha_d \overline{\rho}_c^x \left(\frac{1}{2} \left[\frac{\partial \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t)}{\partial t} - \frac{\partial \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)}{\partial t} + \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) \cdot \nabla \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t) \cdot \nabla \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t) \right] \right.$$

$$\left. - \frac{7}{20} [\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)] \cdot [\nabla \overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \nabla \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)] \right)$$

$$- \overline{\rho}_c^x \left(\frac{2}{5} (\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \cdot (\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \nabla \alpha_d \right.$$

$$+ \frac{9}{20}(\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t))(\overline{\mathbf{v}}_c^{x\rho}(\mathbf{x},t) - \overline{\mathbf{v}}_d^{x\rho}(\mathbf{x},t)) \cdot \nabla \alpha_d \Big)$$

$$+ \overline{\rho}_c^x \frac{7}{20} \nabla(\alpha_d u_d^{Re}) + \frac{\overline{\rho}_c^x}{\overline{\rho}_d^x} \frac{9}{20} \nabla \cdot (\alpha_d \mathbf{T}_d^{Re})$$

It is also possible to calculate the force on a sphere at $\mathbf{z}$ by computing

$$\mathbf{F}_p(\mathbf{z},t) = \int_{\Omega(a)} \mathbf{n}\left(p_0 - \overline{\rho}_c^x \left[\frac{1}{2}|\nabla\phi|^2 + \frac{\partial\phi}{\partial t}\right]\right) d\Omega,$$

where the integration is over the variable $\mathbf{x}'$, with $\mathbf{x} = \mathbf{z} + \mathbf{x}'$. This results in

$$(74) \qquad \mathbf{F}_p(\mathbf{z}) = \frac{4}{3}\pi a^3 \overline{\rho}_c^x \left(\frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{v}_f \cdot \mathbf{e}_f + \frac{1}{2}\left[\frac{\partial \mathbf{v}_f}{\partial t} - \frac{\partial \mathbf{v}_p}{\partial t} + \mathbf{v}_f \cdot \mathbf{e}_f\right]\right).$$

Note that this force agrees with Taylor's (1928) calculation of the force necessary to hold a sphere at rest in an accelerating stream, obtained by setting $\frac{\partial}{\partial t} = 0$ and $\mathbf{v}_p = 0$. The force is

$$(75) \qquad \frac{3}{2}\left(\frac{4}{3}\pi a^3\right)\overline{\rho}_c^x \mathbf{v}_f \cdot \nabla \mathbf{v}_f$$

If we first take the gradients involved in eq. (72), using $\nabla \cdot \overline{\mathbf{v}}_d^{x\rho} = (1/\alpha_d)(\partial \alpha_d/\partial t + \overline{\mathbf{v}}_d^{x\rho} \cdot \nabla \alpha_d)$ and $\nabla \cdot \overline{\mathbf{v}}_c^{x\rho} = -(1/\alpha_c)(\partial \alpha_d/\partial t + \overline{\mathbf{v}}_c^{x\rho} \cdot \nabla \alpha_d)$, then set $\overline{\mathbf{v}}_d^{x\rho} = 0$, $u_d^{Re} = 0$, $\alpha_d = \text{const.}$, and $\mathbf{T}_d^{Re} = 0$; and assume one-dimensional, steady flow, then eq. (72) reduces to eq. (75) in the limit as $\alpha_d \to 0$. Moreover, it is clear that it should. Consider the one-dimensional situation pictured in Fig. 1. The continuum model for the particles between x and $x + \Delta x$ gives the rate of change of the momentum of the particles *and parts of particles* between x and $x + \Delta x$, denoted by $\dot{\mathbf{P}}_d(x, x + \Delta x)$, as the stress force transmitted to the particles by the particle parts outside of the interval, denoted by $(\alpha_d \mathbf{i} \cdot \mathbf{T}_d^x)|_x + (\alpha_d(-\mathbf{i}) \cdot \mathbf{T}_d^x)|_{x+\Delta x}$, plus the force transmitted to the particles through their interface, denoted by $\mathbf{M}_d \Delta x$. Thus,

$$(76) \qquad \dot{\mathbf{P}}_d(x, x + \Delta x) = (\alpha_d \mathbf{i} \cdot \mathbf{T}_d^x)|_x + (\alpha_d(-\mathbf{i}) \cdot \mathbf{T}_d^x)|_{x+\Delta x} + \mathbf{M}_d \Delta x$$

The sum of the forces on all particles with their centers in the interval from x to $x + \Delta x$ is equal to the sum of the pressure forces on all the particles involved. This is

denoted by $\Sigma \int pn dS$. This is equal to the rate of change of the momentum of all the particles, denoted by $\dot{\mathbf{P}}_p$. Thus,

$$(77) \qquad\qquad \dot{\mathbf{P}}_p = \Sigma \int pn dS$$

We note that eqs. (76) and (77) differ in the way they treat the particles being cut by the surfaces at x and $x + \Delta x$. The relation is that the

$$
\begin{aligned}
\dot{\mathbf{P}}_d(x, x + \Delta x) = \dot{\mathbf{P}}_p &+ (\alpha_d \mathbf{i} \cdot \mathbf{T}_d^x)|_x + (\alpha_d(-\mathbf{i}) \cdot \mathbf{T}_d^x)|_{x+\Delta x} \\
&- \Sigma_{\text{cut,in}} \int pn dS|_x + \Sigma_{\text{cut,out}} \int pn dS|_x \\
&- \Sigma_{\text{cut,in}} \int pn dS|_{x+\Delta x} + \Sigma_{\text{cut,out}} \int pn dS|_{x+\Delta x}
\end{aligned}
$$

$$(78)$$

Here

$$\Sigma_{\text{cut,in}} \int pn dS|_x$$

is the sum of the pressure forces on the surfaces of the cut particles at x whose centers are inside the interval from x to $x + \Delta x$,

$$\Sigma_{\text{cut,out}} \int pn dS|_x$$

is the sum of the pressure forces on the surfaces of the cut particles at x whose centers are outside the interval from x to $x + \Delta x$. A similar interpretation is valid for the cut particles at $x + \Delta x$.

The terms on the right hand side of eq. (78) represent the resultants of forces on cut particles. If the approximate equation of motion inside the cut particle is $\nabla \cdot \mathbf{T} = 0$, then the pressure force over the curved side, plus the stress resultant force over the flat side must add up to 0. (Note that if the particles are accelerating, then the forces add up to be the volume of the part of the cut particle, times the acceleration of its center of mass. Presumably, this force is small.)

Conclusion

Consistent forms for the interfacial force, the interfacial pressure, the Reynolds stresses and the particle stress have been derived for the inviscid, irrotational incompressible flow of fluid in a dilute suspension of spheres. The particles are assumed to have a velocity distribution, giving rise to an effective pressure and stress in the particle phase. The velocity fluctuations also contribute in the fluid Reynolds stress and in the (elastic) stress field inside the spheres. The relation of these constitutive equations to the force on an individual sphere is discussed.

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