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A CONFORMING SPECTRAL COLLOCATION STRATEGY FOR STOKES FLOW THROUGH A CHANNEL CONTRACTION

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ON THE COEFFICIENTS OF DIFFERENTIATED EXPANSIONS OF ULTRASPHERICAL POLYNOMIALS

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ABSTRACT

A formula expressing the coefficients of an expansion of ultraspherical polynomials which has been differentiated an arbitrary number of times in terms of the coefficients of the original expansion is proved. The particular examples of Chebyshev and Legendre polynomials are considered.

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1. INTRODUCTION

In spectral methods, the solution of a partial differential equation is represented by a truncated expansion of eigenfunctions of a singular Sturm-Liouville problem. This choice is responsible for the superior approximation properties of spectral methods. This is, of course, most evident for problems possessing smooth solutions in which case the series expansion converges faster than any inverse power of n as $n \rightarrow \infty$. This phenomenon is known as spectral convergence. The method of calculating the expansion coefficients determines the type of spectral approximation: Galerkin, tau, or collocation. Only collocation methods are considered in this paper since they are applicable to a wide class of problems.

Spectral methods are most easily applied to problems defined in rectangular or circular regions in which case Chebyshev or Fourier series, respectively, are appropriate. However, the natural choices of expansion functions for a problem defined in an irregular geometry are unwieldy and inefficient to use and need to be computed for each new irregular region [12]. There are two ways of overcoming these difficulties, namely, mapping and patching [2, 4, 12].

The mapping technique involves transforming the irregular region into a simpler one by using a coordinate transformation. Spectral methods can then be applied in the simpler region using standard expansion techniques [12, 16]. The patching method divides the region into a number of simpler subregions or elements. A spectral approximation to the solution of the differential equation is sought within each element. The representations are patched by imposing continuity conditions across interfaces. This results in a coupling of the expansion coefficients in contiguous elements. Spectral domain decomposition methods combines the flexibility of the finite element method with the superior approximation properties of the spectral method [3, 13, 14].

A number of different spectral domain decomposition techniques have appeared in the literature [3, 5, 7, 10, 13, 14]. The main differences between these variants lie in the choice of trial functions and the treatment of the continuity conditions at element interfaces. In the spectral element method [13] conforming elements are used and C^1 continuity across the interfaces is achieved implicitly through a variational principle. The global element method [3] uses trial functions which are nonconforming. A modified functional is used to ensure that the interface continuity conditions are satisfied. In the present paper, we advocate the use of conforming spectral domain decomposition techniques and describe a collocation strategy for achieving this.

As our test problem, we consider Stokes flow through an abruptly contracting channel with contraction ratio $1 : \alpha$. A conforming spectral domain decomposition of this geometry divides the flow region into three rectangular semi-infinite subdomains with common point $(0, \alpha)$ as shown in Figure 1. In previous work [5, 15], the authors consider nonconforming subregions because of their ease of implementation (see Figure 2). Although this strategy works well for the Stokes problem [5], a lack of interface continuity appears for the Navier-Stokes problem [6] at moderate values of the Reynolds number, eventually causing the method to break down. The resulting spectral approximations are not pointwise continuous across the subregion interfaces. In this paper, we alleviate any possibility of discontinuous solutions and normal derivatives across the interfaces by using a carefully constructed collocation scheme.

Efficient direct methods for solving the collocation equations in rectangularly decompos-

able domains are described in [15] for the two subdomain example. This method, which is basically a block Gauss method, is applied here to efficiently invert the coefficient matrix. The matrices resulting from a spectral collocation discretization assume a block structure in which the non-zero blocks are full. Economically viable solution techniques need to take advantage of this matrix structure.

In Section 2, we define the Stokes problem in the channel contraction and derive the boundary conditions in terms of the stream function. In Section 3, we describe a domain decomposition of the flow region and the spectral approximation within each subregion. The collocation strategy which gives pointwise C^1 continuity of the stream function across the subregion interfaces is developed in Section 4. The solution of the spectral collocation equations using the capacitance matrix technique is described in Section 5. Numerical results are presented in Section 6 and concluding remarks are made in Section 7.

2. The Governing Equations

The governing equations for the planar inertialess flow of an incompressible Newtonian fluid assume the mathematical form

$$\nabla \cdot \mathbf{v} = 0, \quad (2.1)$$

$$\nabla \cdot \boldsymbol{\sigma} = 0, \quad (2.2)$$

where \mathbf{v} denotes the velocity field and $\boldsymbol{\sigma}$ the Cauchy stress tensor. These statements are the conservation of mass and momentum, respectively. For a Newtonian fluid, the extra stress tensor \mathbf{T} and rate of deformation tensor \mathbf{D} are related by

$$\mathbf{T} = 2\eta\mathbf{D}, \quad (2.3)$$

where η is a material constant. For an incompressible fluid, the motion of the continuum determines the stress tensor up to an arbitrary isotropic tensor and thus $\boldsymbol{\sigma}$ and \mathbf{T} are related as follows

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{T}, \quad (2.4)$$

where p is an arbitrary pressure and \mathbf{I} is the identity tensor.

If we define a stream function ψ by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

then (2.1) is satisfied identically. Substitution of \mathbf{T} from (2.3) into (2.4) and then substitution of $\boldsymbol{\sigma}$ from (2.4) into (2.2) results in the equation

$$-\nabla p + 2\eta\nabla \cdot \mathbf{D} = 0. \quad (2.5)$$

The pressure may be eliminated by taking the curl of (2.5) to give a biharmonic equation for the stream function

$$\nabla^4 \psi = 0. \quad (2.6)$$

Consider Stokes flow through the constricted channel defined by $|y| = 1$ ($x < 0$), $|y| = \alpha$ ($x > 0$) and $x = 0$ ($\alpha \leq |y| \leq 1$). The line $y = 0$ is an axis of symmetry and so only the

upper half of the channel needs to be considered. We assume Poiseuille flow at entry and exit which, in terms of the stream function, is defined by

$$\psi(x, y) \rightarrow G(y) \quad \text{as } x \rightarrow -\infty, \quad 0 \leq y \leq 1,$$

$$\psi(x, y) \rightarrow G(y/\alpha) \quad \text{as } x \rightarrow \infty, \quad 0 \leq y \leq \alpha$$

where $G(y) = \frac{1}{2}y(3 - y^2)$. Along the channel walls we have no-slip boundary conditions.

The flow region is truncated on entry and exit at distances h and k from the origin, respectively. These lengths are chosen to be sufficiently large so that the flow is fully developed in the entry and exit sections. In addition, we impose that the normal derivatives of ψ vanishes at entry and exit.

3. Domain Decomposition and Spectral Approximation

The truncated region is divided into three subregions as shown in Figure 1. Within each of the subregions, the solution to the biharmonic equation (2.6) is approximated by a truncated expansion of Chebyshev polynomials. The solutions are patched by applying C^3 continuity conditions in a collocation sense across the subregion interfaces. This is the correct order of weak continuity for this problem. With a judicious choice of collocation strategy we show that the approximations are pointwise C^0 and C^1 continuous across the interfaces.

In region I $\psi(x, y)$ is approximated by $\psi^I(x, y)$ where

$$\psi^I(x, y) = G(y) + \sum_{m=2}^{M_I} \sum_{n=2}^{N_I} a_{mn} P_m^I(x) Q_n^I(y). \quad (3.1)$$

The functions $P_m^I(x)$, $Q_n^I(y)$ are suitable linear combinations of shifted Chebyshev polynomials chosen so that $\psi^I(x, y)$ automatically satisfies the boundary conditions on $x = -h$ ($\alpha \leq y \leq 1$) and $y = 1$ ($-h \leq x \leq 0$). After a little computation we can show that

$$P_m^I(x) = T_m^I(x) + \alpha_m^I T_1^I(x) + \beta_m^I T_0^I(x), \quad 2 \leq m \leq M_I,$$

where $T_m^I(x)$, $0 \leq m \leq M_I$, are the shifted Chebyshev polynomials on $[-h, 0]$ defined by

$$T_m^I(x) = T_m\left(\frac{2x + h}{h}\right),$$

and α_m^I , β_m^I are given by

$$\alpha_m^I = (-1)^m m^2, \quad \beta_m^I = (-1)^m (m^2 - 1), \quad 2 \leq m \leq M_I.$$

Similarly, we can show that

$$Q_n^I(y) = \tilde{T}_n^I(y) + \tilde{\alpha}_n^I \tilde{T}_1^I(y) + \tilde{\beta}_n^I \tilde{T}_0^I(y), \quad 2 \leq n \leq N_I,$$

where $\tilde{T}_n^I(y)$, $0 \leq n \leq N_I$, are the shifted Chebyshev polynomials on $[\alpha, 1]$ defined by

$$\tilde{T}_n^I(y) = T_n\left(\frac{2y - 1 - \alpha}{1 - \alpha}\right),$$

and $\tilde{\alpha}_n^I, \tilde{\beta}_n^I$ are given by

$$\tilde{\alpha}_n^I = -n^2, \quad \tilde{\beta}_n^I = n^2 - 1.$$

In region II, the stream function is approximated by $\psi^{II}(x, y)$ where

$$\psi^{II}(x, y) = G(y) + \sum_{m=2}^{M_{II}} \sum_{n=2}^{N_{II}} b_{mn} P_m^{II}(x) Q_n^{II}(y). \quad (3.2)$$

Since regions I and II share the boundary $y = \alpha$ ($-h \leq x \leq 0$) and we wish the approximations to be conforming, we choose $M_{II} = M_I$. Further, since $\psi^{II}(x, y)$ satisfies the same boundary conditions along $x = -h$ as $\psi^I(x, y)$ we take $P_m^{II}(x) = P_m^I(x)$. As for the y -direction, we define the functions $Q_n^{II}(y)$ so as to satisfy the conditions along the axis of symmetry. Accordingly $Q_n^{II}(y)$ is defined by

$$Q_n^{II}(y) = \tilde{T}_n^{II}(y) + \tilde{\alpha}_n^{II} \tilde{T}_1^{II}(y) + \tilde{\beta}_n^{II} \tilde{T}_0^{II}(y), \quad 2 \leq n \leq N_{II},$$

where $\tilde{T}_n^{II}(y)$, $0 \leq n \leq N_{II}$, are the shifted Chebyshev polynomials on $[0, \alpha]$ defined by

$$\tilde{T}_n^{II}(y) = T_n \left(\frac{2y - \alpha}{\alpha} \right),$$

and $\tilde{\alpha}_n^{II}, \tilde{\beta}_n^{II}$ are given by

$$\tilde{\alpha}_n^{II} = -\frac{1}{12}(-1)^n n^2 (n^2 - 1), \quad \tilde{\beta}_n^{II} = -(-1)^n + \frac{1}{12}(-1)^n n^2 (n^2 - 1), \quad 2 \leq n \leq N_{II}.$$

In region III, the stream function is approximated by $\psi^{III}(x, y)$ where

$$\psi^{III}(x, y) = G(y/\alpha) + \sum_{m=2}^{M_{III}} \sum_{n=4}^{N_{III}} c_{mn} P_m^{III}(x) Q_n^{III}(y). \quad (3.3)$$

The functions $P_m^{III}(x)$, $Q_n^{III}(y)$ are suitable linear combinations of shifted Chebyshev polynomials chosen so that $\psi^{III}(x, y)$ automatically satisfies the boundary conditions on $y = 0$ ($0 \leq x \leq k$), $y = \alpha$ ($0 \leq x \leq k$) and $x = k$ ($0 \leq y \leq \alpha$). As before, we can show that

$$P_m^{III}(x) = T_m^{III}(x) + \alpha_m^{III} T_1^{III}(x) + \beta_m^{III} T_0^{III}(x), \quad 2 \leq m \leq M_{III},$$

where $T_m^{III}(x)$, $0 \leq m \leq M_{III}$, are defined by

$$T_m^{III}(x) = T_m \left(\frac{2x - k}{k} \right),$$

and $\alpha_m^{III}, \beta_m^{III}$ are given by

$$\alpha_m^{III} = -m^2, \quad \beta_m^{III} = m^2 - 1.$$

Since we wish to have polynomials of the same order on both sides of the interface $x = 0$ ($0 \leq y \leq \alpha$) we choose $N_{III} = N_{II}$. The polynomials $Q_n^{III}(y)$ are given by

$$Q_n^{III}(y) = \tilde{T}_n^{III}(y) + \tilde{\alpha}_n^{III} \tilde{T}_3^{III}(y) + \tilde{\beta}_n^{III} \tilde{T}_2^{III}(y) + \tilde{\gamma}_n^{III} \tilde{T}_1^{III}(y) + \tilde{\delta}_n^{III} \tilde{T}_0^{III}(y), \quad 4 \leq n \leq N_{III},$$

where $\tilde{T}_n^{III}(y) = \tilde{T}_n^{II}(y)$, $0 \leq n \leq N_{II}$, and

$$\begin{aligned}\tilde{\alpha}_n^{III} &= \frac{1}{12}[-n^2 - \frac{1}{2}(-1 + (-1)^n) + \frac{1}{3}(-1)^n n^2 (n^2 - 1)], \\ \tilde{\beta}_n^{III} &= 6\tilde{\alpha}_n^{III} - \frac{1}{12}(-1)^n n^2 (n^2 - 1), \\ \tilde{\gamma}_n^{III} &= -\tilde{\alpha}_n^{III} + \frac{1}{2}(-1 + (-1)^n), \\ \tilde{\delta}_n^{III} &= -1 - \tilde{\alpha}_n^{III} - \tilde{\beta}_n^{III} - \tilde{\gamma}_n^{III}.\end{aligned}$$

4. Collocation Strategy

The coefficients in the series expansions (3.1) - (3.3) are determined by collocating the differential equation at certain points in the domain and the solution in the three subregions are patched by imposing the correct order of weak continuity across the subregion interfaces. The points at which the extreme values of the Chebyshev polynomials are attained are well known to give rise to optimal approximation properties of smooth functions. Therefore, we choose the points corresponding to the Chebyshev polynomial of highest degree used in the solution representations as our collocation points in both the x - and y -directions. Thus, in region I, for example, the collocation points are given by

$$x_i^I = \frac{h(x_i - 1)}{2}, \quad y_j^I = \frac{(1 - \alpha)y_j + 1 + \alpha}{2},$$

where

$$\begin{aligned}x_i &= -\cos\left(\frac{i\pi}{M_I}\right), \quad 0 \leq i \leq M_I, \\ y_j &= -\cos\left(\frac{j\pi}{N_I}\right), \quad 0 \leq j \leq N_I.\end{aligned}$$

Boundary conditions

Due to the choice of modified Chebyshev polynomials as trial functions in the expansions (3.1) - (3.3) the boundary conditions are automatically satisfied except along $x = 0$ ($\alpha \leq y \leq 1$). Along this part of the boundary there are $N_I + 1$ collocation points. We deduce from (3.1) that ψ and ψ_x are polynomials of degree N_I along $x = 0$ ($\alpha \leq y \leq 1$) each depending on $N_I - 1$ degrees of freedom. Therefore, collocation of ψ and ψ_x at $N_I - 1$ distinct points ensures that the boundary conditions along $x = 0$ ($\alpha \leq y \leq 1$) are satisfied identically. We collocate these conditions at the points $(0, y_j^I)$, $j = 0, \dots, N_I - 2$.

Interface continuity conditions

Let us first examine the interface $y = \alpha$ ($-h \leq \alpha \leq 0$) between subregions I and II. We impose C^3 continuity of the stream function across this interface, i.e.,

$$\frac{\partial^k \psi^I}{\partial y^k}(x, \alpha) = \frac{\partial^k \psi^{II}}{\partial y^k}(x, \alpha), \quad k = 0, 1, 2, 3, \quad -h \leq x \leq 0. \quad (4.1)$$

Now ψ^I and ψ^{II} are polynomials of degree M_I along $y = \alpha$ each possessing $M_I - 1$ degrees of freedom. We may write the condition (4.1) with $k = 0$ as

$$\sum_{n=2}^{M_{II}} \left[\sum_{m=2}^{N_I} a_{mn} Q_n^I(\alpha) - \sum_{m=2}^{N_I} b_{mn} Q_n^{II}(\alpha) \right] P_m^I(x) = 0. \quad (4.2)$$

We want to collocate at a sufficient number of points to ensure that (4.2) is satisfied identically in which case ψ is pointwise continuous across the interface between subregions I and II. Equation (4.2) is collocated at the points (x_i^I, α) , $i = 2, 3, \dots, M_I - 2$. A further two conditions are required to ensure (4.2) is an identity. Therefore, in addition, we collocate the following conditions

$$\psi^{II}(0, \alpha) = 1, \quad \frac{\partial \psi^{III}}{\partial x}(0, \alpha) = 0. \quad (4.3)$$

Similarly, continuity of ψ_y is obtained by collocating (4.1) with $k = 1$ at the points (x_i^I, α) , $i = 2, 3, \dots, M_I - 2$, as well as the conditions

$$\frac{\partial \psi^{II}}{\partial y}(0, \alpha) = 0, \quad \frac{\partial^2 \psi^{II}}{\partial x \partial y}(0, \alpha) = 0. \quad (4.4)$$

This results in pointwise continuity of ψ_y across the interface $y = \alpha$ ($-h \leq x \leq 0$).

Now consider the continuity conditions on the second and third derivatives of ψ across the interface. We collocate each of the conditions (4.2) with $k = 2$ or $k = 3$ at the $M_I - 3$ points (x_i^I, α) , $i = 2, 3, \dots, M_I - 2$. Thus, these derivatives are not pointwise continuous across the interface. Moffatt [9] shows that the leading singular term in the expansion of ψ about the corner $(0, \alpha)$ is $O(r^\lambda)$ where $\lambda = 1.5445$ and so it would be inconsistent to impose continuity of these higher derivatives.

The same collocation strategy is applied across the interface $x = 0$ ($0 \leq y \leq \alpha$) between subregions II and III. As a result the stream function and its normal derivative are pointwise continuous across the interface whereas the second and third derivatives of ψ are continuous only at the collocation points $(0, y_j^{II})$, $j = 2, 3, \dots, N_{II} - 2$. In addition all tangential derivatives of ψ and ψ_y are pointwise continuous across the interface.

Differential equation

The biharmonic equation is collocated in each subregion at all points on the collocation grid with the exception of those on or one in from each subregion boundary, i.e.,

$$(x_i^k, y_j^k), \quad i = 2, \dots, M_k - 2, \quad j = 2, \dots, N_k - 2,$$

for $k = I, II$, and III .

When the spectral collocation equations resulting from the boundary conditions, interface continuity conditions and differential equations are added together, they yield a total of $[(M_I - 1)(N_I - 1) + (M_{II} - 1)(N_{II} - 1) + (M_{III} - 1)(N_{III} - 3)]$ linear equations which is equal to the number of unknown coefficients a_{mn} , b_{mn} and c_{mn} . Therefore, provided the coefficient matrix is non-singular, this system of equations possesses a unique solution.

The spectral representations we use ((3.1) - (3.3)) are constructed to automatically satisfy some of the boundary conditions. It is possible to construct a collocation scheme in which the basis functions satisfy none of the boundary conditions but which results in the same scheme. This has the advantage of being more flexible but requiring the solution of a slightly larger linear system of equations.

In subregion I, for example, we represent ψ in the form

$$\psi^I(x, y) = G(y) + \sum_{m=0}^M \sum_{n=0}^N a_{mn} T_m^I(x) \tilde{T}_n^I(y). \quad (4.5)$$

Satisfaction of the boundary conditions along $x = -h (\alpha \leq y \leq 1)$ at the collocation points $(-h, y_j^I), j = 0, 1, \dots, N_I$ leads to the following sets of equations:

$$\sum_{n=0}^{N_I} \left\{ \sum_{m=0}^{M_I} (-1)^m a_{mn} \right\} \tilde{T}_n^I(y_j^I) = 0, \quad 0 \leq j \leq N_I, \quad (4.6)$$

$$\sum_{n=0}^{N_I} \left\{ \sum_{m=0}^{M_I} (-1)^m m^2 a_{mn} \right\} \tilde{T}_n^I(y_j^I) = 0, \quad 0 \leq j \leq N_I. \quad (4.7)$$

In effect, we are collocating a polynomial of degree N_I at $N_I + 1$ points and therefore since we are equating it with the zero polynomial, its coefficients must be zero, i.e.,

$$\sum_{m=0}^{M_I} (-1)^m a_{mn} = 0, \quad 0 \leq n \leq N_I, \quad (4.8)$$

$$\sum_{m=0}^{M_I} (-1)^m m^2 a_{mn} = 0, \quad 0 \leq n \leq N_I. \quad (4.9)$$

We may eliminate a_{0n} and a_{1n} from (4.8) and (4.9) to obtain

$$a_{0n} = \sum_{m=2}^{M_I} (-1)^m (m^2 - 1) a_{mn}, \quad a_{1n} = \sum_{m=2}^{M_I} (-1)^m m^2 a_{mn}, \quad 0 \leq n \leq N_I.$$

Thus, we may write (4.5) in the form

$$\psi^I(x, y) = G(y) + \sum_{m=2}^{M_I} \sum_{n=0}^{N_I} a_{mn} P_m^I(x) \tilde{T}_n^I(y), \quad (4.10)$$

where $P_m^I(x)$ is defined in Section 3.

The boundary conditions along $y = 1 (-h \leq x \leq 0)$ leads to

$$\sum_{m=2}^{M_I} \left\{ \sum_{n=0}^{N_I} a_{mn} \right\} P_m^I(x) = 0, \quad (4.11)$$

$$\sum_{m=2}^{M_I} \left\{ \sum_{n=0}^{N_I} n^2 a_{mn} \right\} P_m^I(x) = 0. \quad (4.12)$$

We collocate (4.11) and (4.12) at the collocation points $(x_j^I, 1), j = 2, 3, \dots, M_I$. As before, the coefficients in (4.11) and (4.12) must be zero when collocated at these points, i.e.,

$$\sum_{n=0}^{N_I} a_{mn} = 0, \quad 2 \leq m \leq M_I, \quad (4.13)$$

$$\sum_{n=0}^{N_I} n^2 a_{mn} = 0, \quad 2 \leq m \leq M_I. \quad (4.14)$$

Again, we may eliminate a_{m0} and $a_{m1}, 2 \leq m \leq M_I$, from (4.13) and (4.14) to obtain

$$a_{m0} = \sum_{n=2}^{N_I} (n^2 - 1) a_{mn}, \quad a_{mn} = - \sum_{n=2}^{N_I} n^2 a_{mn}. \quad (4.15)$$

Substitution of (4.15) into (4.10) results in the approximation (3.1). Thus, we have shown the equivalence of the two expansions (3.1) and (4.5) in subregion I when an appropriate collocation strategy is chosen for the boundary conditions. We may show a similar result in the other two subregions.

5. Method of Solution

We describe the capacitance matrix technique [1, 15] for efficiently inverting the linear system of equations derived in the previous section. It is important to incorporate the underlying matrix structure into the solution procedure so that not only do we have an efficient method but also one which is able to solve a large problem without running into storage problems.

The spectral domain decomposition method described in this paper gives rise to a block matrix whose blocks are either full or zero. The full problem may be solved directly, but this process would not be efficient in terms of the number of storage locations and computational time. The linear system of equations for the unknown coefficients is written in the partitioned form

$$\begin{bmatrix} A_1 & 0 & 0 & A_4 & 0 \\ 0 & B_2 & 0 & B_4 & B_5 \\ 0 & 0 & C_3 & 0 & C_5 \\ D_1 & D_2 & 0 & D_4 & D_5 \\ 0 & E_2 & E_3 & E_4 & E_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} p \\ q \\ 0 \\ 0 \\ r \end{bmatrix}. \quad (5.1)$$

The first three blocks of rows correspond to collocation of the biharmonic equation and boundary conditions in subregions I, II, and III, respectively. The last two blocks of rows correspond to the interface continuity conditions between subregions I and II and subregions II and III, respectively. The square matrices A_1, B_2, C_3, D_4 , and E_5 are of orders $n_1, n_2, n_3, 2n_4$, and $2n_5$, respectively, where

$$n_1 = (N_I - 4)(M_I - 4) + 2(N_I - 2),$$

$$n_2 = (N_{II} - 4)(M_{II} - 4) + 4,$$

$$n_3 = (N_{III} - 4)(M_{III} - 4),$$

$$n_4 = 2(M_I - 4),$$

$$n_5 = 2(N_{III} - 4).$$

The vectors x_k are just a partition of the vector of unknown expansion coefficients in the three subregions.

In the capacitance matrix method, we write (5.1) in the natural component form suggested by the partitioning

$$A_1 x_1 + A_4 x_4 = p \quad (5.2)$$

$$B_2 x_2 + B_4 x_4 + B_5 x_5 = q \quad (5.3)$$

$$C_3 x_3 + C_5 x_5 = 0 \quad (5.4)$$

$$D_1 x_1 + D_2 x_2 + D_4 x_4 + D_5 x_5 = 0 \quad (5.5)$$

$$E_2 x_2 + E_3 x_3 + E_4 x_4 + E_5 x_5 = r. \quad (5.6)$$

We write x_1, x_2 , and x_3 in terms of x_4 and x_5 by premultiplying (5.2), (5.3) and (5.4) by the inverses of A_1, B_2 , and C_3 , respectively provided they exist, i.e.,

$$x_1 = A_1^{-1} p - A_1^{-1} A_4 x_4, \quad x_2 = B_2^{-1} q - B_2^{-1} B_4 x_4 - B_2^{-1} B_5 x_5, \quad (5.7)$$

$$x_3 = -C_3^{-1} C_5 x_5.$$

Eliminating x_1, x_2 , and x_3 from (5.5) and (5.6) we obtain the following system for x_4 and x_5 .

$$(D_4 - D_1 A_1^{-1} A_4 - D_2 B_2^{-1} B_4) x_4 + (D_5 - D_2 B_2^{-1} B_5) x_5 = -D_1 A_1^{-1} p - D_2 B_2^{-1} q, \quad (5.8)$$

$$(E_4 - E_2 B_2^{-1} B_4) x_4 + (E_5 - E_2 B_2^{-1} B_5 - E_3 C_3^{-1} C_5) x_5 = r - E_2 B_2^{-1} q.$$

The system (5.8) is one of order $2(n_4 + n_5)$ and can be solved efficiently since it is much smaller than the original system. The coefficient matrix of this system is known as the capacitance matrix. We solve (5.8) simultaneously for x_4 and x_5 and then determine x_1, x_2 , and x_3 from (5.7). Whenever a system of equations needs to be solved in this method, we use a Crout factorization technique from the NAG Library [11].

The following algorithm describes how efficiencies can be made in the solution procedure.

Algorithm

(1) Calculate $A_1^{-1} A_4$ and $A_1^{-1} p$ by solving a system of the form

$$A_1 [W_1 | v_1] = [A_4 | p]. \quad (5.9)$$

Exploit the fact that A_4 has only n_4 nonzero columns, and therefore we need only solve a system with $n_4 + 1$ right-hand sides. Similarly, we calculate $B_2^{-1} B_4, B_2^{-1} B_5, B_2 q$, and $C_3^{-1} C_5$ carefully exploiting any zero columns that B_4, B_5 or C_5 may possess by solving the systems

$$B_2 [W_2 | W_3 | v_2] = [B_4 | B_5 | q], \quad (5.10)$$

$$C_3 [W_4] = [C_5]. \quad (5.11)$$

(2) Evaluate the entries in the capacitance matrix:

$$\begin{bmatrix} D_4 - D_1W_1 - D_2W_2 & D_5 - D_2W_3 \\ E_4 - E_2W_2 & E_5 - E_2W_2 - E_3W_4 \end{bmatrix},$$

and the right-hand side:

$$\begin{bmatrix} -D_1v_1 - D_2v_2 \\ r - E_2v_2 \end{bmatrix},$$

again exploiting any zero columns in W_1, W_2, W_3 and W_4 .

(3) Solve the capacitance matrix system (5.8) for x_4 and x_5 .

(4) Evaluate x_1, x_2 , and x_3 by substitution of x_4 and x_5 in (5.7).

The majority of the work in this algorithm lies in Steps (1) and (3). In Step (1) the solution of the linear systems (5.9), (5.10), and (5.11) requires $kn_1^3 + n_1^2(n_4 + 1)$, $kn_2^3 + n_2^2(2n_4 + n_5)$ and $kn_3^3 + n_3^2(n_5 + 1)$ operations, respectively. The solution of the capacitance matrix system requires $8k(n_4^3 + n_5^3) + 4(n_4^2 + n_5^2)$ operations. The solution of the full system (5.1) would require $kN^3 + N^2$ operations where $N = n_1 + n_2 + n_3 + 2(n_4 + n_5)$. Thus, the use of the capacitance matrix technique has effected savings of $O(n_1^2(n_2 + n_3) + n_2^2(n_1 + n_3) + n_3^2(n_1 + n_2))$.

6. Numerical Results

Numerical experiments are performed for Stokes flow through a 4:1 contraction geometry, i.e., $\alpha = \frac{1}{4}$. We examine the convergence of the approximations as the degree of the trial functions is increased and also the performance of the capacitance matrix method for solving the spectral collocation equations. As in [6, 15], we choose truncation parameters $h = 1.5$ and $k = 0.5$.

To verify the convergence of the approximations (3.1) - (3.3), we give contour plots of the stream function for different numbers of degrees of freedom. In all of these plots, we have $M_I = M_{II} = M_{III} = 14$. In the y -direction, we choose $N_I = N_{II} = 12, N_I = N_{II} = 16$ and $N_I = N_{II} = 20$ in Figures 3, 4, and 5, respectively, with $N_{III} = 8$ in all cases. The smoothness of the contours improves as the number of degrees of freedom is increased. Notice also the continuity of the stream function across the subregion interfaces which is a result of the conforming discretization that we use.

The contours appear a little ragged, especially those defining the vortex in the salient corner. This is due to the contouring routine. Although we obtain a global approximation, the NAG routine for contouring a continuous function fails because of sharp changes in the gradient of the solution in the recirculation region. To overcome this, a uniform mesh is placed over the domain and a NAG routine interpolating the values of the solution at these points is used.

The stream function plot in Figure 5 obtained with a total of 571 degrees of freedom is in qualitative agreement with those in [8] which are obtained using a finite element method with as many as 1326 degrees of freedom. A significant improvement is also observed over the authors' previous work [5, 15] in the continuity achieved across the subregion interfaces which indicates the advantage of using a conforming spectral collocation strategy. The contour plot

of the vorticity is given in Figure 6. This is obtained by differentiating the stream function shown in Figure 5.

In Table I, we compare the direct inversion of (5.1) with the capacitance matrix method in terms of CPU (Amdahl 5890 - 300) time and required storage in Mbytes for different numbers of degrees of freedom. In this comparison, we choose $M_I = M_{II} = M_{III} = N_I = N_{II} = N_{III}$. From the table, we note that the capacitance matrix method becomes more efficient as the number of degrees of freedom increases in terms of both storage requirements and computational effort. Alternatively, with a given storage limit, one may solve, using the capacitance matrix method, a problem using many more degrees of freedom than by the original method which inverts directly the linear system (5.1).

7. Conclusions

A spectral domain decomposition method is described for solving Stokes flow in a channel contraction using the stream function formulation. Spectral approximations are constructed and a collocation strategy devised so that the subregions are conforming. The resulting approximations and their normal derivatives are pointwise continuous across subregion interfaces. The trial functions in these representations are chosen to satisfy some of the boundary conditions. An alternative collocation method in which the trial functions do not satisfy any of the boundary conditions is shown to be equivalent to the former strategy.

Efficient direct methods based on the capacitance matrix technique are used to solve the resulting system of linear equations for the expansion coefficients in the three subregions. Economies are made in terms of storage and computational effort over the original method. This solution procedure can be applied to any rectangularly decomposable domain.

In previous work [6], the use of nonconforming subregions in the solution of the Navier-Stokes equations led to a breakdown in convergence at Reynolds numbers above 200. We aim to use the new conforming subregions to extend the range of Reynolds numbers for which the method converges and this will be reported in a future paper.

Table 1. CPU (Amdahl 5890-300) times in seconds and storage in Mbytes required for solving the 4:1 contraction problem for different numbers of degrees of freedom.

M_I	Degrees of Freedom	Original Method		Capacitance Matrix	
		time	mbytes	time	mbytes
9	176	1.35	0.97	1.44	0.94
11	280	2.57	1.35	1.99	1.19
13	408	5.89	2.06	3.40	1.54

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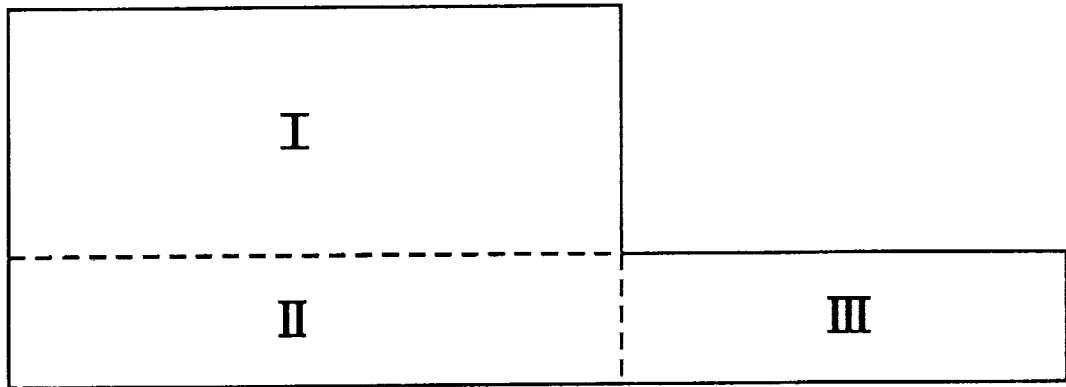


Figure 1. Three subregion decomposition of the flow region.

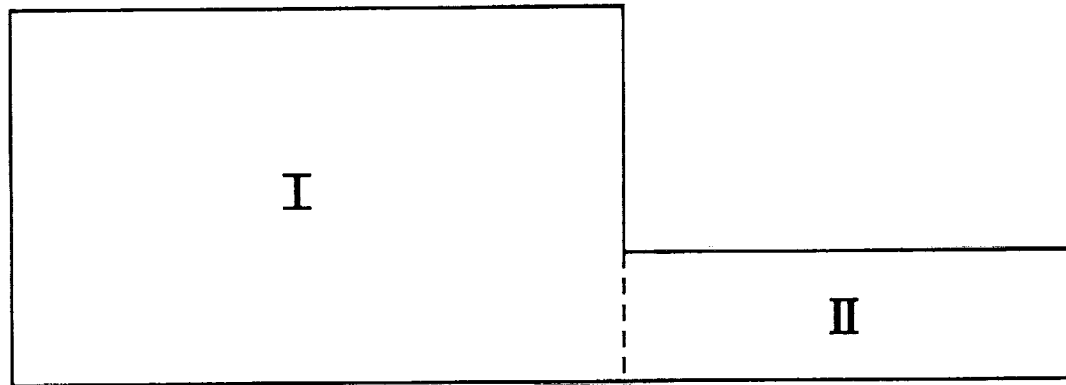


Figure 2. Two subregion decomposition of the flow region.

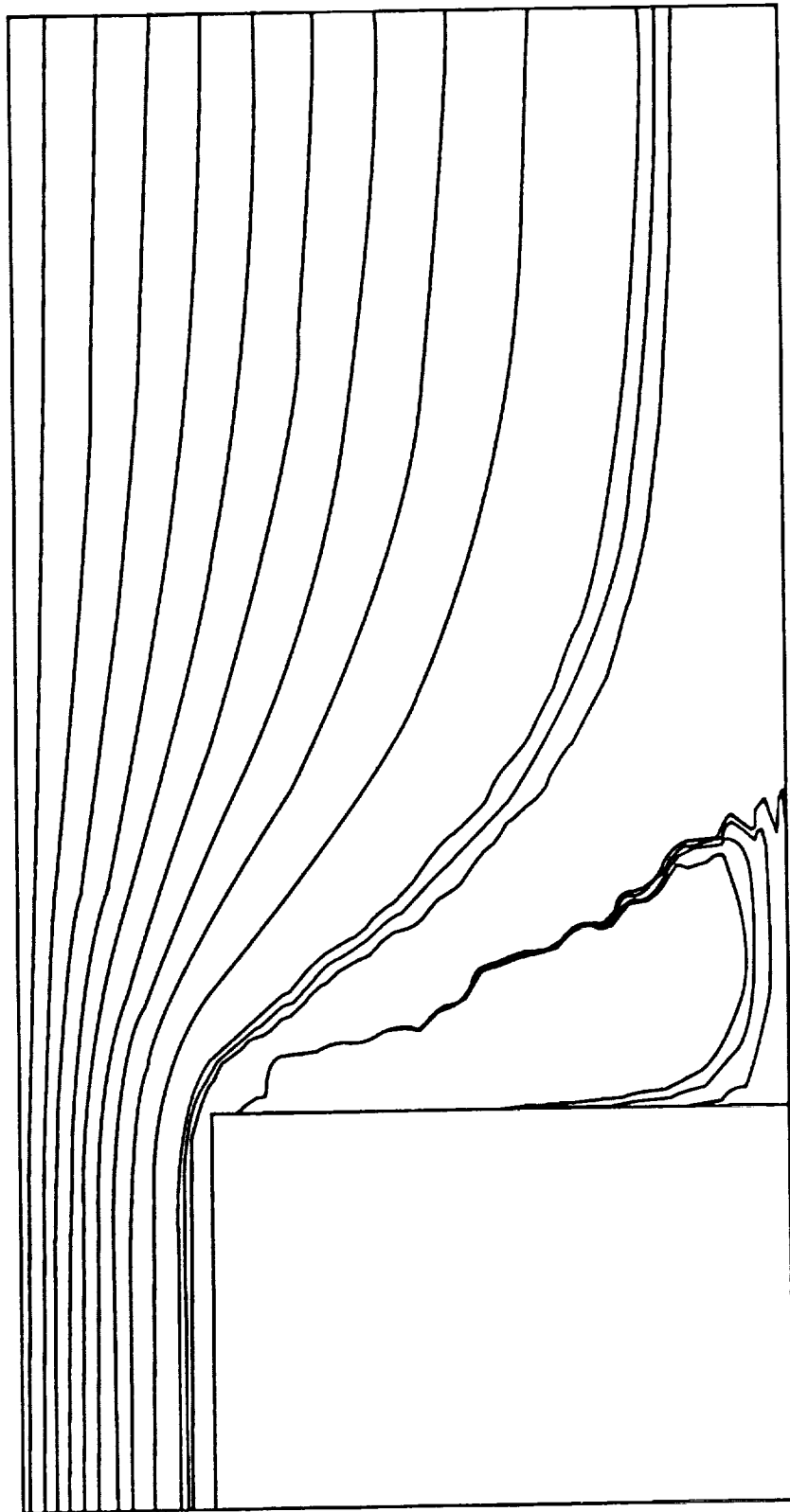


Figure 3. Stream function contours of the 4:1 problem with 363 degrees of freedom.

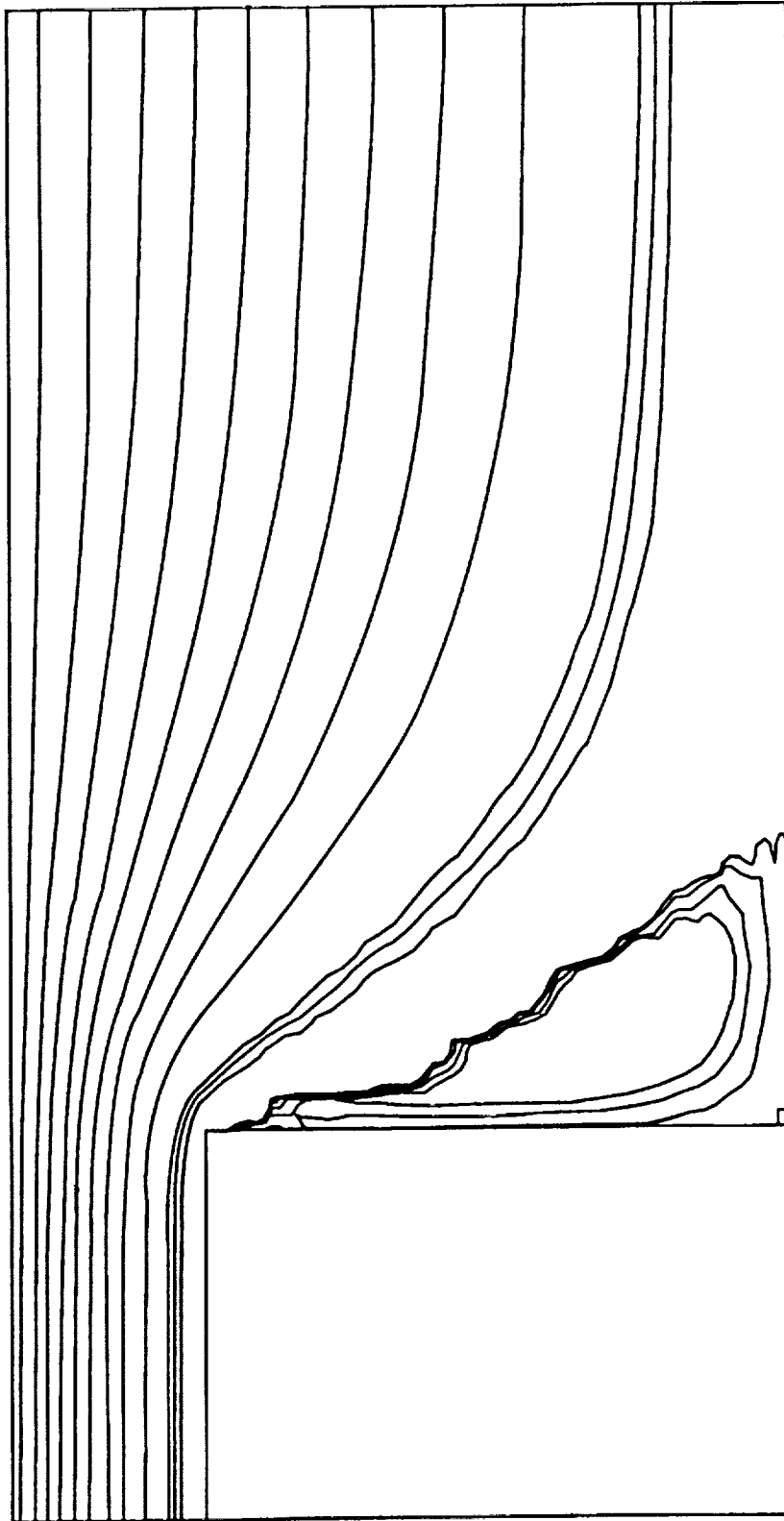


Figure 4. Stream function contours of the 4:1 problem with 467 degrees of freedom.

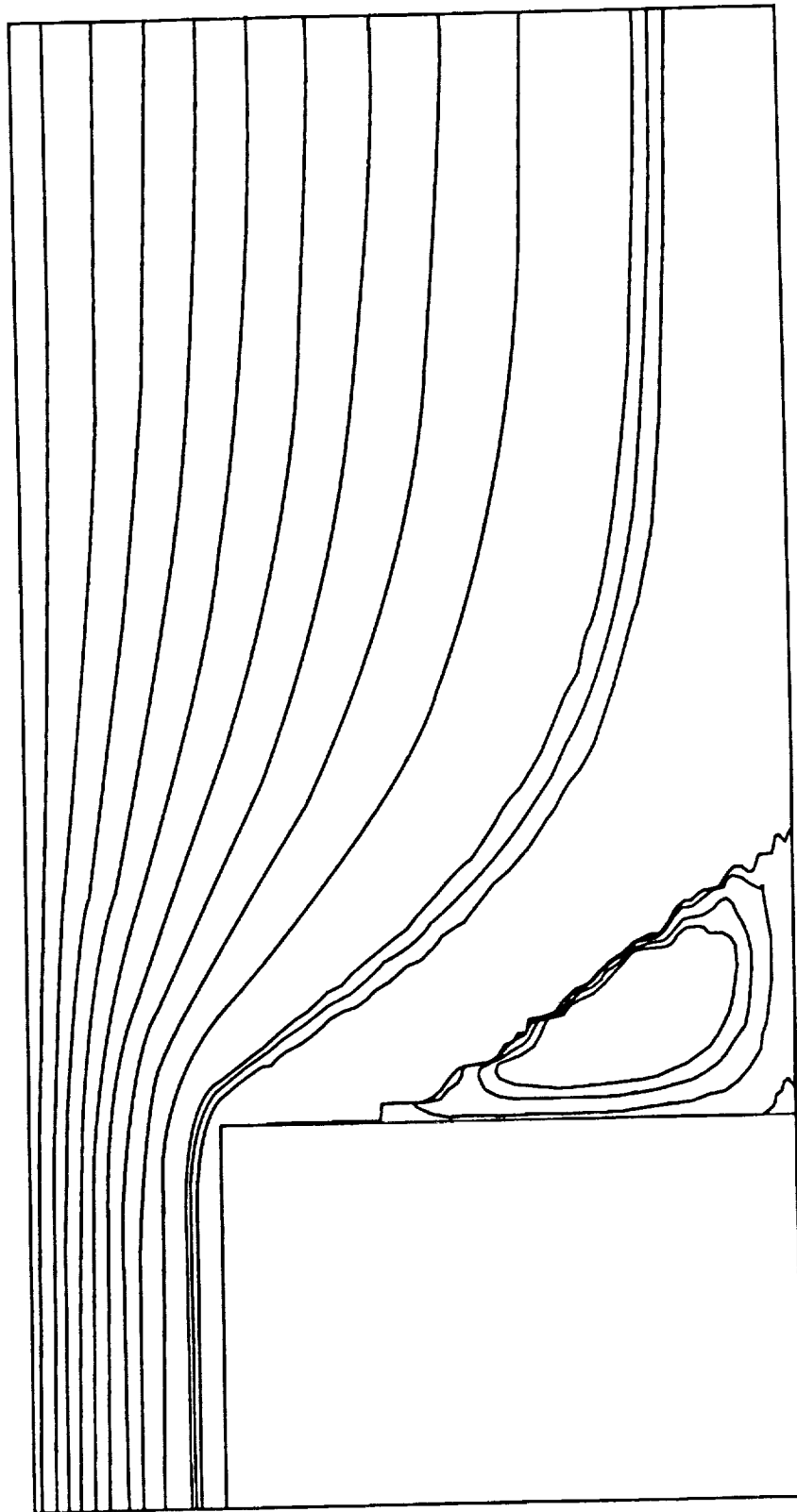


Figure 5. Stream function contours of the 4:1 problem with 571 degrees of freedom.

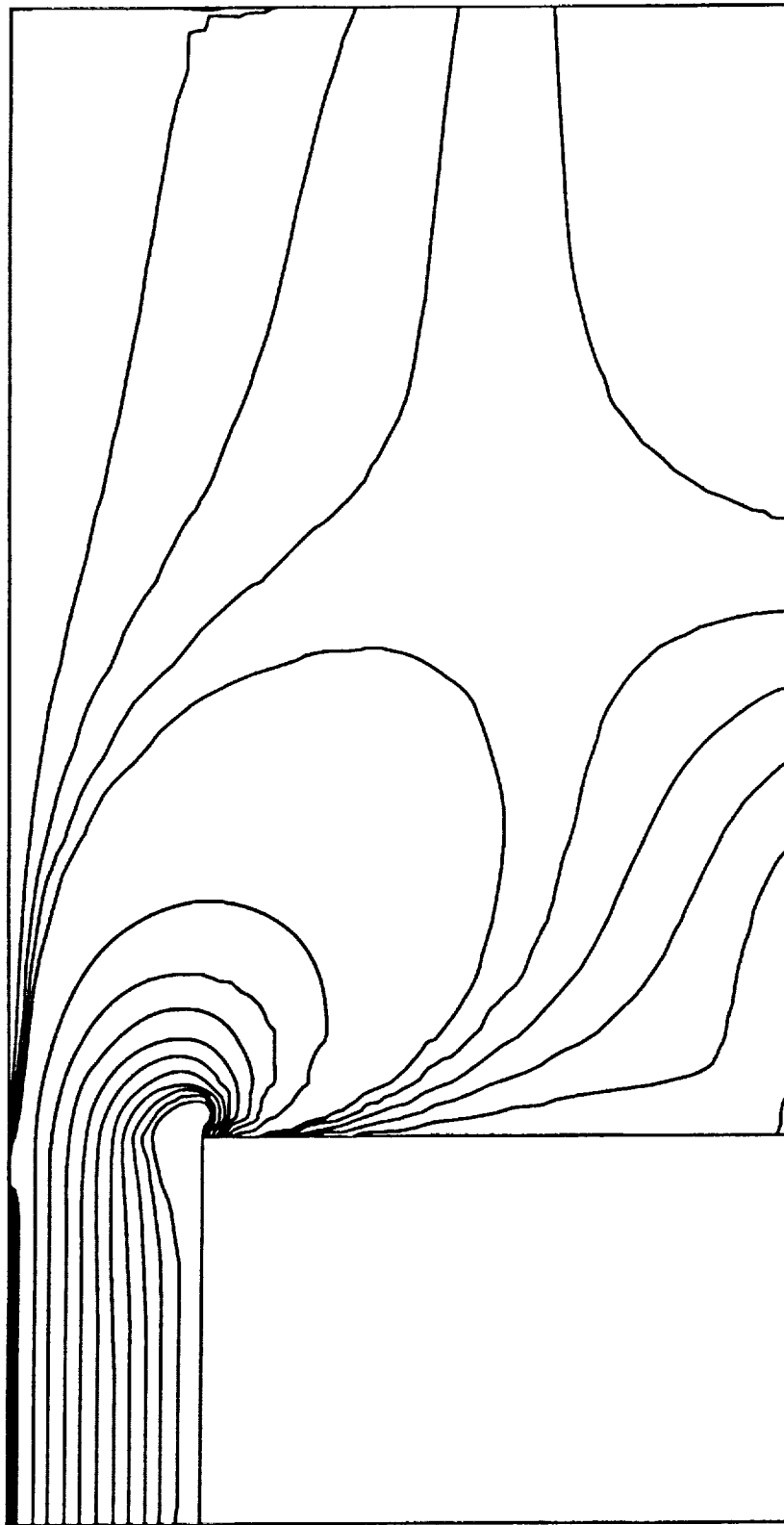


Figure 6. Vorticity contours of the 4:1 problem with 571 degrees of freedom.



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