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Effects of Mistuning and Matrix Structure on the Topology of Frequency Response Curves

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CONTENTS

EFFECTS OF MISTUNING AND MATRIX STRUCTURE ON THE TOPOLOGY OF FREQUENCY RESPONSE CURVES*

ABSTRACT

The stability of a frequency response curve under mild perturbations of the system's matrix is investigated. Using recent developments in the theory of singularities of differentiable maps, it is shown that the stability of a response curve depends on the structure of the system's matrix. In particular, the frequency response curves of a *cyclic* system are shown to be unstable. Consequently, slight parameter variations engendered by mistuning will induce a significant difference in the topology of the forced response curves, if the mistuning transformation crosses the bifurcation set.

1. INTRODUCTION

This investigation is motivated by the need to prevent excessive vibration of blades in jet propulsion systems. Blades are critical components in the machinery for generating motive power for aeronautical and space propulsion. The number of blades in an engine is very large, but the unexpected failure of even one of them is unacceptable. Usually, the blades are arranged on a set of circular wheels, and each wheel is known in the literature as a 'bladed disk assembly'.

Bladed disk assemblies are often modeled either as a rectilinear array (cascade) of blades, or as a cyclic configuration of blades on an axi-symmetric, circular disk. When the global equations of motion are assembled, the structure of the system matrix is usually *banded*, for the rectlinear cascade, and *circulant* (in at least one sub-matrix) for the circular configuration. What we seek to investigate here is whether the structure of a system matrix has any effect on the variation of the forced response amplitudes from one blade to another as small amounts of random mistuning are applied to the system. In other words, will there be a significant difference in results if the same mistuning is applied to two 'similar' systems, one of which has a banded matrix while the other is circulant?

In the structural dynamics literature (see, for instance, Bendiksen (1987), Wei & Pierre 1988) the linear and cyclic chains are sometimes assumed to undergo the same

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qualitative behavior under slight parameter perturbations. Thus, small order perturbations of the tuned system matrix are assumed to lead to no more than small order differences in the system dynamic characteristics relative to the unperturbed case, provided only that the system has "strong coupling".

In this monograph, we show that such an assumption regarding qualitative behavior does not actually hold in the case of cyclic systems; that cyclic systems exhibit a sensitive dependence on parameter perturbation. Thus, a given state of mistuning may produce little or no difference relative to the tuned case, while a considerable difference is induced when a very small change is made in the mistuning. Such unstable behavior arises in cyclic systems, even when there is "strong coupling". We state at the onset that significant differences exist between the 'sensitive dependence on *parameters*' of linear systems, with which we are concerned here, and the 'sensitive dependence on *initial conditions*' of non-linear systems. The unstable—and apparently erratic—behavior induced by cyclicity arises whenever the perfect ordering in a cyclo-symmetric, linear system is destroyed, for which we shall use the term *disordered motion.* This is in consonance with Ziman (1979). On the other hand, what is generally referred to as *chaotic motion* occurs in non-linear systems which exhibit sensitive dependence on initial conditions.

In carrying out this work, we borrow from certain developments in the theory of topological spaces; specifically, from the work of Arnol'd and his co-workers (1968 *et seq.*) in the theory of singularities of differentiable maps.

2. REVIEWS AND MISCELLANEA

2.1 DEFINITION OF MISTUNING.

In turbomachinery dynamics, a turbine bladed disk assembly is said to be *tuned* when all the blades on the disk are assumed to be truly identical. Practical realities of manufacturing processes preclude the existence of exact uniformity among all the blades. When residual differences which exist from one blade to another (no matter how small) are accounted for in the theoretical model, the assembly is termed a *mistuned* bladed disk.

The term 'mistuning' has variously been defined, often implicitly, to mean different things by different authors. In order to avoid any ambiguity, we give an explicit definition. Throughout this work, whenever we use "mistuning" without any qualifier, it should be understood that we mean *a violation of periodicity in a previously periodic system.*

A system is said to be periodic in x, with a period τ, if a characteristic system parameter $v(x)$ has the same value when the origin of x is translated by an integer multiple of τ, i.e. $v(x) = v(x \pm n\tau)$, $n = 1, \cdots, \infty$. In bladed disk assemblies, it is convenient to choose x as the angular displacement from a datum point at the disk rim, and express it as the blade slot number; while $v(x)$ is a measure of mistuning, such as the blade cantilever frequency in a given mode of vibration (1F for the first flexural, 2E for the second edgewise, etc).

In bladed disk vibration, one occasionally comes across terms like "mistuned assemblies with cyclic symmetry", even in reference to a randomly mistuned system. However, it is obvious that once a bladed disk is mistuned, the cyclic symmetry of the tuned state is destroyed. Therefore, in order to avoid any uncertainty here, we emphasize that a *mistuned* system cannot, by definition, have any cyclic symmetry.

The most common form of mistuning in practice is random mistuning, which arbitrarily destroys periodicity. However, several previous investigators have used mistuning as a generalized term to include cases where some form of periodicity is still preserved by the "mistuned" system. We refer in particular to the case of 'alternating mistuning' with an even number of blades on the wheel. In order to avoid any confusion with regard to terminology, we shall use the term "quasi-mistuning" in such cases. In the types of alternating mistuning commonly used in bladed disk modeling, it is true that the initial tuned state is destroyed, so that it is valid to use the term mistuning. However, cyclicity—albeit of a different period—is still preserved by the 'mistuning'.

Although this work has its origin in bladed disk dynamics, the results obtained from here need not be limited to bladed disks. Since we approach the problem from a generalized viewpoint, the conclusions to be drawn will be of relevance to other structures composed of identical substructures which are replicated either in a *uni-axial* chain, or in a closed *cyclic* formation. Therefore, in the sequel, we shall borrow the 'tuned' and 'mistuned' terminology from the bladed disk literature, and apply it to repetitive systems having cyclic or rectilinear periodicity.

2.2 MATHEMATICAL PRELIMINARIES

In general (but not in every case), lower case roman italics x denote vectors; elements of x are identified with subscripts, x_i; upper case roman italics A denote matrices; their elements being subscripted lower case italics a_{ij}; while lower case greek letters α denote scalars.

We use $\mathbb{R} \equiv \mathbb{R}^1$ for the real line, $\mathbb{R}^2$ for the euclidean plane, and $\mathbb{R}^n$ for the generalized n-dimensional space of n-tuples of real numbers. Similarly, $\mathbb{C}^n$ denotes the space of n-tuples of complex numbers. In the complex field, the superscript T denotes a matrix transposition, as in $B = A^T$; while H denotes a complex conjugate transpose (Hermitian transpose) as in $B = A^H$ if $B, A \in \mathbb{C}^{n \times n}$; while an over bar denotes conjugation, as in $\bar{A}$.

The terms "function", "transformation", "operator", and "map" will be used synonimously here, although there are strict differences among the terms; the same for "curve", "surface", and "manifold".

Let $A^{(0)}$ represent the system matrix of a tuned system, while $A^{(\varepsilon)}$ is the matrix of the corresponding mistuned system, which is obtained from $A^{(0)}$ via a suitable transformation, $f: A^{(0)} \rightarrow A^{(\varepsilon)}$. Generally, we assume that mistuning engenders a perturbation leading to the loss of periodicity, and that the perturbations are small, so that the tuned and mistuned system matrices are "close". We express this closeness in a clear and precise manner by taking norms.

$\| x \|_p$ is defined to be a pth vector norm of x, where x belongs to a space of n vectors, ($x \in \mathbb{R}^n$) such that

$$\| x \|_p = \left\{ \sum_{i=1}^{n} | x_i |^p \right\}^{1/p} \qquad 1 \leq p < \infty. \tag{1}$$

The 2-norm ($p=2$) is most often used for a vector because it corresponds to a measure of the vector's length in a euclidean space; but the 1-norm ($p=1$) and ∞-norm ($p=\infty$) are also widely used.

We can define the corresponding matrix norm of A, subordinate to $\| x \|$ as

$$\| A \| = \sup_{x \neq 0} \frac{\| y \|}{\| x \|}, \quad y = A \cdot x. \tag{2}$$

From the foregoing definitions of norms, the tuned and mistuned systems will be regarded as "close" if

$$\lim_{\varepsilon \rightarrow 0} \frac{\| A^{(\varepsilon)} \| - \| A^{(0)} \|}{\| A^{(0)} \|} = 0 \tag{3}$$

where $\| \cdot \|$ is any suitable norm, e.g. as defined above. Similarly, the eigensolutions of the rth mode of the tuned and mistuned systems will be regarded as almost identical if the following three conditions are satisfied:

$$\lim_{\varepsilon \to 0} | \lambda_r^{(\varepsilon)} - \lambda_r^{(0)} | = 0 \tag{4}$$

$$\lim_{\varepsilon \to 0} | (\| u_r^{(\varepsilon)} - u_r^{(0)} \|) | = 0 \tag{5}$$

$$\lim_{\varepsilon \to 0} \frac{< u_r^{(\varepsilon)}, u_r^{(0)} >}{\| u_r^{(\varepsilon)} \| \cdot \| u_r^{(0)} \|} = 1. \tag{6}$$

In the foregoing, $<\cdot,\cdot>$ denotes an inner product, $\lambda_r^{(0)}$ (respectively, $\lambda_r^{(\varepsilon)}$) is the rth eigenvalue of the tuned (mistuned) system, and $u_r^{(0)}$ ($u_r^{(\varepsilon)}$) the corresponding eigenvectors.

A circulant matrix frequently arises in the study of circular systems. An $n \times n$ matrix is said to be circulant when each row is a circular shift (to the right) of the preceding row. Thus, there are only n distinct elements in a matrix of $n \times n$, so that a circulant is completely determined by the elements in any one row. A 3×3 right circulant is of the form:

$$A = \text{circ}\,(a, b, c) = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \tag{7}$$

The above shifting is sometimes known as 'a right circulant'. Left circular shifts give rise to left circulants.

Circulant matrices have very interesting properties that set them apart from matrices of other forms (Davies, 1979). Notable among them is the fact that all circulants commute under multiplication, i.e $BA = AB$, if A and B are circulant matrices, conformable under matrix multiplication. Other properties of circulants that are more relevant for our application here are itemized below, where $A \in \mathbb{C}^{n \times n}$ denotes a circulant matrix.

- Every A is diagonalizable by the Fourier matrix, F; i.e. $A = F^H \Lambda F$ where Λ is a diagonal matrix.
- If A is circulant, then so are A^T, A^H, A^{-1} and αA ($\alpha \in \mathbb{C}$, $\alpha \neq 0$).
- If A is symmetric, then it has a series of degenerate eigenvalues.
- If A is anti-symmetric, its degenerate eigenvalues (if any) occur in complex conjugate pairs.

F is a Fourier matrix if its elements f_{rs} are given by

$$f_{rs} = \frac{1}{\sqrt{n}} e^{(2\pi i)(r-1)(s-1)/n} \tag{8}$$

where i is the unit imaginary number, and e is the natural logarithm base.

A matrix is said to be banded if all the elements in the matrix are zero, except within a narrow band parallel to, and including, the leading diagonal, i.e. $a_{ij} = 0, |i-j| > m$, where m is the semi-bandwidth of A. A special form of banded matrices that is of interest here is the tri-diagonal form $a_{ij} = 0, |i-j| > 1$. In such a matrix, non-zero elements occur in only three parallel diagonals including the leading diagonal.

$A^{(0)}$ and $A^{(\varepsilon)}$ may be regarded as linear operators in a vector space. They may also be regarded as manifolds in a generalized topological space. In the latter case, we can effect a transformation from $A^{(0)}$ to $A^{(\varepsilon)}$ via a map, f. Our aim is to determine if a given map behaves consistently, in a qualitative manner, as different perturbations are imposed on a given $A^{(0)}$ at random. We also want to determine whether certain forms of $A^{(0)}$ exhibit unstable behavior in going from $A^{(0)}$ to $A^{(\varepsilon)}$, via f.

Mathematically, this can be expressed as follows. Let $f: A^{(0)} \to A^{(\varepsilon)}$ denote a linear map in the open region $D \subset \mathbb{R}^n$ of an n-dimensional topological space in which $A^{(0)}$ and $A^{(\varepsilon)}$ are smooth manifolds. We also assume that any vector subspace $N \in \mathbb{R}^k \subset \mathbb{R}^n$ is a smooth manifold in an n dimensional space. It is understood that we are dealing with linear maps and linear systems. A linear map is a transformation

$$f: \mathbb{R}^m \to \mathbb{R}^n \tag{9}$$

such that the following conditions hold:

$$f(x_1 + x_2) = f(x_1) + f(x_2); \tag{10}$$

$$f(\alpha x) = \alpha f(x); \quad x_i \in \mathbb{R}^n, \alpha \in \mathbb{R}. \tag{11}$$

Generally, if the derivative of a map continues up to order k, then we say that f is of class C^k. If f is infinitely differentiable, then it is said to be a smooth map. Thus, a smooth map is of class C^∞.

If, in addition to being a smooth map, f also has an inverse g such that g is also smooth, then f is a diffeomorphism. The geometric effect of a diffeomorphism is to smoothly bend the coordinates associated with $A^{(\varepsilon)}$ compared to those associated with the tuned system, $A^{(0)}$.

If, however, both the invertible map f and its inverse g are merely continuous but not necessarily differentiable, then we say that f is a homeomorphism. A homeomorphism is, in fact, a C^0 diffeomorphism.

The distinction between a homeomorphism and diffeomorphism is important because each leads to a different definition of the "equivalence" of two maps which shall be required presently in our investigation of stability. In order to illustrate the distinction between a diffeomorphism and a homeomorphism let us consider the behavior of the map $f: x \rightarrow x^3$ in $\mathbb{R}$. It is seen by inspection that at $x = 0$, $g = f^{-1}$ is not differentiable (thus, f is not a diffeomorphism), although it is continuous (and hence f is a homeomorphism).

2.3 MODELS OF BLADED DISKS.

It is not an easy matter to set up an accurate, workable model of a bladed disk at modest cost. There are far too many effects to consider if one wanted to make up a "complete" model that included all relevant parameters, so that such a model will be necessarily expensive. It is therefore inevitable that several factors, which are judged to be of insignificant influence, would be ignored in the theoretical modeling process.

Although one cannot always specify in advance the relative importance of the factors to be included when setting up a model, two factors seem to occupy positions of eminence. These are *mistuning* and *cyclicity*. It is surprising, but true, that even a small amount of mistuning can induce considerable qualitative differences between a tuned and mistuned assembly's behavior. Now, in a similar way, will there be a qualitative difference between a model in which cyclicity is accounted for, and one in which it is excluded, on a scale similar to that due to mistuning only?

In most aeroelastic models of bladed discs, it is often assumed that the blades are arranged in a rectilinear cascade (e.g. Whitehead, 1966). Even when aeroelastic effects are ignored, as in some studies of turbine blade packets, a rectilinear arrangement of identical blades is often assumed (Prohl, 1958; Afolabi, 1978). The question now arises: must cyclicity be retained in the analyses in order to obtain a qualitatively realistic modeling? Until now, not much attention has been paid to examine whether the *structure* of the system matrix has any effects on mistuning.

2.3.1 Rectilinear Models of Bladed Disks

Fig 1*a* shows a bladed disk modeled as a cascade of blades on a rigid disk having an infinite radius of curvature. Each blade is treated as an aerofoil, Fig 1 *b*. The resulting system equation is not circulant, since the first blade is not assumed to be coupled with the last blade; hence, the system matrix is merely banded. Consequently, there would be no repeated eigenvalues or 'double modes' in such a system. This is due to the fact that cyclic symmetry, which gives rise to modal degeneracy, is not accounted for. Therefore, any computation carried out using such a model will be "structurally stable" (The concept of structural stability arises from the theory of singularities, and is discussed below).

However, experiments have shown (Jay, MacBain & Burns, 1984; Srinivasan & Cutts, 1985; Mehmed & Kaza, 1986; Mehmed & Murthy, 1988), that double modes exist in most bladed rotors (including prop fans). Moreover, instability of the response curves has been frequently encountered in turbomachinery structural dynamics. This manifests in the form of unequal amplitude distributions among the blades in a mistuned assembly, Ewins (1976).

2.3.2 Cyclic Models of Bladed Disks

A commonly used model of bladed disks is shown in Fig 2*a*, which is a rendition of the basic Dye-Henry (1969) model. The model retains cyclicity due to structural coupling of the first and last blades. Moreover, it can also accommodate mistuning, since it is easy to treat each blade uniquely in the simple model. Although it is very simple, it still captures many important features of an actual bladed disk. Recent experiments conducted by Datko and O'Hara (1987) have confirmed some of the qualitative features predicted (e.g. by Afolabi 1982, 1985) using such a model. However, a major drawback still exists in the model, and that is the inability to account for aerodynamic effects.

The model shown in Fig 2*b* is not an admissible model of a *bladed*-disk, unlike the Dye-Henry model. This is because only the disk subsystem is accounted for, with no accounts being taken of the *blades*. Although some investigators reported that it could be used to model a bladed disk (for instance, Wei & Pierre, 1988), no experimental confirmations of their predictions have been Published. However, because the model exhibits cyclicity, and can also be mistuned, it is a good model for studying the qualitative characteristics of *membranes, rings* or *disk only*.

We now return to the importance of cyclicity in bladed disk modeling. We know from previous work (Afolabi, 1988b) that when cyclicity is accounted for, different response patterns are obtained, not only when different models are used by various authors (Ewins & Han, 1984; Griffin & Hoosac, 1984), but also when the same model is used by the same investigator (Afolabi, 1985). It is then relevant to speculate if the variations in such results might have anything to do with the cyclic form of the bladed disk model used in the analyses. In other words, if the analyses were repeated using a uni-axial model, rather than one which is cyclic, will the inconsistency in results still remain?

In order to provide answers to the foregoing, we shall study a simple system in which the system's dynamical matrix has a *banded* formation, and compare it to the case when the matrix form is *circulant* and verify the structural stability of the response curves of both systems (see section 4).

2.4 ELASTIC STABILITY, GEOMETRIC STABILITY AND STRUCTURAL STABILITY.

A measure of the elastic stability in a system can be established from the quadratic form of the system's energies. Let $< x,y >$ denote an inner product, defined for $x,\ y \in \mathbb{C}^n$ as $y^H x$; T the kinetic energy of the system; U the potential energy; x a coordinate; while M and K are symmetric matrices. Then

$$\mathrm{T} = \tfrac{1}{2} < M\dot{x},\, \dot{x} >, \quad \mathrm{U} = \tfrac{1}{2} < Kx,\, x >, \quad \det\,[K - \lambda M] = 0, \qquad (12,13,14)$$

gives the characteristic polynomial of energies and, therefore, vanishes at values of $\lambda \in \mathbb{C}$, for $K \in \mathbb{C}^{n \times n}$, which are the *eigenvalues*. In component notation, the rth eigenvalue of the system may be written as

$$\lambda_r = \mu_r + i v_r \qquad (15)$$

The system is said to have elastic stablility when $\mu_r < 0$, for all r, i.e. each of the system eigenvalues has a negative real part.

On the other hand, geometric stability can be investigated from the *eigenvectors* of the system. We shall say that a system has geometric stability if the normalized eigenvectors point more or less in the same direction when the parameters of the system are given slight perturbations from their nominally tuned state (see eq. (9)). To the author's knowledge, the term 'geometric stability' has not been previously used in the literature in the context that we use it here.

Structural stability is a concept that has its roots in the theory of the singularity of differentiable maps (Whittney, 1955; Thom, 1972; Poston & Stewart, 1978, Arnol'd *et al*, 1985). The idea will be used when investigating the stability of forced response curves under perturbation. A curve or manifold is said to be "structurally stable" if a small perturbation does not change the topological character of its trajectories. In other words, the *shape* of the manifold must be preserved for structural stability to be said to exist.

2.5 ILL-CONDITIONING DUE TO CYCLICITY

The governing equations of motion of a tuned bladed disk system can be set up using various methods such as the finite element method (Elchuri, 1981), the lumped parameter method (Dye and Henry, 1969) or typical section models (Kaza & Kielb, 1982), Receptance Methods (Ewins, 1973 and El-Bayoumy and Srinivasan, 1975). Usually it is not feasible to account for all the relevant parameters in a given study due to the usual constraints of cost, and the difficulty of setting up the analytical model, etc. A fairly comprehensive model would result in a set of $n\times 1$ complex vector equation of the form

$$M\ddot{x} + (C + \omega_R G)\dot{x} + (K + \omega_R{}^2 R + E)x + iHx = f \tag{16}$$

where i is the unit imaginary number, ω_R is the rotation speed, M is the mass matrix, C the viscous damping matrix, G the coriolis matrix, K the elastic stiffness matrix, R the centrifugal stiffness matrix, E the complex aeroelastic matrix, and H the structural (hysteretic) damping matrix. The vector x and its derivatives represent displacement, velocity and acceleration, and f is the vector of other external forces.

Eq (16) is essentially a set of simultaneous equations, linear or non-linear. Assume we can carry out an admissible linearization. When all of the preparatory work is done, one would be left with a set of linear simultaneous equations of the form

$$(A - \lambda I)\,x = f \tag{17}$$

where I is the identity matrix. The set of equations shown in (17) above has been studied extensively by several investigators, for instance Wilkinson (1965), and Nwokah (1978, 1984) among many others.

In the case when $\lambda = 0$, eq. (17) reduces to the familiar form of simultaneous equations. It is well-known from elementary linear algebra that $Ax = f$ can sometimes be ill-conditioned with respect to solving for x, given A and f. A well-known ill-conditioned matrix is the hilbert matrix, defined as

$$H \in R^{n\times n} \equiv h_{ij} = 1/(i+j-1) \tag{18}$$

The ill-conditioning that arises when solving a set of simultaneous equations is manifest when considerable differences are obtained in computing x, when only a small perturbation is impressed on A, for any given vector of right hand sides, f.

Another type of ill-conditioning that is not as well-known occurs when qualititatively different results are obtained in solving for the eigenvectors x given small perturbations in A with null f. It must be understood that it is not the numerical procedure that is computationally ill-conditioned. Rather, the ill-conditioning is inherent in the physics of the problem, and is not eliminated by using a better algorithm. This type of ill- conditioning is inducible by cyclicity. Thus, the eigenproblem of a tuned cyclic system is ill-conditioned with respect to solving for eigenvectors. It must be noted that ill-conditioning is not a universally fixed characteristic of a given matrix. In other words, a certain A may be perfectly well-conditioned with respect to solving for eigenvalues, but acutely ill-conditioned with respect to solving for eigenvectors, etc; see Section 5 below.

If a system is subject to ill-conditioning with respect to the computation of eigenvectors, then different eigenvectors would be obtained under small (but different) perturbations of A, just as happens when different vectors x are obtained when A is ill-conditioned with respect to the solution of simultaneous equations. Ill-conditioning with respect to the computation of eigenvectors will be demonstrated briefly in section 5, and is treated in more detail in another monograph (Afolabi, 1989b).

Now, the frequency response curve of a given structural system is intimately linked with the system's eigenvectors. Indeed, by using modal analysis, the response curves $x_i(\omega)$ are simply a summation of scalar multiples of the eigenvectors:

$$x_i(\omega) = U^H \cdot [\Lambda - \omega^2 I]^{-1} \cdot U \cdot f_m \tag{19}$$

where Λ is the eigenvalue matrix, U is the eigenvector matrix, f_m is the excitation force (for the mth engine order with an inter-blade phase angle of $\beta_m = 2\pi m/N, f_m = B_m e^{i\beta_m}$).

A series expansion of eq (19) yields

$$x_i(\omega) = \sum_{r=0}^{N} \frac{u_m^H f_m u_r}{\lambda_r - \omega^2 a_r - i b_r} \tag{20}$$

where r denotes the rth resonance, $i = \sqrt{-1}$, $u_m \in \mathbb{C}^n$ is an eigenvector, $\lambda_r \in \mathbb{C}$ is an eigenvalue, and $a_r, b_r \in \mathbb{C}$ are modal constants.

If the geometric configuration of the eigenvectors are subject to sharp re-orientation under small perturbation, then it is to be expected that the resulting response curves will exhibit some form of inconsistency in the sense that different response curves will be obtained under different but similar perturbations. Such inconsistency is exactly what has always been reported by various investigators in blade mistuning research (Ewins, 1973; El-Bayoumy & Srinivasan, 1975; Griffin and Sinha, 1985; Ewins & Han, 1985; Afolabi, 1988a), when cyclicity is admitted into the model. Thus, although all the blades in a tuned assembly have identical response patterns, a small mistuning can lead to considerable changes between the stress or displacement amplitude experienced by each blade, as will be demonstrated shortly.

In some studies using tuned circulant matrices (Afolabi, 1982), the amplitude increase arising from a small mistuning can be both substantial and localized (around only one or two adjacent blades), leading to the premature failure of the affected blades. Those blades, which experience unusually large stresses thereby failing unexpectedly, while all other blades on the same disk are still stress-worthy, are known as "rouge blades", Afolabi (1988a).

2.6 IMPORTANCE OF MATRIX STRUCTURE

We know that a symmetric circulant matrix, $a_{ij} = a_{ji}$, having a series of degenerate eigenvalues, lies on a bifurcation set (Gilmore, 1981). Bifurcations in linear systems, and the importance of the bifurcation set (sometimes known as the catastrophe set) in mistuning research is explained in section 3.4. If a circulant matrix is mistuned, a geometric instability of the degenerate modes will result *if the mistuning leads to a crossing of the bifurcation set* whether the system has so-called strong coupling or not. If however, no crossing of the bifurcation set takes place, then the tuned system will exhibit strong geometric stability when given small amounts of mistuning. In contrast, the tuned banded matrix does not generally exhibit geometric instability until the eigenvalues are pathologically close, Wilkinson (1965).

3. SINGULARITY THEORY AND THE STABILITY OF SMOOTH MAPS

3.1 INTRODUCTION

Since this monograph is written for an engineering audience, we present in this section a brief summary of some basic concepts from the theory of the singularity of differentiable maps, which would be necessary for explaining our results.

Singularity Theory is a relatively new and rapidly evolving branch of mathematics. Its origin can be traced to the work of Whittney (1955). Substantial contributions were made by the French mathematician, René Thom (1972), who undertook a classification of the singularities of smooth maps. The British mathematician, Christopher Zeeman (1976, 1977), coined the colorful tag "Catastrophe Theory", and popularized Catastrophe Theory beyond the circle of a few mathematicians. Tim Poston and Ian Stewart (1978), wrote a very easy to read introduction to Catastrophe Theory. The Catastrophe Theory of

René Thom has now been superseded by singularity theory due in large part to the prolific Russian mathematician, Vladimir I. Arnol'd (1968 *et seq*) whose classification of singularities has extended beyond those established by Thom.

Why do we want to use singularity theory in bladed disk research? The reason is that the theory is very good at handling those situations where 'structural instability' occurs. This is the case when a small change in a system's "control parameter" leads to sudden and sharp changes in the behavior of the system under one situation, while the same degree of change in the control parameter produces little or no change in the behavior of the same system in another instance. Thus, singularity theory is concerned with qualitative properties, rather than quantitative. In a recent article, Zeeman (1987) defines qualitative properties as those that are invariant "under *differentiable* changes of coordinates, as opposed to quantitative properties that are invariant only under *linear* changes of coordinates".

Our basic approach will be to take a given function which typifies the behavior of a structural system, and study how the *shape* of the function changes as we apply small arbitrary perturbations to the system. If the topology of the curve remains basically the same, then the function is said to be stable, in some sense to be specified. If, however, small perturbations in the parameters of the function significantly changes its qualitative character, then it is said to be unstable. This stability may be expressed in different forms: topological stability, differential stability, etc. In essence, what we are interested in is a qualitative measure of how different the mistuned function will be, compared with the original tuned function. Singularity theory offers us a vehicle to do this in a very precise manner.

We can now tie the foregoing concepts into our objective. We seek to examine whether a circulant matrix exhibits any noteworthy qualitative characteristics under slight parameter variation, which are not exhibited by matrices of other forms (e.g. tri-diagonal formation of the linear chain). In order to do this, we shall need to establish some equivalence between various maps. Thus, if we examine a characteristic function (or, map) of the tuned system, and compare it to that of the same system when exposed to an infinitesimal perturbation, then we can compare whether the map 'before' and 'after' are 'close' or not.

One could express the qualitative equivalence between various maps in different forms. The equivalence that we seek between two maps will be based on the stricter requirement of diffeomorphism, rather than homeomorphism. Thus, we seek differential equivalence, rather than mere topological equivalence. When we do this, we then arrive at the concept of C^{∞} equivalence, as recommended by Arnol'd.

Let d_1 and d_2 be diffeomorphisms, while $f_1{}^{(0)}$ and $f_2{}^{(\varepsilon)}$ are linear maps. A C^{∞} equivalence is said to exist between the map

$$f_1{}^{(0)} : A^{(0)}(0) \rightarrow A^{(\varepsilon)}(0) \tag{21}$$

and another map

$$f_2{}^{(\varepsilon)} : A^{(0)}(\omega) \rightarrow A^{(\varepsilon)}(\omega) \tag{22}$$

if there exists diffeomorphisms

$$d_1 : A^{(0)}(0) \rightarrow A^{(0)}(\omega), \quad d_2 : A^{(\varepsilon)}(0) \rightarrow A^{(\varepsilon)}(\omega) \tag{23, 24}$$

such that the following identity holds:

$$f_2{}^{(\varepsilon)} \equiv d_2(f_1{}^{(0)}(d_1{}^{-1})) \tag{25}$$

The identity above can be cast in the form of a commutative diagram as sketched below:

$$\begin{tikzcd}
A^{(0)}(0) \arrow[r, "f_1^{(0)}"] \arrow[d, "d_1"'] & A^{(\varepsilon)}(0) \arrow[d, "d_2"] \\
A^{(0)}(\omega) \arrow[r, "f_2^{(\varepsilon)}"'] & A^{(\varepsilon)}(\omega)
\end{tikzcd}$$

The equivalence between the maps $f_1{}^{(0)}$ and $f_2{}^{(\varepsilon)}$ which transform the tuned frequency response manifolds to the mistuned case is a C^{∞} equivalence if diffeomorphisms d_1 and d_2 can be found. Such a diffeomorphic equivalence is also known as differentiable equivalence (Arnol'd *et al*, 1985).

3.2 STABILITY OF MAPS

We now turn to the question of the stability of maps. Fundamental to this issue is the behavior of the critical points of a smooth map. In the theory of singularity of smooth maps, critical points are also known as singularities. If a map is merely a function of one variable x, we know from elementary calculus that the critical points of that function can be obtained by finding those values of x for which $f'(x) = 0$. There are basically two types of singularities that are of interest to us. These are styled 'generic' or 'degenerate'.

It is easiest to consider the situation when the map f is merely a function of one variable x. Let $D^i f|_u$ denote the ith partial derivative of f evaluated at $x = u$. With this notation, the critical points of f are said to be degenerate when $D^1 f|_u \equiv Df|_u = 0, D^2 f|_u \neq 0$, for a function of one variable). It has been shown that a critical point of a function of one variable is structurally stable if and only if it is a non-degenerate critical point. Another interpretation of this, which is more relevant for our purpose here, is that *every degenerate critical point is structurally unstable* (Poston and Stewart, 1978; italics added).

If a map f is a function of more than one parameter (x_i, $i = 1, \cdots n$), we can define the Jacobian matrix of f in terms of the components of $f \equiv (f_1, f_2, \cdots f_n)$

$$Jf|_u = j_{ij} = \frac{\partial f_i}{\partial x_j}. \tag{26}$$

We can also define the Hessian matrix of f at u as:

$$Hf|_u = h_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad x = u. \tag{27}$$

One may then classify the stability of the critical points of f in terms of $D^1 f|_u$ and $Hf|_u$. The critical points of f are said to be generic if $Df|_u = 0$ and $\det(Hf|_u) = 0$. If all of the critical points of f are non-degenerate, its qualitative behavior is completely characterized by the Morse Lemma. f is then said to be a Morse function. The requirement that $\det(Hf|_u) = 0$ is sometimes known as the *Morse condition.*

There is more than one version of the Morse Lemma. The version given here is stated without proof. (For the interested reader, proofs may be found in Poston & Stewart (1978), Arnol'd (1985), and Rabier (1985) among others. The following statement of Morse Lemma is rephrased from Rabier (*op. cit.*):

> Let f be a map of class C^m ($m \geq 2$) on a neighborhood of the origin such that $f(0) = 0$, $Df|_u(0) = 0$, $\det h_{ij}(0) \neq 0$. Then there exists a C^{m-1} local diffeomorphism d which preserves the origin and transforms the local zero set of the quadratic form $\hat{d} \rightarrow h_{ij}(0)\, \hat{d}^2$ into the local zero set of f, such that $D\hat{d}(0) = I$, and $\hat{d}$ is of class C^m away from the origin.

Now, a Morse function is a very stable function in the qualitative sense. The shape of the function does not change under a slight change in a parameter of the system. It is described by Poston and Stewart (1978) as having a "preservation of type under perturbation". If we choose a suitable function of our matrix $A^{(0)}$ (e.g. a frequency response function under a given excitation, and proceed to investigate its stability, the necessary and sufficient condition for that function to be structurally stable is that it be a Morse function. If, on the other hand, it is not a Morse function, then those critical points of the function that are degenerate will "unfold" under arbitrary perturbation. Several unfoldings are possible depending on the particular degeneracy, and the form of mistuning. The ultimate aim is to obtain a *versal unfolding* of degenerate frequency response curves in a given system, Arnol'd (1972).

We are now in a position to define the specific form of the stability of maps that is of main interest to us here. A map $f^{(0)}$ is said to be diffeomorphically stable if every map $f_i^{(\varepsilon)}$ that is sufficiently close to $f^{(0)}$ is diffeomorphically equivalent to $f^{(0)}$.

The foregoing concepts may be illustrated by considering two functions, each of which is a function of only one parameter. The illustrations we present here are adaptations of the examples used by Arnol'd (1972) and Poston & Stewart (1978), who consider the behavior under perturbation of the functions $f: x \to x^2$ and $f: x \to x^4$ in their treatments of singularity theory. Since our objective is to investigate the stability of forced response functions, their original functions are suitably transformed so that they look like damped vibration response curves. We therefore use the functions shown in eq. (28) and (29) below, rather than the original x^2 and x^4.

Figs 3 $a-c$ show the variation of the "tuned" and "mistuned" quadratic functions. The mistuned function is obtained by adding a perturbation function to obtain a mistuned function that is "close" to the original tuned function. If $f_1^{(0)} = x^n$ represents a tuned smooth function, and $f_2^{(\varepsilon)}$ a mistuned function that is "close" to $f_1^{(0)}$, then the most general perturbation can be written as

$$f_2^{(\varepsilon)} = f_1^{(0)} + \sum_r^n \varepsilon_r x^{n-1}, \ |\, \varepsilon \,| \ll 1$$

with the system variable x and a perturbation parameters ε_r. Note that some of the ε_r may be equal to zero. The sketch in Fig 3a depicts the 'tuned' function

$$f_1^{(0)} \equiv f_2(x) = b_2 - \{(x - a_2)^2\} \qquad (28)$$

while

$$f_2^{(\varepsilon)} \equiv f_2(x, \varepsilon) = b_2 - \{(x-a_2)^2 + \varepsilon(x-a_2)\} \tag{29}$$

is a slightly perturbed variation of the tuned function. Here, a_i, b_i are constants. It is seen in Fig 3*b* that for $\varepsilon<0$, the critical point of f_2 is displaced somewhat, but the overall *shape* of the function f_2 is preserved. The same observation holds when $\varepsilon > 0$, Fig 3c. Thus, the sign of ε has no effect on the shape of the manifold. A very different picture is obtained, however, when we consider the function

$$f_1^{(0)} \equiv f_4(x) = b_4 - \{(x-a_4)^4\} \tag{30}$$

and its suitably perturbed version

$$f_2^{(\varepsilon)} \equiv f_4(x, \varepsilon) = b_4 - \{(x-a_4)^4 + \varepsilon(x-a_4)^2\} \tag{31}$$

The function described by eq. (30) is graphed in Fig 4*a*, while Fig 4*b* shows the mistuned function represented by eq. (31), with $\varepsilon<0$.

It is seen that the degenerate critical points which were previously coincident at $x=a_4$ are now split into two local maxima and an intervening local minimum. On the other hand, when $\varepsilon>0$, the curve shown in Fig 4*c* results. There is now a considerable qualitative difference between the *shapes* of the three curves in Fig 4. This qualitative difference between Figs 4*b* and 4*c* persists, *no matter how small* we make our perturbation parameter, $|\varepsilon|$, provided that $|\varepsilon| \neq 0$. Thus, the critical points of the tuned system are not consistently preserved under arbitrarily small perturbation. It must be noted that the shape of the resulting perturbed curve, Figs 4*a* and *b*, depends on the sign of ε. Functions behaving like f_4 are then said to be unstable, while those with characteristics exhibited by f_2 are stable.

3.3 THE BIFURCATION SET

It can be shown mathematically that degenerate singularities lie on a bifurcation set (see, for instance, Gilmore, 1981). The bifurcation set is the set of all points in a parameter space where the Morse condition (see section 3.3) is not satisfied by the given curve. Consequently, a curve lying on the bifurcation set is unstable.

Intuitively, it has been noted by several investigators in bladed disk research that the double modes of tuned systems become "split" (or, bifurcate) into two close modes when mistuning is admitted (Ewins, 1973; Srinivasan & Cutts, 1975). Since the degenerate tuned modes are delicately perched on the bifurcation set, they are inherently unstable.

The slightest amount of disordering that brakes the cyclic symmetry of the tuned system leads to a bifurcation of the spectrum, and an unfolding of the singularities of the frequency response curves. The bifurcation set occupies a very significant role in mistuning research.

In order to obtain the bifurcation set for a system, we need to first define a system behavior variable x, specify the number of control (mistuning) parameters such as ε_i, and then set up a differentiable function V of all the parameters. V is sometimes known as the potential function, so-called probably because it corresponds to the potential energy in Zeeman's catastrophe machine. The bifurcation set is then obtained from the gradient of the potential function, as shown below for a 2-parameter family of functions. Although one can always identify a system behavior variable (or state variable) x, it is not always easy to set up the potential function for the parameters, or to obtain an equation for the bifurcation set.

For an illustration of the procedure for obtaining the equation of the bifurcation set, we return to the second example given in the previous section. Writing $a \equiv a_4$, and $V \equiv f_4$, we can regard V as a potential function, with a state variable x, and control parameters $\varepsilon_1,\ \varepsilon_2$). Thus, re-writing (31) in a slightly different manner, we get

$$V = b_4 - (x-a)^4 + \varepsilon_1(x-a)^2 + \varepsilon_2(x-a) \tag{32}$$

The control parameter ε_1 is known as the "splitting factor", while ε_2 is the "normal factor", (Saunders, 1980). In eq (31), $\varepsilon_1 = \varepsilon$ while $\varepsilon_2 = 0$.

The equilibrium manifold is obtained by setting the gradient of the potential function, to zero.

$$M \equiv V' = -4(x-a)^3 + 2\varepsilon_1(x-a) + \varepsilon_2 = 0. \tag{33}$$

When $\varepsilon_1 > 0$, then we split the manifold M, and discontinuities can occur in x. The singularity set S is obtained following another partial differentiation:

$$S \equiv 2\varepsilon_1 - 12(x-a)^2 = 0 \tag{34}$$

The bifurcation set B is the complement of the singularity set. It is obtained by eliminating x from the equation for M and S. Solving eqs. (33) and (34) simultaneously, we get

$$\varepsilon_1 = 6x^2 - 12ax + 6a^2 \tag{35}$$

$$\varepsilon_2 = -8x^3 + 24ax^2 - 24a^2x + 8a^3 \tag{36}$$

From eq. (35), we obtain an expression for x

$$x = a \pm \sqrt{\frac{\varepsilon_1}{6}} \tag{37}$$

We accept only the positive root:

$$x = a + \sqrt{\frac{\varepsilon_1}{6}} \tag{38}$$

which, when substituted into (36) yields

$$\varepsilon_2 = -8\left[a + \frac{\sqrt{\varepsilon_1}}{\sqrt{6}}\right]^3 + 24a\left[a + \frac{\sqrt{\varepsilon_1}}{\sqrt{6}}\right]^2 - 24a^2\left[a + \frac{\sqrt{\varepsilon_1}}{\sqrt{6}}\right] + 8a^3 \tag{39}$$

simplifying to

$$\varepsilon_2 = -\frac{2\sqrt{6\varepsilon_1{}^3}}{9} \tag{40}$$

from which we obtain

$$8\varepsilon_1{}^3 - 27\varepsilon_2{}^2 = 0 \tag{41}$$

which is the canonical form of the *cusp catastrophe.* If we plot V versus x for some selected values of ε_1 and ε_2, we obtain the curves shown in Fig 5. Note that when the values of ε_1 and ε_2 satisfy eq. (41) of the bifurcation set, then slight variations in the mistuning parameters ε_1 and ε_2 will induce considerable changes in the topology of the map V.

When we vary the mistuning parameters ε_i over an admissible range in the plane of parameters, and plot the potential V as a function of the behavior variable x, we obtain the illustration in Fig 6. The bifurcation set is demarcated in the figure, and it is seen to be in the form of a cusp. Hence, systems whose characteristic function can be cast in the form of eq. (32) will have a 'cusp catastrophe'.

It must be understood that although the cusp catastrophe is very common, it is not the only one encountered in practice. Bifurcation sets of other shapes exist. The lowest seven categories were obtained by Thom (1972), and these were given picturesque names such as fold, cusp, butterfly, swallow-tail, etc, and were collectively named as the seven elementary catastrophes by Thom. Subsequently, the taxonomy was extended by Arnol'd, who then used the collective name of singularities, after noting some similarities between the abutment diagrams and certain Coxeter-Dynkin groups in crystallography, and certain Lie groups. See Part II of Arnol'd *et al* (1985) for details.

4. STABILITY OF FREQUENCY RESPONSE CURVES

In order to investigate the stability of frequency response curves, we consider a simple model of a ring with three equally spaced masses, Fig 7. The three masses in Fig 7*a* may be considered to be situated at the apogees of an equilateral triangle superimposed on the vibrating ring. If we open up the ring, then a uni-axial chain (simply supported beam) results, Fig 7*b*.

After the equations of motion are derived, the structure of the resulting dynamical matrix is a symmetric circulant. We concentrate on the symmetric case, which arises often in structural dynamics on account of Maxwell's reciprocity laws, excepting cases where factors like gyroscopic effects predominate, for the time being. The case of anti-symmetric and skew symmetric circulants are not treated here in any detail, but will be returned to in subsequent work.

4.1 GENERAL SOLUTION FOR 3 DOF SYSTEM

The static stiffness matrix of each of the two systems shown in Fig 7 is symmetric, the tuned system matrix in either case is of the general form

$$A^{(0)} = \begin{bmatrix} a & -b & -c \\ -b & a & -b \\ -c & -b & a \end{bmatrix} \tag{42}$$

We shall analyze the stability of $A^{(0)}$ in the most general case. Therefore, we assume that all the entries in $A^{(0)}$ are complex elements, i.e. $a,\ b \in \mathbb{C}$.

Let us now apply the following perturbation matrix to eq (42)

$$E = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{43}$$

Without loss of generality, we can assign real values to a and b such that they have physical meanings, in terms of static (and/or dynamic) stiffness. Thus, if we assume that three loads, each of unit mass, are attached to the ring, we can denote the coupling and grounding stiffnesses respectively at each load by k_c and k_g. In that case, elements of the static stiffness matrix are simply those of $A^{(0)}$ (eq (42)), where $a = 2k_c + k_g$, $b = k_c$.

For specific treatments, we can set $c = 0$ for the linear chain, so that $A^{(0)}$ becomes banded; while setting $c = b$ yields the cyclic chain, resulting in a circulant system matrix.

The dynamic stiffness matrix of the tuned system is $D^{(0)} = A^{(0)} - \omega^2 I$:

$$D^{(0)}(\omega) = \begin{bmatrix} 2k_c + k_g - \omega^2 & -k_c & -c \\ -k_c & 2k_c + k_g - \omega^2 & -k_c \\ -c & -k_c & 2k_c + k_g - \omega^2 \end{bmatrix} \tag{44}$$

The dynamic stiffness form of the perturbed matrix, $D^{(\varepsilon)} = A^{(0)} + E - \omega^2 I$, is therefore of the form

$$D^{(\varepsilon)}(\omega) = \begin{bmatrix} 2k_c + k_g + \varepsilon_1 - \omega^2 & -k_c & -c \\ -k_c & 2k_c + k_g + \varepsilon_2 - \omega^2 & -k_c \\ -c & -k_c & 2k_c + k_g - \omega^2 \end{bmatrix} \tag{45}$$

It is seen by inspection that the matrices in eqs (44 and 45) are 'close', by the definition given earlier in section 2.2 in terms of norms, since $|\varepsilon_i| \approx 0$, and $\|E\| \approx 0$. Thus, if we start from *any* mistuned condition, provided that $|\varepsilon_i|$ is small and ultimately made to vanish, in the limit, the mistuned characteristics always approach those of the tuned system as we gradually turn *off* the perturbation, i.e. when $\varepsilon_i = 0$. This is the case for both the banded and circulant matrices alike. However, as shall be shown presently, if we start from the tuned condition, and gradually turn *on* the perturbation there is no telling where we may end up; it all depends on the *type* of mistuning, and more importantly, whether or not we cross the bifurcation set as we increase $|\varepsilon|$.

The frequency response function due to the application of a unit load in a structure is usually known as the *receptance*, if the response we are interested in is defined as:

$$\alpha_{ij} = \frac{x_i}{f_j} \tag{46}$$

where x_i is the displacement at node i due to the application of a unit dynamic load $f_j(\omega)$ at node j. In the case when i and j are coincident, then α_{ii} is termed the *direct receptance* at node i.

For the 3 degree of freedom system under investigation, our interest is to compute the direct receptance expressions for α_{ii}, $(i=1, \cdots, 3)$. These are obtainable from the diagonal elements of the inverse of the dynamic stiffness matrix, eq (45).

In order to keep the subsequent algebraic analysis compact, we use a for diagonal elements, and b or c for the off-diagonals. To obtain the inverse of eq (45), we first obtain its determinant:

$$\begin{aligned}\Delta(\omega, \varepsilon) = & -\omega^6 + 3a\omega^4 - (3a^2 - 2b^2 - c^2)\omega^2 + a^3 - a(2b^2 + c^2) - 2b^2 c \\ & + \varepsilon_1(\omega^4 - 2a\omega^2 + a^2 - b^2) \\ & + \varepsilon_2(\omega^4 - 2a\omega^2 + a^2 - c^2) \\ & + \varepsilon_1\varepsilon_2(\omega^2 - a)\end{aligned} \tag{47}$$

The receptance matrix of the generalized mistuned system is therefore given by

$$[D^{(\varepsilon)}(\omega)]^{-1} = \frac{1}{\Delta(\omega, \varepsilon)} \times \begin{bmatrix} \omega^4 - 2a\omega^2 + a^2 - b^2 - \varepsilon_2(\omega^2 - a) & b(a + c - \omega^2) & b^2 + c(a - \omega^2) + \varepsilon_2 c \\ b(a + c - \omega^2) & \omega^4 - 2a\omega^2 + a^2 - c^2 - \varepsilon_1(\omega^2 - a) & b(a + c - \omega^2) + \varepsilon_1 b \\ b^2 + c(a - \omega^2) + \varepsilon_2 c & b(a + c - \omega^2) + \varepsilon_1 b & \omega^4 - 2a\omega^2 + a^2 - b^2 + (\varepsilon_1 + \varepsilon_2)(\omega^2 - a) - \varepsilon_1\varepsilon_2 \end{bmatrix} \tag{48}$$

From the diagonal elements, we obtain the following expressions for the direct receptance at each of the three nodes:

$$\alpha_{11} = \frac{\omega^4 - 2a\omega^2 + a^2 - b^2 - \varepsilon_2(\omega^2 - a)}{\Delta(\omega, \varepsilon)} \tag{49}$$

$$\alpha_{22} = \frac{\omega^4 - 2a\omega^2 + a^2 - c^2 - \varepsilon_1(\omega^2 - a)}{\Delta(\omega, \varepsilon)} \tag{50}$$

$$\alpha_{33} = \frac{\omega^4 - 2a\omega^2 + a^2 - b^2 + (\varepsilon_1 + \varepsilon_2)(\omega^2 - a) - \varepsilon_1\varepsilon_2}{\Delta(\omega, \varepsilon)} \tag{51}$$

We can, in generalized form, determine the differences in the three receptance expressions by using the first node as reference. We consider two mistuned cases in addition to the tuned datum. These are the circulant and the banded.

CIRCULANT MATRIX

As may be verified from the equations below, the receptance expressions at all coordinates are the same for the tuned circulant case, $b=c$ and $\varepsilon_i = 0$; there being no preferred node for a perfect circle, since all locations are exactly the same. Thus,

- CASE 0: $\varepsilon_1 = 0$; $\varepsilon_2 = 0$.

$$\alpha_{11} - \alpha_{22} = 0 \tag{52}$$

$$\alpha_{11} - \alpha_{33} = 0 \tag{53}$$

- CASE 1: $\varepsilon_1 = \varepsilon$; $\varepsilon_2 = \varepsilon$.

$$\alpha_{11} - \alpha_{22} = 0 \tag{54}$$

$$\alpha_{11} - \alpha_{33} = \frac{\varepsilon(a - \omega^2) + \varepsilon^2}{\Delta(\omega, \varepsilon)} \tag{55}$$

- CASE 2: $\varepsilon_1 = \varepsilon$; $\varepsilon_2 = -\varepsilon$.

$$\alpha_{11} - \alpha_{22} = \frac{2\varepsilon(a - \omega^2)}{\Delta(\omega, \varepsilon)} \tag{56}$$

$$\alpha_{11} - \alpha_{33} = \frac{\varepsilon(a - \omega^2) + \varepsilon^2}{\Delta(\omega, \varepsilon)} \tag{57}$$

BANDED MATRIX

When $b \neq c$, the system is no longer circulant, but is nevertheless periodic in the axial sense. Thus, the periodicity is now rectilinear, resulting in a banded matrix. The direct receptance expressions at each of the nodes, relative to node 1, are therefore as follows:

- CASE 0: $\varepsilon_1 = 0$; $\varepsilon_2 = 0$.

$$\alpha_{11} - \alpha_{22} = \frac{b^2 - c^2}{\Delta(\omega, \varepsilon)} \tag{58}$$

$$\alpha_{11} - \alpha_{33} = 0 \tag{59}$$

where eq. (59) holds on account of the physical symmetry of the system.

- CASE 1: $\varepsilon_1 = \varepsilon$; $\varepsilon_2 = \varepsilon$.

$$\alpha_{11} - \alpha_{22} = \frac{b^2 - c^2}{\Delta(\omega, \varepsilon)} \tag{60}$$

$$\alpha_{11} - \alpha_{33} = \frac{\varepsilon(a-\omega^2) + \varepsilon^2}{\Delta(\omega,\, \varepsilon)} \tag{61}$$

- CASE 2: $\varepsilon_1 = \varepsilon;\ \ \varepsilon_2 = -\varepsilon.$

$$\alpha_{11} - \alpha_{22} = \frac{2\varepsilon(a-\omega^2) - b^2 + c^2}{\Delta(\omega,\, \varepsilon)} \tag{62}$$

$$\alpha_{11} - \alpha_{33} = \frac{\varepsilon(a-\omega^2) + \varepsilon^2}{\Delta(\omega,\, \varepsilon)} \tag{63}$$

For the rectilinearly periodic system the direct receptance expressions at nodes 1 and 3 are then equal when tuned conditions are imposed $\varepsilon_i = 0$, but $\alpha_{11} \neq \alpha_{22}$.

The receptance expressions in the preceding analyses are due to excitation at a single point. In bladed disk research, simultaneous multi-coordinate excitation of the engine order type is common. Thus, in an umbrella-mode type of excitation, all the nodes are excited simultaneously, all the excitations being of equal intensity and in phase.

Applying equal, simultaneous in-phase excitation of unit magnitude to all the three coordinates leads to the following receptance expressions at the different nodes. Here, only one subscript is attached to each receptance, to denote the point of response determination.

$$\alpha_1 = \frac{\omega^4 - (2a+b+c)\omega^2 + a^2 + (a+c)(b+c) + \varepsilon_2(a+c-\omega^2)}{\Delta(\omega,\, \varepsilon)} \tag{64}$$

$$\alpha_2 = \frac{\omega^4 - 2(a+b)\omega^2 + a^2 + 2ba + 2cb - c^2 + \varepsilon_1(a+b-\omega^2)}{\Delta(\omega,\, \varepsilon)} \tag{65}$$

$$\alpha_2 = \frac{\omega^4 - (2a+b+c)\omega^2 + (a+c)(b+c) + \varepsilon_2(a+c-\omega^2) + \varepsilon_1(a+b-\omega^2) + \varepsilon_1\varepsilon_2}{\Delta(\omega,\, \varepsilon)} \tag{66}$$

The difference between the direct receptance elements, relative to that at coordinate 1, is expressed as follows:

$$\alpha_1 - \alpha_2 = \frac{(b-c)\omega^2 - \varepsilon_1(a+b-\omega^2) + (a+c)(c-b) + \varepsilon_2(a+c-\omega^2) + (c-b)a - cb + c^2}{\Delta(\omega,\, \varepsilon)} \tag{67}$$

$$\alpha_1 - \alpha_3 = -\frac{\varepsilon_1(a+b-\omega^2) + \varepsilon_1\varepsilon_2}{\Delta(\omega,\, \varepsilon)} \tag{68}$$

It may be noted that in the foregoing only the diagonal elements were perturbed. However, the results to be derived in terms of structural stability are still applicable when c is regarded as a perturbation parameter; so that as $c \rightarrow 0$, axial periodicity is being gradually turned on, while cyclic symmetry is induced in the neighborhood of $c \simeq b$.

4.2 NUMERICAL IMPLEMENTATION

We consider the case of the so-called 'strong coupling', using the following numerical values: $k_c = 9.5$, $k_g = 1$, $a = 20$, $b = -9.5$, $\varepsilon_3 = 0$, $\varepsilon_2 = -0.1$, $\varepsilon_1 = 0.1$. Clearly, the ratio of mistuning to coupling strength is very small. Now, in order to compute the frequency response curves, we need some damping to obtain finite amplitudes at resonance. Assume hysteretic damping of 0.01 for all cases. Without loss of generality, the response to be computed is the direct receptance, i.e. the response of each node to individual excitation.

For the linear chain, the frequency response of the tuned and mistuned systems are shown in Fig 8. Notice that at the tuned state, the amplitudes of nodes 1 and 3 are equal, on account of symmetry, while that of node 2 is double that magnitude.

Because the system exhibits robust stability, all nodes have almost the same response patterns and magnitudes as in the tuned system. This is also the case when we change the sign of ε_2, from -0.1 to 0.1. The receptance of this system therefore behaves like a Morse function, Fig 3.

When we repeat exactly the same procedure for the circulant ring, a very different picture is obtained. Fig 9 shows the response of individual nodes compared with the tuned case. This case corresponds to a 2-parameter perturbation, with $\varepsilon_1 = 0.1$, $\varepsilon_2 = -0.1$, $\varepsilon_3 = 0$.

Notice that the node with zero mistuning now has a reduction in amplitude of almost 50%. This extremely unequal amplitude distortion is the case no matter how small the magnitude of the perturbation is, so long as we keep the *form* of mistuning.

If we now change the mistuning matrix in a very small way, by making $\varepsilon_2 = 0.1$, we obtain the response curves in Fig 10. We now notice a substantial difference in the topology of the curves in Fig 10, compared to those in Fig 9. Thus, a very small change in the mistuning matrix (eq. 23), now results in a considerable difference in the vibration response at the individual nodes. The question of which node will be most responding, or the one having the least amplitude, is now not as easy as one would have expected. In Fig 9, it is node 2, while it is node 3 in Fig 10. In fact, the amplitude of node 3 has been increased by about 100% from Fig 9 to Fig 10, merely by changing only one entry in the system matrix from 19.9 to 20.1, a change of less than 1% !

The foregoing examples, based on a simple 3 degrees of freedom model of a circular *membrane*, illustrates the instability induced by cyclicity. It is clear that the qualitative conclusions to be drawn from Fig 9 are in conflict with those from Fig 10, no matter how small we make the mistuning. When bladed disk systems are well-modeled to include the effects of blade coupling, blade or disk mistuning and *cyclicity*, more serious distortions can result.

5. STABILITY OF EIGENVECTORS

Frequency response curves are intimately related to eigenvectors, as a modal analysis reveals, eq. (20). Therefore, if cyclicity induces instability in the frequency response curves, as we have just seen, then it is to be expected that cyclicity will also lead to instability of eigenvectors. The subject of eigenvector instability is discussed in greater detail in another monograph (Afolabi, 1989b). Yet, it is appropriate at to present a glimpse of geometric instability arising from modal degeneracy in this place. We give one pair of examples.

Consider the following tuned symmetric circulant, whose elements are complex

$$c_0 = \begin{bmatrix} 200 + i(10) & -95 + i(-5) & -95 + i(-5) \\ -95 + i(-5) & 200 + i(10) & -95 + i(-5) \\ -95 + i(-5) & -95 + i(-5) & 200 + i(10) \end{bmatrix} \tag{69}$$

The eigensolution of c_0 is easily computed (for instance, by the IMSL routine EVCCG), from which we obtain the following eigenvalues

$$\Lambda_0 = \text{diag}\left\{10 + i(0), \quad 295 + i(15), \quad 295 + i(15)\right\} \tag{70}$$

and eigenvectors

$$U_0 = \begin{bmatrix} 1 + i(0) & -0.892 + i(-0.009) & -0.397 + i(-0.001) \\ 1 + i(0) & -0.108 + i(0.009) & 1 + i(0) \\ 1 + i(0) & 1 + i(0) & -0.603 + i(0.001) \end{bmatrix} \tag{71}$$

Note the degenerate roots (at the second and third modes).

In order to investigate the stability of the tuned circulant, we apply two distinct but very similar small perturbations to it, and compare their eigensolutions. The first of the two perturbed matrices are

$$c_1 = \begin{bmatrix} 201 + i(10) & -95 + i(-5) & -95 + i(-5) \\ -95 + i(-5) & 201 + i(10) & -95 + i(-5) \\ -95 + i(-5) & -95 + i(-5) & 200 + i(10) \end{bmatrix} \tag{72}$$

The second mistuned matrix is

$$c_2 = \begin{bmatrix} 201 + i(10.5) & -95 + i(-5) & -95 + i(-5) \\ -95 + i(-5) & 202 + i(9.5) & -95 + i(-5) \\ -95 + i(-5) & -95 + i(-5) & 200 + i(10) \end{bmatrix} \tag{73}$$

It is clear that the two perturbed matrices are 'close' since $\|E\| / \|c_0\| \approx 0$, where

$$E = c_1 - c_2 = \begin{bmatrix} 0 + i(-0.5) & 0 + i(0) & 0 + i(0) \\ 0 + i(0) & -1 + i(0.5) & 0 + i(0) \\ 0 + i(0) & 0 + i(0) & 0 + i(0) \end{bmatrix} \tag{74}$$

and c_0 is defined in eq. (69). The computed eigenvalues of c_1 are:

$$\Lambda_1 = \text{diag}\left\{10.666 + i(0),\quad 295.334 + i(15),\quad 296 + i(15)\right\} \tag{75}$$

The eigenvalues of c_2 are as follows:

$$\Lambda_2 = \text{diag}\left\{10.998 + i(0.00),\quad 295.477 + i(15.16),\quad 296.525 + i(14.84)\right\} \tag{76}$$

Although the eigenvalues of eq (69) are degenerate, those of (72) and (73) are regular. Moreover, as is evident from eqs (75) and (76), the eigenvalues of the two mistuned circulants are close to each other, as they are to those of the original tuned system, c_0. Therefore, the tuned, symmetric circulant given in eq (69) is well-conditioned with respect to eigenvalue extraction.

Notice, however, what happens to the eigenvectors of c_1 (eq. (72)), as a very small change is made using E (eq. 74) to transform it into c_2 to get eq. (73). We first compute the eigenvectors of c_1, followed by those of c_2. The eigenvectors of c_1 are

$$U_1 = \begin{bmatrix} 0.997 + i(0) & -0.502 + i(0) & -1 + i(0) \\ 0.997 + i(0) & -0.502 + i(0) & 1 + i(0) \\ 1 + i(0) & 1 + i(0) & -0 + i(0) \end{bmatrix} \tag{77}$$

while those of c_2 are

$$U_2 = \begin{bmatrix} 0.996 + i(-0.002) & -0.780 + i(0.201) & -0.664 + i(-0.191) \\ 0.993 + i(0.002) & -0.225 + i(-0.203) & 1 + i(0) \\ 1 + i(0) & 1 + i(0) & -0.331 + i(0.187) \end{bmatrix} \tag{78}$$

Notice how a small change in the matrix c_1 induces a significant qualitative difference, especially between the previously degenerate modes 2 and 3 of c_0 and the corresponding eigenvectors of c_1 and c_2.

If we now repeat the procedure using the banded matrix formation, rather than the circulant, no double modes appear. Moreover, the eigenvectors are just as well-conditioned as are the eigenvalues. The first mistuned matrix is:

$$B_1 = \begin{bmatrix} 201 + i(10) & -95 + i(-5) & 0 + i(0) \\ -95 + i(-5) & 201 + i(10) & -95 + i(-5) \\ 0 + i(0) & -95 + i(-5) & 200 + i(10) \end{bmatrix} \tag{79}$$

while the second matrix is given by:

$$B_2 = \begin{bmatrix} 201 + i(10.5) & -95 + i(-5) & 0 + i(0) \\ -95 + i(-5) & 202 + i(9.5) & -95 + i(-5) \\ 0 + i(0) & -95 + i(-5) & 200 + i(10) \end{bmatrix} \tag{80}$$

It is clear that the two perturbed matrices are 'close', as in the case of the circulant.

The eigenvalues of B_1 are:

$$\Lambda_1 = \text{diag}\left\{66.399 + i(2.93),\ \ 200.5 + i(10),\ \ 335.102 + i(17.07)\right\} \tag{81}$$

while those of B_2 are:

$$\Lambda_2 = \text{diag}\left\{66.898 + i(2.81),\ \ 200.500 + i(10.25),\ \ 335.603 + i(16.94)\right\} \tag{82}$$

Note that the eigenvalues of the two mistuned tridiagonal systems are close to each other, as were those of the circulant.

We first compute the eigenvectors of B_1:

$$U_1 = \begin{bmatrix} 0.706 + i(0) & 1 + i(0) & -0.708 + i(0) \\ 1 + i(0) & -0.005 + i(0) & 1 + i(0) \\ 1 + i(0) & 1 + i(0) & -0.703 + i(0) \end{bmatrix} \tag{83}$$

followed by those of B_2:

$$U_2 = \begin{bmatrix} 0.708 + i(-0.003) & 1 + i(0) & -0.706 + i(-0.003) \\ 1 + i(0) & 0.005 + i(0.002) & 1 + i(0) \\ 0.714 + i(-0.001) & -1 + i(0) & -0.701 + i(-.001) \end{bmatrix} \tag{84}$$

Also, the eigenvectors are very stable, unlike in the circulant system.

In closing, we examine the eigenvalues of a quasi-mistuned system (alternating mistuning) having 6 degrees of freedom. The symmetric circulant matrix and its eigensolutions are given below.

The system matrix is given by:

$$P_0 = \begin{bmatrix} 200+i(10) & -95+i(-5) & 0+i(0) & 0+i(0) & 0+i(0) & -95+i(-5) \\ -95+i(-5) & 210+i(11) & -95+i(-5) & 0+i(0) & 0+i(0) & 0+i(0) \\ 0+i(0) & -95+i(-5) & 200+i(10) & -95+i(-5) & 0+i(0) & 0+i(0) \\ 0+i(0) & 0+i(0) & -95+i(-5) & 210+i(11) & -95+i(-5) & 0+i(0) \\ 0+i(0) & 0+i(0) & 0+i(0) & -95+i(-5) & 200+i(10) & -95+i(-5) \\ -95+i(-5) & 0+i(0) & 0+i(0) & 0+i(0) & -95+i(-5) & 210+i(11) \end{bmatrix} \tag{85}$$

while the eigenvalues are

$$\Lambda_0 = \mathrm{diag}\left\{ 14.934+i(0.49) \;\; 109.869+i(5.48) \;\; 109.869+i(5.48) \;\; 300.131+i(15.52) \;\; 300.131+i(15.52) \;\; 395.066+i(20.51) \right\} \tag{86}$$

with corresponding eigenvectors

$$U_0 = \begin{bmatrix} 1+i(0) & 1+i(0) & -0.091+i(0.01) & -0.199+i(-0.03) & 0.949+i(0.00) & -0.974+i(0.00) \\ 0.974+i(0.00) & 0.474+i(0.00) & -0.949+i(0.00) & -0.791+i(0.03) & -0.500+i(0.00) & 1+i(0) \\ 1+i(0) & -0.500+i(0.00) & -0.909+i(-0.01) & 0.949+i(0.00) & -0.474+i(0.00) & -0.974+i(0.00) \\ 0.974+i(0.00) & -0.949+i(0.00) & 0.086+i(-0.01) & -0.209+i(-0.03) & 1+i(0) & 1+i(0) \\ 1+i(0) & -0.500+i(0.00) & 1+i(0) & -0.750+i(0.03) & -0.474+i(0.00) & -0.974+i(0.00) \\ 0.974+i(0.00) & 0.474+i(0.00) & 0.863+i(0.01) & 1+i(0) & -0.500+i(0.00) & 1+i(0) \end{bmatrix} \tag{87}$$

It may be noted that most of the eigenvalues occur in doublets, as in the perfectly tuned state.

6. CONCLUSIONS

- A distinction exists between the *elastic stability* and *geometric stability* of a system. The former can be predicted from the real part of the system eigenvalues, while the latter are obtainable from the direction of eigenvectors as small perturbations of the system parameters are gradually turned on or off.

- The system matrix of a tuned circular disk or membrane is circulant. For a tuned bladed disk, at least one of the sub-matrices is circulant. The reduction of such matrices to Jordan canonical form is unstable under small parameter perturbation, as proved by Arnol'd (1968).

- Consequent upon this instability is the "erratic" behavior of the frequency response curve, which exhibits sensitive dependence on parameter perturbation, or mistuning.

- If the *form* of mistuning (not just the magnitude of mistuning) moves the system matrix away from the bifurcation set, then geometric stability is enhanced.

- However, even the slightest parameter variation from the delicately balanced cyclo-symmetric form will destabilize the eigenvectors, if the map goes across a bifurcation set. Consequently, in such cases, the topology of the frequency response curves will exhibit significant instability under a very small perturbation.

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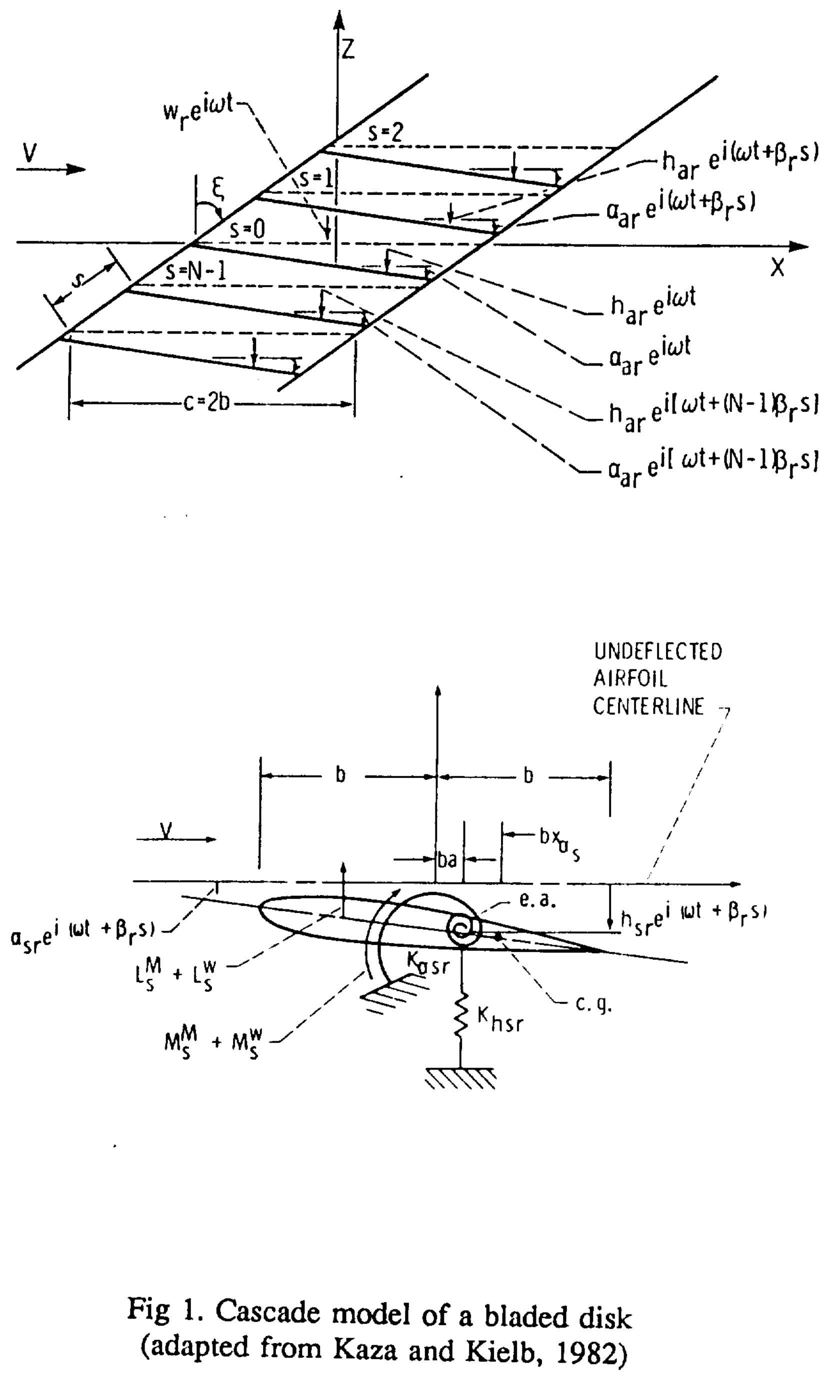

Fig 1. Cascade model of a bladed disk
(adapted from Kaza and Kielb, 1982)

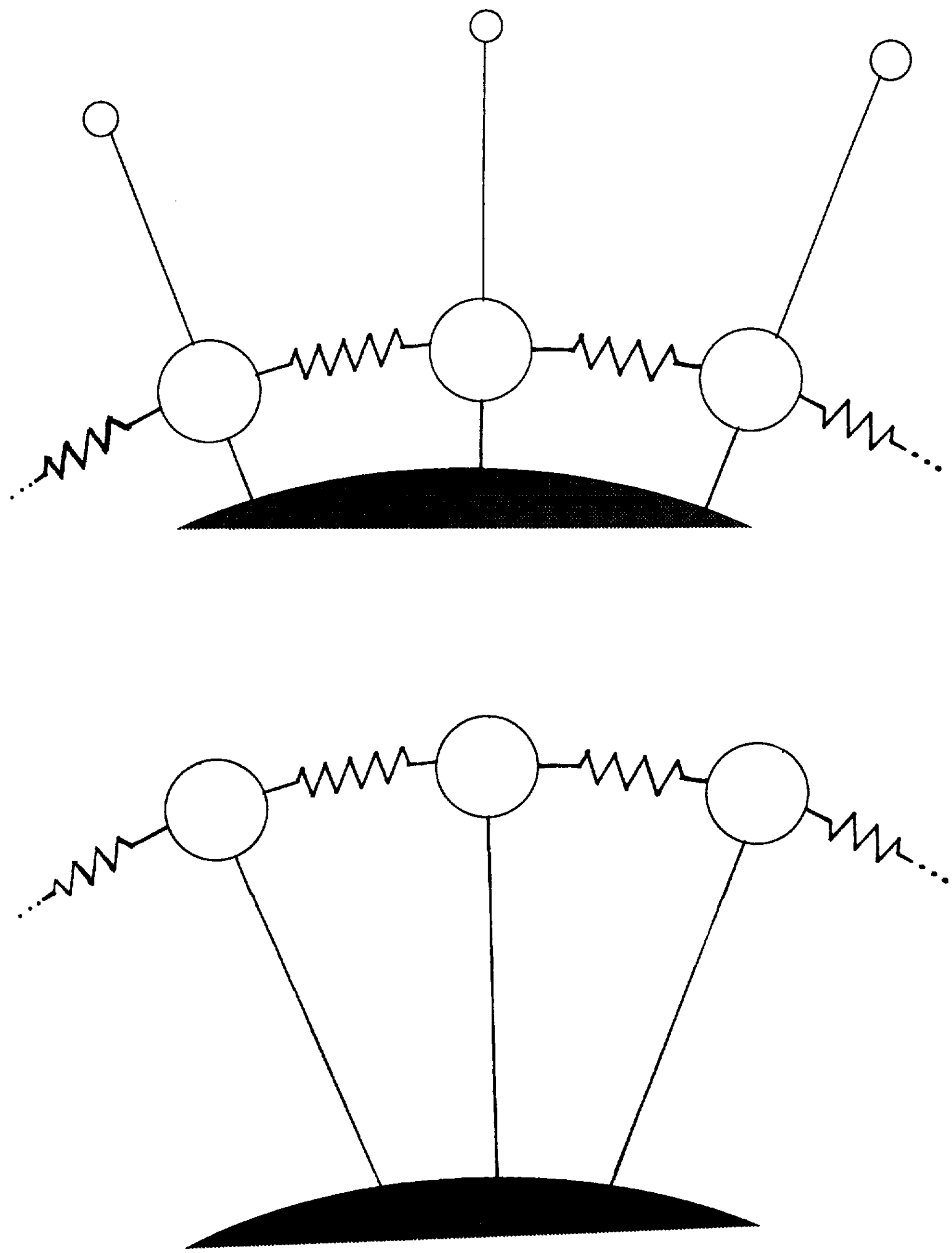

Fig 2. Cyclic models of
(a) bladed disk (b) membrane or disk only

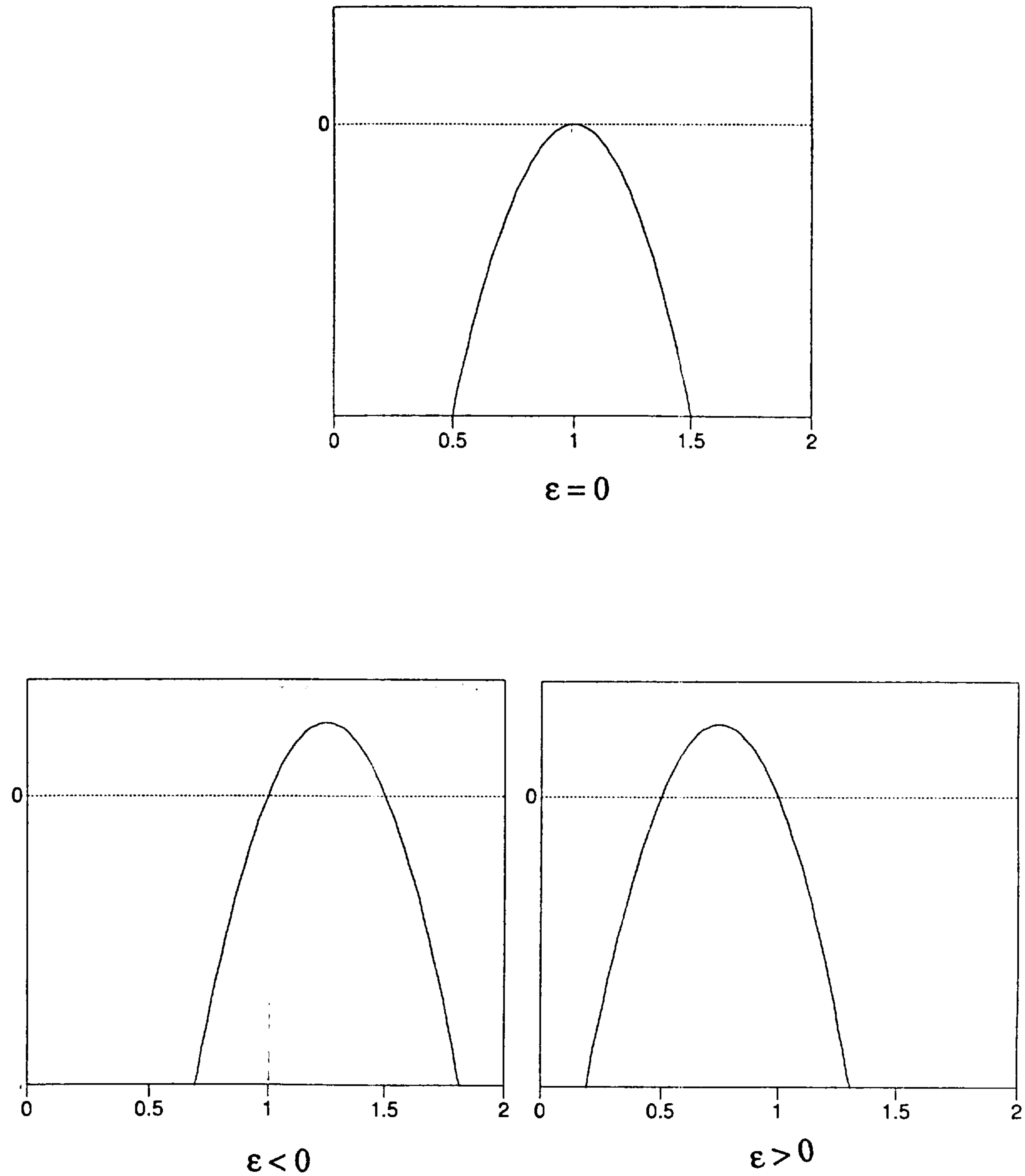

Fig 3. Effect of mistuning parameter ε on the generic singularity of

$$f: x \to b_2 - \{(x-a_2)^2 + \varepsilon x - a_2\}$$

f is Morse, and preserves its shape under mild perturbation

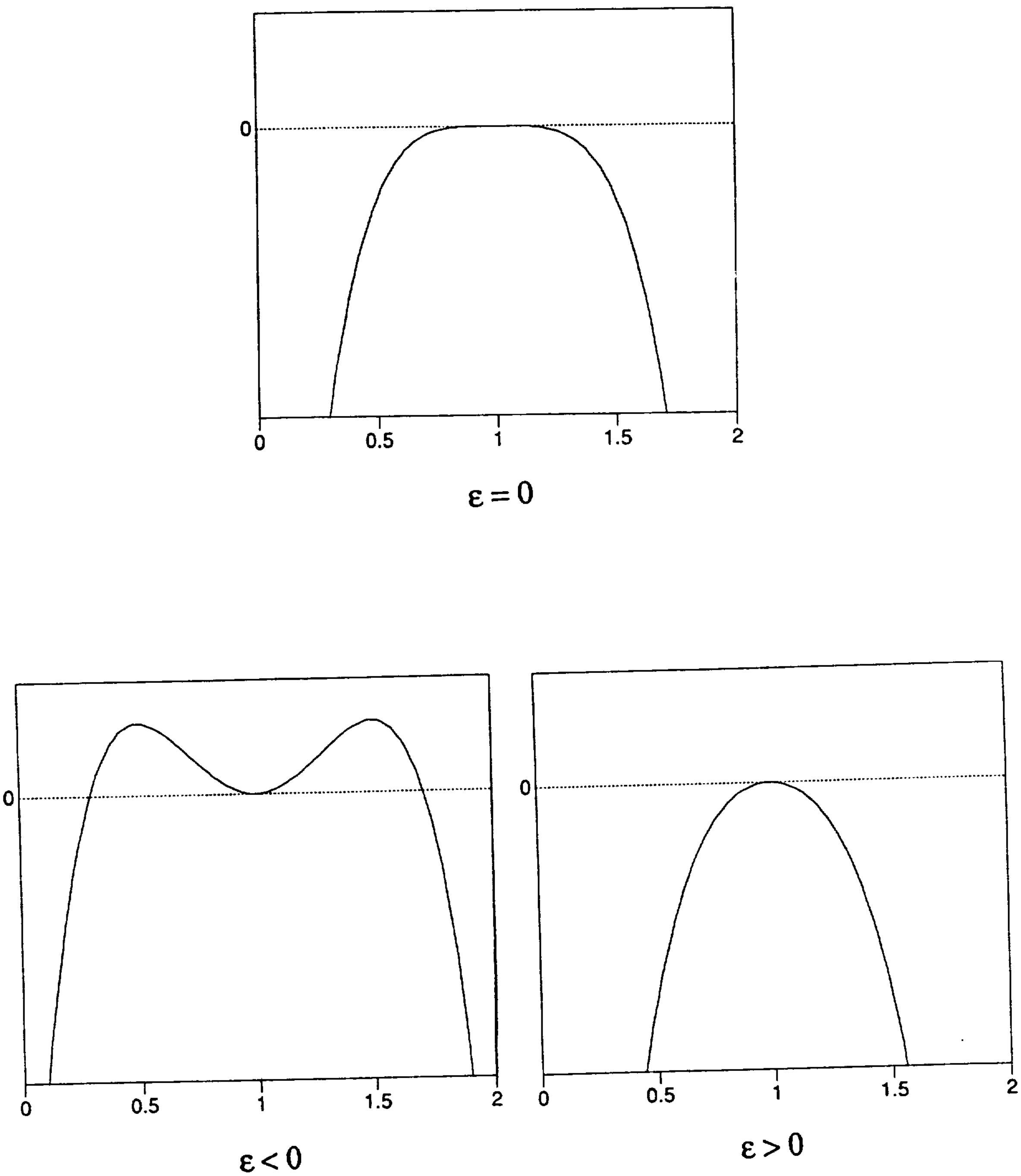

Fig 4. Effect of mistuning parameter ε on the degenerate singularity of

$$f : x \to b_4 - \{(x-a_4)^4 + \varepsilon(x-a_4)^2\}$$

Note the topological unfolding of the singularity due to structural instability of the tuned function; the shape of f depends on the *sign* of ε.

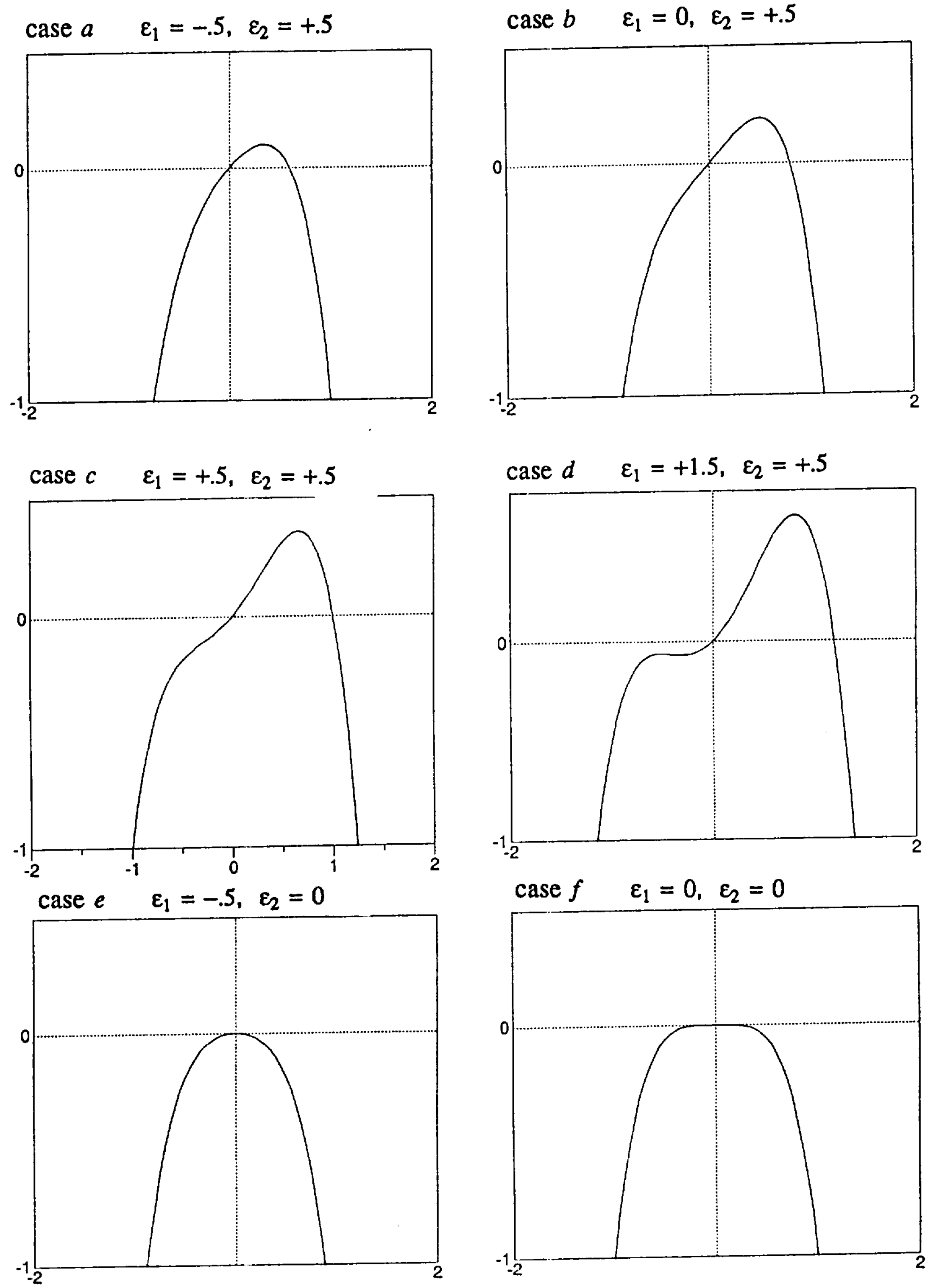

Fig 5. Topological trajectories of V for various values of the mistuning parameters; these figures refer to eq. (32), setting $b_4 = 0$, $a_4 = 1.5$

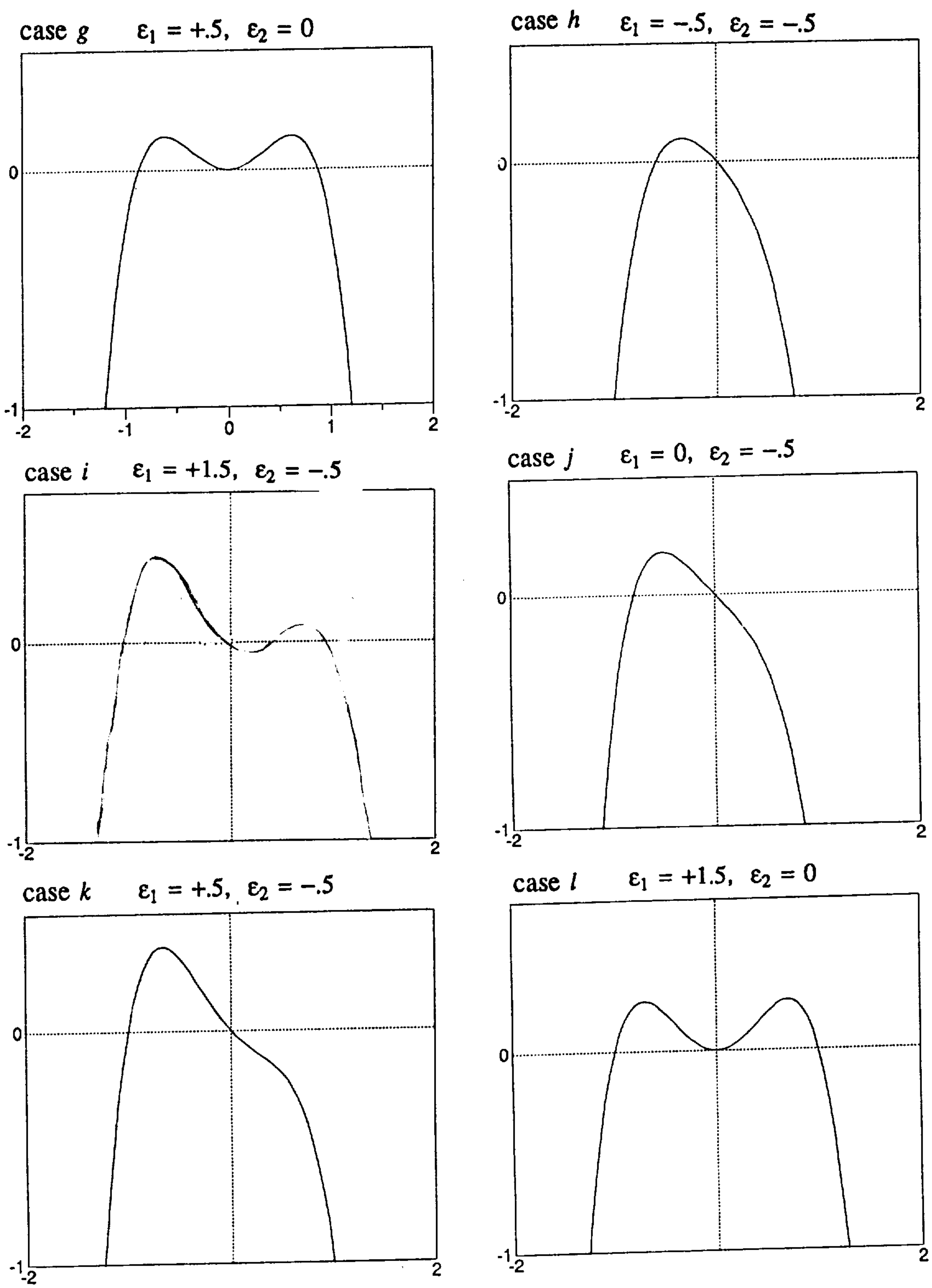

Fig 5 concluded

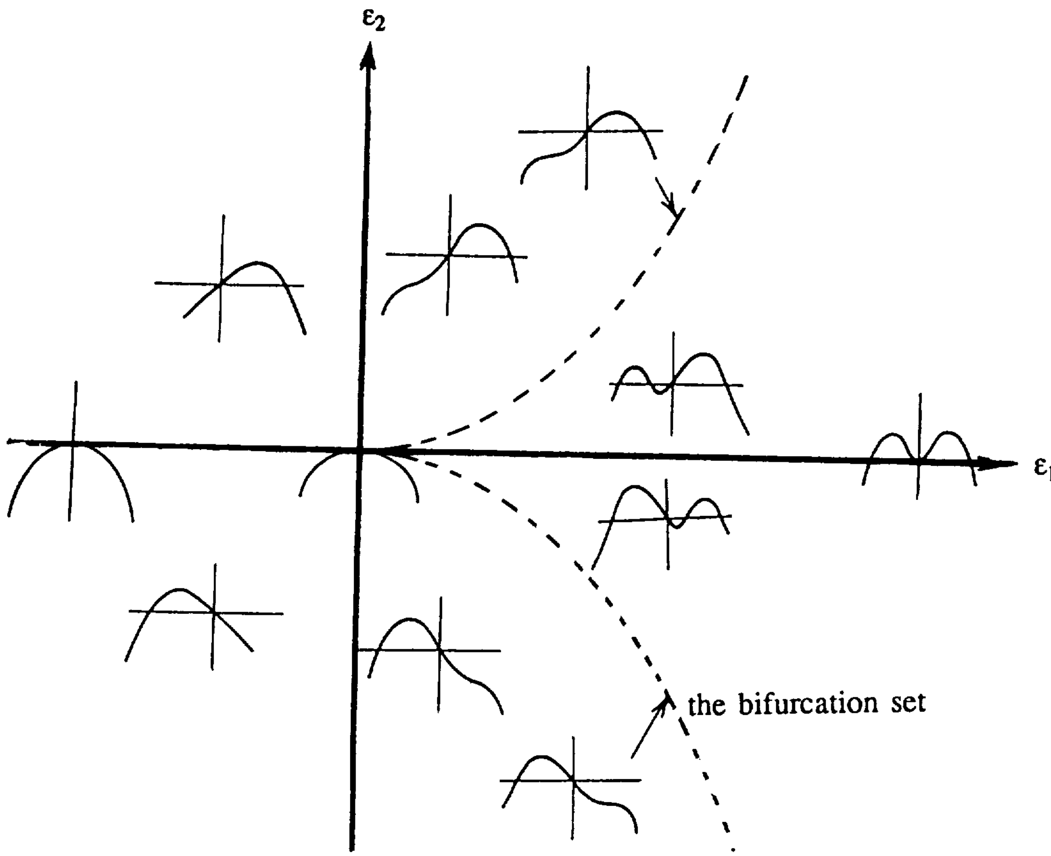

Fig 6. Topological unfolding of the singularity of eq (32) and the bifurcation set. Note that within the bifurcation set, there are two resonance peaks; outside, only one peak; for the tuned system, the two peaks coalesce into a degenerate singularity.

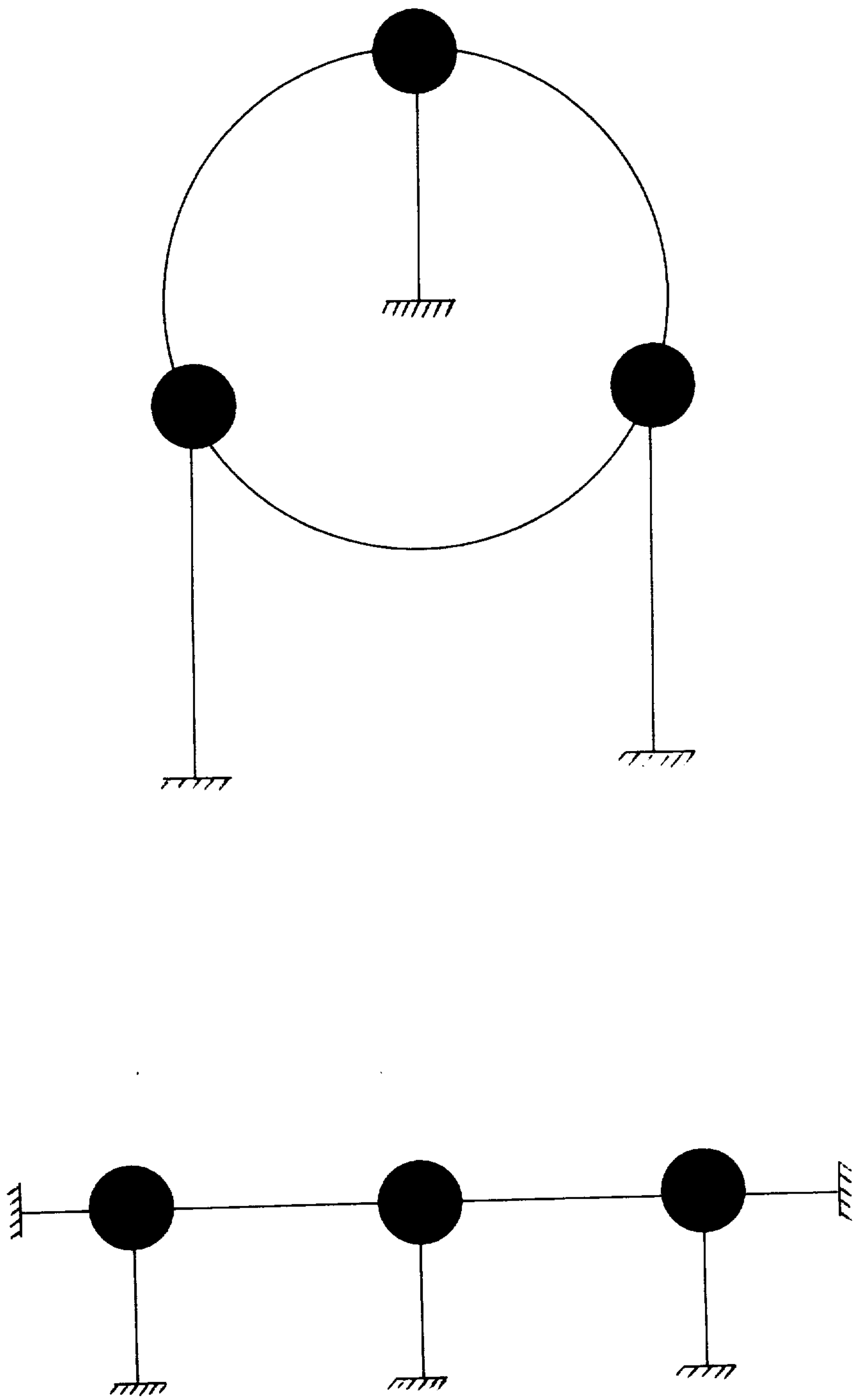

Fig 7. Models of (a) the cyclic membrane, (b) the linear chain with three degrees of freedom.

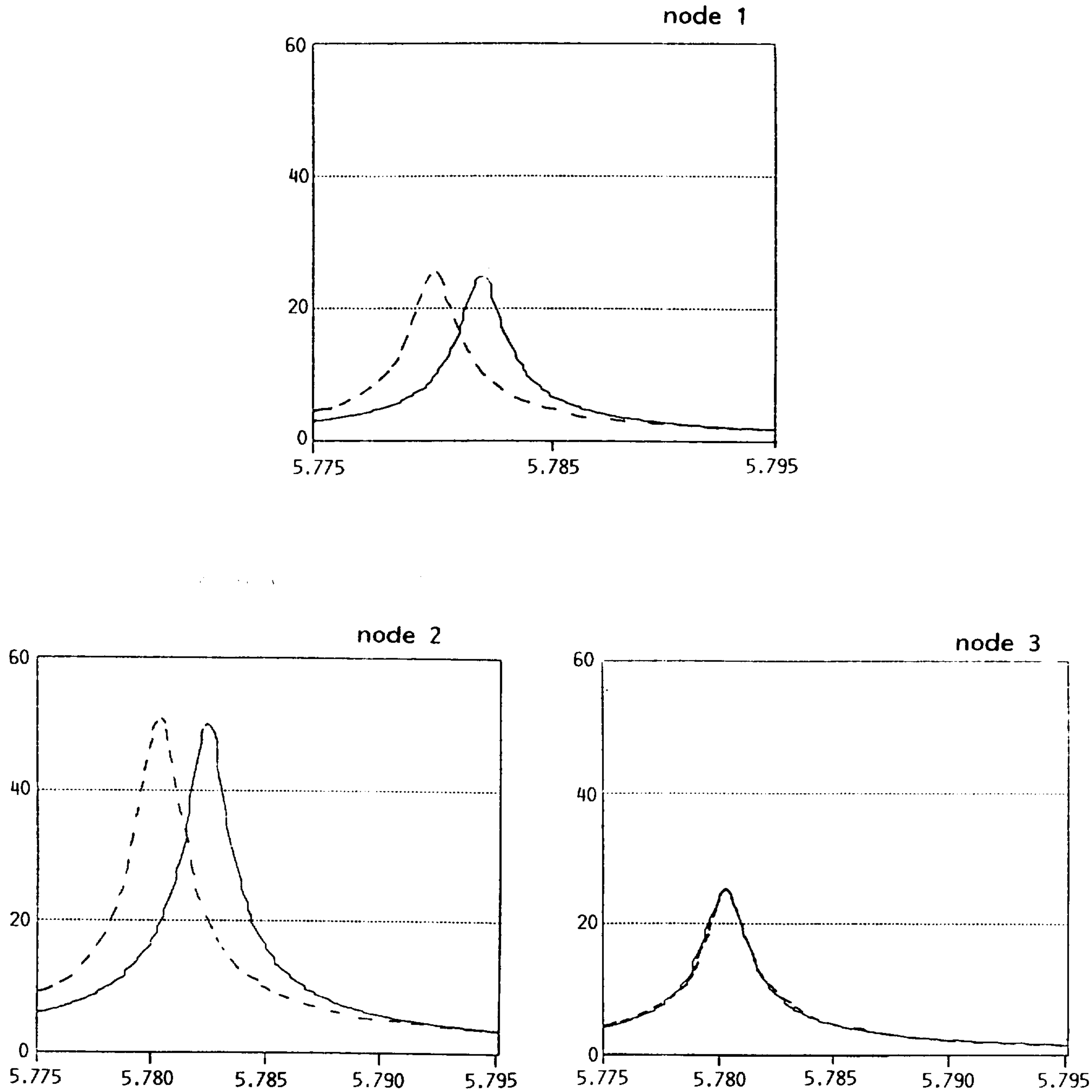

Fig 8. Effect of mistuning on the response curve of the linear chain. Note the preservation of shape, and the minimal difference in the peak amplitudes of the tuned and mistuned system.

(– – – *tuned system;* _____ *mistuned system*)

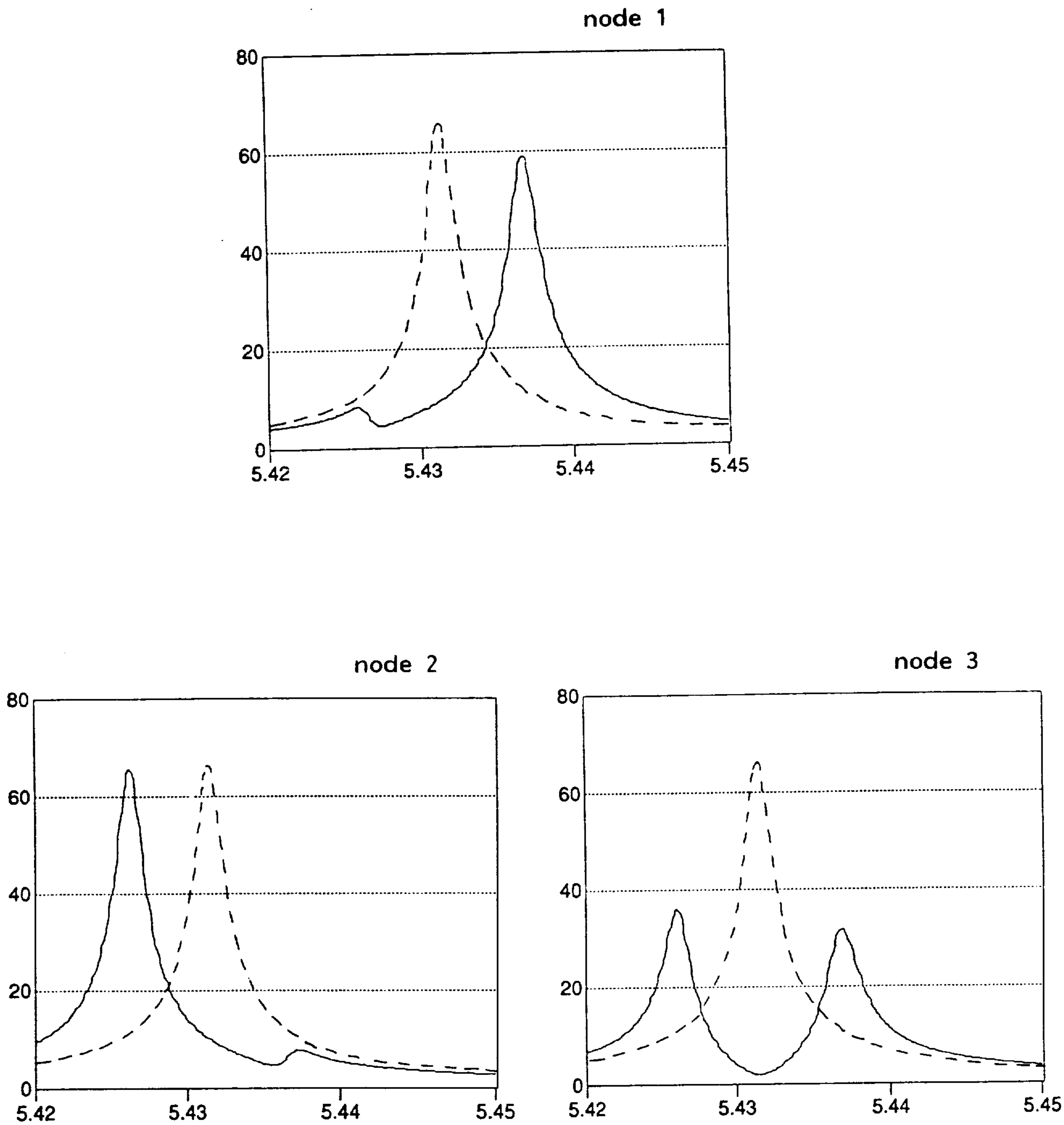

Fig 9. Effect of two-parameter mistuning on the response curve of the cyclic membrane. Note the severe reduction in the amplitude at node 3, which is only 50% of the tuned system

(– – – *tuned system;* ______ *mistuned system*)

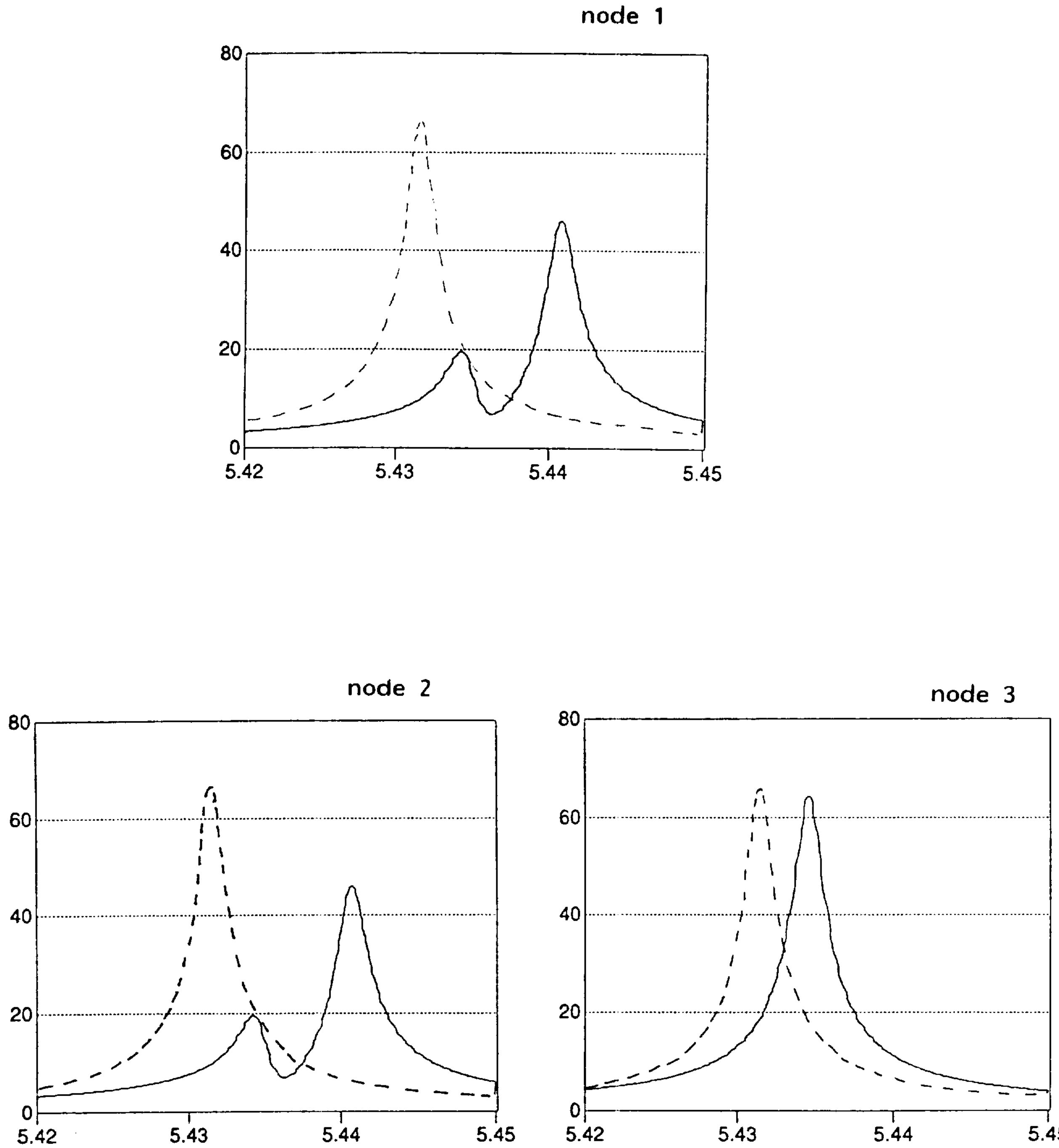

Fig 10. Effect of one-parameter mistuning on the response curve of the cyclic membrane. Note the symmetrical unfolding of the degenerate singularity.
(– – – *tuned system;* ______ *mistuned system*)

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16. Abstract

The stability of a frequency response curve under mild perturbations of the system's matrix is investigated. Using recent developments in the theory of singularities of differentiable maps, it is shown that the stability of a response curve depends on the structure of the system's matrix. In particular, the frequency response curves of a *cyclic* system are shown to be unstable. Consequently, slight parameter variations engendered by mistuning will induce a significant difference in the topology of the forced response curves, if the mistuning transformation crosses the bifurcation set.

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