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THE NONCONVEX MULTI-DIMENSIONAL RIEMANN PROBLEM FOR HAMILTON-JACOBI EQUATIONS

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ABSTRACT

We present simple inequalities for the Riemann problem for a Hamilton-Jacobi equation in N space dimension when neither the initial data nor the Hamiltonian need be convex (or concave). The initial data is globally continuous, affine in each orthant, with a possible jump in normal derivative across each coordinate plane, $x_i = 0$. The inequalities become equalities wherever a "maxmin" equals a "minmax" and thus an exact closed form solution to this problem is then obtained.

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We shall be concerned with solutions to the following differential equation

(H-J) $$\varphi_t + H(D_x\varphi) = 0 \quad \text{in} \quad R^N \times (0,\infty)$$

where $H \varepsilon C(R^N)$ and $D_x\varphi = (\varphi_{x_1}, \ldots, \varphi_{x_n})$ is the spatial gradient of φ.

We shall take special (Riemann) initial data. Let u_i^+, u_i^- be constants and define

(1) $$\begin{aligned} u_i(x) &= u_i^+ \quad \text{if} \quad x_i > 0 \\ u_i(x) &= u_i^- \quad \text{if} \quad x_i < 0 \end{aligned}$$

for $i = 1, \ldots, N$. Then take:

(2) $$\varphi_0(x) = A + \sum_{i=1}^{N} x_i \; u_i(x) = A + x \cdot u(x)$$

We wish to solve H-J with initial data (2) in the class of viscosity solutions as defined in [2]. The four properties of viscosity solutions that we shall need here (apart from existence, uniqueness derived in [2]) are:

(P1) The solution $\varphi(x,t)$ is a non-decreasing function of the initial data.

(P2) The partial derivatives φ_{x_i} satisfy a maximum principle at points of continuity, i.e. for $i = 1, \ldots, N$:

$$\min(u_i^-, u_i^+) \leq \varphi_{x_i} \leq \max(u_i^-, u_i^+).$$

(P3) The speed of propagation is finite.

(P4) If $\psi(x_2, \ldots, x_N, t)$ is a viscosity solution of

$$\psi_t + H(v_1, \psi_{x_2}, \ldots, \psi_{x_N}) = 0$$

for a constant v_1 then

$$\varphi(x,t) = v_1 x_1 + \psi(x_2, \dots, x_N, t)$$

is a viscosity solution to (H-J).

Let

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$$

where for $i = 1, \dots, N$:

$$\Omega_i = \{v / \min(u_i^-, u_i^+) \le v \le \max(u_i^-, u_i^+)\}$$

Finally we let, for $i = 1, \dots, N$:

$$\chi_i = \text{sign } (u_i^+ - u_i^-)$$

For convenience only, we order the indices so that

$$\chi_i = 1, \qquad i = 1, 2, \dots, j$$

$$\chi_i = -1, \qquad i = j+1, \dots, N$$

(j might be 0 or $N+1$).

We now state:

THEOREM 1. *The viscosity solution to* (H-J) *with initial data* (2) *satisfies*:

$$\begin{aligned}
&A + \max_{v_1 \epsilon \Omega_1} \max_{v_2 \epsilon \Omega_2} \cdots \max_{v_j \epsilon \Omega_j} \min_{v_{j+1} \epsilon \Omega_{j+1}} \cdots \min_{v_N \epsilon \Omega_N} [x \cdot v - tH(v)] \\
&\quad \le \varphi(x,t) \le \qquad\qquad (3) \\
&A + \min_{v_{j+1} \epsilon \Omega_{j+1}} \cdots \min_{v_N \epsilon \Omega_N} \max_{v_1 \epsilon \Omega_1} \cdots \max_{v_j \epsilon \Omega_j} [x \cdot v - tH(v)]
\end{aligned}$$

We note that if all the $\chi_i = 1$, then this solution is just $\max_{v \epsilon \Omega}[x \cdot v - tH(v)]$, if all the $\chi_i = -1$ it is $\min_{v \epsilon \Omega}[x \cdot v - tH(v)]$. Otherwise we have a pointwise inequality which gives the exact solution whenever the first and last terms in (3) are equal. This occurs e.g. if $H(v) = H_1(v_1, \ldots, v_j) + H_2(v_{j+1}, \ldots, v_N)$, i.e., if the Hamiltonian separates, and in many other cases.

The rest of this paper consists of the proof of this theorem, and some remarks about both conservation laws and numerical approximations to H-J.

It is easy to see that the solution to the Cauchy problem satisfies

$$\varphi(x,t) = tg(\frac{x}{t}) + A = tg(\zeta) + A \tag{4}$$

where g satisfies:

$$g = \zeta \cdot D_\zeta g - H(D_\zeta g) = -H^1(g_1(\zeta), g_2(\zeta), \ldots, g_N(\zeta)) \tag{5}$$

where $D_\zeta g$ is continuous.

In H-J, we let $\tau = t, y_i = x_i - \zeta_i t$ for ζ fixed. H-J becomes

$$\begin{aligned} &\varphi_\tau + H(D_y\varphi) - \zeta D_y\varphi \\ &= \varphi_\tau + H^1(D_y\varphi) = 0 \end{aligned} \tag{H-J^1}$$

with the same initial data (2).

Thus, by (5), to evaluate $g(\zeta)$ we need only evaluate $-H^1(D_y g)$ at $y = 0$, for any $t > 0$. From (P1) above we know that $(D_y g)_{y=0}$ lies in Ω for $t > 0$. Moreover, if we integrate $(\text{H-J})^1$ from $t = 0$ to $t = \Delta t$ we have

$$\begin{aligned} \varphi(0,\Delta t) &= A - \Delta t H^1((D_y g)_{y=0}) \\ &= \varphi_0(0) - \Delta t \tilde{H}^1(D_+^{x_1}\varphi_0(0), D_-^{x_1}\varphi_0(0); D_+^{x_2}\varphi_0(0), D_-^{x_2}\varphi_0(0); \cdots \\ &\qquad \cdots; D_+^{x_N}\varphi_0(0), D_-^{x_N}\varphi_0(0)) \end{aligned} \tag{6}$$

Here:

$$D_\pm^{x_i}\varphi_0(0) = \pm\frac{(\varphi_0(\pm h e_i) - \varphi_0(0))}{h} = u_i^\pm \tag{7}$$

where $e_i = \{0, 0, \ldots, 1, 0, \ldots, \}$, the i^{th} unit vector, and $\tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \ldots; u_N^+, u_N^-)$ is determined by (6).

This formula can be interpreted as a numerical algorithm. Suppose we are given a grid

$$x^i_{j_i} = j_i h, i = 1, \ldots, N; \; j_i = 0, \pm 1, \ldots$$

and values of a discrete function $\psi_j = \psi_{j_1 j_2 \ldots j_N}$. Then for each j, we construct the piecewise affine function which, in each of the 2^N orthants centered at j, interpolates ψ_j and its N nearest neighbors, $\psi_{j \pm e_i}$ for $i = 1, \ldots, N$. From (P3) above, if

$$\text{(8)} \qquad \text{(CFL)} \qquad \frac{\Delta t}{h} \max_{\substack{v \in \Omega(j) \\ i=1,\ldots,N}} |H^1_{u_i}| \leq \frac{1}{N^{\frac{1}{2}}}$$

where $\Omega^{(j)}$ is the same as Ω with each u_i^-, u_i^+ replaced by $D_-^{x_i}\psi_j, D_+^{x_i}\psi_j$, then the solution to the initial value problem (H-J)^1 with the above affine initial data in the diamond centered at j when evaluated at $x = x_j$ and $t = \Delta t$ is independent of the values of the initial data outside of this diamond.

Thus (6) (with $\varphi_0(0)$ replaced by ψ_j^n and $\varphi(0, \Delta t)$ by ψ_j^{n+1}), gives us a monotone finite difference scheme approximating (H-J)^1 which is in differenced form with numerical Hamiltonian $\tilde{H}^1$. These concepts were introduced in [3]. The scheme is monotone, which means that the right side of (6) is an increasing function of all the $\varphi_{j \pm e_i}$, because of property (P1). The function $\tilde{H}^1$ is called Godunov's Hamiltonian by analogy with the definition of Godunov's scheme for conservation laws in one space dimension [5]. The scheme is consistent, which means

$$\tilde{H}^1(u_1, u_1; u_2, u_2; \ldots; u_N, u_N) = H^1(u_1, u_2, \ldots, u_N)$$

Monotonicity implies that

$$\tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; , \ldots; u_N^+, u_N^-)$$

is a nonincreasing function of all the u_i^+ and a non-decreasing function of all the u_i^-. In particular, for $N = 1$, this means for any $v_1 \varepsilon \Omega = \Omega_1$:

$$
\begin{aligned}
(9) \qquad & \operatorname{sgn}(u_1^+ - u_1^-)[\tilde{H}^1(u_1^+, u_1^-) - H^1(v_1)] \\
&= \operatorname{sgn}(u_1^+ - v_1)[\tilde{H}^1(u_1^+, u_1^-) - \tilde{H}^1(v_1, u_1^-)] \\
&+ \operatorname{sgn}(v_1 - u_1^-)[\tilde{H}^1(v_1, u_1^-) - \tilde{H}^1(v_1, v_1)] \\
&\leq 0
\end{aligned}
$$

But, by (P2), $\tilde{H}^1(u_1^+, u_1^-) = H^1(\tilde{u}_1)$ for some $\tilde{u}_1$ in Ω. Thus we have

$$
(10) \qquad \tilde{H}^1(u_1^+, u_1^-) = \chi_1 \min_{v_1 \varepsilon \Omega_1} \chi_1 H^1(v_1)
$$

(This formula was obtained earlier in [6]). Now we proceed inductively. Suppose, for $N \leq M - 1$, we have

$$
\begin{aligned}
& \max_{v_{j+1} \varepsilon \Omega_{j+1}} \cdots \max_{v_N \varepsilon \Omega_N} \min_{v_1 \varepsilon \Omega_1} \cdots \min_{v_j \varepsilon \Omega_j} H^1(v_1, v_2, \ldots, v_N) \\
(11) \qquad & \leq \tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \ldots; u_N^+, u_N^-) \\
& \leq \min_{v_1 \varepsilon \Omega_1} \cdots \min_{v_j \varepsilon \Omega_j} \max_{v_{j+1} \varepsilon \Omega_{j+1}} \cdots \max_{v_N \varepsilon \Omega_N} H^1(v_1, v_2, \ldots, v_N)
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_i &= 1, \qquad i = 1, \ldots, j \\
\chi_i &= -1, \qquad i = j+1, \ldots, N
\end{aligned}
$$

Next we have, $N = M$ and for any $v_1 \varepsilon \Omega_1$:

$$
\begin{aligned}
(12) \qquad & \chi_1[\tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \ldots; u_M^+, u_M^-) \\
& - \tilde{H}^1(v_1, v_1; u_2^+, u_2^-; \ldots; u_M^+, u_M^-)] \\
& \leq 0
\end{aligned}
$$

using the same argument as in (9).

Now, for any fixed v_1, $\tilde{H}^1(v_1, v_1; u_2^+ u_2^-, \ldots; u_M^+, u_M^-)$ is Godunov's Hamiltonian when the initial data for (H-J)^1 has a constant x_1 derivative,

$$\frac{\partial \varphi_0}{\partial x_1}(x) \equiv v_1$$

Then it follows from (P4) that

$$g(\frac{x}{t}) = \frac{x_1}{t} v_1 + \tilde{g}(\frac{x_2}{t}, \frac{x_3}{t}, \ldots \frac{x_M}{t})$$

(where $\tilde{g}$ also depends on v_1).

By the induction hypothesis, this means we have

$$\begin{aligned}
(13) \qquad & \chi_1 \tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \ldots; u_M^+, u_M^-) \\
& \le \chi_1 \tilde{H}^1(v_1, v_1; u_2^+, u_2^-; \ldots; u_M^+, u_M^-) \\
& = -\chi_1 \tilde{g}(0, 0, \ldots, 0) \\
& = \chi_1 \chi_2 \min_{v_2 \varepsilon \Omega_2} \chi_2 \cdots \chi_M \min_{v_M \varepsilon \Omega_M} \chi_M \; H^1(v_1, v_2 \ldots v_M) \\
& = \chi_1 H^1(v_1, \tilde{v}_2, \ldots, \tilde{v}_M)
\end{aligned}$$

where the extrema is taken on at $\tilde{v}_2, \ldots, \tilde{v}_M$, which depends on v_1. The vector $(v_1, \tilde{v}_2, \ldots, \tilde{v}_N) \varepsilon \Omega_1$ where $v_1 \varepsilon \Omega_1$ is arbitrary. We next take $\min\limits_{v_1}$ of the expression in (13); If all the $\chi_i \equiv 1$ or all the $\chi_i \equiv -1$ we have equality by (P2). Otherwise $\chi_i \equiv 1,\ 1 \le i \le j,\ \chi_i \equiv -1,\ j+1 \le i \le M$ and we have the right hand inequality in (11). Next we have, for any $v_{j+1} \varepsilon \Omega_{j+1}$, following the argument above:

(14)

$$\begin{aligned}
& \tilde{H}^1(u_1^+, u_1^-; u_2^+, u_2^-; \ldots; u_M^+, u_M^-) \\
& \quad \ge \tilde{H}^1(u_1^+, u_1^-; \ldots; v_{j+1}, v_{j+1}; \ldots; u_M^+, u_M^-) \\
& \quad \ge \chi_{j+2} \min_{v_{j+2} \varepsilon \Omega_{j+2}} \chi_{j+2} \cdots \chi_M \min_{v_M \varepsilon \Omega_M} \chi_M \chi_1 \min_{v_1 \varepsilon \Omega_1} \chi_1 \cdots \chi_j \min_{v_j \varepsilon \Omega_j} \chi_j \; H^1(v_1, v_2, \ldots v_M)
\end{aligned}$$

We next take the $\max\limits_{v_{j+1}}$ of the expression in (P4) which gives us the left hand inequality in (11).

We have now obtained formula (11) for any N; using (4) and (5) give us Theorem 1.

We note that (11) validates the conjecture about Godunov's Hamiltonian in [7] when the inequalities in (13) and 14) become equalities. That paper also discusses the high-order accurate non-oscillatory numerical solution of (H-J) in some detail.

If we take the space gradient of (H-J) and call $u_1 = \varphi_{x_1}, u_2 = \varphi_{x_2}$, etc., we arrive at the system of conservation laws

$$(u_i)_t + \frac{\partial}{\partial_{x_i}} H(u_1, \ldots, u_N) = 0, \quad i = 1, \ldots, N \tag{14}$$

with intial data:

$$\begin{aligned} u_i(x,0) &= u_i^+ \text{ if } x_i > 0 \\ &= u_i^- \text{ if } x_i < 0 \\ i &= 1, \ldots, n \end{aligned}$$

Then taking the space gradient of (3) gives us information about the solution to this special Riemann problem for a special system of conservation laws.

We finally remark that if the initial data is convex (concave) or if $H(u_1, \ldots, u_N)$ is convex (concave) then the Hopf formulas [1] for this problem apply. In the case $N = 1$ it was shown in [1] that these formulas give the solution (3) originally derived in [6]. (In the one dimensional case the Riemann initial data is automatically convex or concave). The same must be true in the multi-dimensional convex or concave case. Our general nonconvex results presumably follow from the rather complicated formulas in [4], although the connection seems to be unclear. P. Sougandis has verified for us (private communication) that the left and right sides of (3) are always viscosity sub and super solutions for (H-J).

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