

N-62-1R  
25507  
278

# Efficient Diagnosis of Multiprocessor Systems under Probabilistic Models\*

*Douglas M. Blough*

Department of Electrical and Computer Engineering  
University of California  
Irvine, CA 92717

*Gregory F. Sullivan*

Department of Computer Science  
The Johns Hopkins University  
Baltimore, MD 21218

*Gerald M. Masson*

Department of Computer Science  
The Johns Hopkins University  
Baltimore, MD 21218

## Abstract

In this paper, the problem of fault diagnosis in multiprocessor systems is considered under a probabilistic fault model. This work focuses on minimizing the number of tests that must be conducted in order to correctly diagnose the state of every processor in the system with high probability. A diagnosis algorithm that can correctly diagnose the state of every processor with probability approaching one in a class of systems performing slightly greater than a linear number of tests is presented. A nearly matching lower bound on the number of tests required to achieve correct diagnosis in arbitrary systems is also proven. Lower and upper bounds on the number of tests required for regular systems are also presented. A class of regular systems which includes hypercubes is shown to be correctly diagnosable with high probability. In all cases, the number of tests required under this probabilistic model is shown to be significantly less than under a bounded-size fault set model. Because the number of tests that must be conducted is a measure of the diagnosis overhead, these results represent a dramatic improvement in the performance of system-level diagnosis techniques.

**Index Terms:** Algorithms, fault diagnosis, hypercube, multiprocessor systems, permanent faults, probabilistic models.

---

\*This research was supported in part by NASA grant 1442.

## 1 Introduction

Highly parallel computer systems, *i.e.* computer systems containing a large number of distinct processing elements, are being utilized in a growing number of applications. For systems with a large number of processors, automatic fault diagnosis is an attractive method of reducing maintenance costs as well as increasing system availability. Previous work on multiprocessor system fault diagnosis has been primarily concerned with worst-case fault scenarios, leading to overly pessimistic assessments of diagnostic capability. The work presented in this paper focuses on evaluation of diagnosis strategies under a probabilistic model in which processors are faulty with independent and identical probabilities. This approach yields a more realistic assessment of diagnostic capability but at the same time increases the complexity of the corresponding analysis.

The problem of multiprocessor system diagnosis has been addressed previously from a probabilistic viewpoint in [3,4,6,7,13,15,17,18,19,20]. The first paper concerning probabilistic diagnosis [13] examined heterogeneous systems in which each processor has an associated probability of failure. The authors examined the class of systems known as  $p$ -probabilistically diagnosable systems in which any set of faulty processors that has *a priori* probability greater than or equal to  $p$  of occurring is uniquely diagnosable. The problem of determining whether a given system is  $p$ -probabilistically diagnosable has been shown to be co-NP-complete [20] while an  $O(n^3)$  algorithm has been given [6] for determining the most likely fault set of a system in the closely related weighted model. In related work, Blount presented a method of achieving optimal diagnosis (diagnosis which is correct with maximum probability) in a general probabilistic model [4]. Unfortunately, this optimal diagnosis requires exponential time and it was not determined how the quality of diagnosis varies with the number of tests conducted.

In  $p$ -probabilistically diagnosable systems, fault sets with probability of occurrence slightly less than  $p$  can exist. Hence, the most likely fault set may be only slightly more probable than other fault sets, meaning that the probability of choosing an incorrect fault set may be high. The same may be true even when optimal diagnosis can be achieved. In [18], the author examined systems for which the correct fault set can be identified with high probability. The model utilized applies to homogeneous systems in which each processor has a common probability of failure  $p$ . An efficient diagnosis algorithm was presented that correctly diagnoses a class of systems containing  $cn \log n$  tests, for  $c > \frac{1}{\log \frac{1}{p}}$ , with probability approaching one. It was also claimed in [18] that this result was the best possible, *i.e.* all algorithms must have probability approaching zero of achieving correct diagnosis in systems containing  $o(n \log n)$  tests. Unfortunately, due to a subtle flaw in the proof, this

result is untrue. This result was also used in [3] to prove a similarly flawed lower bound in a more general probabilistic model.

In this paper, we utilize the model presented in [18]. A counterexample to the lower bound claimed in [18] is given in which correct diagnosis is achieved with constant probability in a sequence of digraphs with  $n - 1$  tests. Next, a diagnosis algorithm that produces correct diagnosis with probability approaching one in digraphs containing slightly more than a linear number of tests is given. A nearly matching lower bound on the number of tests required to achieve correct diagnosis with probability approaching one is then proven. Finally, the problem of diagnosis in regular systems is considered. A class of systems conducting  $\Theta(n \log n)$  tests in which correct diagnosis can be achieved with probability approaching one is presented. This class contains the systems given in [18] as well as the important class of hypercubes. It is also shown that for regular systems possessing  $o(n \log n)$  tests, all diagnosis algorithms perform poorly. This final result implies that for the important class of fixed-degree regular systems, weaker forms of diagnosis must be considered.

## 2 Preliminaries

The multiprocessor system model utilized in this paper was proposed in [16]. In this model a system is represented as a directed graph with vertices of the digraph representing processors in the system and edges of the digraph representing tests performed by one processor on another processor. In this section, all basic quantities related to this model are defined and a measure of diagnosis algorithm performance is presented.

### 2.1 Basic Definitions

For a system composed of  $n$  processors, the set of processors is represented by  $U = \{u_1, \dots, u_n\}$ . It is assumed that these processors are capable of performing tests on one another. This situation is represented by a digraph  $G(U, E)$ , where the vertex set  $U$  corresponds to the set of processors of the system and  $(u, v) \in E$  if and only if processor  $u$  tests processor  $v$  in the system. Associated with each  $(u, v) \in E$  is a *test outcome*. This outcome is a 1(0) if  $u$  evaluates  $v$  as faulty (fault-free). A complete collection of test outcomes constitutes a *syndrome*. Below syndromes, fault sets, and other fundamental concepts are defined.

**Definition 1** For a digraph  $G(U, E)$ , a syndrome is a function from  $E$  to  $\{0, 1\}$ .

**Definition 2** For a digraph  $G(U, E)$ , a fault set is a subset of the vertex set  $U$ .

For a processor  $u$ , the *tester set* consists of the processors that test  $u$ , the *failure set* consists of the processors that fail  $u$ , and the *neighbor set* consists of the processors that test  $u$  along with those that  $u$  tests. These quantities are defined below.

**Definition 3** For a digraph  $G(U, E)$  and  $u \in U$ , the tester set of  $u$ , denoted by  $\Gamma^{-1}(u)$ , is given by

$$\Gamma^{-1}(u) = \{v \in U : (v, u) \in E\}$$

**Definition 4** For a digraph  $G(U, E)$ , a syndrome  $S$ , and  $u \in U$ , the failure set of  $u$ , denoted by  $\text{fail}_{in}(u)$ , is given by

$$\text{fail}_{in}(u) = \{v \in \Gamma^{-1}(u) : S((v, u)) = 1\}$$

**Definition 5** For a digraph  $G(U, E)$  and  $u \in U$ , the neighbor set of  $u$ , denoted by  $N(u)$ , is given by

$$N(u) = \{v \in U : (u, v) \in E \text{ or } (v, u) \in E\}$$

## 2.2 Diagnosis Algorithm Evaluation

A fundamental problem in multiprocessor systems is to identify the faulty processors in a system given a syndrome. An algorithm for this problem is referred to as a *diagnosis algorithm*. A diagnosis algorithm takes a syndrome as input and outputs a subset of the processors in the system. This subset contains exactly the processors diagnosed as faulty by the algorithm. Thus, for a set of faulty processors and a syndrome it is possible to evaluate if the output of a deterministic algorithm is correct by comparing the algorithm's output with the set of faulty processors. Syndrome, fault set pairs are therefore used as the basic element in the subsequent probabilistic analysis of diagnosis algorithm performance. Before proceeding with this analysis however, the notion of correct diagnosis must be defined. For a syndrome  $S$  from a digraph  $G(U, E)$ , and a deterministic algorithm  $A$ , let

$$\text{Faulty}_A(S) = \{u \in U : \text{Algorithm } A \text{ diagnoses } u \text{ as faulty when run on } S\}$$

Thus,  $\text{Faulty}_A(S)$  represents the output of Algorithm  $A$  when run on syndrome  $S$ . Using this, the diagnosis quality of an algorithm on a syndrome, fault set pair is characterized in Definition 6.

**Definition 6** For a syndrome, fault set pair  $(S, F)$  from a digraph  $G(U, E)$ , a deterministic algorithm  $A$  is said to produce

- correct diagnosis if and only if  $Faulty_A(S) = F$ ,
- partial diagnosis if and only if  $Faulty_A(S) \subset F$ , and
- false alarm diagnosis if and only if  $Faulty_A(S) \not\subseteq F$ .

Note that Definition 6 differs from that used in some previous work, e.g. [21], where correct diagnosis may allow faulty processors to be identified as fault-free so long as no fault-free processor is identified as faulty. In Definition 6, diagnosis is correct only when each fault-free processor is identified as fault-free *and* each faulty processor is identified as faulty. One of the goals of this paper is to provide a rigorous foundation for the analysis of the diagnosis problem. To achieve this goal we take great care in defining a proper measure of diagnosis algorithm performance as well as a probabilistic fault model under which this performance can be evaluated. This probabilistic model is presented in the following section.

### 3 Probabilistic Model

In much of the previous work in the system-level diagnosis area, diagnosis algorithm evaluation has focused on worst-case performance. Under a bounded-size fault set model, correct diagnosis can be guaranteed if the number of faulty processors in the system is no greater than some value  $t < n/2$ . Such a model allows any set of  $t$  or fewer processors in a system to be faulty, including sets that may be extremely rare in practice. This approach can therefore lead to an overly pessimistic view of diagnosis algorithm performance. In this paper, we present a probabilistic model for the faults in a system and we use, as a measure of performance, the probability that a diagnosis algorithm correctly identifies the faulty processors in the system. This approach yields a more realistic assessment of diagnosis algorithm performance by accounting for the likelihood of occurrence of the fault sets in a system. In our probabilistic fault model, processors are faulty with probability  $p$  independently of one another, fault-free processors always produce the correct outcome when performing a test, and no assumptions are made concerning the outcomes of tests performed by faulty processors. It will be shown in this paper that in contrast to the bounded-size fault set model, correct diagnosis can be achieved with high probability in this model at relatively low cost.

Some comments concerning the behavior of faulty processors under this model are in order. We make *no* assumptions concerning the outcomes of tests performed by faulty processors. Thus, faulty processors can pass or fail other processors in virtually any manner. For example, faulty processors can:

1. always fail other processors,
2. always pass other processors,
3. fail other processors with some probability,
4. collaborate with other faulty processors through their test outcomes in an attempt to confuse the diagnosis algorithm, or
5. combine any or all of the above behaviors.

Since these as well as any other behaviors are allowed under this model, this is equivalent to assuming that the faulty processors produce test outcomes in the most detrimental manner. The diagnosis algorithms we present in this paper are shown to produce correct diagnosis with high probability under *any* of these faulty processor behaviors and are therefore very robust. We also show that the set of systems for which these algorithms work contains systems which are very nearly the sparsest possible under this model and hence, significant improvements can only be achieved by restricting the behavior of faulty processors. With this in mind, we now present the probability model.

For a digraph  $G(U, E)$ , the sample space  $\Omega_{G(U, E)}$  of this probability model consists of all syndrome, fault set pairs in that digraph, *i.e.*

$$\Omega_{G(U, E)} = \{(S, F) : F \subseteq U \text{ and } S \text{ is a function from } E \text{ to } \{0, 1\}\}.$$

Since no assumptions are made concerning the outcomes of tests performed by faulty processors, the probability of a particular syndrome given a fault set may not be specified in this model. The basic events of the model consist of sets of syndrome, fault set pairs which have the same fault set and whose syndromes are identical except for the labels on edges out of faulty processors. Formally, a syndrome, fault set pair  $(S', F')$  is contained in a basic event  $B$  defined as follows:

$$B = \{(S, F) : F = F' \text{ and } \forall (u, v) \in E \text{ with } u \in U - F, S((u, v)) = S'((u, v))\}$$

Note that there is a unique fault set associated with each basic event but that each event may contain many distinct syndrome, fault set pairs. Now, let

$$\mathcal{B}_{G(U, E)} = \{B : B \text{ is a basic event of } G(U, E)\}.$$

The family of events  $\mathcal{F}_{G(U, E)}$  in this probability space is the set of all subsets of  $\mathcal{B}_{G(U, E)}$ .

**Definition 7** A syndrome, fault set pair  $(S, F)$  in a digraph  $G(U, E)$  is said to be incompatible if and only if  $\exists u, v \in U$  such that  $u \in U - F$ ,  $(u, v) \in E$ , and

1.  $v \in U - F$  and  $S((u, v)) = 1$ , or
2.  $v \in F$  and  $S((u, v)) = 0$ .

A syndrome, fault set pair which is not incompatible is said to be *compatible*. A basic event is said to be incompatible if its syndrome, fault set pairs are incompatible, otherwise it is compatible. The probability of a basic event  $B$  in a digraph  $G(U, E)$  is defined as follows:

$$P_G(B) = \begin{cases} 0 & \text{if } B \text{ is incompatible} \\ p^{|F|}(1-p)^{n-|F|} & \text{otherwise} \end{cases}$$

where  $F$  represents the unique fault set associated with  $B$ . Clearly,

$$\sum_{B \in \mathcal{B}_{G(U, E)}} P_G(B) = 1$$

and, hence, this is a legitimate probability measure.

The primary measure of the performance of a diagnosis algorithm used in this paper is the probability that the algorithm produces correct diagnosis. For a digraph  $G(U, E)$  and a deterministic algorithm  $A$ , let

$$\text{Correct}_G(A) = \{(S, F) : \text{Faulty}_A(S) = F\}$$

and let  $\text{NotCorrect}_G(A)$  represent the complement of  $\text{Correct}_G(A)$ . Thus,  $\text{Correct}_G(A)$  represents the set of all syndrome, fault set pairs in a digraph for which Algorithm  $A$  produces correct diagnosis. Note that it may be the case that  $\text{Correct}_G(A) \notin \mathcal{F}_{G(U, E)}$  in which case  $P_G(\text{Correct}_G(A))$  will not be defined. The output of a particular diagnosis algorithm may depend on the outcomes of tests performed by faulty processors and thus, the probability of correct diagnosis for the algorithm cannot be determined until a probability distribution on these edges is specified.

For a digraph  $G(U, E)$ , let  $P'_G$  be a probability function defined on  $\Omega_{G(U, E)}$  such that the family of events is equal to all subsets of  $\Omega_{G(U, E)}$  and  $\forall B \in \mathcal{B}_{G(U, E)}$ ,  $P'_G(B) = P_G(B)$ . Such a probability function will be referred to as a *refinement* of  $P_G$ . Now, let  $\mathcal{P}_G$  represent the set of all refinements of  $P_G$ . Since any type of behavior of the faulty processors is allowed in this model, the probability of correct diagnosis for a deterministic algorithm  $A$  in a digraph  $G(U, E)$ , denoted by  $\text{DiagProb}_G(A)$  is defined to be

$$\text{DiagProb}_G(A) = \min_{P'_G \in \mathcal{P}_G} P'_G(\text{Correct}_G(A)) = \min_{P'_G \in \mathcal{P}_G} \sum_{(S, F) \in \text{Correct}_G(A)} P'_G((S, F))$$

Thus, when calculating the probability of correct diagnosis for an algorithm it is assumed that the faulty processors perform their tests in the manner most detrimental to the algorithm. We may also define this diagnosis probability for probabilistic diagnosis algorithms. Given a syndrome  $S$ , a probabilistic diagnosis algorithm  $A$  chooses a fault set  $F$  with some probability call it  $p_{A,S}(F)$  where  $\sum_{F \subseteq U} p_{A,S}(F) = 1$ . Thus, for a digraph  $G(U, E)$  and a probabilistic diagnosis algorithm  $A$ , the probability of correct diagnosis for Algorithm  $A$  is defined to be

$$\text{DiagProb}_G(A) = \min_{P'_G \in \mathcal{P}_G} \sum_{(S,F) \in \Omega_G} P'_G((S,F)) \cdot p_{A,S}(F)$$

## 4 Diagnosis Using n-1 Tests

In [18], an efficient diagnosis algorithm that achieves correct diagnosis with probability approaching one in sequences of digraphs containing  $cn \log n$  edges, for  $c > \frac{1}{\log \frac{1}{p}}$ , was presented. It was also claimed in [18] that all diagnosis algorithms must have a probability of correct diagnosis that approaches zero for digraphs containing  $o(n \log n)$  edges. In this section, a sequence of digraphs containing  $n - 1$  edges is exhibited for which a simple diagnosis algorithm can achieve correct diagnosis with constant probability, thereby providing a counter-example to this claim.

Consider a sequence of digraphs  $G_n(U_n, E_n)$  with  $U_n = \{u_1, \dots, u_n\}$  and  $E_n$  defined as follows:

$$E_n = \{(u_1, u_2), (u_1, u_3), \dots, (u_1, u_{n-1}), (u_1, u_n)\},$$

*i.e.*  $u_1$  tests all other processors. Now, consider the following simple diagnosis algorithm.

### Algorithm Naive

*Input:* A syndrome  $S$  in a digraph  $G(U, E)$ .

*Output:* A set  $F \subseteq U$ .

$F \leftarrow \emptyset$

for each  $v \in \{u_2, u_3, \dots, u_n\}$

if  $S((u_1, v)) = 1$  then  $F \leftarrow F \cup \{v\}$

Algorithm Naive simply assumes that  $u_1$  is fault-free and diagnoses a processor as faulty if and only if it is failed by  $u_1$ . Clearly, if  $u_1$  is faulty, Algorithm Naive



incorrectly diagnoses  $u_1$  itself. If  $u_1$  is fault-free however, Algorithm Naive produces correct diagnosis. Thus,  $\forall P'_{G_n} \in \mathcal{P}_{G_n}$

$$\begin{aligned} P'_{G_n}(\text{Correct}_{G_n}(\text{Naive})) &= P'_{G_n}(\{(S, F) : u_1 \text{ is fault-free}\}) \\ &= 1 - p \end{aligned}$$

and therefore

$$\text{DiagProb}_{G_n}(\text{Naive}) = 1 - p.$$

Thus, this simple diagnosis algorithm produces correct diagnosis with constant probability in a sequence of digraphs containing exactly  $n - 1$  edges.

## 5 A Majority-Vote Algorithm

In this section, a simple yet powerful diagnosis algorithm known as Algorithm Majority is presented. In Algorithm Majority a processor is diagnosed as faulty if and only if it is failed by more than  $1/2$  the processors in its tester set.

### Algorithm Majority

*Input:* A syndrome  $S$  in a digraph  $G(U, E)$ .

*Output:* A set  $F \subseteq U$ .

$F \leftarrow \emptyset$

for each  $u \in U$

if  $|\text{fail}_{in}(u)| > \frac{|\Gamma^{-1}(u)|}{2}$  then  $F \leftarrow F \cup \{u\}$

**Theorem 1** *For a digraph  $G(U, E)$ , Algorithm Majority has a time complexity of  $O(|E|)$  and a space complexity of  $O(|E|)$ .*

**Proof:** The failure set cardinalities as well as the tester set cardinalities can be calculated in a single traversal of the labeled adjacency lists of the digraph. This requires  $O(|E|)$  time. The only storage requirement for the algorithm aside from the input and output is a set of temporary variables to hold these values as they are calculated. Hence, the space complexity is also  $O(|E|)$ . ■

Algorithm Majority is slightly more sophisticated than Algorithm Naive. Rather than blindly believing the test outcomes of a single processor, it relies on a majority-vote among the processors in the tester set of a given processor. It should be noted that for the special class of systems in which one processor tests every other processor and no other tests are conducted, Algorithms Naive and Majority are equivalent.

## 6 Diagnosis in Sparse Systems

In this section, we examine the problem of correctly diagnosing multiprocessor systems having sparse communication networks. First, it is shown that for a class of irregularly structured systems utilizing a number of tests growing just faster than  $n$ , Algorithm Majority correctly diagnoses every processor with probability approaching one. Next, the probability of correct diagnosis of Algorithm Majority is evaluated on some fixed systems which utilize a modest number of tests. Finally, it is proven that a linear number of tests are required for any diagnosis algorithm to be capable of producing correct diagnosis with high probability.

### 6.1 An Upper Bound on the Number of Tests Necessary for Correct Diagnosis

Consider a class of systems in which there is a set of processors known as the *testers*. The systems are such that any processor which is a tester tests all other processors in the system (including the other testers). Any processor that is not a tester conducts no tests. Thus, a (small) fraction of the processors are relied upon to satisfy all the testing requirements of the system. Such a digraph will be referred to as a *tester digraph*, formally defined below.

**Definition 8** A digraph  $G(U, E)$  is said to be a tester digraph if and only if  $\exists T_G \subseteq U$  such that

$$E = \{(u, v) : u \in T_G, v \in U, \text{ and } u \neq v\}.$$

The set  $T_G$  is known as the testing set of  $G$ .

Figure 1 is an example of a tester digraph with 3 testers and 8 vertices.

For a tester digraph  $G(U, E)$  with testing set  $T_G$ , let

$$\text{GoodMaj}_G = \{(S, F) : |T_G \cap (U - F)| > \frac{|T_G|}{2} \text{ and } (S, F) \text{ is compatible}\}$$

Thus,  $\text{GoodMaj}_G$  represents the set of compatible syndrome, fault set pairs in which more than  $1/2$  the testers are fault-free. The following lemma shows that if the majority of testers in a tester digraph are fault-free, then the diagnosis of Algorithm Majority will be correct.

**Lemma 1** For a tester digraph  $G(U, E)$ ,  $\text{GoodMaj}_G \subseteq \text{Correct}_G(\text{Majority})$ .

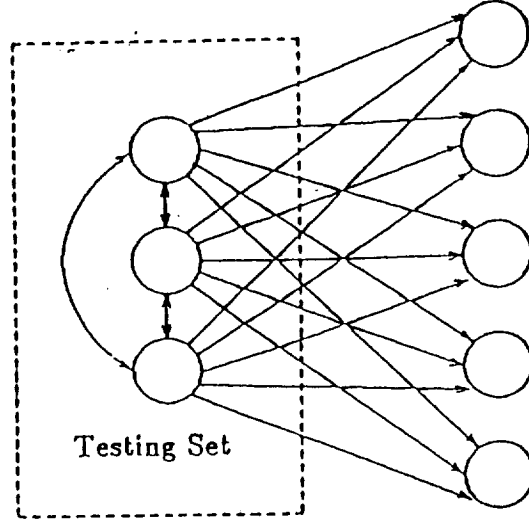


Figure 1: A Tester Digraph

**Proof:** We will show that if  $(S, F) \in \text{GoodMaj}_G$ , then  $(S, F) \in \text{Correct}_G(\text{Majority})$  and therefore,  $\text{GoodMaj}_G \subseteq \text{Correct}_G(\text{Majority})$ .

Consider any  $(S, F) \in \text{GoodMaj}_G$  and any  $u \in U$ .

*case 1 :*  $u \in (U - T_G) \cap (U - F)$

Because  $(S, F)$  is compatible,  $u$  must be passed by more than  $1/2$  the testers, implying  $u \notin \text{Faulty}_{\text{Majority}}(S)$ . Recall that  $\text{Faulty}_{\text{Majority}}(S)$  is the set of processors diagnosed as faulty by Algorithm Majority when run on  $S$ .

*case 2 :*  $u \in (U - T_G) \cap F$

Similarly,  $u$  must be failed by more than  $1/2$  the testers implying  $u \in \text{Faulty}_{\text{Majority}}(S)$ .

*case 3 :*  $u \in T_G \cap (U - F)$

Here,  $u$  can be failed by *at most*  $1/2$  the remaining testers. Since Algorithm Majority diagnoses a unit as faulty only when it is failed by a *strict majority* of its tester set,  $u \notin \text{Faulty}_{\text{Majority}}(S)$ .

*case 4 :*  $u \in T_G \cap F$

In this case,  $u$  must be failed by more than  $1/2$  the remaining testers, implying  $u \in \text{Faulty}_{\text{Majority}}(S)$ .

Hence,  $\text{Faulty}_{\text{Majority}}(S) = F$  and therefore  $(S, F) \in \text{Correct}_G(\text{Majority})$ . ■

Thus, if more than  $1/2$  the testers in a tester digraph are fault-free, Algorithm Majority produces correct diagnosis. Theorem 2 shows that if the number of testers is given by any unbounded function, this condition will be achieved with probability approaching one and hence the probability of correct diagnosis for Algorithm Majority approaches one.

**Theorem 2** *Let  $\omega(n)$  be any unbounded function. If  $p < 1/2$ , then for any sequence of tester digraphs on  $n$  vertices having  $\omega(n)$  testers, the probability of correct diagnosis for Algorithm Majority approaches one as  $n \rightarrow \infty$ .*

**Proof:** We must show that for any sequence satisfying the theorem condition,  $\text{DiagProb}_{G_n}(\text{Majority}) \rightarrow 1$  as  $n \rightarrow \infty$ . If we let  $X$  be a random variable representing the number of faulty units in the testing set of a tester digraph  $G$ , then

$$\text{GoodMaj}_G = \{(S, F) : X < \frac{|T_G|}{2} \text{ and } (S, F) \text{ is compatible}\}$$

Now,  $X$  is a binomial random variable with parameters  $|T_G|$  and  $p$ . It follows from Lemma 1 that  $\forall P'_{G_n} \in \mathcal{P}_{G_n}$

$$\begin{aligned} & P'_{G_n}(\text{Correct}_{G_n}(\text{Majority})) \\ & \geq P'_{G_n}(\text{GoodMaj}_{G_n}) \\ & = P'_{G_n}(\{(S, F) : X < \frac{|T_G|}{2}\}) \\ & = 1 - P'_{G_n}(\{(S, F) : X \geq \frac{|T_G|}{2}\}) \\ & = 1 - P'_{G_n}(\{(S, F) : \frac{X}{|T_{G_n}|} - p \geq \frac{1}{2} - p\}) \end{aligned}$$

Now, since  $p < 1/2$ ,  $\frac{1}{2} - p > 0$ , and by the Weak Law of Large Numbers [9],

$$P'_{G_n}(\text{Correct}_{G_n}(\text{Majority})) \rightarrow 1$$

and therefore

$$\text{DiagProb}_{G_n}(\text{Majority}) \rightarrow 1.$$

Thus, Algorithm Majority produces correct diagnosis with probability approaching one in a class of digraphs containing a number of edges given by  $n \cdot \omega(n)$ , where  $\omega(n)$  is any function that goes to infinity (arbitrarily slowly) with  $n$ . Under a bounded-size fault set model a quadratic number of tests are required to withstand a linear number of faults while this result shows that in this probabilistic model a

$p$	$ T_G $
0.001	3
0.005	5
0.010	5
0.050	11
0.100	19
0.200	41
0.300	105

**Table 1: Size of Testing Set Required for Correct Diagnosis Probability of 0.99**

linear expected number of faults can be tolerated with a number of tests that is arbitrarily close to linear. The maximum degree of the vertices in this class of digraphs is large, however, which may be a problem in some applications. This motivates us to study the problem of diagnosis in sparse regular systems in Section 7.

## 6.2 Performance of Algorithm Majority on Fixed Systems

In this section, the number of tests required to achieve a given probability of correct diagnosis in tester digraphs using Algorithm Majority is examined. For a tester digraph  $G(U, E)$  with testing set  $T_G$

$$\text{DiagProb}_G(\text{Majority}) \geq \sum_{i=0}^{\lfloor \frac{|T_G|}{2} \rfloor - 1} \binom{|T_G|}{i} p^i (1-p)^{|T_G|-i} \quad (1)$$

Note that the probability of correct diagnosis depends only on the testing set cardinality and not on  $n$ . For a given probability of failure, Inequality 1 can be used to determine the number of testers needed for Algorithm Majority to achieve a specified probability of correct diagnosis. The size of the testing set required to achieve a correct diagnosis probability of 0.99999 for various values of  $p$  is shown in Table 1. If the probability of failure of a processor is 0.001, Algorithm Majority can achieve correct diagnosis with a probability of 0.99999 using three tests per processor regardless of the number of processors in the system. For a probability of failure of 0.005 or 0.010 the tester set need only be of cardinality five for Algorithm Majority to achieve a probability of correct diagnosis of 0.99999. Thus, when the probability of failure

$n$	$p$	Bounded-size	Probabilistic
100	0.01	400	99
100	0.10	1800	495
100	0.30	4100	3069
1000	0.01	18,000	999
1000	0.10	123,000	4995
1000	0.30	334,000	30,969
10,000	0.01	1,240,000	9999
10,000	0.10	10,700,000	49,995
10,000	0.30	31,070,000	309,969

**Table 2: Total Number of Tests Necessary for Correct Diagnosis Probability of 0.99**

is small correct diagnosis can be achieved with extremely high probability using a total number of tests that is near  $n$ . When  $p$  is larger, more tests are necessary. As indicated in Table 1, for a probability of failure of 0.300, more than 100 tests per processor are required to achieve correct diagnosis with probability 0.99999. Since a large fraction of the processors in the system will be faulty in this situation it is to be expected that a larger number of tests are required. The important point is that the total number of tests remains proportional to  $n$  regardless of the value of  $p$ .

In Table 2, we compare the number of tests required under the bounded-size fault set model to the number required by Algorithm Majority in order to achieve a correct diagnosis probability of 0.99. The number of tests required under the bounded-size fault set model was calculated in the following manner. For a given  $n$  and  $p$ , determine  $t$  such that the probability of more than  $t$  out of the  $n$  processors being faulty is no greater than 0.01. Table 2 shows the results of this comparison for various values of  $n$  and  $p$ . For large  $n$  and small  $p$  the number of tests required under the probabilistic model is dramatically lower than the number required under the bounded-size fault set model. For example, when  $n = 10,000$  and  $p = 0.10$ , the number of tests required in the probabilistic model is reduced by a factor of 214 over the bounded-size fault set model.

### 6.3 A Lower Bound on the Number of Tests Necessary for Correct Diagnosis

In this section, a lower bound on the number of tests necessary to achieve correct diagnosis with high probability is proven. It is shown that if the number of edges in an arbitrary sequence of digraphs grows slower than  $n$ , then all diagnosis algorithms have probability approaching zero of achieving correct diagnosis. This result implies that Algorithm Majority achieves a probability approaching one of correct diagnosis on systems that are very nearly as sparse as possible. Thus, this relatively simple diagnosis algorithm is indeed extremely powerful.

When the number of edges in a sequence of digraphs grows slower than  $n$ , isolated processors, *i.e.* processors which have no incident edges must exist. Intuitively, no diagnosis algorithm should be capable of correctly identifying the state of all these isolated processors with high probability, making diagnosis in such situations impossible. This is formally proven in Theorem 3. The essence of the proof of Theorem 3 can be explained quite simply. To prove that a deterministic diagnosis algorithm  $A$  has a probability approaching zero of achieving correct diagnosis in a sequence of digraphs  $G_n(U_n, E_n)$ , a set of  $(S, F)$  pairs disjoint from  $\text{Correct}_{G_n}(A)$  must be exhibited that has a probability dominating the probability of  $\text{Correct}_{G_n}(A)$ . For a given syndrome from a system with isolated processors, it can be shown that so long as the number of isolated processors approaches infinity, the probability of that syndrome and a fault set with a particular labeling of the isolated processors is dominated by the probability of that syndrome and the fault sets in which the isolated processors are relabeled in all possible ways. Thus, for any  $(S, F) \in \text{Correct}_{G_n}(A)$ , a set of syndrome, fault set pairs disjoint from  $\text{Correct}_{G_n}(A)$  can be exhibited that has probability dominating the probability of  $(S, F)$ . It is also shown that there exists a deterministic diagnosis algorithm that has performance at least as good as the performance of any probabilistic algorithm, thus completing the proof.

**Theorem 3** *Let  $A$  be any probabilistic or deterministic diagnosis algorithm. If  $0 < p < 1$ , then for any sequence of digraphs on  $n$  vertices having  $o(n)$  edges, the probability of correct diagnosis for Algorithm  $A$  approaches zero as  $n \rightarrow \infty$ .*

**Proof:** We must show that for any probabilistic or deterministic diagnosis algorithm  $A$  and any sequence of digraphs  $G_n(U_n, E_n)$  having  $|E_n| \in o(n)$ ,  $\text{DiagProb}_{G_n}(A) \rightarrow 0$  as  $n \rightarrow \infty$ . Assume faulty processors pass all processors they test. This yields a refinement  $P'_{G_n} \in \mathcal{P}_{G_n}$ , where

$$P'_{G_n}((S, F)) = \begin{cases} 0 & \text{if } (S, F) \text{ is incompatible or } \exists u \in F, v \in U \text{ with } S((u, v)) = 1 \\ p^{|F|}(1-p)^{n-|F|} & \text{otherwise} \end{cases}$$

Now, let  $ISO_{G_n} \subseteq U_n$  represent the set of isolated processors, i.e. processors which have no incident edges, in  $G_n(U_n, E_n)$ . Clearly,

$$|ISO_{G_n}| \geq n - 2|E_n| \rightarrow \infty.$$

For a syndrome, fault set pair  $(S, F) \in \text{Correct}_{G_n}(A)$  let

$$\text{Relabel}_{(S,F)} = \{(S', F') : S' = S, F' \neq F, \text{ and } F - ISO_{G_n} = F' - ISO_{G_n}\}$$

and let

$$\text{AllLabel}_{(S,F)} = \text{Relabel}_{(S,F)} \cup \{(S, F)\}.$$

Thus,  $\text{Relabel}_{(S,F)}$  consists of the syndrome, fault set pairs in which the processors of  $ISO_{G_n}$  are relabeled in all possible ways. Clearly,

$$\begin{aligned} & P'_{G_n}(\text{NotCorrect}_{G_n}(A)) \\ & \geq \sum_{(S,F) \in \text{Correct}_{G_n}(A)} P'_{G_n}(\text{Relabel}_{(S,F)}) \\ & = \sum_{(S,F) \in \text{Correct}_{G_n}(A)} [P'_{G_n}(\text{AllLabel}_{(S,F)}) - P'_{G_n}((S, F))] \end{aligned}$$

and since all processors in the set  $ISO_{G_n}$  are isolated,

$$P'_{G_n}((S, F)) = p^{|ISO_{G_n} \cap F|} (1-p)^{|ISO_{G_n} \cap (U_n - F)|} P'_{G_n}(\text{AllLabel}_{(S,F)}).$$

Therefore,

$$\begin{aligned} & \sum_{(S,F) \in \text{Correct}_{G_n}(A)} P'_{G_n}(\text{AllLabel}_{(S,F)}) \\ & = \sum_{(S,F) \in \text{Correct}_{G_n}(A)} \frac{P'_{G_n}((S, F))}{p^{|ISO_{G_n} \cap F|} (1-p)^{|ISO_{G_n} \cap (U_n - F)|}} \\ & \geq \frac{\sum_{(S,F) \in \text{Correct}_{G_n}(A)} P'_{G_n}((S, F))}{[\max(p, 1-p)]^{|ISO_{G_n}|}} \end{aligned}$$

and thus

$$\begin{aligned} & P'_{G_n}(\text{NotCorrect}_{G_n}(A)) \\ & \geq \left( \frac{1}{[\max(p, 1-p)]^{|ISO_{G_n}|}} - 1 \right) \sum_{(S,F) \in \text{Correct}_{G_n}(A)} P'_{G_n}((S, F)) \end{aligned}$$



Therefore,

$$P'_{G_n}(\text{Correct}_{G_n}(A)) \leq \frac{[\max(p, 1-p)]^{|\text{ISO}_{G_n}|}}{1 - [\max(p, 1-p)]^{|\text{ISO}_{G_n}|}} \cdot P'_{G_n}(\text{Correct}_{G_n}(A))$$

$$\rightarrow 0$$

as  $n \rightarrow \infty$ . Thus, for any algorithm  $A$ ,  $\text{DiagProb}_{G_n}(A) \rightarrow 0$ , as well. Now, consider any probabilistic diagnosis algorithm  $A$ . Then,  $\forall P'_{G_n} \in \mathcal{P}_{G_n}$

$$\text{DiagProb}_{G_n}(A) \leq \sum_{(S,F) \in \Omega_{G_n}} P'_{G_n}((S, F)) \cdot p_{A,S}(F)$$

Consider the deterministic algorithm  $A'$  that for any syndrome  $S$  chooses fault set  $F$  such that  $\forall F' \subseteq U_n$

$$P'_{G_n}((S, F)) \geq P'_{G_n}((S, F')).$$

Then, if  $\mathcal{S}$  represents the set of all syndromes in  $G_n$

$$\begin{aligned} \text{DiagProb}_{G_n}(A) &\leq \sum_{(S,F) \in \Omega_{G_n}} P'_{G_n}((S, \text{Faulty}_{A'}(S))) \cdot p_{A,S}(F) \\ &= \sum_{S \in \mathcal{S}} \sum_{F \subseteq U_n} P'_{G_n}((S, \text{Faulty}_{A'}(S))) \cdot p_{A,S}(F) \\ &= \sum_{S \in \mathcal{S}} P'_{G_n}((S, \text{Faulty}_{A'}(S))) \sum_{F \subseteq U_n} p_{A,S}(F) \\ &= P'_{G_n}(\text{Correct}_{G_n}(A')) \\ &\rightarrow 0 \end{aligned}$$

A few comments concerning this result are in order. While the theorem implies only that under some behavior of the faulty processors, correct diagnosis with high probability is impossible to achieve, the result can be shown to hold for all “natural” faulty processor behaviors using virtually the same proof. The key to the proof lies in the fact that the isolated processors of the system can be relabeled in arbitrary ways without affecting the probability of any test outcomes in the system or the status of other processors. This holds as long as outcomes of tests performed by faulty processors do not depend on the status of these isolated processors. Thus, correct diagnosis with high probability cannot be achieved unless the faulty processors are, in some sense, clairvoyant.

## 7 Diagnosis in Regular Systems

The study of regular systems is important for several reasons. First, regular designs are more easily and efficiently implementable than irregular designs. Furthermore,

the majority of existing multiprocessor systems possess a regular structure. Finally, assuming the tests are conducted in a set of rounds, the maximum number of tests conducted by any processor is a measure of the overhead required to achieve fault tolerance. For a fixed total number of tests, regular systems require the minimum overhead using this measure. In this section, we examine the diagnosis problem for regular systems under our probabilistic model.

## 7.1 Upper Bound

In [18], it was shown that correct diagnosis can be achieved with probability approaching one in a class of systems, known as  $D_{1,c \log n}$  systems, for  $c > \frac{1}{\log \frac{1}{p}}$ . The systems from this class conduct  $cn \log n$  tests. In this section, we present a class of systems conducting  $cn \log n$  tests which contains the class given in [18] and for which Algorithm Majority achieves correct diagnosis with probability approaching one. This class of systems contains many useful systems, *e.g.* hypercubes, which are not contained in the  $D_{1,c \log n}$  class.

The systems studied in this section are those for which every processor in the system is tested by at least  $c \log_2 n$  other processors, for  $c$  sufficiently large. This includes regular systems with  $\Theta(n \log_2 n)$  tests, of sufficiently large degree. Theorem 4 shows that for any sequence of these systems, Algorithm Majority will produce correct diagnosis with probability approaching one. In order to prove this and subsequent results, the following corollary [1,8] to a theorem proved by Chernoff [5] is needed.

**Corollary 1** *Let  $Y$  be a binomial random variable with parameters  $n$  and  $p$ . Then*

$$P(Y \leq anp) = \sum_{i=0}^{\lfloor anp \rfloor} \binom{n}{i} p^i (1-p)^{n-i} \leq e^{-(1-a)^2 np/2}, \quad 0 < a \leq 1$$

$$P(Y \geq anp) = \sum_{i=\lceil anp \rceil}^n \binom{n}{i} p^i (1-p)^{n-i} \leq e^{n(1-ap) \log_e \frac{1-p}{1-ap} + anp \log_e \frac{1}{a}}, \quad a \geq 1$$

**Theorem 4** *Let  $c$  be any constant such that  $c > \left\{ \frac{\log_2 e}{2} \left[ \log_e \frac{1}{2(1-p_1)} + \log_e \frac{1}{2p_1} \right] \right\}^{-1}$ . If  $p < 1/2$ , then for any sequence of digraphs on  $n$  vertices having a tester set of size at least  $c \log_2 n$  for every processor, the probability of correct diagnosis for Algorithm Majority approaches one as  $n \rightarrow \infty$ .*

**Proof:** We need to show that for any sequence satisfying the theorem condition,  $\text{DiagProb}_{G_n}(\text{Majority}) \rightarrow 1$  as  $n \rightarrow \infty$ . Intuitively, the worst performance of

Algorithm Majority is obtained when faulty processors fail all fault-free processors and pass all faulty processors. Let  $P'_{G_n} \in \mathcal{P}_{G_n}$  be the refinement corresponding to this faulty processor behavior. Consider  $(S, F) \in \text{Correct}_{G_n}(\text{Majority})$  such that  $P'_{G_n}((S, F)) > 0$ . Let  $B$  be the basic event with  $(S, F) \in B$ . Then,  $\forall (S', F') \in B$ ,  $(S', F') \in \text{Correct}_{G_n}(\text{Majority})$ . Thus,  $\forall P''_{G_n} \in \mathcal{P}_{G_n}$ ,

$$P'_{G_n}(\text{Correct}_{G_n}(\text{Majority})) \leq P''_{G_n}(\text{Correct}_{G_n}(\text{Majority}))$$

and therefore

$$\text{DiagProb}_{G_n}(\text{Majority}) = P'_{G_n}(\text{Correct}_{G_n}(\text{Majority}))$$

Now, let  $X$  be a random variable representing the number of units whose tester set does not contain a majority of fault-free units. Clearly,

$$\begin{aligned} & \{(S, F) : X = 0 \text{ and } (S, F) \text{ is compatible}\} \\ & \subseteq \text{Correct}_{G_n}(\text{Majority}) \end{aligned}$$

Therefore,

$$\begin{aligned} & P'_{G_n}(\text{Correct}_{G_n}(\text{Majority})) \\ & \geq P'_{G_n}(\{(S, F) : X = 0\}) \\ & \geq 1 - P'_{G_n}(\{(S, F) : X > 0\}) \\ & \geq 1 - E[X] \end{aligned}$$

Now,  $X = \sum_{i=1}^n X_i$ , where

$$X_i = \begin{cases} 1 & \text{if } u_i \text{ is tested by at least} \\ & |\Gamma^{-1}(u_i)|/2 \text{ faulty units} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} E[X] &= \sum_{i=1}^n E[X_i] \\ &= \sum_{i=1}^n P'_{G_n}(\{(S, F) : X_i = 1\}) \\ &= \sum_{i=1}^n \sum_{j=\lceil \frac{|\Gamma^{-1}(u_i)|}{2} \rceil}^{|\Gamma^{-1}(u_i)|} \binom{|\Gamma^{-1}(u_i)|}{j} p^j (1-p)^{|\Gamma^{-1}(u_i)|-j} \end{aligned}$$

Now,  $\left\lceil \frac{|\Gamma^{-1}(u_i)|}{2} \right\rceil = \lceil ap|\Gamma^{-1}(u_i)| \rceil$ , where  $a = \frac{1}{2p} > 1$ , and thus, by Corollary 1,

$$E[X] \leq \sum_{i=1}^n e^{\frac{|\Gamma^{-1}(u_i)|}{2} [\log_e 2(1-p) + \log_e 2p]}$$

Since  $p < 1/2$ ,  $\log_e 2(1-p) + \log_e 2p < 0$  and so

$$\begin{aligned} E[X] &\leq n e^{-\frac{c \log_2 n}{2} [\log_e \frac{1}{2(1-p)} + \log_e \frac{1}{2p}]} \\ &= n^{1 - \frac{c \log_2 e}{2} [\log_e \frac{1}{2(1-p)} + \log_e \frac{1}{2p}]} \\ &\rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , since  $c > \left\{ \frac{\log_2 e}{2} \left[ \log_e \frac{1}{2(1-p_1)} + \log_e \frac{1}{2p_1} \right] \right\}^{-1}$ . ■

## 7.2 A Special Case – Hypercube Systems

In this section, we examine the consequences of Theorem 4 for hypercube systems. In a binary hypercube architecture, the constant  $c$  is equal to one. Hence, in order for the hypercube to be diagnosable with probability approaching one the probability of failure  $p$  must satisfy

$$\left\{ \frac{\log_2 e}{2} \left[ \log_e \frac{1}{2(1-p_1)} + \log_e \frac{1}{2p_1} \right] \right\}^{-1} < 1.$$

This implies that  $p$  must be less than approximately 0.067. This condition is likely to be satisfied in the majority of fault environments. The probability of failure can be higher in many of the other members of the hypercube family which have  $c > 1$ .

Most of the previous work in the diagnosis area has utilized a bounded-size fault set model where it is assumed that no more than  $t$  faults occur in the system. A system is said to be *t*-diagnosable if any combination of  $t$  faulty units in the system can be uniquely diagnosed. It is well known that a  $k$ -dimensional hypercube is  $k$ -diagnosable but not  $(k+1)$ -diagnosable. Since,  $k = \log_2 n$ , where  $n$  is the number of vertices of the cube, the assumptions of the bounded-size fault set model are satisfied only when the number of faults is less than or equal to the logarithm of the number of units. It is unlikely that this condition will be met in large systems. Under the probabilistic model, however, a number of faults that is linear in the number of units can be tolerated.

Table 3 illustrates the diagnosis performance difference between the two models on hypercube systems for probabilities of failure of 0.002 and 0.020. The fourth column of this table lists the expected number of faulty processors for the corresponding system and failure probability.  $P_k$  represents the probability that no more

$k$	$n$	$p$	Exp. # faulty	$P_k$	$P_{Maj}$
6	64	0.002	0.13	1.0000	1.0000
6	64	0.020	1.28	0.9997	0.9999
8	256	0.002	0.51	1.0000	1.0000
8	256	0.020	5.12	0.9258	1.0000
10	1024	0.002	2.05	1.0000	1.0000
10	1024	0.020	20.48	0.0079	1.0000
12	4096	0.002	8.19	0.9267	1.0000
12	4096	0.020	81.92	0.0000	1.0000
14	16384	0.002	32.77	0.0002	1.0000
14	16384	0.020	327.68	0.0000	1.0000
16	65536	0.002	131.07	0.0000	1.0000
16	65536	0.020	1310.72	0.0000	1.0000
20	1048576	0.002	2097.15	0.0000	1.0000
20	1048576	0.020	20971.52	0.0000	1.0000

Table 3: Diagnosis Probability on a  $k$ -dimensional,  $n$ -node Hypercube

than  $k$  units are faulty and  $P_{Maj}$  represents a lower bound on the probability of correct diagnosis for Algorithm Majority. Since the diagnosis algorithms proposed for the bounded-size fault set model can only guarantee correct diagnosis in a  $k$ -dimensional hypercube when the number of faults is less than or equal to  $k$ ,  $P_k$  is an estimate for the probability of correct diagnosis for those algorithms.

It can be seen from Table 3 that performance under the bounded-size fault set model degrades rapidly as the size of the hypercube increases. The probability of correct diagnosis for Algorithm Majority, however, is very nearly one for all the hypercubes studied, even when the probability of failure of a processor is as large as 0.02. Consider the case where  $k = 16$  and the probability of failure is 0.02. In this situation, the expected number of faults is greater than 1300 and yet Algorithm Majority still produces correct diagnosis with a probability that is very nearly one. Under the bounded-size fault set model, the number of faults is limited to 16 for this situation. When  $k = 16$ , the number of processors is 65,536. While this may seem large, a system containing this many processors, namely the Connection Machine [11], has been built.

### 7.3 Lower Bound

While hypercubes are an important class of system, systems with even fewer connections are expected to see increased use in future multiprocessor applications. We are therefore interested in determining a lower bound on the total number of tests necessary to achieve correct diagnosis with high probability. Such a lower bound was proven in [2] for regular systems. This result states that all diagnosis algorithms must have a probability of correct diagnosis that approaches zero in regular systems with  $o(n \log n)$  tests. This more general probability model contains the model utilized in this paper as a special case and hence this result holds for this model as well. Thus, for the important class of regular systems the algorithm given in [18] as well as Algorithm Majority are both optimal to within a constant factor. This result also demonstrates that the irregular structure of the tester digraphs studied in this paper is a crucial factor in making them amenable to diagnosis.

Of special interest due to their widespread use are multiprocessor systems which are regular and of fixed degree. Included in this class of systems are rings, torii, and hexagonal meshes. This somewhat pessimistic result implies that weaker forms of diagnosis must be considered for these systems.

## 8 Diagnosis using a Linear Number of Tests

It has been shown that Algorithm Majority can achieve correct diagnosis with probability approaching one in digraphs containing  $n\omega(n)$  edges, while all algorithms must have probability approaching zero of correct diagnosis in digraphs possessing  $o(n)$  edges. These results leave open the question of what can be achieved using  $cn$  edges, for some positive constant  $c$ . In this section, it is shown that with  $cn$  edges Algorithm Majority can achieve a probability of correct diagnosis that is a constant arbitrarily close to one. It is also shown that a constant probability less than one is the best that any algorithm can hope to achieve in this situation, meaning that Algorithm Majority is optimal for digraphs with a linear number of edges.

The following theorem characterizes the performance of Algorithm Majority on digraphs with a linear number of edges.

**Theorem 5** *Let  $\epsilon$  be any real number such that  $0 < \epsilon \leq 1$ . If  $p < 1/2$ ,  $\exists c > 0$  such that for all sufficiently large tester digraphs having at least  $c$  testers, the probability of correct diagnosis for Algorithm Majority is at least  $1 - \epsilon$ .*

**Proof:** We must show that, for any  $\epsilon$  with  $0 < \epsilon \leq 1$ ,  $\exists c > 0, n_0$  such that if  $G_n(U_n, E_n)$  is a sequence of tester digraphs with  $|T_{G_n}| \geq c$ , then  $\forall n \geq n_0$ ,

$\text{DiagProb}_{G_n}(\text{Majority}) \geq 1 - \epsilon$ . Let  $a = \frac{1}{2(1-p)} < 1$ . Then,  $\forall P'_{G_n} \in \mathcal{P}_{G_n}$ ,

$$\begin{aligned} P'_{G_n}(\text{Correct}_{G_n}(\text{Majority})) &\geq 1 - \sum_{i=0}^{\lfloor \frac{|T_{G_n}|}{2} \rfloor} \binom{|T_{G_n}|}{i} (1-p)^i p^{|T_{G_n}|-i} \\ &\geq 1 - \left[ e^{-(1-a)^2/2} \right]^{(1-p)c} \end{aligned}$$

by Corollary 1. Now, if  $c$  is chosen such that

$$c \geq \frac{-2 \ln \epsilon}{(1-a)^2(1-p)}$$

then

$$\begin{aligned} P'_{G_n}(\text{Correct}_{G_n}(\text{Majority})) &\geq 1 - e^{\ln \epsilon} \\ &= 1 - \epsilon \end{aligned}$$

■

Thus, Algorithm Majority can achieve correct diagnosis with probability arbitrarily close to one in sequences of digraphs having a linear number of edges. The following theorem shows that all diagnosis algorithms must have a probability of correct diagnosis that is bounded away from one by a positive constant in this situation.

**Theorem 6** *Let  $c$  be any positive constant. If  $0 < p < 1$ ,  $\exists \epsilon > 0$  such that for any probabilistic or deterministic diagnosis algorithm  $A$  and any sufficiently large digraph on  $n$  vertices having no more than  $cn$  edges, the probability of correct diagnosis for Algorithm  $A$  is no greater than  $1 - \epsilon$ .*

**Proof:** We must show that, for any  $c > 0$ ,  $\exists \epsilon > 0$ ,  $n_0$  such that if  $G_n(U_n, E_n)$  is a sequence of digraphs with  $|E_n| < cn$ , then  $\forall n \geq n_0$ ,  $\text{DiagProb}_{G_n}(A) \leq 1 - \epsilon$ . Let  $P'_{G_n} \in \mathcal{P}_{G_n}$  be such that faulty processors fail all other processors. Now, let  $u_{\min_{G_n}} \in U_n$  be any vertex of  $G_n$  such that  $\forall u \in U_n$ ,  $|N(u_{\min_{G_n}})| \leq |N(u)|$ . Thus,  $u_{\min_{G_n}}$  is a processor having minimum size neighbor set in  $G_n$ . Clearly,  $|N(u_{\min_{G_n}})| \leq 2c$ . Now, let

$$\text{Surr}_{G_n} = \{(S, F) : N(u_{\min_{G_n}}) \cap (U - F) = \emptyset\}.$$

Thus,  $\text{Surr}_{G_n}$  represents the set of syndrome, fault set pairs in which  $u_{\min_{G_n}}$  has only faulty processors in its neighbor set. Then,

$$\begin{aligned} P'_{G_n}(\text{Surr}_{G_n}) &= p^{|N(u_{\min_{G_n}})|} \\ &\geq p^{2c} \end{aligned}$$







Now, if  $Surr_{G_n}^c$  represents the complement of  $Surr_{G_n}$ , then for any deterministic diagnosis algorithm  $A$

$$\begin{aligned}
P'_{G_n}(\text{Correct}_{G_n}(A) \cap Surr_{G_n}) &= 1 - P'_{G_n}(\text{NotCorrect}_{G_n}(A) \cup Surr_{G_n}^c) \\
&\geq 1 - P'_{G_n}(\text{NotCorrect}_{G_n}(A)) - P'_{G_n}(Surr_{G_n}^c) \\
&\geq p^{2c} - P'_{G_n}(\text{NotCorrect}_{G_n}(A))
\end{aligned}$$

Now, consider any  $(S, F) \in \text{Correct}_{G_n}(A) \cap Surr_{G_n}$  and  $(S, F')$  such that if  $u_{\min_{G_n}} \in F$ , then  $F' = F - \{u_{\min_{G_n}}\}$ , otherwise  $F' = F \cup \{u_{\min_{G_n}}\}$ . Thus,  $(S, F')$  is identical to  $(S, F)$  except for the label on  $u_{\min_{G_n}}$ . Since faulty processors fail all other processors, each edge incident on  $u_{\min_{G_n}}$  will be a one regardless of the state of  $u_{\min_{G_n}}$ . Thus,

$$P'_{G_n}((S, F')) \geq \min\left(\frac{p}{1-p}, \frac{1-p}{p}\right) P'_{G_n}((S, F))$$

and therefore,

$$\begin{aligned}
P'_{G_n}(\text{NotCorrect}_{G_n}(A)) &\geq \min\left(\frac{p}{1-p}, \frac{1-p}{p}\right) P'_{G_n}(\text{Correct}_{G_n}(A) \cap Surr_{G_n}) \\
&\geq \min\left(\frac{p}{1-p}, \frac{1-p}{p}\right) [p^{2c} - P'_{G_n}(\text{NotCorrect}_{G_n}(A))]
\end{aligned}$$

or

$$P'_{G_n}(\text{NotCorrect}_{G_n}(A)) \left[1 + \min\left(\frac{p}{1-p}, \frac{1-p}{p}\right)\right] \geq \min\left(\frac{p}{1-p}, \frac{1-p}{p}\right) p^{2c}$$

and

$$P'_{G_n}(\text{NotCorrect}_{G_n}(A)) \geq \frac{\min\left(\frac{p}{1-p}, \frac{1-p}{p}\right) p^{2c}}{1 + \min\left(\frac{p}{1-p}, \frac{1-p}{p}\right)} = \epsilon > 0$$

so long as  $0 < p < 1$ . Now, consider any probabilistic diagnosis algorithm  $A$ . Then,  $\forall P'_{G_n} \in \mathcal{P}_{G_n}$

$$\text{DiagProb}_{G_n}(A) \leq \sum_{(S,F) \in \Omega_{G_n}} P'_{G_n}((S, F)) \cdot p_{A,S}(F)$$

Consider the deterministic algorithm  $A'$  that for any syndrome  $S$  chooses fault set  $F$  such that  $\forall F' \subseteq U_n$

$$P'_{G_n}((S, F)) \geq P'_{G_n}((S, F')).$$

Then, if  $S$  represents the set of all syndromes in  $G_n$

$$\begin{aligned}
\text{DiagProb}_{G_n}(A) &\leq \sum_{(S,F) \in \Omega_{G_n}} P'_{G_n}((S, \text{Faulty}_{A'}(S))) \cdot p_{A,S}(F) \\
&= \sum_{S \in S} \sum_{F \subseteq U_n} P'_{G_n}((S, \text{Faulty}_{A'}(S))) \cdot p_{A,S}(F) \\
&= \sum_{S \in S} P'_{G_n}((S, \text{Faulty}_{A'}(S))) \sum_{F \subseteq U_n} p_{A,S}(F) \\
&= P'_{G_n}(\text{Correct}_{G_n}(A')) \\
&\leq 1 - \epsilon
\end{aligned}$$

■

## 9 Conclusion

A probabilistic fault model for multiprocessor systems in which processors are faulty with probability  $p$  has been studied. It has been shown that correct diagnosis can be achieved with probability approaching one in a class of systems that conducts slightly more than a linear number of tests using a simple and efficient diagnosis algorithm. This algorithm also produces a probability of correct diagnosis that is arbitrarily close to one in systems conducting a linear number of tests. It has also been shown that this result is the best possible, *i.e.* in systems for which the number of tests grows more slowly than  $n$ , all diagnosis algorithms, whether they be deterministic or probabilistic in nature, must have a probability approaching zero of correct diagnosis and furthermore, in systems containing a linear number of tests, all algorithms have a probability of correct diagnosis bounded above by some constant less than one. In addition, this algorithm has been shown to work with high probability on a class of regular systems which contains hypercubes as a special case. This result is nearly the best possible as it is known that no algorithm can achieve diagnosis with high probability on regular systems of degree  $o(\log n)$ .

## References

- [1] Angluin, D., and L. Valiant, "Fast Probabilistic Algorithms for Hamiltonian Circuits and Matchings," *Journal of Computer and System Sciences*, vol. 18, pp. 155–193, April 1979.
- [2] Blough, D. M., *Fault Detection and Diagnosis in Multiprocessor Systems*, Ph. D. Dissertation, The Johns Hopkins University, 1988.

- [3] Blough, D. M., G. F. Sullivan, and G. M. Masson, "Almost Certain Diagnosis for Intermittently Faulty Systems," *Digest of the 18th International Symposium on Fault-Tolerant Computing*, IEEE Computer Society Press, pp. 260–265, 1988.
- [4] Blount, M., "Probabilistic Treatment of Diagnosis in Digital Systems," *Digest of the 7th International Symposium on Fault-Tolerant Computing*, IEEE Computer Society Press, pp. 72–77, 1977.
- [5] Chernoff, H., "A Measure of Asymptotic Efficiency for Tests of a Hypothesis Based on the Sum of Observations," *Annals of Mathematical Statistics*, vol. 23, pp. 493–507, 1952.
- [6] Dahbura, A. T., "An Efficient Algorithm for Identifying the Most Likely Fault Set in a Probabilistically Diagnosable System," *IEEE Transactions on Computers*, vol. C-35, pp. 354–356, April 1986.
- [7] Dahbura, A. T., K. Sabnani, and L. King, "The Comparison Approach to Multiprocessor Fault Diagnosis," *IEEE Transactions on Computers*, vol. C-36, pp. 373–378, March 1987.
- [8] Erdős, P., and P. Spencer, *Probabilistic Methods in Combinatorics*, New York: Academic Press, 1974.
- [9] Feller, W., *An Introduction to Probability Theory and its Applications*, New York: John Wiley & Sons, Inc., 1968.
- [10] Hakimi, S. L., and A. T. Amin, "Characterization of the Connection Assignment of Diagnosable Systems," *IEEE Transactions on Computers*, vol. C-23, pp. 86–88, January 1974.
- [11] Hillis, W. Daniel, *The Connection Machine*, Cambridge: The MIT Press, 1985.
- [12] Kuhl, J., and S. Reddy, "Distributed Fault-Tolerance for Large Multiprocessor Systems," *Proceedings 7th Annual Symposium on Computer Architecture*, pp. 23–30, May 1980.
- [13] Maheshwari, S. N., and S. L. Hakimi, "On Models for Diagnosable Systems and Probabilistic Fault Diagnosis," *IEEE Transactions on Computers*, vol. C-25, pp. 228–236, March 1976.
- [14] Mallela, S., and G. M. Masson, "Diagnosis Without Repair for Hybrid Fault Situations," *IEEE Transactions on Computers*, vol. C-29, pp. 461–470, June 1980.

- [15] Pelc, A., "Undirected Graph Models for System-Level Fault Diagnosis," Département d'Informatique Research Report #10, Université du Québec à Hull, 1988.
- [16] Preparata, F., G. Metze, and R. Chien, "On the Connection Assignment Problem of Diagnosable Systems," *IEEE Transactions on Electronic Computers*, vol. EC-16, pp. 848-854, December 1967.
- [17] Rangarajan, S., and D. Fussell, "A Probabilistic Method for Fault Diagnosis of Multiprocessor Systems," *Digest of the 18th International Fault-Tolerant Computing Symposium*, pp. 278-283, IEEE Computer Society Press, June 1988.
- [18] Scheinerman, E., "Almost Sure Fault Tolerance in Random Graphs," *SIAM Journal on Computing*, vol. 16, pp. 1124-1134, December 1987.
- [19] Somani, A., V. Agarwal, and D. Avis, "A Generalized Theory for System-Level Diagnosis," *IEEE Transactions on Computers*, vol. C-36, pp. 538-546, May 1987.
- [20] Sullivan, G. F., "System-Level Fault-Diagnosability in Probabilistic and Weighted Models," *Digest of the 17th International Symposium on Fault-Tolerant Computing*, IEEE Computer Society Press, pp. 190-195, 1987.
- [21] Yang, C.-L., and Masson, G. M., "A Fault Identification Algorithm for  $t_i$ -diagnosable Systems," *IEEE Transactions On Computers*, vol. C-35, pp. 503-510, June 1986.

**NASA  
FORMAL  
REPORT**

**FFNo 665 Aug 65**