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AN EIGENVALUE ANALYSIS OF FINITE-DIFFERENCE APPROXIMATIONS FOR HYPERBOLIC IBVPs[1]

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SUMMARY

The eigenvalue spectrum associated with a linear finite-difference approximation plays a crucial role in the stability analysis and in the actual computational performance of the discrete approximation. We investigate the eigenvalue spectrum associated with the Lax-Wendroff scheme applied to a model hyperbolic equation. For an initial-boundary-value problem (IBVP) on a finite domain, the eigenvalue or normal mode analysis is analytically intractable. A study of auxiliary problems (Dirichlet and quarter-plane) leads to asymptotic estimates of the eigenvalue spectrum and to an identification of individual modes as either *benign* or *unstable*. The asymptotic analysis establishes an intuitive as well as quantitative connection between the algebraic tests in the theory of Gustafsson, Kreiss, and Sundström and Lax-Richtmyer L_2 stability on a finite domain.

1. INTRODUCTION

A classical method for carrying out a stability analysis of a discrete hyperbolic IBVP is the normal mode analysis of Gustafsson, Kreiss, and Sundström (GKS) [1]. The GKS theory avoids the analytical intractability of the finite-domain normal mode analysis by analyzing *related* quarter-plane problems. On the other hand, when one performs numerical experiments to verify stability and/or accuracy predictions, the computations are on a finite domain and one typically uses the discrete L_2 norm and not the GKS norm used to prove stability. Thus in practice, we have the dichotomy of *analyzing* quarter-plane problems with GKS norms and *computing* on finite domains with L_2 norms.

The goal of this paper is twofold. First we present asymptotic limits for the normal modes of the discrete (Lax-Wendroff) IBVP on a finite domain. These limits lead to a delineation of the normal modes of the finite-domain problem into three classes. Next we use the asymptotic estimates to make a direct algebraic connection between the normal modes of the GKS quarter-plane analysis and the classes of normal modes of the finite-domain problem. This leads to an interpretation of (unstable) GKS modes which is readily understandable in terms of the Lax-Richtmyer stability in the L_2 norm. In this paper we give only a brief outline of our analysis. A detailed exposition is given in [2].

2. IBVP FOR A MODEL HYPERBOLIC EQUATION

We consider the scalar hyperbolic equation

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x}, \quad 0 \le x \le L, \quad t > 0 \tag{2.1}$$

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where $u = u(x,t)$ and c is a real constant. For a well-posed IBVP on a finite domain one must specify initial data, $u(x,0) = f(x)$, $0 \leq x \leq L$, and an *analytical* boundary condition at $x = L$,

$$u(L,t) = g(t) \quad \text{for} \quad c > 0. \tag{2.2}$$

3. A PROTOTYPE FINITE-DIFFERENCE APPROXIMATION

To obtain a difference approximation of the model equation (2.1) a mesh is introduced in (x,t) space with increments Δx and Δt and indexing defined by $x = j\Delta x$ and $t = n\Delta t$. The spatial domain $0 \leq x \leq L$ is divided into J equally spaced increments, i.e., $J\Delta x = L$. As a prototype (explicit) finite-difference approximation for the model equation (2.1), we consider the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n + \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \tag{3.1}$$

where $\nu = c\Delta t/\Delta x$ is defined to be the Courant number. In our analysis the analytical boundary condition (2.2) for the difference approximation is assumed to be homogeneous, i.e.,

$$u_J^n = 0. \tag{3.2}$$

If we apply the Lax-Wendroff scheme at the outflow boundary ($j = 0$), the computational stencil protrudes one point to the left of the boundary. It is clear that an additional *numerical boundary scheme* (NBS) is required to calculate u_0^{n+1}, i.e., the solution on the outflow boundary at time level $n+1$. As a prototype NBS we choose the spatially one-sided scheme:

$$u_0^{n+1} = u_0^n + \nu[-\alpha u_2^n + (1+2\alpha)u_1^n - (1+\alpha)u_0^n] \tag{3.3}$$

where α is a (real) parameter. If $\alpha = 0$, then (3.3) is simply

$$u_0^{n+1} = u_0^n + \nu(u_1^n - u_0^n). \tag{3.4}$$

The Lax-Wendroff scheme (3.1) together with the analytical boundary condition (3.2) and the NBS (3.3) is called a *discrete* IBVP. For our purposes it is convenient to rewrite the NBS (3.3) as an equivalent space extrapolation formula [2]:

$$h(E)u_{-1}^n = 0 \quad \text{where} \quad h(E) = (E-1)^2[2\alpha E - (1-\nu)]. \tag{3.5}$$

The shift operator E is defined by $Eu_j = u_{j+1}$ and $h(E)$ is a polynomial in E.

4. LAX-RICHTMYER STABILITY OF A DISCRETE IVP OR IBVP

For the stability analysis of a discrete initial value problem (IVP) or IBVP with requisite homogeneous boundary conditions, it is appropriate to write the discrete approximation in vector-matrix form:

$$\mathbf{u}^{n+1} = \mathbf{C}\mathbf{u}^n. \tag{4.1}$$

The Lax-Wendroff scheme (3.1) with the analytical boundary condition (3.2) and NBS (3.3) can be written in vector-matrix form (4.1) where

$$\mathbf{u}^n = \begin{bmatrix} u_0^n \\ u_1^n \\ \cdot \\ \cdot \\ \cdot \\ u_{J-2}^n \\ u_{J-1}^n \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} s & v & w & & & & \\ p & q & r & & & O & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & O & & & p & q & r \\ & & & & & p & q \end{bmatrix} \tag{4.2a,b}$$

and

$$p = -\nu(1-\nu)/2, \quad q = 1-\nu^2, \quad r = \nu(1+\nu)/2 \tag{4.3a}$$

$$s = 1-\nu(1+\alpha), \quad v = \nu(1+2\alpha), \quad w = -\nu\alpha. \tag{4.3b}$$

Here the matrix size is $J \times J$.

The determination of the *eigensolutions* of the first-order system (4.1) is sometimes called the *normal mode* analysis. If $\mathbf{C}$ has a complete set of eigenvectors, then the general solution of (4.1) can be written as

$$\mathbf{u}^n = \sum_{\ell=0}^{J-1} \alpha_\ell z_\ell^n \boldsymbol{\phi}_\ell \tag{4.4}$$

where z_ℓ and $\boldsymbol{\phi}_\ell$ denote the ℓth eigenvalue and eigenvector of the matrix $\mathbf{C}$ and the α_ℓ's are complex constants determined from a specified initial vector $\mathbf{u}^0$. Thus the eigensolutions are the *normal modes* and it is obvious from (4.4) that they act independently.

A discrete IVP or IBVP represented by (4.1) is Lax-Richtmyer stable if there exists a constant $K \geq 1$ such that for any initial condition $\mathbf{u}^0$

$$\|\mathbf{u}^n\| \leq K\|\mathbf{u}^0\| \tag{4.5}$$

for all $n \geq 0$, $0 \leq n\Delta t \leq T$ with T fixed and $\Delta t/\Delta x$ fixed. A necessary condition for Lax-Richtmyer stability is that there exists a nonnegative constant w such that

$$\rho(C) \leq 1 + \frac{w}{J} \tag{4.6}$$

for all $n \geq 0$, $0 \leq n\Delta t \leq T$ with T fixed and $\Delta t/\Delta x$ fixed. Here $\rho(\mathbf{C})$ denotes the spectral radius of $\mathbf{C}$. Inequality (4.6) is referred to as the *spectral radius condition.*

5. EIGENVALUE SPECTRUM OF A DISCRETE IVP

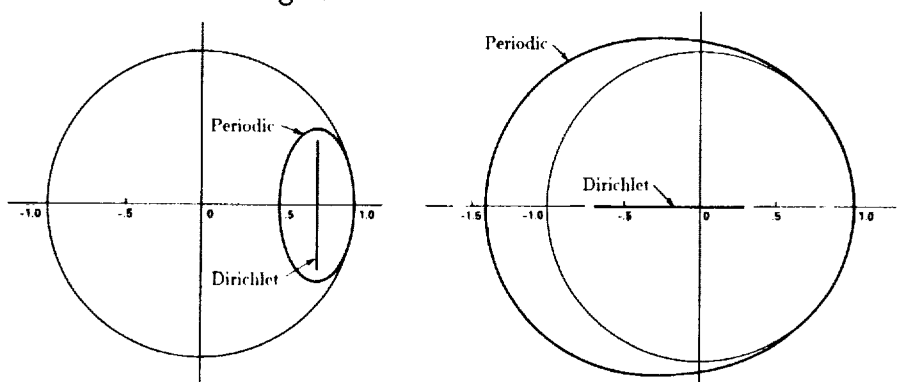
A necessary condition for the stability of a discrete IBVP is the stability of the corresponding pure IVP or Cauchy problem. In this section we review the eigenvalue spectrum of the IVP for the Lax-Wendroff scheme. The solution of the discrete IVP is assumed to be spatially periodic with period $L = J\Delta x$, and hence

$$u_j^n = u_{j+J}^n. \tag{5.1}$$

Consequently, the Lax-Wendroff scheme can be written in vector-matrix form (4.1) where $\mathbf{C}$ is a $J \times J$ circulant matrix and the eigenvalues z_ℓ are given analytically by

$$z_\ell = 1 - 2\nu^2 \sin^2(\theta_\ell/2) + i\nu \sin\theta_\ell, \quad \ell = 0, 1, \cdots, J-1 \tag{5.2}$$

where $\theta_\ell = 2\ell\pi/J$ and $\nu = c\Delta t/\Delta x$ is the Courant number. The eigenvalue locus given by (5.2) is an ellipse in the complex z-plane. (The eigenvalue *locus* is defined to be a curve through the eigenvalues in the complex z-plane.) The eigenvalue loci for $\nu = 0.5$ and $\nu = 1.1$ are shown in Fig. 5.1.



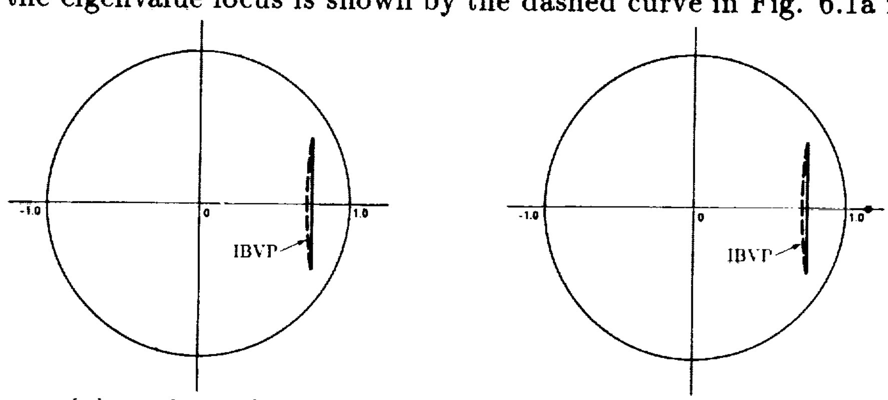
(a) z–plane $(\nu = 0.5)$ (b) z–plane $(\nu = 1.1)$

Fig. 5.1. Eigenvalue locus for periodic and Dirichlet boundary conditions.

The ellipse is contained within the unit circle for $|\nu| < 1$ (Fig. 5.1a) and if $|\nu| = 1$, the locus is the unit circle. If $|\nu| > 1$, the ellipse contains the unit circle (Fig. 5.1b). Since a circulant matrix is normal, the spectral radius condition (4.6) is necessary and sufficient for stability. Consequently, as is well-known, the Lax-Wendroff scheme is stable for the IVP (i.e., Cauchy stable) in the L_2 norm for $|\nu| \leq 1$.

6. EIGENVALUE SPECTRUM OF A DISCRETE IBVP

Before we present asymptotic limits for the eigenvalues of the IBVP matrix (4.2b) it is advantageous to examine the eigenvalue spectrum by computing the eigenvalues numerically. As an example if we choose $\alpha = 0$ in (3.3) we obtain the NBS (3.4). A sketch of the eigenvalue locus is shown by the dashed curve in Fig. 6.1a for $\nu = 0.5$.



(a) z–plane $(\alpha = 0)$ (b) z–plane $(\alpha = -1)$

Fig. 6.1. Dashed curve is eigenvalue locus of the matrix (4.2b) for ν=0.5.

If we increase the number of spatial increments J one finds that the eigenvalue locus approaches the solid vertical line for increasing J.

As a second numerical example, we consider $\alpha = -1$ in (3.3). This NBS leads to an unstable discrete IBVP. The eigenvalue locus is shown in Fig. 6.1b. The eigenvalue locus is qualitatively similar to the locus shown in Fig. 6.1a except for a single eigenvalue outside the unit circle shown by the solid symbol in the figure. To plotting accuracy the eigenvalue indicated by the solid symbol appears fixed for increasing values of J. The eigenvalue locus indicated by the dashed curve approaches the solid vertical line for increasing values of J. If one does a GKS quarter-plane analysis, one finds that the lone eigenvalue outside the unit circle is a GKS eigenvalue.

7. AN AUXILIARY DIRICHLET PROBLEM

The GKS stability analysis involves three auxiliary problems: the Cauchy problem and the left- and right-quarter plane problems. In this section we consider a fourth auxiliary problem which we call the Dirichlet problem on a finite domain. The importance of the Dirichlet problem accrues from the fact that, with the exception of isolated eigenvalues detected by the GKS quarter-plane analysis, the eigenvalue spectrum of a discrete IBVP is a perturbation of the eigenvalue spectrum of the auxiliary Dirichlet problem.

The auxiliary Dirichlet problem is constructed by equating to zero any grid function value u_j^n which is required by the interior difference approximation but falls outside the computational domain $0 \le x \le L$. Hence the auxiliary Dirichlet problem for the Lax-Wendroff scheme can be written in the vector-matrix form (4.1) where the matrix operator $\mathbf{C}$ is a $J \times J$ tridiagonal matrix with elements p, q, r. The eigenvalues of $\mathbf{C}$ can be determined analytically:

$$z_\ell = (1-\nu^2) + i\nu\sqrt{1-\nu^2}\ \cos\theta_\ell, \qquad \ell = 1, 2, \cdots, J \tag{7.1}$$

where $\theta_\ell = \ell\pi/(J+1)$. If $|\nu| < 1$, the eigenvalues are complex and the eigenvalue locus is a vertical line centered at the point $(1-\nu^2, 0)$ in the complex z-plane. If $|\nu| = 1$ all the eigenvalues degenerate to the single point $z_\ell = 0$. For $|\nu| > 1$ the eigenvalues are real. The eigenvalue spectra of the pure IVP (periodic boundary conditions) and the auxiliary Dirichlet problem are compared in Fig. 5.1. By some elementary calculations [2] one finds the rather remarkable result that the eigenvalue locus of the auxiliary Dirichlet problem is simply a straight line segment joining the foci of the ellipse which is the eigenvalue locus of the IVP.

8. NORMAL MODE ANALYSIS

In this section the relevant formulas for the normal mode analysis of the finite-domain problem and the quarter-plane problem are summarized. A detailed analysis is given in [2].

8.1 Finite-Domain Normal-Mode Analysis – Summary. An eigensolution or normal mode of the finite-domain IBVP is determined by looking for a solution of the form

$$u_j^n = z^n \phi_j \tag{8.1}$$

which satisfies the Lax-Wendroff scheme (3.1) with the analytical boundary condition (3.2) and the NBS (3.3) written as an extrapolation formula (3.5).

The eigenvalue z is given by

$$z = 1 + \frac{\nu}{2}(\kappa - \frac{1}{\kappa}) + \frac{\nu^2}{2}(\kappa - 2 + \frac{1}{\kappa}) \tag{8.2}$$

and the components ϕ_j of the eigenvector $\boldsymbol{\phi}$ are

$$\phi_j = a[\kappa^j - (-\kappa^2/\zeta)^J(-\zeta/\kappa)^j] \tag{8.3}$$

where $\kappa = \sqrt{\zeta}\hat{\kappa}$ and $\hat{\kappa}$ is a root of the characteristic equation

$$h(\sqrt{\zeta}\hat{\kappa}) - (-\hat{\kappa}^2)^{J+1}h(-\sqrt{\zeta}/\hat{\kappa}) = 0. \tag{8.4}$$

The parameter ζ, which is positive for $-1 < \nu < 1$, is defined by

$$\zeta = \frac{1-\nu}{1+\nu}. \tag{8.5}$$

The coefficient a on the right-hand side of (8.3) is an arbitrary constant. The polynomial $h(\sqrt{\zeta}\hat{\kappa})$ depends solely on the NBS (3.5), i.e., $h(\sqrt{\zeta}\hat{\kappa})$ is the polynomial associated with the NBS written as an extrapolation formula. If one could solve for the roots of the characteristic equation (8.4), then the eigenvalues z and the eigenvectors $\boldsymbol{\phi}$ would follow directly from (8.2) and (8.3). The normal mode analysis on a finite domain is, in general, analytically intractable because one cannot solve for the roots of the characteristic equation (8.4).

For the auxiliary Dirichlet problem, the polynomial $h(\sqrt{\zeta}\hat{\kappa})$ is unity and (8.4) reduces to

$$(-\hat{\kappa}^2)^{J+1} = 1. \tag{8.6}$$

One can solve (8.6) by using the roots of unity formula and the normal modes can be found analytically.

8.2 Quarter-Plane Normal-Mode Analysis – Summary. For the right quarter-plane problem one also looks for a solution of the form (8.1) which satisfies the Lax-Wendroff scheme (3.1) and the NBS (3.5). The details of the GKS normal mode analysis are given in [2, Appendix A]. The eigenvalue z is given by (8.2) and the components ϕ_j of the eigenvector $\boldsymbol{\phi}$ are

$$\phi_j = a\kappa^j \tag{8.7}$$

where $\kappa = \sqrt{\zeta}\hat{\kappa}$ and $\hat{\kappa}$ is a root of the quarter-plane characteristic equation

$$h(\sqrt{\zeta}\hat{\kappa}) = 0. \tag{8.8}$$

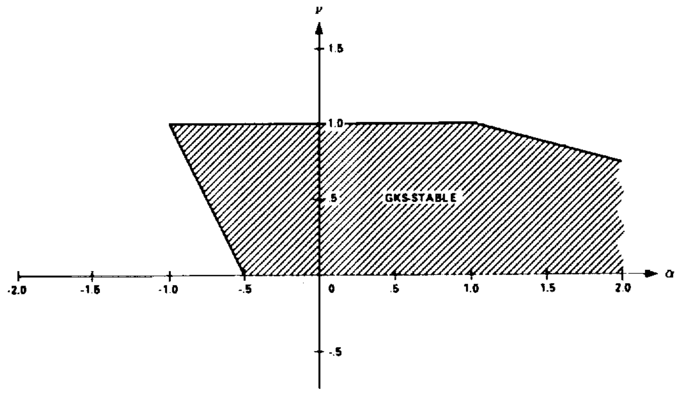


Fig. 8.1. GKS-stability region for Lax-Wendroff (3.1) with NBS (3.3).

The (α, ν) parameter space for which the discrete IBVP is GKS stable is shown by the cross-hatched region of Fig. 8.1.

9. ASYMPTOTIC ROOTS OF CHARACTERISTIC EQUATION

There exists a small class of discrete IBVP's which are sometimes called *borderline* cases. Borderline cases are unstable according to the GKS theory but they may be Lax-Richtmyer stable or unstable in the L_2 norm on a finite domain. Borderline approximations can be characterized by the presence of a *stationary* mode for the finite-domain problem. A stationary mode is defined to have the property that a $\hat{\kappa}$ root of the characteristic equation (8.4) is independent of J, i.e., $\hat{\kappa}$ remains fixed in the complex plane as J increases. It is important to note that the detection of a stationary mode requires no asymptotic (large J) analysis because there is no J dependence.

Although there can be stationary modes which are independent of J, almost all roots of the characteristic equation (8.4) depend upon J and we write $\hat{\kappa} = \hat{\kappa}(J)$. One can show that there is no loss in generality in assuming $|\hat{\kappa}| < 1$ and we write

$$|\hat{\kappa}| = |\hat{\kappa}(J)| = 1 - \epsilon, \quad 0 < \epsilon(J) < 1 \tag{9.1}$$

where

$$\text{either} \quad \epsilon(J) \geq \delta > 0 \quad \text{or} \quad \epsilon(J) \to 0 \qquad \text{as} \quad J \to \infty. \tag{9.2a,b}$$

In particular we are interested in the conditions under which the characteristic equation (8.4) reduces to the quarter-plane characteristic equation (8.8) in the limit $J \to \infty$. Obviously this depends on the asymptotic behavior of $|\hat{\kappa}(J)|^J$. There are only two possibilities, either

$$I: \quad \lim_{J\to\infty} |\hat{\kappa}(J)|^J = \text{constant} > 0, \qquad \text{or} \quad II: \quad \lim_{J\to\infty} |\hat{\kappa}(J)|^J = 0. \tag{9.3a,b}$$

In case II it is obvious that (8.4) reduces to (8.8) as $J \to \infty$. In case I one can show that the roots of (8.4) asymptotically approach the roots of the auxiliary Dirichlet problem (8.6) as $J \to \infty$. The details of the asymptotic estimates are in [2].

10. CLASSIFICATION OF NORMAL MODES

The normal modes of a discrete IBVP can be divided into three classes according to their asymptotic behavior as $J \to \infty$. We associate the following nomenclature with the three classes:

$$\begin{aligned} &I. \text{ Dirichlet-}\textit{like} \text{ modes} \\ &II. \text{ Quarter-plane-}\textit{like} \text{ modes} \\ &III. \text{ Stationary modes.} \end{aligned} \tag{10.1}$$

The adjective *like* is used to imply that one can identify for finite J a particular mode that becomes either a Dirichlet mode or quarter-plane mode in the limit $J \to \infty$.

For a given difference approximation almost all the normal modes are in class I. These modes have a generic eigenvalue spectrum which is easy to describe; to wit, for a given Courant number ν and finite J the spectrum is simply a perturbation of the Dirichlet spectrum. A typical generic case is shown in Fig. 6.1a. The solid vertical line is the Dirichlet locus. The dotted line slightly to the left is the eigenvalue locus of the class I modes of the difference approximation to the IBVP. One can show [2] that the dotted eigenvalue locus approaches the Dirichlet locus at least as fast as $O(1/J)$. Hence as the mesh is refined, i.e., $J \to \infty$, the dotted locus collapses onto the Dirichlet locus. Hence the terminology Dirichlet-*like* modes.

The class I modes are always *benign* in the sense that they do not introduce unstable modes into a difference approximation which is Cauchy stable. Only modes of class II and III can introduce *unstable* modes into a difference approximation which is Cauchy stable. If modes of class II and III exist, they are created by the NBS.

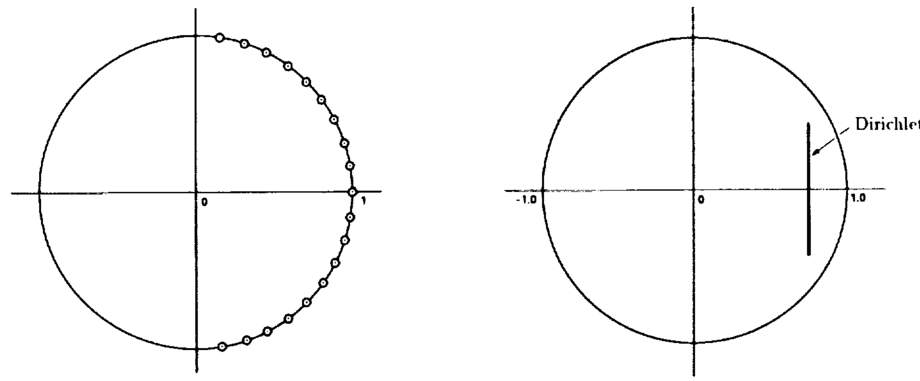
The modes in class II are related to the GKS stability theory in the sense that they become quarter-plane modes as $J \to \infty$, i.e., as the the mesh is refined. Finally, the stationary modes which constitute class III are common to both the finite-domain problem and the quarter-plane problems.

11. SKETCHES OF ROOT AND EIGENVALUE DISTRIBUTIONS

In this section we give a pictorial description of the roots of the characteristic equation (8.4) and the corresponding eigenvalue spectra associated with each of the three classes (10.1). The $\hat{\kappa}$ roots plotted in the examples were computed numerically from the characteristic equation (8.4) and the corresponding eigenvalues were computed using (8.2).

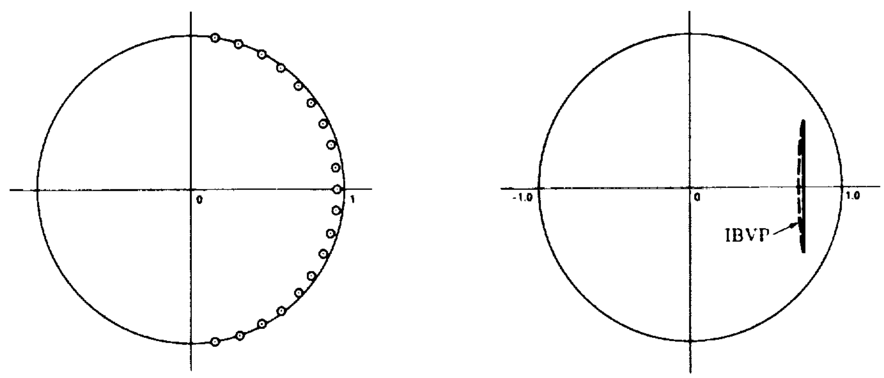
The roots for the characteristic equation (8.6) of the auxiliary Dirichlet problem are plotted in Fig. 11.1a for $J = 19$. The corresponding eigenvalue locus is the vertical line shown in Fig. 11.1b. In this figure and the figures to follow we plot the eigenvalue locus rather than individual eigenvalues because of the small size of the figures.

11.1 Dirichlet-*like* modes. The roots of the characteristic equation (8.4) which correspond to modes in class I have a generic root locus in the complex $\hat{\kappa}$-plane which is simply a perturbation (inside the unit circle) of the Dirichlet root locus which is on the unit circle. As an example we consider a stable case from the shaded region of Fig. 8.1 by choosing parameter values $\alpha = 0$ and $\nu = 0.5$. The roots of (8.4) for $J = 19$ are plotted in Fig. 11.2a.



(a) $\hat{\kappa}$–plane (b) z–plane

Fig. 11.1. Roots of characteristic polynomial (8.6) and eigenvalue locus for $\nu = 0.5$.



(a) $\hat{\kappa}$–plane $(\alpha = 0)$ (b) z–plane $(\alpha = 0)$

Fig. 11.2. Roots of characteristic polynomial (8.4) and eigenvalue locus for $\nu = 0.5$.

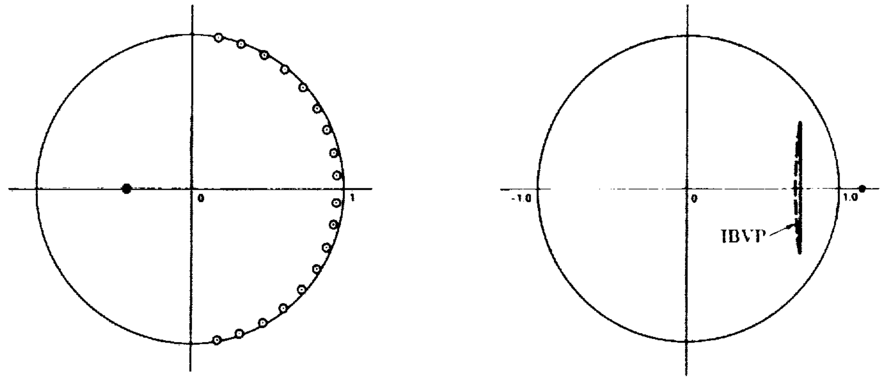
The eigenvalue locus is indicated by the dashed curve of Fig. 11.2b. The solid vertical line is the eigenvalue locus of the auxiliary Dirichlet problem. As J increases the root locus approaches the unit circle and the eigenvalue locus moves toward the Dirichlet locus at least as fast as $O(1/J)$.

11.2 Quarter-plane-*like* modes. The previous example had only class I modes, i.e., Dirichlet-*like* modes, while the examples of this section have both class I and class II modes. Modes in class II are related to the GKS theory in the sense that they become quarter-plane modes as $J \to \infty$. In addition, there are only a few modes in this class and the maximum number is known exactly.

The following two examples are unstable discrete IBVP's. Even though a discrete IBVP is unstable, there is no *dramatic* change in the eigenvalue locus in the sense that it remains a perturbation of the Dirichlet eigenvalue locus but with the addition of one or two eigenvalues near or strictly outside the unit circle. These additional eigenvalues correspond to GKS eigenvalues or generalized eigenvalues in the limit $J \to \infty$.

11.2a Unstable quarter-plane-*like* mode – GKS eigenvalue. For the first example we choose $\alpha = -1$ and $\nu = 0.5$ from the unshaded region of Fig. 8.1. The roots of (8.4) are shown in Fig. 11.3a for $J = 19$.

From the figure it is apparent that there is a single isolated root indicated by the solid symbol in the figure. The corresponding eigenvalue is indicated by the solid symbol in Fig. 11.3b. This single eigenvalue remains strictly outside the unit circle as $J \to \infty$

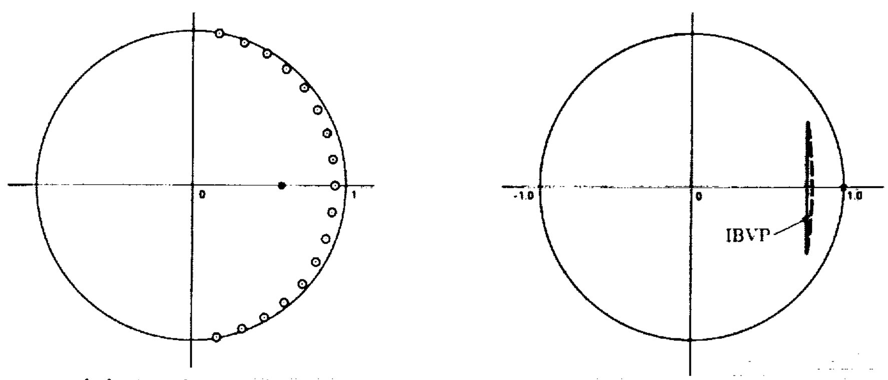


(a) $\hat{\kappa}$–plane $(\alpha = -1)$ (b) z–plane $(\alpha = -1)$

Fig. 11.3. Roots of characteristic polynomial (8.4) and eigenvalue locus for $\nu = 0.5$.

and consequently the approximation is unstable (GKS eigenvalue).

11.2b Unstable quarter-plane-*like* mode – GKS generalized eigenvalue. One should expect the discrete IBVP to be unstable for $\nu < 0$ for otherwise one would have a stable approximation for an ill-posed IBVP. As an example we choose $\alpha = 0$ and $\nu = -0.5$ from the unshaded region of of Fig. 8.1. The roots of (8.4) are depicted in Fig. 11.4a for $J = 19$.

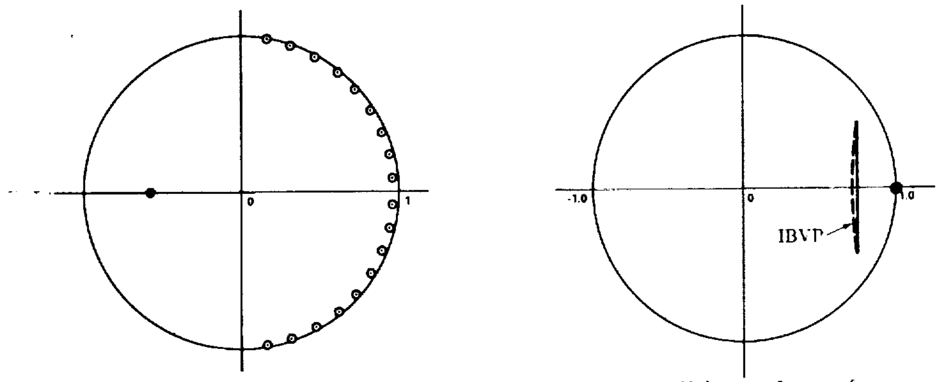


(a) $\hat{\kappa}$–plane $(\alpha = 0)$ (b) z–plane $(\alpha = 0)$

Fig. 11.4. Roots of characteristic polynomial (8.4) and eigenvalue locus for $\nu = -0.5$.

There are two (nearly coincident) isolated roots as shown by the solid symbol of the figure. The corresponding eigenvalues are $z \approx 1$ in Fig. 11.4b. For finite J, one of these eigenvalues is slightly inside the unit circle and the other is slightly outside. In this example the origin of the instability is rather subtle. The instability is not due to a violation of the spectral radius condition (4.6) but rather is due to the introduction of a solution (proportional to the eigenvector) whose norm cannot be uniformly bounded by the norm of the initial data as the mesh is refined, i.e., there is algebraic growth. In the nomenclature of the GKS theory, there is a generalized eigenvalue.

11.3 Stationary mode. As a final example we consider a *borderline* case. For parameter values we pick $\alpha = -0.75$ and $\nu = 0.5$ which is a point on the *left* boundary between stability and instability in Fig. 8.1. The roots of (8.4) are shown in Fig. 11.5a for $J = 19$.



(a) $\hat{\kappa}$–plane ($\alpha = -0.75$) (b) z–plane ($\alpha = -0.75$)

Fig. 11.5. Roots of characteristic polynomial (8.4) and eigenvalue locus for $\nu = 0.5$.

There is one isolated root shown by the solid symbol of the figure. The corresponding eigenvalue is $z = 1$. This eigenvalue of unity is independent of J. Since $z^n = 1$, the stability or instability of the difference approximation devolves to the behavior of the corresponding eigenvector as the mesh is refined. This approximation happens to be Lax-Richtmyer stable but GKS unstable. The details of this example are worked out in [3].

12. CONCLUSIONS

We have investigated the eigenvalue spectrum for the Lax-Wendroff scheme applied to a model hyperbolic IBVP. For the discrete IBVP on a finite domain, the normal mode analysis is analytically intractable even for the simple prototype difference approximation. On the basis of an asymptotic normal mode analysis (large J), we have classified the normal modes of the finite-domain problem into three classes. The resulting classification leads to a simple description of the asymptotic eigenvalue distribution. For a given Courant number, the spectrum is simply a perturbation of the spectrum of the auxiliary Dirichlet problem plus whatever eigenvalues are present in the GKS normal mode analysis for the related quarter-plane problems.

Almost all the modes are Dirichlet-*like* modes and are *benign* in the sense that they do not introduce unstable modes into a difference approximation which is Cauchy stable. Only quarter-plane-*like* modes and stationary modes can introduce *unstable* modes into an approximation which is Cauchy stable. Consequently, if an instability exists it is caused by the NBS and is detected in the GKS analysis.

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16. Abstract

The eigenvalue spectrum associated with a linear finite-difference approximation plays a crucial role in the stability analysis and in the actual computational performance of the discrete approximation. We investigate the eigenvalue spectrum associated with the Lax-Wendroff scheme applied to a model hyperbolic equation. For an initial-boundary-value problem (IBVP) on a finite domain, the eigenvalue or normal mode analysis is analytically intractable. A study of auxiliary problems (Dirichlet and quarter-plane) leads to asymptotic estimates of the eigenvalue spectrum and to an identification of individual modes as either *benign* or *unstable*. The asymptotic analysis establishes an intuitive as well as quantitative connection between the algebraic tests in the theory of Gustafsson, Kreiss, and Sundström and Lax-Richtmyer L_2 stability on a finite domain.

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