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SPACECRAFT ATTITUDE DETERMINATION USING THE EARTH'S MAGNETIC FIELD

David G. Simpson

OAO Corporation, Greenbelt, Maryland 20770

A method is presented by which the attitude of a low-Earth orbiting spacecraft may be determined using a vector magnetometer, a digital Sun sensor, and a mathematical model of the Earth's magnetic field. The method is currently being implemented for the Solar Maximum Mission spacecraft (as a backup for the failing star trackers) as a way to determine roll gyro drift.

I. INTRODUCTION

For centuries sailors have used the Earth's magnetic field to guide their ships though the oceans of the world by means of the magnetic compass. Today it is possible for spacecraft to navigate themselves in much the same way, with the mariner's compass replaced by the modern magnetometer. In this paper I describe how a vector magnetometer, in conjunction with a digital Sun sensor, can be used to determine the attitude of a low-Earth orbiting spacecraft.

This work was motivated by the failure of one of the two star trackers on the Solar Maximum Mission (SMM) spacecraft in 1987. The complement of attitude sensors on SMM provides only gyroscopes, two star trackers, and vector magnetometers for determining the roll attitude. As currently written, the on-board computer flight software uses the gyros to determine the roll attitude of the spacecraft, with the star tracker used only to calculate the roll gyro drift. The magnetometers are not currently used for attitude determination on SMM.

Should the remaining star tracker fail, however, this would leave the magnetometers as the only means of determining an *absolute* roll attitude, since the gyros only measure changes in the attitude with respect to inertial space. The work described in this paper is a result of an effort to determine how SMM's magnetometers might be used as a replacement for the remaining star tracker in the event that it fails.

The approach here will be to find the components of two vectors (the geomagnetic induction and Sun vectors) in each of two coordinate frames (the spacecraft frame and a reference frame); we then solve for the rotation matrix between the two frames to determine the spacecraft attitude. These calculations will be performed by a computer on the ground using data telemetered from the spacecraft; the ground computer will calculate roll gyro drift coefficients which will be periodically uplinked to the on-board computer. Calculating the gyro drift coefficients on the ground will permit ground personnel to select data which was sampled while the geomagnetic field was relatively quiet, thus giving the most accurate results.

Section II of this paper describes how the Earth's magnetic field vector at the spacecraft position can be calculated from a mathematical model. Section III describes the calculation of the Sun vector, and Sections IV and V describe how these two vectors may be combined with sensor data to determine the spacecraft attitude. Section VI is a summary of the paper, and Section VII is a short discussion of associated Legendre functions.

II. MODELING THE GEOMAGNETIC FIELD

In order to determine the spacecraft attitude from the magnetometers, one must first generate an accurate mathematical model of the Earth's magnetic field. Ampere's law at the spacecraft position $\vec{r}$ is (SI units):

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + \frac{\partial \vec{D}(\vec{r})}{\partial t} \tag{1}$$

where $\vec{H}(\vec{r})$ is the geomagnetic intensity at $\vec{r}$, $\vec{J}(\vec{r})$ is the electric current density, and $\partial\vec{D}(\vec{r})/\partial t$ is the displacement current. Since there is no current density at $\vec{r}$ and the geomagnetic field is approximately static, we may take $\vec{J}$ and $\partial\vec{D}/\partial t$ to both be zero. Eq. (1) then becomes

$$\nabla \times \vec{H}(\vec{r}) = \vec{0} \tag{2}$$

The constitutive relation for the magnetic induction $\vec{B}$ is

$$\frac{1}{\mu_0} \vec{B}(\vec{r}) = \vec{H}(\vec{r}) + \vec{M}(\vec{r}) \tag{3}$$

where μ_0 is the permeability of free space ($4\pi \times 10^{-7}$ N A^{-2}), and $\vec{M}$ is the magnetization, which is zero at $\vec{r}$. Eq. (3) then becomes

$$\frac{1}{\mu_0} \vec{B}(\vec{r}) = \vec{H}(\vec{r}) \tag{4}$$

Substituting Eq. (4) for $\vec{H}$ into Eq. (2) we get

$$\nabla \times \frac{1}{\mu_0} \vec{B}(\vec{r}) = \vec{0} \tag{5}$$

since the curl of any gradient is zero, this means that $\vec{B}$ can be written as the gradient of a magnetic scalar potential V:[1]

$$\vec{B}(\vec{r}) = -\mu_0 \nabla V(\vec{r}) \tag{6}$$

It is conventional in geomagnetism to model the geomagnetic field by expanding the magnetic scalar potential $V(\vec{r})$ into a Laplace series of spherical harmonics with real eigenfunctions:[2]

$$V(r,\theta,\lambda) = \frac{a}{\mu_0}\sum_{n=1}^{k}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left[g^{nm}\cos m\lambda + h^{nm}\sin m\lambda\right]P^{nm}(\cos\theta) \qquad (7)$$

where

r is the distance of the spacecraft from the center of the Earth;

θ is the co-elevation of the sub-satellite point (90° minus the north latitude;

λ is the east longitude of the sub-satellite point;

a is the radius of the Earth, taken to be 6371.2 km;

$P^{nm}(\cos\theta)$ are the Gauss-normalized associated Legendre functions of the first kind;

g^{nm} and h^{nm} are the Gauss-normalized coefficients of the expansion.

The n=0 terms in this expansion are absent because they would represent a magnetic monopole component of the field; the n=1 terms represent the dipole component, the n=2 terms represent the quadrupole component, etc.

The expansion coefficients g^{nm} and h^{nm} are found empirically; they are updated every five years and published along with their time derivatives (the *secular variation*) by the International Association of Geomagnetism and Aeronomy (IAGA). These published coefficients are Schmidt normalized and may be used to calculate the geomagnetic scalar potential $V(\vec{r})$ using Eq. (7) if the Schmidt-normalized associated Legendre functions P_n^m are substituted for the Gauss-normalized functions P^{nm}. Using Gauss normalization will save about 7% in computer time, however,[3] so for convenience in computer work, Gauss normalization will be used throughout this paper.

Table I shows the Gauss-normalized coefficients g^{nm} and h^{nm} for the International Geomagnetic Reference Field (IGRF) 1985. These were calculated from the Schmidt-normalized coefficients published by the IAGA and can safely be extended to 1990 with the secular variation coefficients in the last two columns.

TABLE I. International Geomagnetic Reference Field, IGRF 1985.
Coefficients are *Gauss*-normalized.

n	m	g^{nm} (nT)	h^{nm} (nT)	$\dot{g}^{nm}$ (nT/yr)	$\dot{h}^{nm}$ (nT/yr)
1	0	-29877		23.2	
1	1	-1903	5497	10.0	-24.5
2	0	-3110		-20.6	
2	1	5274	-3795	5.9	-19.9
2	2	1464	-268	6.1	-17.5
3	0	3250		12.8	
3	1	-6761	-955	-14.1	16.2
3	2	2409	550	-1.2	4.5
3	3	660	-234	0.1	-8.5
4	0	4099		0.4	
4	1	4317	1289	-3.3	21.0
4	2	1420	-978	-30.5	8.6
4	3	-891	142	-2.9	5.2
4	4	125	-220	-5.0	0.7
5	0	-1693		10.2	
5	1	3619	478	1.0	1.0
5	2	1944	1137	-11.5	-1.5
5	3	-442	-729	-15.1	-0.5
5	4	-357	-166	0.2	1.3
5	5	-34	67	-0.1	0.0
6	0	751		20.2	
6	1	1229	-302	-5.7	-7.6
6	2	747	1345	25.4	-16.4
6	3	-1853	687	6.0	-8.0
6	4	22	-273	0.0	-12.6
6	5	40	-9	2.1	-1.2
6	6	-69	13	0.8	-0.1
7	0	2011		5.4	
7	1	-2164	-2909	-21.3	7.1
7	2	58	-753	-14.5	29.0
7	3	491	-20	16.4	22.5
7	4	-74	284	12.3	23.5
7	5	25	105	2.5	1.9
7	6	22	-51	-1.2	0.5
7	7	0	-4	-0.1	0.6
8	0	1056		35.2	
8	1	402	469	0.0	6.7
8	2	0	-1178	16.8	-56.1
8	3	-456	207	16.6	4.1
8	4	-241	-668	-8.0	-21.4
8	5	30	163	-4.4	3.0
8	6	27	82	0.7	-5.5
8	7	10	-40	-1.3	-0.3
8	8	-4	-6	-0.5	0.8

TABLE I (cont.)

n	m	g^{nm} (nT)	h^{nm} (nT)	$\dot{g}^{nm}$ (nT/yr)	$\dot{h}^{nm}$ (nT/yr)
9	0	475			
9	1	1274	-2675		
9	2	109	1738		
9	3	-996	747		
9	4	507	-282		
9	5	-101	-202		
9	6	-17	157		
9	7	53	75		
9	8	5	-16		
9	9	-3	1		
10	0	-722			
10	1	-973	243		
10	2	421	0		
10	3	-826	496		
10	4	-234	701		
10	5	370	-296		
10	6	124	0		
10	7	20	-20		
10	8	16	33		
10	9	8	0		
10	10	0	-4		

When the coefficients in Table I are substituted into Eq. (7), we get the geomagnetic scalar potential $V(\vec{r})$; substituting this $V(\vec{r})$ into Eq. (6) yields the geomagnetic induction vector $\vec{B}(\vec{r})$ in a reference frame fixed in the Earth (which will be referred to as the EB (Earth-based) frame). The EB frame has its origin at the center of the Earth, its x axis pointing out of the intersection of the equator with the prime meridian, its z axis pointing out of the Earth's geographic north pole, and its y axis in the $\hat{x} \times \hat{z}$ direction.

If we calculate the gradient in Eq. (6) in spherical polar coordinates,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\lambda}\hat{e}_\lambda \tag{8}$$

the resulting spherical components of $\vec{B}$ may be used to easily calculate the standard geomagnetic elements:[2]

X	$= -B_\theta$	(Northward component)
Y	$= B_\lambda$	(Eastward component)
Z	$= -B_r$	(Downward component)
H	$= (X^2 + Y^2)^{1/2}$	(Horizontal intensity)
V	$= \|Z\|$	(Vertical intensity)
F	$= (X^2 + Y^2 + Z^2)^{1/2}$	(Total intensity)
D	$= \arctan(Y/X)$	(Magnetic declination)
I	$= \arctan(Z/H)$	(Magnetic inclination)

A computer program which calculates $\vec{B}(\vec{r})$ can then be checked by comparing the geomagnetic elements it calculates with the elements found in charts and tables in the literature.[5]

For spacecraft attitude determination, we will need to know the components of the modeled geomagnetic induction vector $\vec{B}(\vec{r})$ in the *geocentric inertial* (GCI) reference frame rather than the EB frame. The GCI frame is fixed with respect to the stars and has its origin at the Earth's center, its x axis toward the vernal equinox, its z axis out of the Earth's geographic north pole, and its y axis in the $\hat{z} \times \hat{x}$ direction.

The GCI frame differs from the EB frame only by a rotation about their common z axis. Specifically, in cartesian coordinates

$$\vec{B}_{GCI} = R\,\vec{B}_{EB} \qquad (9)$$

where the rotation matrix R is given by

$$R = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (10)$$

and where γ is the Greenwich hour angle of the vernal equinox, which is equal to the sidereal time at Greenwich (GST) and is given by[6]

$$\gamma = \mathrm{LST} - (\lambda/15^\circ) \qquad (11)$$

where LST is the local sidereal time and λ the east longitude of any convenient point on the Earth's surface. An expression for γ which is often more convenient is[6]

$$\gamma = 99^\circ.6910 + 36000^\circ.7689T + 0^\circ.0004T^2 + \mathrm{UTC} \qquad (12)$$

where T is the time (in Julian centuries of 36525 days) since 1900 and UTC is Coordinated Universal Time expressed in *degrees*.

Alternatively, if we work in spherical polar coordinates, Eqs. (9) and (10) may be replaced with

$$\begin{aligned} B_{r_{GCI}} &= B_{r_{EB}} \\ B_{\theta_{GCI}} &= B_{\theta_{EB}} \\ B_{\lambda_{GCI}} &= B_{\lambda_{EB}} - \gamma \end{aligned} \qquad (13)$$

III. SUN VECTOR CALCULATION

The geomagnetic induction vector $\vec{B}(\vec{r})$ modeled in Section II is by itself insufficient to determine the spacecraft attitude, since if the spacecraft is rotated about the $\vec{B}$ vector it will still yield the same components of $\vec{B}$ in a reference frame fixed in the spacecraft; hence one degree of freedom is left unspecified.[7] It is therefore necessary to know the components of one more vector (not parallel to $\vec{B}(\vec{r})$) in order to specify the spacecraft attitude completely. For this discussion we choose the Earth-to-Sun vector (or simply the "Sun vector") $\hat{S}$, and in this section I discuss how to calculate the components of $\hat{S}$ in the GCI frame. (The cartesian GCI components of $\hat{S}$ will be used along with the cartesian GCI components of $\vec{B}$ from Section II to determine the spacecraft attitude in Sections IV and V.) The Sun unit vector $\hat{S}$ is given in cartesian GCI coordinates approximately by (ignoring the small corrections for parallax and light aberration)[8]

$$\hat{S}(t) = \begin{bmatrix} \cos L \\ \sin L \;\; \cos \varepsilon \\ \sin L \;\; \sin \varepsilon \end{bmatrix} \tag{14}$$

where $L(t)$ is the mean longitude of the Sun and ε is the mean inclination of the ecliptic from the Earth's equatorial plane.

The mean longitude of the Sun $L(t)$ may be calculated from[8]

$$L(t) = L(t_0) + M(t) + 2e \sin M(t) + \beta t \tag{15}$$

where t_0 is a reference time, e is the eccentricity of the Earth's orbit (e = 0.016722), $M(t)$ is the mean anomaly of the Sun, and β is defined by

$$\beta = \frac{360^\circ}{\tau_{sy}} - \frac{360^\circ}{\tau_{ay}} \tag{16}$$

where τ_{sy} is the length of the sidereal year (3.1558149548×10^7 seconds) and τ_{ay} is the length of the anomalistic year (3.1558433×10^7 seconds).

The mean anomaly of the Sun $M(t)$ is given by[8]

$$M(t) = M(t_0) + \frac{360^\circ}{\tau_{ay}}(t - \sigma) \tag{17}$$

where σ is the time it takes light to travel from the Sun to the Earth, about 499 seconds.

Finally, the mean inclination of the ecliptic ε is given by[8]

$$\varepsilon = 23^\circ 27' \, 8.26'' - 46.845'' \, T \tag{18}$$

where T is the time in Julian centuries since 1900.

IV. GENERAL ATTITUDE DETERMINATION

Knowing the cartesian components of the geomagnetic induction unit vector $\vec{B}$ and the Sun unit vector $\hat{S}$ in the GCI reference frame (as described in Sections II and III), we can now determine the spacecraft attitude if we also know the components of these two vectors in the *spacecraft* (SC) reference frame (i.e. along the roll (x), pitch (y), and yaw (z) axes). The essence of the problem of determining the spacecraft attitude is then this: we are given the components of two vectors in each of these two reference frames which have a common origin, and we must solve for the rotation matrix between the two frames. There are several ways of doing this; I will describe here one of the simplest methods, known as the *algebraic method*.[9] Let $\vec{\xi}$ be some vector whose components we wish to transform from the GCI reference frame to the SC frame. The two sets of components are related by

$$\vec{\xi}_{SC} = A \, \vec{\xi}_{GCI} \tag{19}$$

where $\vec{\xi}_{SC}$ and $\vec{\xi}_{GCI}$ are column vectors containing the SC and GCI components, respectively, and A is the rotation matrix we wish to solve for. *Formally*, we could solve this equation of A by post-multiplying both sides by the inverse of the matrix $\vec{\xi}_{GCI}$:

$$A = \vec{\xi}_{SC} \, \vec{\xi}_{GCI}^{\;-1}$$

Unfortunately, $\vec{\xi}_{GCI}$ is not a square matrix, so we cannot take its inverse directly (without a somewhat messy diversion into pseudo-inverses).

In the algebraic method, we use the two known vectors $\hat{B}$ and $\hat{S}$ to *construct* an orthogonal triad of vectors ($\hat{a}$, $\hat{b}$, and $\hat{c}$) in each reference frame (SC and GCI). (This will work only if $\hat{B}$ and $\hat{S}$ are not parallel.) We then define a matrix M for each frame which has the vectors $\hat{a}$, $\hat{b}$, and $\hat{c}$ as its columns; the columns of this matrix will transform from GCI to SC coordinates just like the vector $\vec{\xi}$ above, *and* we will be able to take its inverse (since it will be a square matrix), thus allowing us to solve for the rotation matrix A.

Specifically, let us define the vectors $\hat{a}$, $\hat{b}$, and $\hat{c}$ in the GCI and SC reference frames as follows:

GCI Frame	SC Frame	
$\hat{a}_{GCI} = \hat{S}_{GCI}$	$\hat{a}_{SC} = \hat{S}_{SC}$	
$\hat{b}_{GCI} = \dfrac{\hat{S}_{GCI} \times \hat{B}_{GCI}}{\lvert \hat{S}_{GCI} \times \hat{B}_{GCI} \rvert}$	$\hat{b}_{SC} = \dfrac{\hat{S}_{SC} \times \hat{B}_{SC}}{\lvert \hat{S}_{SC} \times \hat{B}_{SC} \rvert}$	(20)
$\hat{c}_{GCI} = \hat{a}_{GCI} \times \hat{b}_{GCI}$	$\hat{c}_{SC} = \hat{a}_{SC} \times \hat{b}_{SC}$	

where:

$\hat{B}_{GCI}$ is the geomagnetic induction unit vector in GCI cartesian coordinates, calculated as in Section II.

$\hat{S}_{GCI}$ is the Sun unit vector in GCI cartesian coordinates, calculated as in Section III.

$\hat{B}_{SC}$ is the geomagnetic induction unit vector in SC cartesian coordinates, from the spacecraft magnetometers.

$\hat{S}_{SC}$ is the Sun unit vector in SC cartesian coordinates, from the spacecraft digital Sun sensor.

We now construct two 3×3 square matrices M_{GCI} and M_{sc} whose columns are the cartesian components of $\hat{a}$, $\hat{b}$, and $\hat{c}$:

$$
\begin{aligned}
M_{GCI} &= \left[\hat{a}_{GCI} \mid \hat{b}_{GCI} \mid \hat{c}_{GCI} \right] \\
M_{SC} &= \left[\hat{a}_{SC} \mid \hat{b}_{SC} \mid \hat{c}_{SC} \right]
\end{aligned}
\qquad (21)
$$

Since the rotation matrix A rotates column vectors from the GCI to the SC reference frame, it will also rotate each column of M_{GCI} into the corresponding column of M_{SC}:

$$M_{SC} = A\ M_{GCI}$$

Since M_{GCI} is a square matrix, we can now solve for the rotation matrix A by post-multiplying both sides by the inverse of M_{GCI}:

$$A = M_{SC}\ M_{GCI}^{-1}$$

Furthermore, since M_{GCI} was defined to be an orthogonal matrix, its inverse is equal to its transpose and so

$$A = M_{SC}\ M_{GCI}^{T} \qquad (22)$$

The matrix A given by Eq. (22) rotates any vector from the GCI to the spacecraft reference frame and thus determines the spacecraft attitude.

V. SMM ROLL ATTITUDE DETERMINATION

On SMM, the roll, pitch, and yaw angles are measured with respect to the somewhat whimsically named "SUN" reference frame, defined by the three orthonormal vectors $\hat{S}$, $\hat{U}$, and $\hat{N}$:

$\hat{S}$ is the Sun unit vector, calculated as in Section III.

$$\hat{U} = \frac{\hat{P} \times \hat{S}}{|\hat{P} \times \hat{S}|} \qquad \text{(}\hat{U}\text{ points out of the east limb of the Sun, along the solar equator.)} \tag{23}$$

$$\hat{N} = \hat{S} \times \hat{U} \qquad \text{(}\hat{N}\text{ is }\hat{P}\text{ projected onto the plane normal to the line of sight.)}$$

where $\hat{P}$ is the solar spin axis unit vector. In order to determine the spacecraft attitude from the magnetometers and the Fine Pointing Sun Sensor (FPSS), we will need to know the components of the vectors $\hat{S}$, $\hat{U}$, and $\hat{N}$ in the GCI reference frame.

We first need to calculate the cartesian components of the solar spin axis unit vector $\hat{P}$ in the GCI frame. We begin by working in the *ecliptic* (ECL) frame, defined by the unit vectors x_{ECL}, y_{ECL}, and z_{ECL}:

x_{ECL} points in the direction of the vernal equinox;
y_{ECL} points in the direction of $\hat{z}_{ECL} \times \hat{x}_{ECL}$;
z_{ECL} points toward the ecliptic north pole.

The vector $\hat{P}$ in the ecliptic frame has cartesian components[8]

$$\hat{P}_{ECL} = \begin{bmatrix} \sin\Omega \ \sin i \\ -\cos\Omega \ \sin i \\ \cos i \end{bmatrix} \tag{24}$$

where Ω is the longitude of the ascending node of the solar equator on the ecliptic, given by[8]

$$\Omega = 73^\circ\ 40' + 50^\circ\ 25'\ \Lambda \tag{25}$$

where Λ is the time in years since 1850; i is the inclination of the solar equator to the ecliptic:

$$i = 7^\circ\ 15' \tag{26}$$

The ecliptic and GCI reference frames differ only by a rotation of magnitude $-\varepsilon$ about their common x axis, where ε is the inclination of the ecliptic (Eq. (18)). Hence the GCI cartesian components of $\hat{P}$ are

$$\hat{P}_{GCI} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} \hat{P}_{ECL} \tag{27}$$

Having found the cartesian components of the solar spin axis unit vector $\hat{P}$ in the GCI frame, and knowing the components of $\hat{S}$ from section III, we may now use Eq. (23) to determine the cartesian components of the $\hat{U}$ and $\hat{N}$ vectors in the GCI frame; these will be used in Eq. (29) below to determine the SMM roll attitude.

It is especially convenient to determine SMM's roll angle ρ when the spacecraft's FPSS is pointed at Sun center (so that the pitch and yaw angles are zero). In this case the x axis of the spacecraft reference frame (the roll axis) will coincide with the $\hat{S}$ vector of the SUN frame. The two reference frames will then differ only by a rotation about their common x-$\hat{S}$ axis (Fig. 1). The roll angle ρ is then given by[8]

$$\rho = \phi_1 - \phi_2 \tag{28}$$

where ϕ_1 is calculated from the magnetic field model and solar ephemeris (Sections II-V):

$$\phi_1 = \tan^{-1} \frac{-\hat{B}_{GCI} \cdot \hat{U}_{GCI}}{\hat{B}_{GCI} \cdot \hat{N}_{GCI}} \tag{29}$$

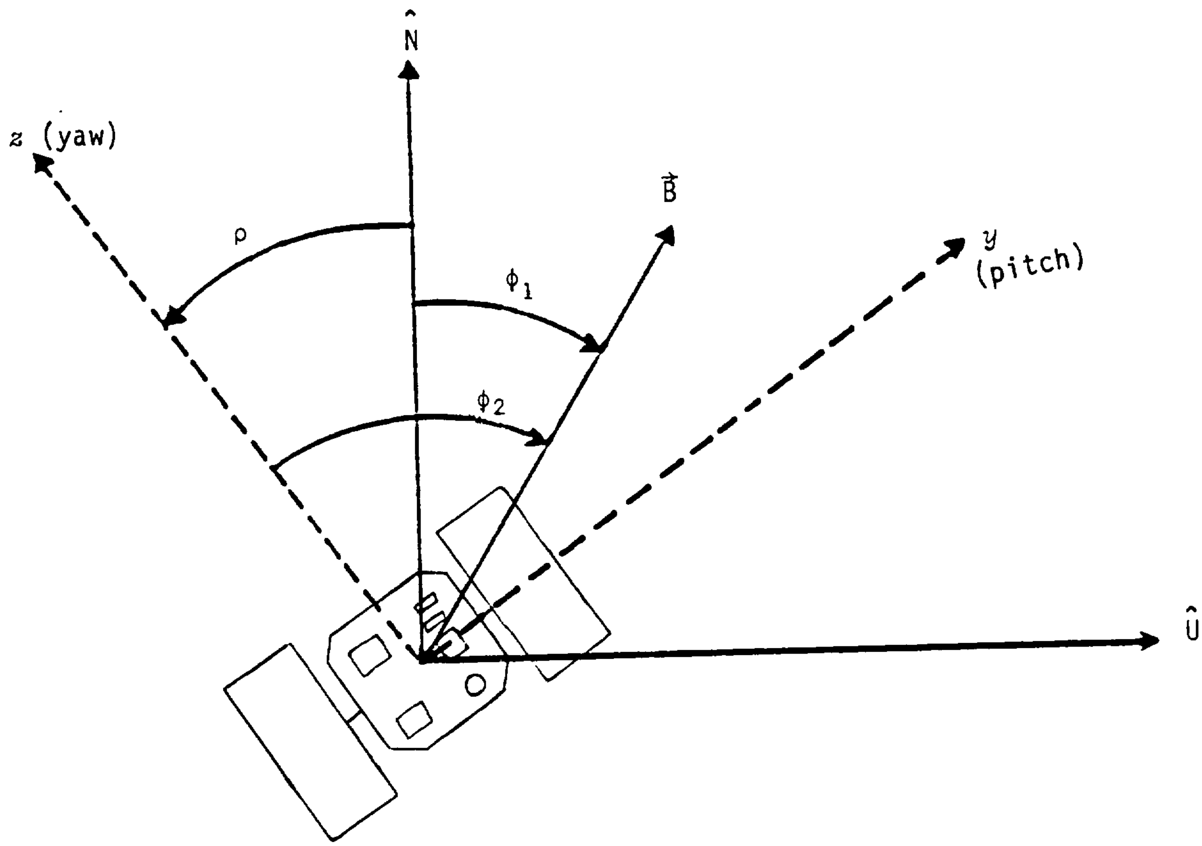

Fig. 1. Determination of SMM roll attitude. This view is looking from the Sun down onto the SMM top face plate. (After [8].)

and ϕ_2 is found from magnetometer data:

$$\phi_2 = \tan^{-1} \frac{-\vec{B}_{mag_{pitch}}}{\vec{B}_{mag_{yaw}}} \tag{30}$$

Substituting Eqs. (29) and (30) into Eq. (28) then determines the SMM roll attitude. Note that we do not require any FPSS data in this case; the Sun vector $\hat{S}_{SC}$ was tacitly assumed to lie along the $\hat{S}$ axis of the SUN frame since the FPSS is pointed at Sun center.

VI. SUMMARY

Two methods for spacecraft attitude determination using the Earth's magnetic field have been presented: the *algebraic method* for spacecraft in general (Section IV), and a simpler method specific to SMM (Section V). Both methods compare the Earth's magnetic field as calculated by a mathematical model (Eq. (7)) with magnetometer measurements; in addition, the Sun vector calculated by Eq. (14) and measured with a digital Sun sensor is used to completely specify the spacecraft attitude.

VII. APPENDIX: ASSOCIATED LEGENDRE FUNCTIONS

The associated Legendre functions of the first kind form a complete orthogonal set of functions over the interval $\theta = [0, \pi]$; it is this property which makes them a useful basis in which to expand the geomagnetic scalar potential $V(\vec{r})$.

Although a variety of normalization conventions for the associated Legendre functions are in common use, the three most common are the so-called Neumann, Schmidt, and Gauss normalization conventions. Neumann normalization is the convention most often found in mathematics textbooks.[10-12] The geomagnetic coefficients g and h given in the literature are usually defined for the Schmidt-normalized associated Legendre functions; Schmidt normalization has the advantage that the normalization constants are independent of m (for $m \neq 0$) for any given n, so the relative strengths of the different terms can be easily judged. Gauss normalization is useful because it saves about 7% in computation time on a computer[3]; the g and h coefficients in this paper have been converted to Gauss normalization for ready use in computer work.

The various normalization conventions are defined as follows:

Neumann normalization (P_{nm}):

$$\int_0^\pi P_{nm}(\cos\theta)\, P_{\ell m}(\cos\theta)\, \sin\theta\, d\theta = \frac{2}{2n+1}\, \frac{(n+m)!}{(n-m)!}\, \delta_{n\ell}$$

Schmidt normalization (P_n^m):

$$\int_0^\pi P_n^m(\cos\theta)\, P_\ell^m(\cos\theta)\, \sin\theta\, d\theta = \frac{2\,(2-\delta_{m0})}{2n+1}\, \delta_{n\ell}$$

Gauss normalization (P^{nm}):

$$\int_0^\pi P^{nm}(\cos\theta)\, P^{\ell m}(\cos\theta)\, \sin\theta\, d\theta = \frac{2\,(n-m)!\,(n+m)!}{(2n+1)\,[(2n-1)!!]^2}\, \delta_{n\ell}$$

One may easily calculate the conversion factors for converting between Schmidt and Gauss normalizations by simply taking the square root of the quotient of the respective normalization constants. In particular, if the conversion factors S_{nm} are defined by[13]

$$P_n^{\ m} = S_{nm} \ P^{nm}$$

then

$$S_{nm} = \left[\frac{\dfrac{2(2-\delta_{m0})}{2n+1} \ \delta_{n\ell}}{\dfrac{2(n-m)! \ (n+m)!}{(2n+1) \ [(2n-1)!!]^2} \ \delta_{n\ell}} \right]^{\frac{1}{2}}$$

or

$$S_{nm} = \left[\frac{(2-\delta_{m0}) \ (n-m)!}{(n+m)!} \right]^{\frac{1}{2}} \ \frac{(2n-1)!!}{(n-m)!}$$

Table II lists the explicit values of these conversion factors up to n=12. To convert a table of Schmidt-normalized coefficients $g_n^{\ m}$ and $h_n^{\ m}$ (such as those usually found in the literature) from Schmidt to Gauss normalization, use

$$g^{nm} = S_{nm} \ g_n^{\ m}$$
$$h^{nm} = S_{nm} \ h_n^{\ m}$$

and analogous expressions for the secular variation coefficients.

TABLE II. Conversion factors S_{nm} between Schmidt- and Gauss- normalized associated Legendre functions of the first kind.

n	m	S_{nm}
1	0	1
1	1	1
2	0	3/2
2	1	$\sqrt{3}$
2	2	$(1/2)\sqrt{3}$
3	0	5/2
3	1	$(5/4)\sqrt{6}$
3	2	$(1/2)\sqrt{15}$
3	3	$(1/4)\sqrt{10}$
4	0	35/8
4	1	$(7/4)\sqrt{10}$
4	2	$(7/4)\sqrt{5}$
4	3	$(1/4)\sqrt{70}$
4	4	$(1/8)\sqrt{35}$
5	0	63/8
5	1	$(21/8)\sqrt{15}$
5	2	$(3/4)\sqrt{105}$
5	3	$(9/16)\sqrt{70}$
5	4	$(3/8)\sqrt{35}$
5	5	$(3/16)\sqrt{14}$
6	0	231/16
6	1	$(33/8)\sqrt{21}$
6	2	$(33/32)\sqrt{210}$
6	3	$(11/16)\sqrt{210}$
6	4	$(33/16)\sqrt{7}$
6	5	$(3/16)\sqrt{154}$
6	6	$(1/32)\sqrt{462}$
7	0	429/16
7	1	$(429/32)\sqrt{7}$
7	2	$(143/32)\sqrt{42}$
7	3	$(143/32)\sqrt{21}$
7	4	$(13/16)\sqrt{231}$
7	5	$(13/32)\sqrt{231}$
7	6	$(1/32)\sqrt{6006}$
7	7	$(1/32)\sqrt{429}$
8	0	6435/128
8	1	2145/32
8	2	$(429/64)\sqrt{70}$
8	3	$(39/32)\sqrt{1155}$
8	4	$(195/64)\sqrt{77}$
8	5	$(15/32)\sqrt{1001}$
8	6	$(15/64)\sqrt{858}$
8	7	$(3/32)\sqrt{715}$
8	8	$(3/128)\sqrt{715}$

n	m	S_{nm}
9	0	12155/128
9	1	$(7293/128)\sqrt{5}$
9	2	$(663/64)\sqrt{110}$
9	3	$(221/128)\sqrt{2310}$
9	4	$(51/64)\sqrt{5005}$
9	5	$(255/128)\sqrt{286}$
9	6	$(17/64)\sqrt{4290}$
9	7	$(51/256)\sqrt{1430}$
9	8	$(3/128)\sqrt{12155}$
9	9	$(1/256)\sqrt{24310}$
10	0	46189/256
10	1	$(4199/128)\sqrt{55}$
10	2	$(4199/256)\sqrt{165}$
10	3	$(323/128)\sqrt{4290}$
10	4	$(323/128)\sqrt{2145}$
10	5	$(323/128)\sqrt{858}$
10	6	$(323/512)\sqrt{4290}$
10	7	$(19/256)\sqrt{72930}$
10	8	$(19/256)\sqrt{12155}$
10	9	$(1/256)\sqrt{461890}$
10	10	$(1/512)\sqrt{92378}$
11	0	88179/256
11	1	$(29393/512)\sqrt{66}$
11	2	$(2261/256)\sqrt{2145}$
11	3	$(969/512)\sqrt{30030}$
11	4	$(969/128)\sqrt{1001}$
11	5	$(6783/512)\sqrt{143}$
11	6	$(399/512)\sqrt{14586}$
11	7	$(133/512)\sqrt{36465}$
11	8	$(7/256)\sqrt{692835}$
11	9	$(21/512)\sqrt{46189}$
11	10	$(1/512)\sqrt{1939938}$
11	11	$(1/512)\sqrt{88179}$
12	0	676039/1024
12	1	$(52003/512)\sqrt{78}$
12	2	$(7429/512)\sqrt{3003}$
12	3	$(7429/512)\sqrt{2002}$
12	4	$(22287/2048)\sqrt{2002}$
12	5	$(1311/512)\sqrt{17017}$
12	6	$(3059/1024)\sqrt{4862}$
12	7	$(161/512)\sqrt{138567}$
12	8	$(161/1024)\sqrt{138567}$
12	9	$(23/512)\sqrt{323323}$
12	10	$(23/1024)\sqrt{176358}$
12	11	$(1/512)\sqrt{2028117}$
12	12	$(1/2048)\sqrt{1352078}$

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