N90-13435

CALCULATION OF DOUBLE-LUNAR SWINGBY TRAJECTORIES: I. KEPLERIAN FORMULATION*

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ABSTRACT

Scientific satellites may require translunar orbits aligned with the Sun-Earth line, with most of the period spent in either the sunward or antisunward direction. To maintain alignment, the orbit's line of apsides must rotate at a rate equal to mean angular motion of the Earth about the Sun. To maintain this rotation of the line of apsides by use of fuel onboard the spacecraft is prohibitively expensive. Farquhar and Dunham (Reference 1) proposed a method for maintaining the desired alignment by gaining momentum at the expense of the Moon during a close approach—a lunar swingby—as the spacecraft passes beyond lunar orbit, then returning the momentum at the second lunar swingby as the spacecraft returns within the lunar orbit. The cycle of double-lunar swingbys may then be repeated. Dunham (Reference 2) presented the orbit parameters necessary to achieve double-lunar swingby orbits which will maintain Sun-Earth line alignment. The details of the Keplerian approach to calculation of these parameters are presented. Methods for solution of the necessary equations for these parameters are presented.

* This work was supported by the National Aeronautics and Space Administration (NASA)/Goddard Space Flight Center (GSFC), Greenbelt, Maryland, under Contract NAS 5-31500.

1. DEFINITION OF A DOUBLE-LUNAR SWINGBY CYCLE

Consider a Keplerian orbit about the Earth with its line of apsides pointing toward the Sun. After 3 months, ignoring perturbations, the line of apsides would be perpendicular to a line drawn from the Sun to the Earth; to maintain its Sun-pointing line of apsides, this line must rotate about the Earth at a rate equal to the mean angular motion of the Earth about the Sun. Farquhar and Dunham (Reference 1) described a method to achieve such an apsidal rotation rate for translunar orbits. The method requires two close lunar encounters, swingbys, per cycle. The first swingby occurs as the spacecraft, moving away from the Earth, crosses the lunar orbit with the Moon to its left. This "trailing edge" swingby increases the energy of the spacecraft's orbit and hence increases the semimajor axis. We call the orbit with the larger semimajor axis the outer-segment loop or simply "outer loop," and the original orbit the inner-segment loop or simply "inner loop." The spacecraft's outer loop orbit period is such that more than one lunar month passes following the first lunar swingby before the spacecraft again crosses the Moon's orbit, this time moving toward the Earth. The second swingby occurs at this crossing, again with the Moon to the left of the spacecraft. Thus a "leading edge" swingby occurs, removing energy from the spacecraft's orbit and reducing its semimajor axis to its original length—the length before the first lunar swingby. Next, slightly less than one lunar month passes, the spacecraft is now ready for another outward crossing of the Moon's orbit, and the Moon and spacecraft are at the same relative position as for the first lunar swingby. This defines one complete "double-lunar swingby" cycle. Note that at the first swingby the Moon's pull rotated the line of apsides counterclockwise. The second swingby, occurring as the spacecraft moved toward the Earth, also resulted in a counterclockwise rotation of the line of apsides. If, then, the sum of these two rotations divided by the time for one complete cycle equals the mean angular motion of the Sun, the spacecraft's line of apsides will continue its sunward alignment.

Figure 1 shows one complete double-lunar swingby cycle. The Moon's positions at the first, second, and third (first) swingbys are shown as S_1, S_2, and S_3. The true anomaly of the spacecraft at the time of the first swingby is shown as f_I for the inner loop orbit and f_O for the outer loop orbit. Thus the apsidal rotation is $2(f_I - f_O)$, and occurs in a time equal to the time from S_1 to S_3. We can then write the first necessary condition for a lunar swingby as

$$\Delta\dot{\omega} \equiv \underbrace{\frac{2(f_I - f_O)}{2t_a + t_s}}_{\substack{\text{rate of rotation}\\ \text{of line of apsides}\\ \text{for one double-}\\ \text{lunar swingby}\\ \text{cycle}}} - \underbrace{\frac{2\pi}{T_E}}_{\substack{\text{mean angular}\\ \text{motion of Earth}\\ \text{about the Sun}}} = 0 \qquad (1)$$

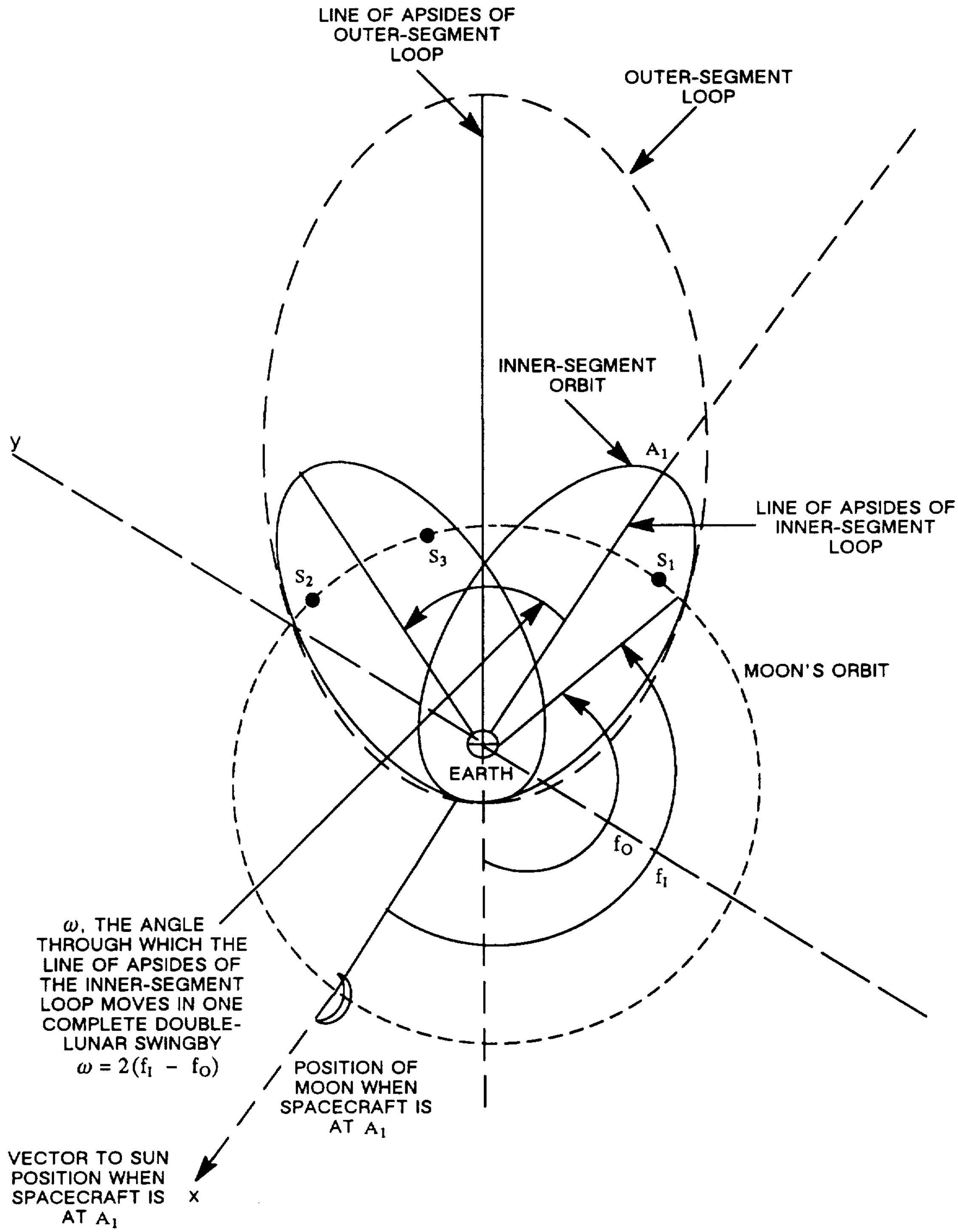

Figure 1. A Double-Lunar Swingby Trajectory

where

T_E = Earth's orbital period

$2t_a$ = time spent in outer loop from S_1 to S_2

t_s = twice the time from perigee of inner loop to S_1, plus one complete inner loop period

Two more conditions relate the spacecraft's orbital parameters to the Moon's motion:

$$\Delta\theta_1 \equiv \underbrace{\left\{2(f_I - f_O) + 2\left(\pi - f_I - \frac{r_m}{A_m}\right)\right\}}_{\text{angle traveled by Moon from } S_1 \text{ to } S_2 \text{ expressed in inner and outer loop true anomalies}} - \underbrace{\frac{2\pi}{T_m}(2t_a)}_{\text{angle traveled by Moon (mod } 2\pi) \text{ between } S_1 \text{ and } S_2 \text{ expressed in outer loop time, } 2t_a\text{, from } S_1 \text{ to } S_2} = 0 \tag{2}$$

$$\Delta\theta_2 \equiv \underbrace{\left\{2\left(\pi - f_I - \frac{r_m}{A_m}\right)\right\}}_{\text{angle traveled by Moon from } S_2 \text{ to } S_3 \text{ expressed in inner loop true anomaly}} - \underbrace{\left\{2\pi - \frac{2\pi}{T_m} t_s\right\}}_{\text{angle traveled by Moon from } S_2 \text{ to } S_3 \text{ expressed in terms of inner loop period}} = 0 \tag{3}$$

where

A_m = radius of Moon's orbit

r_m = swingby distance of spacecraft from Moon

T_m = Moon's orbital period

These three equations specify the geometry constraints for a complete double-lunar swingby cycle.

2. REFORMULATION OF THE NECESSARY EQUATIONS

In this Keplerian formulation, the transfer from the inner orbit segment to the outer orbit segment is assumed to occur instantaneously when the spacecraft crosses the lunar orbit. We also assume a circular lunar orbit.

The three equations may be expressed solely in terms of three variables; a_i, P_i, and α, where

α ≡ bend angle; angle through which velocity of spacecraft is changed at S_1

a_i ≡ apogee distance of inner loop

P_i ≡ perigee distance of inner loop

The three nonlinear equations may then be solved numerically to determine these three unknowns. A description of the solution process is given in Section 3. We now show all the relations which allow Equations (1), (2), and (3) to be expressed in terms of α, a_i, and P_i.

For Equation (1), we require expressions for f_I, f_O, t_a, and t_s in terms of α, a_i, and P_i.

μ_E = gravitational parameter for Earth

$$a_I = \frac{1}{2}(a_i + P_i) \text{ = semimajor axis of inner loop}$$

$$e_I = \frac{(a_i - a_I)}{a_I} \text{ = eccentricity of inner loop}$$

$$\cos f_I = \frac{\left\{\frac{a_I}{A_m}(1 - e_I^2) - 1\right\}}{e_I}$$

$$f_I = \tan^{-1}\left\{\frac{\sin f_I}{\cos f_I}\right\} \text{ = true anomaly of inner loop at } S_1$$

$$v_s = \left\{\mu_E\left(\frac{2}{A_m} - \frac{1}{a_I}\right)\right\}^{1/2}, \text{ spacecraft speed at } S_1, \text{ in inner loop}$$

$$v_m = \frac{2\pi A_m}{T_m}, \text{ velocity of Moon in its orbit}$$

$$\left.\begin{aligned} v_{xm} &= -v_m \sin f_I \\ v_{ym} &= +v_m \cos f_I \end{aligned}\right\} \text{ x and y components of Moon's velocity}$$

$$v_x = -\sin f_I$$

$$v_y = e_I + \cos f_I$$

$$v_{xs} = v_s\left\{\frac{v_x}{(v_x^2 + v_y^2)^{1/2}}\right\}$$

$$v_{ys} = v_s\left\{\frac{v_y}{(v_x^2 + v_y^2)^{1/2}}\right\}$$

$$\left.\begin{aligned} v_{xO} &= v_{xm} + (v_{xs} - v_{xm}) \cos \alpha - (v_{ys} - v_{ym}) \sin \alpha \\ v_{yO} &= v_{ym} + (v_{ys} - v_{ym}) \cos \alpha + (v_{xs} - v_{xm}) \sin \alpha \end{aligned}\right\} \begin{array}{l}\text{rotation of spacecraft}\\ \text{velocity vector with}\\ \text{respect to Moon by}\\ \text{an angle } \alpha\end{array}$$

$$v_O = \left(v_{xO}^2 + v_{yO}^2\right)^{1/2}, \text{ spacecraft speed, at } S_1, \text{ in outer loop}$$

$$a_O = \left(\frac{2}{A_m} - \frac{v_O^2}{\mu_E}\right)^{-1}, \text{ semimajor axis of outer loop}$$

$$E_I = 2 \tan^{-1} \left\{\left(\frac{1 - e_I}{1 + e_I}\right)^{1/2} \tan \frac{f_I}{2}\right\}, \text{ eccentric anomaly of inner loop at } S_1$$

$$r_x = A_m \cos f_I$$

$$r_y = A_m \sin f_I$$

$$h = r_x v_{yO} - r_y v_{xO}, \text{ angular momentum of outer loop orbit}$$

$$f_O = \tan^{-1} \left\{\frac{h(r_x v_{xO} + r_y v_{yO})}{h^2 - \mu_E A_m}\right\}, \text{ true anomaly of outer loop, at } S_1$$

$$e_O = \left\{1 - \frac{h^2}{\mu_E a_O}\right\}^{1/2}, \text{ eccentricity of outer loop}$$

$$E_O = 2 \tan^{-1} \left\{\left(\frac{1 - e_O}{1 + e_O}\right)^{1/2} \tan \frac{f_O}{2}\right\}, \text{ eccentric anomaly of outer loop, at } S_1$$

$$T_O = \frac{2\pi}{\sqrt{\mu_E}} a_O^{3/2}, \text{ period of outer loop}$$

$$T_I = \frac{2\pi}{\sqrt{\mu_E}} a_I^{3/2}, \text{ period of inner loop}$$

$$t_a = \frac{1}{2} T_O - \frac{E_O - e_O \sin E_O}{2\pi/T_O}, \text{ time from } S_1 \text{ to apogee of outer loop}$$

$$t_s = 2\left\{\frac{E_I - e_I \sin E_I}{2\pi/T_I}\right\} + T_I, \text{ one inner loop period plus twice the time from perigee of inner loop to } S_1$$

For Equation (2), we express r_m in terms of previously defined quantities.

μ_m = gravitational parameter for Moon

$$v_\infty = \{(v_{xs} - v_{xm})^2 + (v_{ys} - v_{ym})^2\}^{1/2}$$

$$r_m = \frac{\mu_m}{v_\infty^2}\left(\frac{1}{\sin\frac{\alpha}{2}} - 1\right)$$

For Equation (3), all variables have been related previously to α, a_i, and P_i.

3. SOLUTION OF THE EQUATIONS

Writing the equations as

$$\Delta\dot{\omega} = f_1(\alpha, a_i, P_i) = 0$$

$$\Delta\theta_1 = f_2(\alpha, a_i, P_i) = 0$$

$$\Delta\theta_2 = f_3(\alpha, a_i, P_i) = 0$$

or

$$\vec{F}(\vec{x}) = \begin{Bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ f_3(\vec{x}) \end{Bmatrix} = 0, \quad \vec{x} \equiv \begin{Bmatrix} \alpha \\ a_i \\ P_i \end{Bmatrix}$$

we use the Newton-Raphson method to find the required solution vector, $\vec{x}$.

The method is to guess a new iterate, $\vec{x}^{N+1}$, from the previous iterate, $\vec{x}^{N}$, by

$$\vec{x}^{N+1} = \vec{x}^{N} - \frac{\vec{F}(\vec{x}^{N})}{J(\vec{x}^{N})}$$

where $J(\vec{x})$ is the Jacobian matrix

$$J(\vec{x}) = \begin{Bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{Bmatrix}$$

In fact, we define

$$\vec{Z}^{N+1} \equiv -(\vec{x}^{N+1} - \vec{x}^{N})$$

and write

$$J(\vec{x}^{N})\,\vec{Z}^{N+1} = \vec{F}(\vec{x}^{N})$$

This linear system can be solved for $\vec{Z}^{N+1}$ once J and $\vec{F}$ are evaluated at $\vec{x}^{N}$. Then $\vec{x}^{N+1}$ is found from

$$\vec{x}^{N+1} = -\vec{Z}^{N+1} + \vec{x}^{N}$$

The partial derivatives are estimated numerically via central differences

$$\frac{\partial f_i}{\partial x_j} = \frac{f_i(x_j + h_{ij}) - f_i(x_j - h_{ij})}{2h_{ij}}$$

The step size h_{ij} is chosen such that

$$|h_{ij}| \left| \frac{\partial f_i}{\partial x_j} \right| \approx \sqrt{\epsilon}\, |f_i|$$

where ϵ is a machine constant. For IBM double precision, $\epsilon = 2.2 \times 10^{-16}$. For IBM single precision, $\epsilon = 9.5 \times 10^{-7}$.

4. SAMPLE CALCULATION

Computer output for a sample solution of a double-lunar swingby orbit is shown in the appendix. First, the initial guess values for the three independent variables are shown.

$$\alpha = 0.2 \text{ radians} = 11.46 \text{ degrees}$$

$$a_i = 700,000 \text{ km}$$

$$P_i = 40,000 \text{ km}$$

Next, values for h, used to calculate partial derivatives, are displayed. Then the results of each iteration are shown in the form

$$J(\vec{x})\,\vec{Z}(\vec{x}) = \vec{F}(\vec{x})$$

After seven iterations, the final values of the independent variables are shown: α = 19.4205 degrees, a_i = 549,888 km, and P_i = 37,432 km. These results agree with those given on page 2-2 of Reference 2.

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APPENDIX — SAMPLE DOUBLE-LUNAR SWINGBY CALCULATION

The following shows details for a solution of a double-lunar swingby orbit.

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         PERIOD OF SATELLITE:     27.3217 DAYS,
            PERIOD OF PLANET:    365.2500 DAYS,
      PLANET GRAV. PARAMETER:   3.986E+05 KM**3/SEC**2,
   SATELLITE GRAV. PARAMETER: ( 3.986E+05 /      81.3700) KM**3/SEC**2
    SATELLITE ORBITAL RADIUS:     384399. KM


   INITIAL VALUES FOR X :

                  2.00000E-01   7.00000E+05   4.00000E+04

   ERROR TOLERANCES FOR SUCCESSIVE ITERATES :

                  1.00000E-06   1.00000E-06   1.00000E-06


   FROM HSET, H VALUES ARE:

     2.00000E-06   3.50000E+00   2.02500E+00
     5.00000E-07   4.37500E-01   2.02500E+00
     6.75000E-06   8.75000E-01   4.00000E-01



***** ITERATION NUMBER  1 *****

THE SOLUTION TO THE FOLLOWING MATRIX EQUATION,

|  3.02452E-07 -2.30101E-13  2.28045E-13 |  | Z(1) |       |-9.61618E-08 |
| -4.53911E+01 -4.36530E-05  5.97961E-06 |  | Z(2) |   =   |-2.51620E+00 |
|  1.13706E+00  1.24872E-05  2.72502E-05 |  | Z(3) |       | 1.74578E+00 |

 IS  Z :               -1.44445E-01   2.04596E+05  -2.36627E+04

ITERATION  1, X :   3.44445E-01   4.95404E+05   6.36627E+04
F-NORM =  2.62326E+00



***** ITERATION NUMBER  2 *****

THE SOLUTION TO THE FOLLOWING MATRIX EQUATION,

|  5.51903E-07 -3.29366E-13  3.67081E-13 |  | Z(1) |       | 4.00435E-08 |
| -1.15460E+01 -3.90971E-05  2.20171E-06 |  | Z(2) |   =   | 2.61698E+00 |
|  7.55482E-01  1.17588E-05  2.18728E-05 |  | Z(3) |       |-6.28797E-03 |

 IS  Z :                8.56683E-03  -6.74558E+04   3.56809E+04

ITERATION  2, X :   3.35878E-01   5.62860E+05   2.79819E+04
F-NORM =  8.89039E-01
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*****  ITERATION NUMBER  3 *****

THE SOLUTION TO THE FOLLOWING MATRIX EQUATION,

|  2.35513E-07 -5.67579E-13  6.72816E-13 |  | Z(1) |       |-1.38927E-08 |
|                                        |  |      |       |             |
| -3.32048E+01 -6.23036E-05  1.39124E-05 |  | Z(2) |   =   |-7.87616E-01 |
|                                        |  |      |       |             |
|  4.51755E-01  1.10362E-05  2.66817E-05 |  | Z(3) |       |-1.01423E-01 |

 IS  Z :            -3.44420E-03   1.24880E+04  -8.90822E+03

ITERATION  3, X :    3.39323E-01   5.50372E+05   3.68901E+04
F-NORM =  5.10358E-02



*****  ITERATION NUMBER  4 *****

THE SOLUTION TO THE FOLLOWING MATRIX EQUATION,

|  2.90579E-07 -5.71015E-13  5.48010E-13 |  | Z(1) |       |-4.62181E-10 |
|                                        |  |      |       |             |
| -2.78361E+01 -5.74992E-05  9.34668E-06 |  | Z(2) |   =   |-4.30255E-02 |
|                                        |  |      |       |             |
|  5.00948E-01  1.11918E-05  2.51765E-05 |  | Z(3) |       |-8.01032E-03 |

 IS  Z :             3.71385E-04   4.80824E+02  -5.39298E+02

ITERATION  4, X :    3.38951E-01   5.49891E+05   3.74294E+04
F-NORM =  2.36043E-04



*****  ITERATION NUMBER  5 *****

THE SOLUTION TO THE FOLLOWING MATRIX EQUATION,

|  2.94573E-07 -5.68936E-13  5.40713E-13 |  | Z(1) |       |-3.21530E-12 |
|                                        |  |      |       |             |
| -2.75352E+01 -5.71133E-05  9.10926E-06 |  | Z(2) |   =   |-2.13269E-04 |
|                                        |  |      |       |             |
|  5.05424E-01  1.12043E-05  2.51065E-05 |  | Z(3) |       |-2.27742E-05 |

 IS  Z :            -1.49764E-08   3.35773E+00  -2.40526E+00

ITERATION  5, X :    3.38951E-01   5.49888E+05   3.74318E+04
F-NORM =  4.30244E-09
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*****  ITERATION NUMBER  6 *****

THE SOLUTION TO THE FOLLOWING MATRIX EQUATION,

|  2.94591E-07 -5.68930E-13  5.40685E-13 |   | Z(1) |       |-4.24080E-17 |
|                                        |   |      |       |             |
| -2.75337E+01 -5.71117E-05  9.10824E-06 |   | Z(2) |   =   |-3.82426E-09 |
|                                        |   |      |       |             |
|  5.05440E-01  1.12043E-05  2.51062E-05 |   | Z(3) |       |-4.78174E-10 |

 IS  Z :              2.49561E-11   4.83695E-05  -4.11348E-05

ITERATION  6, X :     3.38951E-01   5.49888E+05   3.74318E+04
F-NORM =  1.01585E-14




*****  ITERATION NUMBER  7 *****

THE SOLUTION TO THE FOLLOWING MATRIX EQUATION,

|  2.94591E-07 -5.68930E-13  5.40685E-13 |   | Z(1) |       |-1.05879E-22 |
|                                        |   |      |       |             |
| -2.75337E+01 -5.71117E-05  9.10824E-06 |   | Z(2) |   =   |-8.21565E-15 |
|                                        |   |      |       |             |
|  5.05440E-01  1.12043E-05  2.51062E-05 |   | Z(3) |       |-1.94289E-15 |

 IS  Z :              5.44511E-17   9.81024E-11  -1.22264E-10

ITERATION  7, X :     3.38951E-01   5.49888E+05   3.74318E+04
F-NORM =  5.16254E-15

SOLUTION CONVERGED IN  7 ITERATIONS.

X :
     3.38951E-01   5.49888E+05   3.74318E+04
Z :
     5.44511E-17   9.81024E-11  -1.22264E-10
F :
     2.64698E-23   3.99680E-15   1.16573E-15



                   ALPHA, BEND ANGLE =   19.4205 DEGREES,
              APOGEE OF INNER ORBIT  =  5.4988764E+05 KM,
              PERIGEE OF INNER ORBIT =  3.7431801E+04 KM,
         ECCENTRICITY OF INNER ORBIT =  0.8725334,
              APOGEE OF OUTER ORBIT  =  8.9892445E+05 KM,
              PERIGEE OF OUTER ORBIT =  1.0420415E+05 KM,
         ECCENTRICITY OF OUTER ORBIT =  0.7922417,
                    SWINGBY DISTANCE =  2.7638929E+04 KM.
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REFERENCES

1. R. W. Farquhar and D. W. Dunham, "A New Trajectory Concept for Exploring the Earth's Geomagnetic Tail," *Journal of Guidance and Control 4*, 1981, pp. 192-196

2. D. W. Dunham and S. A. Davis, *Catalog of Double-Lunar Swingby Orbits for Exploring the Earth's Geomagnetic Tail*, Computer Sciences Corporation, CSC/TM-80/6322, October 1980