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OPTIMAL TRAJECTORIES FOR THE AEROASSISTED FLIGHT EXPERIMENT, PART 1, EQUATIONS OF MOTION IN AN EARTH-FIXED SYSTEM

by

A. MIELE, Z. G. ZHAO, AND W. Y. LEE

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Optimal Trajectories for the Aeroassisted Flight Experiment, Part 1, Equations of Motion in an Earth-Fixed System[1]

by

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Abstract. This report is the first of a series dealing with the determination of optimal trajectories for the aeroassisted flight experiment (AFE). The AFE refers to the study of the free flight of an autonomous spacecraft, shuttle-launched and shuttle-recovered. Its purpose is to gather atmospheric entry environmental data for use in designing aeroassisted orbital transfer vehicles (AOTV).

It is assumed that: the spacecraft is a particle of constant mass; the Earth is rotating with constant angular velocity; the Earth is an oblate planet, and the gravitational potential depends on both the radial distance and the latitude; however, harmonics of order higher than four are ignored; the atmosphere is at rest with respect to the Earth.

Under the above assumptions, the equations of motion for hypervelocity atmospheric flight (which can be used not only for AFE problems, but also for AOT problems and space shuttle problems) are derived in an Earth-fixed system. Transformation relations are supplied which allow one to pass from quantities computed in an Earth-fixed system to quantities computed in an inertial system, and viceversa.

Key Words. Flight mechanics, hypervelocity flight, atmospheric flight, coordinate systems, equations of motion, transformation techniques, optimal trajectories, aeroassisted flight experiment, aeroassisted orbital transfer, space shuttle reentry.

1. Introduction

This report is the first of a series dealing with the determination of optimal trajectories for the aeroassisted flight experiment (AFE). The AFE refers to the study of the free flight of an autonomous spacecraft, shuttle-launched and shuttle-recovered. Its purpose is to gather atmospheric entry environmental data for use in designing aeroassisted orbital transfer vehicles (AOTV).

It is assumed that: (a) the spacecraft is a particle of constant mass; (b) the Earth is rotating with constant angular velocity; (c) the atmosphere is at rest with respect to the Earth; (d) the Earth is an oblate planet, and the gravitational potential depends on both the radial distance and the latitude; however, harmonics of order higher than four are ignored.

Under the above assumptions, the equations of motion for hypervelocity atmospheric flight (which can be used not only for AFE problems, but also for AOT problems and space shuttle problems) are derived in an Earth-fixed system. Transformation relations are supplied which allow one to pass from quantities computed in an Earth-fixed system to quantities computed in an inertial system, and viceversa.

Previous Research. Previous research on the topics covered here can be found in Refs. 1-11. For the general theory of flight paths and coordinate systems, see Refs.1-2; for the equations of flight over a spherical Earth, see Refs. 1-3; for the perturbed motion about an oblate Earth, see Ref. 4; for AFE problems, see Ref. 5; for reentry problems, see Ref. 6; for methods of orbit determination, see Refs. 7-8; for the values of the astrophysical quantities, see Ref. 9; for the values of the characteristic constants of the oblate Earth, see Refs. 10-11.

Outline. Section 2 contains the notations, and Section 3 defines the basic coordinate systems. The relations between coordinate systems are discussed in Section 4,and the angular velocity (or evolutory velocity) is introduced in Section 5. The kinematical equations for an Earth-fixed system are derived in Section 6, and the dynamical equations are obtained in Section 7. Section 8 summarizes the results, and Section 9 presents the transformation relations which allow one to pass from quantities computed in an Earth-fixed system to quantities computed in an inertial system, and viceversa.

2. Notations

Throughout the paper, the following notations are employed:

a = acceleration, m/sec^2;

A = aerodynamic force, N;

C_D = drag coefficient;

C_L = lift coefficient;

C_Q = side force coefficient;

D = drag force, N;

f = latitudinal component of the gravitational acceleration, m/sec^2;

g = radial component of the gravitational acceleration, m/sec^2;

J_2 = characteristic constant of the Earth's gravitational field;

J_3 = characteristic constant of the Earth's gravitational field;

J_4 = characteristic constant of the Earth's gravitational field;

L = lift force, N;

m = mass, kg;

M = Mach number;

Q = side force, N;

r = radial distance, m;

r_e = equatorial radius, m;

r_p = polar radius, m;

R_e = Reynolds number;

S = reference surface area, m^2;

T = thrust force, N;

U = Earth's gravitational potential, m^2/sec^2;

V = velocity, m/sec;

W = gravitational force, N;

x = Cartesian coordinate, m;

y = Cartesian coordinate, m;

z = Cartesian coordinate, m;

α = angle of attack, rad;

γ = path inclination, rad;

θ = longitude, rad;

μ = bank angle, rad;

μ_e = Earth's gravitational constant, m^3/sec^2;

ρ = air density, kg/m^3;

σ = sideslip angle, rad;

ϕ = latitude, rad;

χ = heading angle, rad;

ω = angular velocity of the Earth with respect to an inertial system, rad/sec;

ω_{he} = angular velocity of the local horizon system with respect to the Earth axes system, rad/sec.

Subscripts

b = body axes system;

e = Earth axes system;

h = local horizon system;

i = inertial system;

w = wind axes system.

Superscripts

$\cdot$ = derivative with respect to time;

$\rightarrow$ = vector quantity.

3. Basic Coordinate Systems

The basic coordinate systems for flight over a spherical Earth are the Earth axes system $Ox_e y_e z_e$, the local horizon system $Px_h y_h z_h$, the wind axes system $Px_w y_w z_w$, and the body axes system $Px_b y_b z_b$.

3.1. Earth Axes System. The Earth axes system $Ox_e y_e z_e$ is a Cartesian reference frame which is rigidly attached to the Earth. Its origin O is the center of the Earth; the z_e-axis is aligned with the axis of rotation of the Earth and is positive northward; the axes x_e, y_e are orthogonal to the z_e-axis and are directed radially; the trihedral $Ox_e y_e z_e$ is right-handed. In particular, the plane x_e, y_e contains the fundamental parallel (the Equator); and the plane x_e, z_e contains the fundamental meridian (the Greenwich meridian). The symbols $\vec{i}_e, \vec{j}_e, \vec{k}_e$ denote the unit vectors of the Earth axes system.

3.2. Local Horizon System. The local horizon system $Px_h y_h z_h$ is a Cartesian reference frame defined as follows. Its origin P is identical with the instantaneous position of the spacecraft; the z_h-axis is directed radially (that is, vertical) and is positive downward; the axes x_h, y_h are orthogonal to the z_h-axis (therefore, they are tangent to the spherical surface through P; they form the so-called local horizon plane); the trihedral $Px_h y_h z_h$ is right-handed. In particular, the x_h-axis is tangent to the local parallel through P and is positive eastward; the y_h-axis is tangent to the local meridian through P and is positive southward. The symbols $\vec{i}_h, \vec{j}_h, \vec{k}_h$ denote the unit vectors of the local horizon system.

3.3. Wind Axes System. The wind axes system $Px_w y_w z_w$ is a Cartesian reference frame defined as follows. Its origin P is identical with the instantaneous position of the spacecraft; the x_w-axis is tangent to the flight path (relative velocity) and is positive forward; the axes y_w, z_w

are orthogonal to the x_w-axis and are such that the trihedral $Px_wy_wz_w$ is right-handed. In particular, the z_w-axis is contained in the plane of symmetry of the spacecraft and is positive downward for the normal flight attitude of the spacecraft; the y_w-axis is positive rightward for the normal flight attitude of the spacecraft. The symbols $\vec{i}_w, \vec{j}_w, \vec{k}_w$ denote the unit vectors of the wind axes system.

3.4. Body Axes System. The body axes system $Px_by_bz_b$ is a Cartesian reference frame defined as follows. Its origin P is identical with the instantaneous position of the spacecraft; the y_b-axis is orthogonal to the plane of symmetry of the spacecraft and is positive rightward; the axes x_b, z_b are orthogonal to the y_b-axis, are contained in the plane of symmetry, and are such that the trihedral $Px_by_bz_b$ is right-handed. In particular, the x_b-axis is positive forward, the y_b-axis is positive rightward, and the z_b-axis is positive downward for the normal flight attitude of the spacecraft. The symbols $\vec{i}_b, \vec{j}_b, \vec{k}_b$ denote the unit vectors of the body axes system.

4. Relations between Coordinate Systems

In this section, the relationships between the different coordinate systems are derived; more specifically, attention is focused on the following system pairs: Earth axes-local horizon; local horizon-wind axes; and wind axes-body axes.

We recall that, in the Earth axes system, a point P can be described via its Cartesian coordinates x_e, y_e, z_e. Alternatively, P can be described via its spherical coordinate r, θ, ϕ. Here, r is the radial distance from the center of the Earth; θ is the longitude, positive eastward; and ϕ is the latitude, positive northward.

4.1. Transformation from Earth Axes to Local Horizon. The local horizon system $Px_hy_hz_h$ can be obtained from the Earth axes system $Ox_ey_ez_e$ by means of the combination of four rotations and one translation. This requires the definition of four intermediate coordinate systems: the system $Ox_1y_1z_1$; the system $Ox_2y_2z_2$; the system $Px_3y_3z_3$; and the system $Px_4y_4z_4$.

The system $Ox_1y_1z_1$ is obtained from the Earth axes system $Ox_ey_ez_e$ by means of the counterclockwise rotation θ around the z_e-axis. Note that the z_1-axis is the same as the z_e-axis, that the axes x_1, y_1 are contained in the equatorial plane, and that the axes x_1, z_1 are contained in a meridian plane. The symbols $\vec{i}_1, \vec{j}_1, \vec{k}_1$ denote the unit vectors of the system $Ox_1y_1z_1$.

The system $Ox_2y_2z_2$ is obtained from the system $Ox_1y_1z_1$ by means of the clockwise rotation ϕ around the y_1-axis. Note that the y_2-axis is the same as the y_1-axis, that the axes x_2, z_2 are contained in a meridian plane, and that the axes y_2, z_2 are contained in a plane parallel to the local horizon plane. The symbols $\vec{i}_2, \vec{j}_2, \vec{k}_2$ denote the unit vectors of the system $Ox_2y_2z_2$.

The system $Px_3y_3z_3$ is obtained from the system $Ox_2y_2z_2$ by means of the radial translation r, leading from point 0 to point P. Since there is no rotation, the axes x_3,y_3,z_3 are parallel to the axes x_2,y_2,z_2; in particular, the axes x_3,z_3 are contained in a meridian plane, while the axes y_3,z_3 are contained in the local horizon plane. The symbols $\vec{i}_3,\vec{j}_3,\vec{k}_3$ denote the unit vectors of the system $Px_3y_3z_3$.

The system $Px_4y_4z_4$ is obtained from the system $Px_3y_3z_3$ by means of the counterclockwise rotation $\pi/2$ around the z_3-axis. Note that the z_4-axis is the same as the z_3-axis, that the axes y_4,z_4 are contained in a meridian plane, while the axes x_4,z_4 are contained in the local horizon plane. The symbols $\vec{i}_4,\vec{j}_4,\vec{k}_4$ denote the unit vectors of the system $Px_4y_4z_4$.

The local horizon system $Px_hy_hz_h$ is obtained from the system $Px_4y_4z_4$ by means of the clockwise rotation $\pi/2$ around the x_4-axis. Note that the x_h-axis is the same as the x_4-axis, that the axes y_h,z_h are contained in a meridian plane, and the axes x_h,y_h are contained in the local horizon plane.

In vector-matrix notation, the successive transformations leading from one coordinate system to another can be expressed as follows:

$$\begin{bmatrix} \vec{i}_1 \\ \vec{j}_1 \\ \vec{k}_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{i}_e \\ \vec{j}_e \\ \vec{k}_e \end{bmatrix}, \tag{1a}$$

$$\begin{bmatrix} \vec{i}_2 \\ \vec{j}_2 \\ \vec{k}_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \vec{i}_1 \\ \vec{j}_1 \\ \vec{k}_1 \end{bmatrix}, \tag{1b}$$

and

$$\begin{bmatrix} \vec{i}_3 \\ \vec{j}_3 \\ \vec{k}_3 \end{bmatrix} = \begin{bmatrix} \vec{i}_2 \\ \vec{j}_2 \\ \vec{k}_2 \end{bmatrix}, \tag{2a}$$

$$\begin{bmatrix} \vec{i}_4 \\ \vec{j}_4 \\ \vec{k}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{i}_3 \\ \vec{j}_3 \\ \vec{k}_3 \end{bmatrix}, \tag{2b}$$

$$\begin{bmatrix} \vec{i}_h \\ \vec{j}_h \\ \vec{k}_h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{i}_4 \\ \vec{j}_4 \\ \vec{k}_4 \end{bmatrix}. \tag{2c}$$

Equations (1) imply that

$$\begin{bmatrix} \vec{i}_2 \\ \vec{j}_2 \\ \vec{k}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \cos\phi & \sin\theta \cos\phi & \sin\phi \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta \sin\phi & -\sin\theta \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \vec{i}_e \\ \vec{j}_e \\ \vec{k}_e \end{bmatrix}, \tag{3}$$

while Eqs. (2) imply that

$$\begin{bmatrix} \vec{i}_h \\ \vec{j}_h \\ \vec{k}_h \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{i}_2 \\ \vec{j}_2 \\ \vec{k}_2 \end{bmatrix}. \tag{4}$$

Therefore, upon combining Eqs. (3)-(4), we see that

$$\begin{bmatrix} \vec{i}_h \\ \vec{j}_h \\ \vec{k}_h \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ \cos\theta\,\sin\phi & \sin\theta\,\sin\phi & -\cos\phi \\ -\cos\theta\,\cos\phi & -\sin\theta\,\cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \vec{i}_e \\ \vec{j}_e \\ \vec{k}_e \end{bmatrix}, \tag{5a}$$

with the implication that

$$\begin{bmatrix} \vec{i}_e \\ \vec{j}_e \\ \vec{k}_e \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta\,\sin\phi & -\cos\theta\,\cos\phi \\ \cos\theta & \sin\theta\,\sin\phi & -\sin\theta\,\cos\phi \\ 0 & -\cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \vec{i}_h \\ \vec{j}_h \\ \vec{k}_h \end{bmatrix}. \tag{5b}$$

4.2. Transformation from Local Horizon to Wind Axes. The wind axes system $Px_wy_wz_w$ can be obtained from the local horizon system $Px_hy_hz_h$ by means of the combination of three rotations. This requires the definition of two intermediate coordinate systems: the system $Px_5y_5z_5$ and the system $Px_6y_6z_6$.

The system $Px_5y_5z_5$ is obtained from the local horizon system $Px_hy_hz_h$ by means of the counterclockwise rotation χ around the z_h-axis. Note that the z_5-axis is the same as the z_h-axis, that the axes x_5,y_5 are contained in the local horizon plane, and that the axes x_5,z_5 are contained in the plane $(\vec{OP},\vec{V})$, where $\vec{OP}$ is the radius vector connecting the points O and P and $\vec{V}$ is the spacecraft velocity vector. Also note that the axis x_5 has the direction of the projected velocity vector $\vec{V}_p$; this is the projection of $\vec{V}$ on the local horizon. The angle χ is called the heading angle and is positive if the projected velocity vector $\vec{V}_p$ is directed outward with respect to the local parallel. The symbols $\vec{i}_5,\vec{j}_5,\vec{k}_5$ denote the unit vectors of the system $Px_5y_5z_5$.

The system $Px_6y_6z_6$ is obtained from the system $Px_5y_5z_5$ by means of the counterclockwise rotation γ around the y_5-axis. Note that the y_6-axis is the same as the y_5-axis and that the axes x_6, z_6 are contained in the plane $(\vec{OP}, \vec{V})$. Also note that the x_6-axis is positive forward and that the z_6-axis is positive downward. The angle γ is called the path inclination and is positive if the velocity vector $\vec{V}$ is inclined upward with respect to the local horizon. The symbols $\vec{i}_6, \vec{j}_6, \vec{k}_6$ denote the unit vectors of the system $Px_6y_6z_6$.

The wind axes system $Px_wy_wz_w$ is obtained from the system $Px_6y_6z_6$ by means of the counterclockwise rotation μ around the x_6-axis. Note that the x_w-axis is the same as the x_6-axis. Also, note that the x_w-axis is positive forward, the y_w-axis is positive rightward, and the z_w-axis is positive downward and is contained in the plane of symmetry of the spacecraft. The angle μ is called the angle of bank and is positive if the spacecraft is banked to the right.

In vector-matrix notation, the successive transformations leading from one coordinate system to another can be expressed as follows:

$$\begin{bmatrix} \vec{i}_5 \\ \vec{j}_5 \\ \vec{k}_5 \end{bmatrix} = \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{i}_h \\ \vec{j}_h \\ \vec{k}_h \end{bmatrix}, \tag{6a}$$

$$\begin{bmatrix} \vec{i}_6 \\ \vec{j}_6 \\ \vec{k}_6 \end{bmatrix} = \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} \vec{i}_5 \\ \vec{j}_5 \\ \vec{k}_5 \end{bmatrix}, \tag{6b}$$

$$\begin{bmatrix} \vec{i}_w \\ \vec{j}_w \\ \vec{k}_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{bmatrix} \begin{bmatrix} \vec{i}_6 \\ \vec{j}_6 \\ \vec{k}_6 \end{bmatrix}. \tag{6c}$$

Equations (6) lead to

$$\begin{bmatrix} \vec{i}_w \\ \vec{j}_w \\ \vec{k}_w \end{bmatrix} = \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma \\ \sin\mu\sin\gamma\cos\chi - \cos\mu\sin\chi & \sin\mu\sin\gamma\sin\chi + \cos\mu\cos\chi & \sin\mu\cos\gamma \\ \cos\mu\sin\gamma\cos\chi + \sin\mu\sin\chi & \cos\mu\sin\gamma\sin\chi - \sin\mu\cos\chi & \cos\mu\cos\gamma \end{bmatrix} \begin{bmatrix} \vec{i}_h \\ \vec{j}_h \\ \vec{k}_h \end{bmatrix}, \tag{7a}$$

with the implication that

$$\begin{bmatrix} \vec{i}_h \\ \vec{j}_h \\ \vec{k}_h \end{bmatrix} = \begin{bmatrix} \cos\gamma\cos\chi & \sin\mu\sin\gamma\cos\chi - \cos\mu\sin\chi & \cos\mu\sin\gamma\cos\chi + \sin\mu\sin\chi \\ \cos\gamma\sin\chi & \sin\mu\sin\gamma\sin\chi + \cos\mu\cos\chi & \cos\mu\sin\gamma\sin\chi - \sin\mu\cos\chi \\ -\sin\gamma & \sin\mu\cos\gamma & \cos\mu\cos\gamma \end{bmatrix} \begin{bmatrix} \vec{i}_w \\ \vec{j}_w \\ \vec{k}_w \end{bmatrix}. \tag{7b}$$

4.3. Transformation from Wind Axes to Body Axes. The body system $Px_by_bz_b$ can be obtained from the wind axes system $Px_wy_wz_w$ by means of the combination of two rotations. This requires the definition of one intermediate coordinate system, the system $Px_7y_7z_7$.

The system $Px_7y_7z_7$ is obtained from the wind axes system $Px_wy_wz_w$ by means of the counterclockwise rotation σ around the z_w-axis. Note that the z_7-axis is the same as the z_w-axis and that the axes x_7, z_7 are contained in the plane of symmetry of the spacecraft. Also note that the axis x_7

is positive forward, the axis y_7 is positive rightward, and the axis z_7 is positive downward. The angle σ is called the sideslip angle and is positive if the velocity vector $\vec{V}$ is directed leftward with respect to the plane of symmetry of the spacecraft.

The body axes system $Px_by_bz_b$ is obtained from the system $Px_7y_7z_7$ by means of the counterclockwise rotation α around the y_7-axis. Note that the y_b-axis is the same as the y_7-axis and that the axes x_b, z_b are contained in the plane of symmetry of the spacecraft. Also note that the axis x_b is positive forward, the axis y_b is positive rightward, and the axis z_b is positive downward. The angle α is called the angle of attack and is positive if the velocity vector $\vec{V}$ is directed downward with respect to the x_b-axis of the spacecraft.

In vector-matrix notation, the successive transformations leading from one coordinate system to another can be expressed as follows:

$$\begin{bmatrix} \vec{i}_7 \\ \vec{j}_7 \\ \vec{k}_7 \end{bmatrix} = \begin{bmatrix} \cos\sigma & \sin\sigma & 0 \\ -\sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{i}_w \\ \vec{j}_w \\ \vec{k}_w \end{bmatrix}, \tag{8a}$$

$$\begin{bmatrix} \vec{i}_b \\ \vec{j}_b \\ \vec{k}_b \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \vec{i}_7 \\ \vec{j}_7 \\ \vec{k}_7 \end{bmatrix}, \tag{8b}$$

with the implication that

$$\begin{bmatrix} \vec{i}_b \\ \vec{j}_b \\ \vec{k}_b \end{bmatrix} = \begin{bmatrix} \cos\alpha\ \cos\sigma & \cos\alpha\ \sin\sigma & -\sin\alpha \\ -\sin\sigma & \cos\sigma & 0 \\ \sin\alpha\ \cos\sigma & \sin\alpha\ \sin\sigma & \cos\alpha \end{bmatrix} \begin{bmatrix} \vec{i}_w \\ \vec{j}_w \\ \vec{k}_w \end{bmatrix}, \tag{9a}$$

and that

$$\begin{bmatrix} \vec{i}_w \\ \vec{j}_w \\ \vec{k}_w \end{bmatrix} = \begin{bmatrix} \cos\alpha\ \cos\sigma & -\sin\sigma & \sin\alpha\ \cos\sigma \\ \cos\alpha\ \sin\sigma & \cos\sigma & \sin\alpha\ \sin\sigma \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \vec{i}_b \\ \vec{j}_b \\ \vec{k}_b \end{bmatrix}. \tag{9b}$$

5. Angular Velocity

In this section, we compute the angular velocity (or evolutory velocity) of the local horizon system with respect to the Earth axes system. To do so, consider the behavior of the spacecraft between the time instants t and t + dt, and denote by $d\vec{\Omega}_{he}$ the infinitesimal vectorial rotation of the local horizon system with respect to the Earth axes system. This infinitesimal vectorial rotation can be decomposed into partial rotations as follows:

$$d\vec{\Omega}_{he} = d\vec{\Omega}_{h4} + d\vec{\Omega}_{43} + d\vec{\Omega}_{32} + d\vec{\Omega}_{21} + d\vec{\Omega}_{1e}, \tag{10}$$

with

$$d\vec{\Omega}_{h4} = 0, \tag{11a}$$

$$d\vec{\Omega}_{43} = 0, \tag{11b}$$

$$d\vec{\Omega}_{32} = 0, \tag{11c}$$

$$d\vec{\Omega}_{21} = -d\phi\ \vec{j}_1, \tag{11d}$$

$$d\vec{\Omega}_{1e} = d\theta\ \vec{k}_e. \tag{11e}$$

Here, $d\theta$ denotes the infinitesimal change of the longitude and $d\phi$ denotes the infinitesimal change of the latitude. Note that the rotation $d\theta$ occurs around the z_e-axis and is positive counterclockwise and that the rotation $d\phi$ occurs around the y_1-axis and is positive clockwise. This explains the difference in the signs appearing on the right-hand sides of Eqs. (11d) and (11e).

Upon combining Eqs. (10)-(11), we see that the infinitesimal vectorial rotation of the local horizon system with respect to the Earth axes system can be written as

$$d\vec{\Omega}_{he} = d\theta\ \vec{k}_e - d\phi\ \vec{j}_1. \tag{12}$$

As a consequence, the angular velocity of the local horizon system with respect to the Earth axes system is given by

$$\vec{\omega}_{he} = d\vec{\Omega}_{he}/dt = \dot{\theta}\vec{k}_e - \dot{\phi}\vec{j}_1. \tag{13}$$

In the light of Eqs. (1)-(5), the unit vectors $\vec{k}_e$ and $\vec{j}_1$ can be expressed in terms of the unit vectors of the local horizon system as follows:

$$\vec{k}_e = -\cos\phi\ \vec{j}_h - \sin\phi\ \vec{k}_h, \tag{14a}$$

$$\vec{j}_1 = \vec{i}_h, \tag{14b}$$

so that

$$\vec{\omega}_{he} = -\dot{\phi}\vec{i}_h - \dot{\theta}\cos\phi\ \vec{j}_h - \dot{\theta}\sin\phi\ \vec{k}_h. \tag{15}$$

Next, Poisson's formulas are employed to compute the derivatives of the unit vectors of the local horizon system with respect to time:

$$d\vec{i}_h/dt = \vec{\omega}_{he} \times \vec{i}_h, \tag{16a}$$

$$d\vec{j}_h/dt = \vec{\omega}_{he} \times \vec{j}_h, \tag{16b}$$

$$d\vec{k}_h/dt = \vec{\omega}_{he} \times \vec{k}_h. \tag{16c}$$

Upon combining Eqs. (15)-(16), we obtain the relations

$$d\vec{i}_h/dt = -(\dot{\theta}\sin\phi)\vec{j}_h + (\dot{\theta}\cos\phi)\vec{k}_h, \tag{17a}$$

$$d\vec{j}_h/dt = (\dot{\theta}\sin\phi)\vec{i}_h - \dot{\phi}\vec{k}_h, \tag{17b}$$

$$d\vec{k}_h/dt = -(\dot{\theta}\cos\phi)\vec{i}_h + \dot{\phi}\vec{j}_h, \tag{17c}$$

whose vector-matrix form is the following:

$$(d/dt)\begin{bmatrix}\vec{i}_h\\ \vec{j}_h\\ \vec{k}_h\end{bmatrix} = \begin{bmatrix}0 & -\dot{\theta}\sin\phi & \dot{\theta}\cos\phi\\ \dot{\theta}\sin\phi & 0 & -\dot{\phi}\\ -\dot{\theta}\cos\phi & \dot{\phi} & 0\end{bmatrix}\begin{bmatrix}\vec{i}_h\\ \vec{j}_h\\ \vec{k}_h\end{bmatrix}. \tag{18}$$

It is interesting to note that Eq. (18) can also be obtained by taking the time derivative of Eq. (5a) and using Eq. (5b).

6. Kinematical Equations

In this section, we derive the scalar relationships corresponding to the vectorial equation

$$d\vec{OP}/dt = \vec{V}. \tag{19}$$

Here, $\vec{V}$ denotes the velocity of the spacecraft with respect to the Earth and $\vec{OP}$ denotes the position vector joining the center of the Earth O with the spacecraft position P.

First, we observe that the position vector $\vec{OP}$ is given by

$$\vec{OP} = -r\vec{k}_h, \tag{20}$$

where r is the radial distance from the center of the Earth and $\vec{k}_h$ is the third unit vector of the local horizon system. As a consequence, the time derivative of $\vec{OP}$ can be written as

$$d\vec{OP}/dt = -\dot{r}\vec{k}_h - r(d\vec{k}_h/dt), \tag{21}$$

where $d\vec{k}_h/dt$ is given by Eq. (17c). Therefore, upon combining Eqs. (17c) and (21), we obtain the relation

$$d\vec{OP}/dt = \dot{\theta}r\cos\phi\ \vec{i}_h - \dot{\phi}r\vec{j}_h - \dot{r}\vec{k}_h. \tag{22}$$

Next, we observe that the velocity vector $\vec{V}$ is given by

$$\vec{V} = V\vec{i}_w, \tag{23}$$

where V is the velocity modulus and $\vec{i}_w$ is the first unit vector of the wind axes system. In the light of Eq. (7a), the unit vector $\vec{i}_w$ can be written as

$$\vec{i}_w = \cos\gamma \, \cos\chi \, \vec{i}_h + \cos\gamma \, \sin\chi \, \vec{j}_h - \sin\gamma \, \vec{k}_h. \tag{24}$$

Therefore, upon combining Eqs.(23) and (24), we obtain the relation

$$\vec{V} = V\cos\gamma \, \cos\chi \, \vec{i}_h + V\cos\gamma \, \sin\chi \, \vec{j}_h - V\sin\gamma \, \vec{k}_h. \tag{25}$$

Finally, upon combining Eqs.(19), (22), and (25), and upon projecting the resulting vectorial equation on the axes of the local horizon system, we obtain the following scalar form of the kinematical equations:

$$\dot{\theta} = V\cos\gamma \, \cos\chi / r\cos\phi, \tag{26a}$$

$$\dot{\phi} = -V\cos\gamma \, \sin\chi / r, \tag{26b}$$

$$\dot{r} = V\sin\gamma. \tag{26c}$$

7. Dynamical Equations

In this section, we derive the scalar relationships corresponding to the vectorial equation

$$\vec{T} + \vec{A} + \vec{W} = m\vec{a}_i, \tag{27}$$

where $\vec{T}$ is the thrust, $\vec{A}$ is the aerodynamic force, $\vec{W}$ is the gravitational force, m is the mass of the spacecraft, and $\vec{a}_i$ is the inertial acceleration. We consider the case where the engine is shut-off, so that

$$\vec{T} = 0, \tag{28}$$

and the mass of the spacecraft is constant. Hence, Eq. (27) is written as

$$\vec{A} + \vec{W} = m\vec{a}_i. \tag{29}$$

Because of the theorem of composition of the accelerations, the inertial acceleration can be written as the sum of the relative acceleration $d\vec{V}/dt$ (acceleration with respect to the Earth), the Coriolis acceleration $2\vec{\omega} \times \vec{V}$, and the transport acceleration $\vec{\omega} \times (\vec{\omega} \times \vec{OP})$:

$$\vec{a}_i = d\vec{V}/dt + 2\vec{\omega} \times \vec{V} + \vec{\omega} \times (\vec{\omega} \times \vec{OP}). \tag{30}$$

Here, $\vec{\omega}$ is the angular velocity of the Earth with respect to an inertial system. Note that $\vec{\omega}$ is constant and is aligned with the axis of rotation of the Earth. Therefore, upon combining Eqs. (29)-(30), we obtain the vectorial equation

$$\vec{A} + \vec{W} = m[d\vec{V}/dt + 2\vec{\omega} \times \vec{V} + \vec{\omega} \times (\vec{\omega} \times \vec{OP})]. \tag{31}$$

We now compute the components of the vectors appearing in Eq. (31) on the axes of the local horizon system.

7.1. Aerodynamic Force. The components of the aerodynamic force on the wind axes are the drag D, the side force Q, and the lift L. Therefore, the aerodynamic force can be written as

$$\vec{A} = -D\vec{i}_w - Q\vec{j}_w - L\vec{k}_w. \tag{32}$$

No special significance is implied in the signs appearing on the right-hand side of Eq. (32). These signs merely reflect the conventions adopted in this report with regard to the positive values for the drag, the side force, and the lift.

In vector-matrix form, Eq. (32) can be rewritten as follows:

$$\vec{A} = -\begin{bmatrix} D & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & L \end{bmatrix}\begin{bmatrix} \vec{i}_w \\ \vec{j}_w \\ \vec{k}_w \end{bmatrix}. \tag{33}$$

Therefore, upon combining Eqs. (7a) and (33), we obtain the relation

$$\vec{A} = \begin{bmatrix} -D\cos\gamma\cos\chi & -D\cos\gamma\sin\chi & D\sin\gamma \\ -Q\sin\mu\sin\gamma\cos\chi + Q\cos\mu\sin\chi & -Q\sin\mu\sin\gamma\sin\chi - Q\cos\mu\cos\chi & -Q\sin\mu\cos\gamma \\ -L\cos\mu\sin\gamma\cos\chi - L\sin\mu\sin\chi & -L\cos\mu\sin\gamma\sin\chi + L\sin\mu\cos\chi & -L\cos\mu\cos\gamma \end{bmatrix}\begin{bmatrix} \vec{i}_h \\ \vec{j}_h \\ \vec{k}_h \end{bmatrix}. \tag{34}$$

7.2. Gravitational Force. Here, we assume that the Earth is an oblate planet and that its mass has radial symmetry with respect to the axis of rotation. Because the equatorial radius r_e is larger than the polar radius r_p, the gravity force $\vec{W}$ has two components: the radial component mg, directed toward the center of the Earth, and the latitudinal component mf, tangent to the local meridian and directed toward the Equator. Therefore, the gravity force can be written as

$$\vec{W} = mf\vec{j}_h + mg\vec{k}_h. \tag{35}$$

The radial component g and the latitudinal component f of the acceleration of gravity are related to the Earth's gravitational potential U by the expressions

$$g = \partial U/\partial r, \quad f = (1/r)\partial U/\partial \phi, \tag{36}$$

where

$$U = -(\mu_e/r)[1+J_2(r_e/r)^2 H_2 + J_3(r_e/r)^3 H_3 + J_4(r_e/r)^4 H_4], \tag{37a}$$

$$H_2 = 1/2 - (3/2)\sin^2\phi, \tag{37b}$$

$$H_3 = (3/2)\sin\phi - (5/2)\sin^3\phi, \tag{37c}$$

$$H_4 = -(3/8) + (30/8)\sin^2\phi - (35/8)\sin^4\phi. \tag{37d}$$

Here, μ_e is the Earth's gravitational constant, r_e is the equatorial radius, and J_2, J_3, J_4 denote the characteristic constants of the Earth's gravitational field. Note that the expression for U is approximate, since harmonics of order higher than four are ignored.

Upon combining Eqs. (36)-(37), we see that the components of the acceleration of gravity can be written as

$$g = (\mu_e/r^2)[1 + J_2(r_e/r)^2G_2 + J_3(r_e/r)^3G_3 + J_4(r_e/r)^4G_4], \tag{38a}$$

$$G_2 = 3/2 - (9/2)\sin^2\phi, \tag{38b}$$

$$G_3 = 6\sin\phi - 10\sin^3\phi, \tag{38c}$$

$$G_4 = -15/8 + (150/8)\sin^2\phi - (175/8)\sin^4\phi, \tag{38d}$$

and

$$f = (\mu_e/r^2)[J_2(r_e/r)^2F_2 + J_3(r_e/r)^3F_3 + J_4(r_e/r)^4F_4], \tag{39a}$$

$$F_2 = 3\sin\phi\cos\phi, \tag{39b}$$

$$F_3 = -(3/2)\cos\phi + (15/2)\sin^2\phi\cos\phi, \tag{39c}$$

$$F_4 = -(15/2)\sin\phi\cos\phi + (35/2)\sin^3\phi\cos\phi. \tag{39d}$$

7.3. Relative Acceleration. Let V_{xh}, V_{yh}, V_{zh} denote the components of the relative velocity on the local horizon system,

$$V_{xh} = V\cos\gamma\cos\chi, \tag{40a}$$

$$V_{yh} = V\cos\gamma\sin\chi, \tag{40b}$$

$$V_{zh} = -V\sin\gamma. \tag{40c}$$

With this understanding, the relative velocity (25) can be rewritten as

$$\vec{V} = V_{xh}\vec{i}_h + V_{yh}\vec{j}_h + V_{zh}\vec{k}_h. \tag{41}$$

Therefore, the relative acceleration is given by

$$d\vec{V}/dt = \dot{V}_{xh}\vec{i}_h + \dot{V}_{yh}\vec{j}_h + \dot{V}_{zh}\vec{k}_h$$
$$+ V_{xh}(d\vec{i}_h/dt) + V_{yh}(d\vec{j}_h/dt) + V_{zh}(d\vec{k}_h/dt). \tag{42}$$

If we combine Eqs. (17) and (26), the time derivatives of the unit vectors of the local horizon system can be written as

$$d\vec{i}_h/dt = -(V\cos\gamma\ \cos\chi\ \tan\phi/r)\vec{j}_h + (V\cos\gamma\ \cos\chi/r)\vec{k}_h, \tag{43a}$$

$$d\vec{j}_h/dt = (V\cos\gamma\ \cos\chi\ \tan\phi/r)\vec{i}_h + (V\cos\gamma\ \sin\chi/r)\vec{k}_h, \tag{43b}$$

$$d\vec{k}_h/dt = -(V\cos\gamma\ \cos\chi/r)\vec{i}_h - (V\cos\gamma\ \sin\chi/r)\vec{j}_h. \tag{43c}$$

Upon combining Eqs. (40), (42), (43), the relative acceleration becomes

$$d\vec{V}/dt = (\dot{V}_{xh} + V^2\cos^2\gamma\ \cos\chi\ \sin\chi\ \tan\phi/r + V^2\cos\gamma\ \sin\gamma\ \cos\chi/r)\vec{i}_h$$
$$+ (\dot{V}_{yh} - V^2\cos^2\gamma\ \cos^2\chi\ \tan\phi/r + V^2\cos\gamma\ \sin\gamma\ \sin\chi/r)\vec{j}_h$$
$$+ (\dot{V}_{zh} + V^2\cos^2\gamma/r)\vec{k}_h. \tag{44}$$

7.4. <u>Coriolis Acceleration</u>. The angular velocity of the Earth with respect to an inertial system is given by

$$\vec{\omega} = \omega\vec{k}_e. \tag{45}$$

Here, $\vec{k}_e$ is the third unit vector of the Earth axes system, which is given by [see Eq. (5b)]

$$\vec{k}_e = -\cos\phi\ \vec{j}_h - \sin\phi\ \vec{k}_h. \tag{46}$$

Hence, Eq. (45) becomes

$$\vec{\omega} = -\omega\cos\phi\ \vec{j}_h - \omega\sin\phi\ \vec{k}_h. \tag{47}$$

Next,we recall Eq. (25),

$$\vec{V} = V\cos\gamma\ \cos\chi\ \vec{i}_h + V\cos\gamma\ \sin\chi\ \vec{j}_h - V\sin\gamma\ \vec{k}_h. \tag{48}$$

Therefore, the Coriolis acceleration becomes

$$\begin{aligned} 2\vec{\omega} \times \vec{V} &= 2\omega V(\sin\gamma\ \cos\phi + \cos\gamma\ \sin\chi\ \sin\phi)\vec{i}_h \\ &\quad - 2\omega V\cos\gamma\ \cos\chi\ \sin\phi\ \vec{j}_h + 2\omega V\cos\gamma\ \cos\chi\ \cos\phi\ \vec{k}_h. \end{aligned} \tag{49}$$

7.5. Transport Acceleration. We recall that the vector connecting the center of the Earth with the instantaneous position of the spacecraft is given by [see Eq. (20)]

$$\vec{OP} = -r\vec{k}_h. \tag{50}$$

We also recall that the angular velocity of the Earth is given by [see Eq. (47)]

$$\vec{\omega} = -\omega\cos\phi\ \vec{j}_h - \omega\sin\phi\ \vec{k}_h. \tag{51}$$

Therefore, the transport acceleration becomes

$$\vec{\omega} \times (\vec{\omega} \times \vec{OP}) = -\omega^2 r\ \cos\phi\ \sin\phi\ \vec{j}_h + \omega^2 r\cos^2\phi\ \vec{k}_h. \tag{52}$$

7.6. Scalar Equations. Next, we combine Eqs. (31), (34), (35), (44), (49), (52). Upon projecting the resulting vectorial equation on the axes of the local horizon system, we obtain the following scalar equations:

$$\dot{V}_{xh} = -(D/m)\cos\gamma\ \cos\chi + (Q/m)(\cos\mu\ \sin\chi - \sin\mu\ \sin\gamma\ \cos\chi)$$

$$- \ (L/m)(\sin\mu\ \sin\chi + \cos\mu\ \sin\gamma\ \cos\chi)$$

$$- \ (V^2/r)(\cos^2\gamma\ \cos\chi\ \sin\chi\ \tan\phi + \cos\gamma\ \sin\gamma\ \cos\chi)$$

$$- \ 2\omega V(\sin\gamma\ \cos\phi + \cos\gamma\ \sin\chi\ \sin\phi), \tag{53a}$$

$$\dot{V}_{yh} = -(D/m)\cos\gamma\ \sin\chi - (Q/m)(\cos\mu\ \cos\chi + \sin\mu\ \sin\gamma\ \sin\chi)$$

$$+ \ (L/m)(\sin\mu\ \cos\chi - \cos\mu\ \sin\gamma\ \sin\chi) + f$$

$$+ \ (V^2/r)(\cos^2\gamma\ \cos^2\chi\ \tan\phi - \cos\gamma\ \sin\gamma\ \sin\chi)$$

$$+ \ 2\omega V\cos\gamma\ \cos\chi\ \sin\phi + \omega^2 r\cos\phi\ \sin\phi, \tag{53b}$$

$$\dot{V}_{zh} = \ (D/m)\sin\gamma - (Q/m)\sin\mu\ \cos\gamma - (L/m)\cos\mu\ \cos\gamma + g$$

$$- \ (V^2/r)\cos^2\gamma - 2\omega V\cos\gamma\ \cos\chi\ \cos\phi - \omega^2 r\cos^2\phi. \tag{53c}$$

We recall that the components of the relative velocity on the axes of the local horizon system are given by [see Eqs. (40)]

$$V_{xh} = V\cos\gamma\ \cos\chi, \tag{54a}$$

$$V_{yh} = V\cos\gamma\ \sin\chi, \tag{54b}$$

$$V_{zh} = -V\sin\gamma, \tag{54c}$$

with the implication that

$$\begin{bmatrix} \dot{V}_{xh} \\ \dot{V}_{yh} \\ \dot{V}_{zh} \end{bmatrix} = \begin{bmatrix} \cos\gamma\ \cos\chi & -\sin\gamma\ \cos\chi & -\sin\chi \\ \cos\gamma\ \sin\chi & -\sin\gamma\ \sin\chi & \cos\chi \\ -\sin\gamma & -\cos\gamma & 0 \end{bmatrix} \begin{bmatrix} \dot{V} \\ V\dot{\gamma} \\ V\cos\gamma\ \dot{\chi} \end{bmatrix}, \tag{55}$$

and that

$$\begin{bmatrix} \dot{V} \\ V\dot{\gamma} \\ V\cos\gamma\ \dot{\chi} \end{bmatrix} = \begin{bmatrix} \cos\gamma\ \cos\chi & \cos\gamma\ \sin\chi & -\sin\gamma \\ -\sin\gamma\ \cos\chi & -\sin\gamma\ \sin\chi & -\cos\gamma \\ -\sin\chi & \cos\chi & 0 \end{bmatrix} \begin{bmatrix} \dot{V}_{xh} \\ \dot{V}_{yh} \\ \dot{V}_{zh} \end{bmatrix}. \tag{56}$$

The final step consists of combining Eqs. (53) and (56). This leads to the following scalar form of the dynamical equations:

$$\dot{V} = -D/m - g\sin\gamma + f\cos\gamma\ \sin\chi$$

$$+ \ \omega^2 r(\sin\gamma\ \cos^2\phi + \cos\gamma\ \sin\chi\ \cos\phi\ \sin\phi), \tag{57a}$$

$$V\dot{\gamma} = \ (L/m)\cos\mu + (Q/m)\sin\mu + (V^2/r - g)\cos\gamma - f\sin\gamma\ \sin\chi$$

$$+ \ 2\omega V\cos\chi\ \cos\phi + \omega^2 r(\cos\gamma\ \cos^2\phi - \sin\gamma\ \sin\chi\ \cos\phi\ \sin\phi), \tag{57b}$$

$$V\cos\gamma\ \dot{\chi} = \ (L/m)\sin\mu - (Q/m)\cos\mu + (V^2/r)\cos^2\gamma\ \cos\chi\ \tan\phi + f\cos\chi$$

$$+ 2\omega V(\cos\gamma\ \sin\phi + \sin\gamma\ \sin\chi\ \cos\phi) + \omega^2 r\cos\chi\ \cos\phi\ \sin\phi, \tag{57c}$$

which can be rewritten as

$$\dot{V} = -D/m - g\sin\gamma + f\cos\gamma\ \sin\chi$$

$$+ \ \omega^2 r(\sin\gamma\ \cos^2\phi + \cos\gamma\ \sin\chi\ \cos\phi\ \sin\phi), \tag{58a}$$

$$\dot{\gamma} = (L/mV)\cos\mu + (Q/mV)\sin\mu + (V/r - g/V)\cos\gamma - (f/V)\sin\gamma \sin\chi$$

$$+ 2\omega\cos\chi \cos\phi + (\omega^2 r/V)(\cos\gamma \cos^2\phi - \sin\gamma \sin\chi \cos\phi \sin\phi), \tag{58b}$$

$$\dot{\chi} = (L/mV)\sin\mu/\cos\gamma - (Q/mV)\cos\mu/\cos\gamma$$

$$+ (V/r)\cos\gamma \cos\chi \tan\phi + (f/V)\cos\chi/\cos\gamma$$

$$+ 2\omega(\sin\phi + \tan\gamma \sin\chi \cos\phi) + (\omega^2 r/V)\cos\chi \cos\phi \sin\phi/\cos\gamma. \tag{58c}$$

8. Summary of Results

In this report, we have derived the equations of motion of a spacecraft under the following assumptions: (a) the spacecraft is a particle of constant mass; (b) the Earth is rotating with constant angular velocity; (c) the atmosphere is at rest with respect to the Earth; (d) the Earth is an oblate planet, and the gravitational potential depends on both the radial distance and the latitude; however, harmonics of order higher than four are ignored.

An Earth-fixed system has been used, and the following kinematical and dynamical equations have been obtained:

$$\dot{\theta} = V\cos\gamma\ \cos\chi/r\cos\phi, \tag{59a}$$

$$\dot{\phi} = -V\cos\gamma\ \sin\chi/r, \tag{59b}$$

$$\dot{r} = V\sin\gamma, \tag{59c}$$

and

$$\dot{V} = -D/m - g\sin\gamma + f\cos\gamma\ \sin\chi$$
$$+\ \omega^2 r(\sin\gamma\ \cos^2\phi + \cos\gamma\ \sin\chi\ \cos\phi\ \sin\phi), \tag{60a}$$

$$\dot{\gamma} = (L/mV)\cos\mu + (Q/mV)\sin\mu + (V/r - g/V)\cos\gamma - (f/V)\sin\gamma\ \sin\chi$$
$$+\ 2\omega\cos\chi\ \cos\phi + (\omega^2 r/V)(\cos\gamma\ \cos^2\phi - \sin\gamma\ \sin\chi\ \cos\phi\ \sin\phi), \tag{60b}$$

$$\dot{\chi} = (L/mV)\sin\mu/\cos\gamma - (Q/mV)\cos\mu/\cos\gamma$$
$$+\ (V/r)\cos\gamma\ \cos\chi\ \tan\phi + (f/V)\cos\chi/\cos\gamma$$
$$+\ 2\omega(\sin\phi + \tan\gamma\ \sin\chi\ \cos\phi) + (\omega^2 r/V)\cos\chi\ \cos\phi\ \sin\phi/\cos\gamma. \tag{60c}$$

8.1. Aerodynamic Force. In Eqs. (60), the drag, the side force, and the lift are given by

$$D = (1/2)C_D\rho SV^2, \tag{61a}$$

$$Q = (1/2)C_Q\rho SV^2, \tag{61b}$$

$$L = (1/2)C_L\rho SV^2, \tag{61c}$$

where C_D is the drag coefficient, C_Q is the side force coefficient, C_L is the lift coefficient, ρ is the air density, and S is a reference surface area. In turn, the aerodynamic coefficients are functions of the form

$$C_D = C_D(\alpha,\sigma,M,R_e), \tag{62a}$$

$$C_Q = C_Q(\alpha,\sigma,M, R_e), \tag{62b}$$

$$C_L = C_L(\alpha,\sigma, M,R_e), \tag{62c}$$

where α is the angle of attack, σ is the sideslip angle, M is the Mach number, and R_e is the Reynolds number.

8.2. Gravitational Force. In Eqs. (60), the radial component and the latitudinal component of the acceleration of gravity are given by

$$g = (\mu_e/r^2)[1 + J_2(r_e/r)^2G_2 + J_3(r_e/r)^3G_3 + J_4(r_e/r)^4G_4], \tag{63a}$$

$$G_2 = 3/2 - (9/2)\sin^2\phi, \tag{63b}$$

$$G_3 = 6\sin\phi - 10\sin^3\phi, \tag{63c}$$

$$G_4 = -15/8 + (150/8)\sin^2\phi - (175/8)\sin^4\phi, \tag{63d}$$

and

$$f = (\mu_e/r^2)[J_2(r_e/r)^2F_2 + J_3(r_e/r)^3F_3 + J_4(r_e/r)^4F_4], \tag{64a}$$

$$F_2 = 3\sin\phi\ \cos\phi, \tag{64b}$$

$$F_3 = -(3/2)\cos\phi + (15/2)\sin^2\phi\ \cos\phi, \tag{64c}$$

$$F_4 = -(15/2)\sin\phi\ \cos\phi + (35/2)\sin^3\phi\ \cos\phi. \tag{64d}$$

8.3. Physical Constants. The major physical constants appearing in the system (59)-(64) have the following values:

$$\omega = 0.729211595 \quad E-04 \quad \text{rad/sec}, \tag{65a}$$

$$\mu_e = 0.39860064 \quad E+15 \quad m^3/sec^2, \tag{65b}$$

$$J_2 = 0.10826271 \quad E-02, \tag{65c}$$

$$J_3 = -0.25358868 \quad E-05, \tag{65d}$$

$$J_4 = -0.1624618 \quad E-05, \tag{65e}$$

$$r_e = 0.6378164 \quad E+07 \quad m, \tag{65f}$$

$$r_p = 0.6356755 \quad E+07 \quad m, \tag{65g}$$

Here, ω is the Earth's angular velocity; μ_e is the Earth's gravitational constant; J_2, J_3, J_4 are the characteristic constants of the Earth's gravitational field; r_e is the Earth's equatorial radius; and r_p is the Earth's polar radius. Note that the Earth's sea-level radius $r_{s\ell}$ varies with the latitude ϕ according to the relation

$$r_{s\ell} = (1/2)(r_e + r_p) + (1/2)(r_e - r_p)\cos(2\phi). \tag{66}$$

8.4. Spacecraft Data. For the AFE vehicle, it is assumed that

$$m = 0.16782918 \quad E+04 \quad kg, \tag{67a}$$

$$S = 0.14314 \quad E+02 \quad m^2, \tag{67b}$$

$$\alpha = 0.17000 \quad E+02 \quad deg, \tag{67c}$$

$$C_L = -0.370696 \quad E+00, \tag{67d}$$

$$C_D = 0.131452 \quad E+01. \tag{67e}$$

Here, m is the spacecraft mass at atmospheric entry; S is the reference surface area; α is the angle of attack; C_L is the lift coefficient; and C_D is the drag coefficient. Note that, for the aeroassisted flight experiment, the angle of attack is kept constant; the aerodynamic coefficients are assumed to be independent of the Mach number and the Reynolds number; and the spacecraft is controlled via the angle of bank.

9. Transformation Relations

In this section, we supply some transformation relations which allow one to pass from (i) quantities computed in an Earth-fixed system to (ii) quantities computed in an inertial system, and viceversa.

9.1. Spacecraft Position. Let r, θ, ϕ denote the spherical coordinates of the spacecraft P in the Earth-fixed system $Ox_ey_ez_e$. Let r_i, θ_i, ϕ_i denote the spherical coordinates of the same spacecraft in the inertial system $Ox_iy_iz_i$. Assume that the axes of the Earth-fixed system coincide with the axes of the inertial system at time instant $t = 0$. Then, the following transformation relations hold:

$$r_i = r, \tag{66a}$$

$$\theta_i = \theta + \omega t, \tag{66b}$$

$$\phi_i = \phi. \tag{66c}$$

Equations (66) imply the following inverse relations:

$$r = r_i, \tag{67a}$$

$$\theta = \theta_i - \omega t, \tag{67b}$$

$$\phi = \phi_i. \tag{67c}$$

9.2. Spacecraft Velocity. Let V, γ, χ denote the velocity modulus, the path inclination, and the heading angle in the Earth-fixed system $Ox_ey_ez_e$. Let V_i, γ_i, χ_i denote the velocity modulus, the path inclination, and the heading angle in the inertial system $Ox_iy_iz_i$. Let $\vec{V}$ denote the velocity vector in the Earth-fixed system; and let $\vec{V}_i$ denote the velocity vector in the inertial system.

We employ the theorem of composition of velocities, which states that the inertial velocity $\vec{V}_i$ is the sum of the relative velocity $\vec{V}$ and the transport velocity $\vec{\omega} \times \vec{OP}$,

$$\vec{V}_i = \vec{V} + \vec{\omega} \times \vec{OP}. \tag{68}$$

The vectors appearing in Eq. (68) can be written in terms of their components on the local horizon system $Px_hy_hz_h$ as follows [see Eqs. (48), (50),(51)]:

$$\vec{V}_i = V_i \cos\gamma_i \cos\chi_i \, \vec{i}_h + V_i \cos\gamma_i \sin\chi_i \, \vec{j}_h - V_i \sin\gamma_i \, \vec{k}_h, \tag{69a}$$

$$\vec{V} = V\cos\gamma \cos\chi \, \vec{i}_h + V\cos\gamma \sin\chi \, \vec{j}_h - V\sin\gamma \, \vec{k}_h, \tag{69b}$$

$$\vec{\omega} \times \vec{OP} = \omega r \cos\phi \, \vec{i}_h, \tag{69c}$$

with the implication that

$$V_i \cos\gamma_i \cos\chi_i = V\cos\gamma \cos\chi + \omega r \cos\phi, \tag{70a}$$

$$V_i \cos\gamma_i \sin\chi_i = V\cos\gamma \sin\chi, \tag{70b}$$

$$V_i \sin\gamma_i = V\sin\gamma. \tag{70c}$$

Laborious manipulations, omitted for the sake of brevity, lead to the following transformation relations:

$$V_i = \sqrt{[V^2 + 2\omega r V\cos\gamma \cos\chi \cos\phi + (\omega r \cos\phi)^2]}, \tag{71a}$$

$$\tan\gamma_i = V\sin\gamma/\surd[(V\cos\gamma)^2 + 2\omega rV\cos\gamma\ \cos\chi\ \cos\phi + (\omega r\ \cos\phi)^2], \tag{71b}$$

$$\tan\chi_i = V\cos\gamma\ \sin\chi/(V\cos\gamma\ \cos\chi + \omega r\cos\phi). \tag{71c}$$

Equations (71) imply the following inverse relations:

$$V = \surd[V_i^2 - 2\omega r_i V_i\cos\gamma_i\ \cos\chi_i\ \cos\phi_i + (\omega r_i\cos\phi_i)^2], \tag{72a}$$

$$\tan\gamma = V_i\sin\gamma_i/\surd[(V_i\cos\gamma_i)^2 - 2\omega r_i V_i\cos\gamma_i\ \cos\chi_i\ \cos\phi_i + (\omega r_i\cos\phi_i)^2], \tag{72b}$$

$$\tan\chi = V_i\cos\gamma_i\ \sin\chi_i/(V_i\cos\gamma_i\ \cos\chi_i - \omega r_i\cos\phi_i). \tag{72c}$$

9.3. Cartesian Coordinates. After the spacecraft position is known in spherical coordinates, the corresponding Cartesian coordinates can be computed. The following transformation relations hold:

$$x_e = r\cos\theta\ \cos\phi = r_i\cos(\theta_i - \omega t)\cos\phi_i, \tag{73a}$$

$$y_e = r\sin\theta\ \cos\phi = r_i\sin(\theta_i - \omega t)\cos\phi_i, \tag{73b}$$

$$z_e = r\sin\phi \qquad = r_i\sin\phi_i. \tag{73c}$$

Equations (73) imply the following inverse relations:

$$x_i = r_i\cos\theta_i\ \cos\phi_i = r\cos(\theta + \omega t)\cos\phi, \tag{74a}$$

$$y_i = r_i\sin\theta_i\ \cos\phi_i = r\sin(\theta + \omega t)\cos\phi, \tag{74b}$$

$$z_i = r_i\sin\phi_i \qquad = r\sin\phi. \tag{74c}$$

10. Conclusions

This report is the first of a series dealing with the determination of optimal trajectories for the aeroassisted flight experiment (AFE). The AFE refers to the study of the free flight of an autonomous spacecraft, shuttle-launched and shuttle-recovered. Its purpose is to gather atmospheric entry environmental data for use in designing aeroassisted orbital transfer vehicles (AOTV).

It is assumed that: the spacecraft is a particle of constant mass; the Earth is rotating with constant angular velocity; the Earth is an oblate planet, and the gravitational potential depends on both the radial distance and the latitude; however, harmonics of order higher than four are ignored; the atmosphere is at rest with respect to the Earth.

Under the above assumptions, the equations of motion for hypervelocity atmospheric flight (which can be used not only for AFE problems, but also for AOT problems and space shuttle problems) are derived in an Earth-fixed system. Transformation relations are supplied which allow one to pass from quantities computed in an Earth-fixed system to quantities computed in an inertial system,and viceversa.

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