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LOCAL ERROR ESTIMATES FOR DISCONTINUOUS SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS

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ABSTRACT

Let $u(x,t)$ be the possibly discontinuous entropy solution of a nonlinear scalar conservation law with smooth initial data. Suppose $u_\varepsilon(x,t)$ is the solution of an approximate viscosity regularization, where $\varepsilon > 0$ is the small viscosity amplitude. We show that by post-processing the small viscosity approximation u_ε, we can recover pointwise values of u and its derivatives with an error as close to ε as desired.

The analysis relies on the adjoint problem of the forward error equation, which in this case amounts to a backward linear transport equation with discontinuous coefficients. The novelty of our approach is to use a (generalized) E-condition of the forward problem in order to deduce a $W^{1,\infty}$ energy estimate for the discontinuous backward transport equation; this, in turn, leads us to ε-uniform estimate on moments of the error $u_\varepsilon - u$.

Our approach does not 'follow the characteristics' and, therefore, applies mutatis mutandis to other approximate solutions such as E-difference schemes.

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1. INTRODUCTION

Consider the scalar, genuinely nonlinear conservation law

$$\frac{\partial}{\partial t}[u(x,t)] + \frac{\partial}{\partial x}[f(u(x,t))] = 0. \tag{1.1}$$

It is well-known, e.g. [8], that (1.1) may admit many possible weak solutions. To guarantee uniqueness (and in fact L^1-stability), we therefore have to restrict our attention to a subclass of possible weak solutions. Namely, we select weak solutions of (1.1) which are realizable as small viscosity solutions of

$$\frac{\partial}{\partial t}[u_\varepsilon(x,t)] + \frac{\partial}{\partial x}[f(u_\varepsilon(x,t))] = \varepsilon\frac{\partial^2}{\partial x^2}[Q(u_\varepsilon(x,t))], \quad \varepsilon Q' \downarrow 0. \tag{1.2}$$

With this in mind, we recall that $u_\varepsilon(\cdot,t)$ converges strongly in L^1 to $u(\cdot,t)$, where $u(\cdot,t)$ is the unique, so-called entropy solution of (1.1). The L^1 convergence rate in this case is upper bounded by

$$\|u_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^1} \leq \text{Const.}\sqrt{\varepsilon}. \tag{1.3}$$

Consult [6], [12] for the discrete analogue of monotone difference schemes, and [14] for spectral viscosity approximations.

We are not satisfied with this error estimate for two related reasons.

1. Ideally, one would like to recover the entropy solution, $u(x,t)$, within $O(\varepsilon)$ error. Although the estimate (1.3) is sharp, it fails to do so because of the following reason.

2. The error estimate (1.3) is a global one, while in many practical cases one is interested in the local behavior of $u(x,t)$. Consequently, when the error is measured by the L^1-norm, there is a loss of information due to the 'poor' resolution of shock (and, in the general case, also contact) waves in $u(x,t)$, by the corresponding viscous layers in $u_\varepsilon(x,t)$.

In this paper we are concerned with the local convergence rate of the small viscosity solutions, $u_\varepsilon(x,t)$, towards the entropy solution, $u(x,t)$. Assume that initially, at $t=0$, the initial conditions of the small viscosity problem, $u_\varepsilon(\cdot,0)$, are consistent with smooth initial conditions of the conservation law (1.1). Then for any $t>0$ we show, in Section 4, that by post-processing the small viscosity solution $u_\varepsilon(\cdot,t)$, we can recover pointwise values of the (possibly discontinuous) entropy solution $u(\cdot,t)$ and its derivatives, with error as close to ε as desired. It should be emphasized that our approach does not "follow the characteristics" and therefore could be extended to certain discrete approximations for nonlinear conservation laws such as E-difference schemes [10]. Indeed, the present study was originally motivated by recent numerical experiments reported in [17]. By post-processing of Spectral Viscosity (SV) approximations [16], we were able to recover the pointwise values of discontinuous conservative solutions with spectral accuracy, in agreement with the formally spectrally small viscosity regularization of the SV method.

The paper is organized as follows. In Section 2 we study linear transport equations with possibly discontinuous coefficients. We show that these discontinuous transport equations are well-posed in $W^{1,\infty}$, provided their coefficients are one-sided Lipschitz continuous.

Such (backward) transport equations arise as the adjoint problems for the forward error equation governing the difference $u_\varepsilon - u$. The coefficients of these backward transport equations are indeed upper-sided Lipschitz continuous, in view of Oleinik's E-condition which characterizes the entropy solution. This enables us, in Section 3, to derive ε-uniform estimate on the $W^{1,\infty}$-moments of the error $u_\varepsilon - u$, which in turn, leads to the local recovery of the entropy solution discussed in Section 4. Finally, we note that our $W^{1,\infty}$ upper-bound on the moments of the error provides an independent one-dimensional proof of the usual $O(\sqrt{\varepsilon})$ - L^1-convergence rate mentioned earlier in (1.3).

2. LINEAR EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

In this section we study the linear transport equation

$$\frac{\partial}{\partial t}[\phi(x,t)] + a(x,t)\frac{\partial}{\partial x}[\phi(x,t)] = 0, \tag{2.1}$$

subject to prescribed initial conditions

$$\phi(x,t=0) = \phi_0(x). \tag{2.2}$$

Assume that the initial data, $\phi_0(x)$, are Lipschitz continuous. The standard theory tells us that if $a(\cdot,t)$ is sufficiently smooth, say C^1, then there exists a unique generalized solution, $\phi(\cdot,t)$, of (2.1), (2.2), which remains Lipschitz for all time,

$$\|\phi(\cdot,t)\|_{\mathrm{Lip}} \le \mathrm{Const}_T \cdot \|\phi_0(\cdot)\|_{\mathrm{Lip}}, \quad -T < t < T. \tag{2.3}$$

The key issue that we address in this section is, roughly speaking, the following question:

What is the minimal degree of smoothness required from $a(\cdot,t)$ in order to retain the Lipschitz continuity of the solution $\phi(\cdot,t)$ in (2.1)?

The answer provided in our next theorem is in the heart of the matter.

THEOREM 2.1. *Consider the linear transport equation (2.1) with Lipschitz continuous initial data (2.2). We assume that*

I. (Uniform Boundedness). $a(x,t)$ is uniformly bounded.

II. (OSLC). $a(x,t)$ satisfies the following One-Sided Lipschitz Condition.

$$L^-[a(\cdot,t)] \equiv ess \inf_{x\neq y} \left(\frac{a(x,t)-a(y,t)}{x-y}\right)^- \geq -m(t), \quad m\epsilon L^1[0,T]. \tag{2.4}$$

Then for $t>0$, there exists a unique Lipschitz continuous solution, $\phi(x,t)$, of (2.1), (2.2), such that the following estimate holds

$$\|\phi(\cdot,t)\|_{W^{1,\infty}} \leq \|\phi_0(\cdot)\|_{W^{1,\infty}} \cdot e^{M(t)}, \quad M(t) \equiv \int_0^t m(\tau)d\tau, \quad t>0. \tag{2.5}$$

Before we turn to the proof of this theorem, a couple of remarks is in order.

1. Theorem 2.1 allows the coefficient $a(\cdot,t)$ to be discontinuous; in fact, increasing jumps are permitted. The OSLC assumption (2.4) requires only the decreasing part of $a(\cdot,t)$ to be Lipschitz.

2. Theorem 2.1 asserts that the transport equation (2.1) is well-posed only for positive time. In general, linear hyperbolic equation with smooth coefficients is a prototype of reversible process. However, since in our case the augmenting OSLC assumption (2.4) is irreversible, so is the final conclusion of the theorem. Indeed, simple counterexamples can be constructed (see below), which demonstrate that the backward solution $\phi(\cdot,t)$ may cease to be Lipschitz in a finite negative time $t<0$.

PROOF. The proof consists of the usual three steps of regularization, a priori energy estimate and compactness arguments.

Step 1. (Regularization). Let $\zeta_\delta(x) = \frac{1}{\delta}\zeta(\frac{x}{\delta})$ be a standard positive C_0^∞-mollifier with unit mass. We regularize $a(\cdot,t)$ by spatial convolution, $a_\delta(\cdot,t) \equiv a(\cdot,t) * \zeta_\delta$, and consider the regularized equation

$$\frac{\partial}{\partial t}[\phi_\delta(x,t)] + a_\delta(x,t)\frac{\partial}{\partial x}[\phi_\delta(x,t)] = 0, \quad t \geq 0 \tag{2.6}$$

$$\phi_\delta(x,t=0) = \phi_{0\delta}(x), \quad \phi_{0\delta} \equiv \phi_0 * \zeta_\delta. \tag{2.7}$$

Since we now have a smooth and uniformly bounded coefficient, $a_\delta(\cdot,t)$, there exists a classical C^∞ solution, $\phi_\delta(\cdot,t)$, of (2.6), (2.7). Since this solution propagates with finite speed, a further approximation by truncation (which is omitted) can be used, so that we may restrict our attention to the compactly supported case where $\phi_\delta(\cdot,t)\epsilon C_0^\infty$. Clearly, we have

$$\|\phi_\delta(\cdot,t)\|_{L^\infty} \leq \|\phi_0(\cdot)\|_{L^\infty}. \tag{2.8}$$

Step 2. ($W^{1,\infty}$-energy estimate). We want to show that $\frac{\partial\phi_\delta}{\partial x}$ is uniformly bounded w.r.t. x, t, and δ. To this end we shall carefully iterate on the L^p-norms of $\frac{\partial}{\partial x}\phi_\delta(\cdot,t)$. Differentiation of (2.6) implies

$$\frac{\partial}{\partial t}[\psi_\delta(x,t)] + \frac{\partial}{\partial x}[a_\delta(x,t)\psi_\delta(x,t)] = 0, \quad \psi_\delta \equiv \frac{\partial\phi_\delta}{\partial x},$$

or, equivalently

$$\frac{\partial}{\partial t}[\psi_\delta(x,t)] + a_\delta(x,t)\frac{\partial}{\partial x}[\psi_\delta(x,t)] = -\frac{\partial}{\partial x}[a_\delta(x,t)]\psi_\delta(x,t). \tag{2.9}$$

Integrating (2.9) against $p\psi_\delta^{p-1}, p$ even, over the compact support, we find

$$\frac{d}{dt}\|\psi_\delta(\cdot,t)\|_{L^p}^p + \int_x a_\delta(x,t)\frac{\partial}{\partial x}[\psi_\delta^p(x,t)]dx = p\int_x -\frac{\partial}{\partial x}[a_\delta(x,t)]\psi_\delta^p(x,t)dx,$$

and after further integration by parts on the left we arrive at

$$\frac{d}{dt}\|\psi_\delta(\cdot,t)\|_{L^p}^p = (p-1)\int_x -\frac{\partial}{\partial x}[a_\delta(x,t)]\psi_\delta^p(x,t)dx. \tag{2.10}$$

We now invoke the OSLC assumption on the coefficient $a(\cdot,t)$. Since ζ_δ was chosen as a positive mollifier, $a_\delta = a * \zeta_\delta$ satisfies the same OSLC (2.4), namely

$$-\frac{\partial}{\partial x}[a_\delta(x,t)] \leq m(t). \tag{2.11}$$

Inserting this into the RHS of (2.10) yields for p even

$$\frac{d}{dt}\|\psi_\delta(\cdot,t)\|_{L^p}^p \leq (p-1)m(t)\|\psi_\delta(\cdot,t)\|_{L^p}^p. \tag{2.12}$$

Therefore,

$$\|\psi_\delta(\cdot,t)\|_{L^p} \leq \|\psi_\delta(\cdot,t=0)\|_{L^p}\cdot e^{\frac{p-1}{p}M(t)}, \quad M(t) = \int_0^t m(\tau)d\tau, \quad t>0, \tag{2.13}$$

and by letting $p\uparrow\infty$ we conclude

$$\|\frac{\partial}{\partial x}\phi_\delta(\cdot,t)\|_{L^\infty} \leq \|\frac{\partial}{\partial x}\phi_{0\delta}(\cdot)\|_{L^\infty}\cdot e^{M(t)} \leq \|\phi_0(\cdot)\|_{\text{Lip}}\cdot e^{M(t)}, \quad t>0. \tag{2.14}$$

Since a (and likewise a_δ) is assumed to be uniformly bounded, we can use the equation (2.6) to upper-bound the temporal derivative as well,

$$\|\frac{\partial}{\partial t}\phi_\delta(\cdot,t)\|_{L^\infty} \leq \|a(\cdot,t)\|_{L^\infty}\cdot\|\phi_0(\cdot)\|_{\text{Lip}}\cdot e^{M(t)}, \quad t>0. \tag{2.15}$$

Step 3. (Compactness). By (2.14), (2.15), the uniformly bounded family $\{\phi_\delta\}$ is equicontinuous – in fact equiLipschitz, and shares a common compact support. Therefore, we can extract a subsequence, still denoted by $\phi_\delta(x,t)$, which converges uniformly to a Lipschitz limit function, $\phi(x,t)$. We observe that

$$\begin{aligned} \frac{\partial}{\partial t}\phi_\delta(\cdot,t) &\rightharpoonup \frac{\partial}{\partial t}\phi(\cdot,t) \quad \text{weak} - * \text{ in } L^\infty, \\ \frac{\partial}{\partial x}\phi_\delta(\cdot,t) &\rightharpoonup \frac{\partial}{\partial x}\phi(\cdot,t) \quad \text{weak} - * \text{ in } L^\infty, \\ a_\delta(\cdot,t) &\rightarrow a(\cdot,t) \qquad \text{strongly in } L^1. \end{aligned}$$

Passing to the limit $\delta \downarrow 0$ in (2.6) we conclude that $\phi(x,t)$ is a Lipschitz (generalized) solution of the transport equation (2.1), (2.2). Similarly (2.5) follows from (2.8), (2.14) and we are done.

REMARKS.

1. The $W^{1,\infty}$-energy estimate revisited. Consider the L^p a priori estimates in (2.13). The case $p = 1$ (odd p's can be justified by further approximation argument which we omit), leads to usual BV-contraction of the solution operator associated with (2.6). The proof of Theorem 2.1 hinges on the observation that with the help of the OSLC assumption 2.4, one can iterate on higher L^p norms at the expense of an additional bounded (w.r.t. δ) exponent. An alternative proof of this essential L^∞ bound, (2.14), is provided by the following duality argument. If $\lambda_\delta(x,t)$ solves the adjoint equation of (2.6)

$$\frac{\partial}{\partial t}[\lambda_\delta(x,t)] + \frac{\partial}{\partial x}[a_\delta(x,t)\lambda_\delta(x,t)] = 0, \quad t \le T,$$

then $\Lambda_\delta(x,t) = \int^x \lambda_\delta(\xi,t)d\xi$ satisfies the backward transport equation

$$\frac{\partial}{\partial t}[\Lambda_\delta(x,t)] + a_\delta(x,t)\frac{\partial}{\partial x}[\Lambda_\delta(x,t)] = 0, \quad t \le T. \tag{2.16}$$

Integrating this against $\text{sgn}\Lambda_\delta(x,t)$ we obtain – in view of the OSLC assumption (2.4),

$$\|\Lambda_\delta(\cdot,t)\|_{L^1} \ge \|\Lambda_\delta(\cdot,0)\|_{L^1} \cdot e^{-M(t)}, \quad t > 0, \tag{2.17}$$

which is the dual estimate of (2.14).

2. Characteristics. The transport equation (2.1) is governed by the evolution of the characteristics

$$\dot{x} = a(x,t). \tag{2.18}$$

If $a(\cdot,t)$ is at least Lipschitz continuous, then the ODE (2.18) leads to a 'nice' unique reversible flow. For our purpose, however, we need less. Namely, in order for $\phi(\cdot,t)$ to remain Lipschitz for $t > 0$, we have to guarantee that while tracing characteristics backward in time, different characteristics are not pulled apart. The OSLC assumption is sufficient to guarantee that (2.18) generates such a 'nice' Lipschitz flow backward in time. The following examples demonstrate our point.

Example 1. We set

$$a(x,t) = \mathrm{sgn}x. \tag{2.19}$$

Then the Lipschitz continuous solution of (2.1) is given by

$$\phi(x,t) = \phi_0((x-t)^+ + (x+t)^-). \tag{2.20}$$

Example 2. We reverse the sign in (2.19),

$$a(x,t) = -\mathrm{sgn}x. \tag{2.20}$$

Now $a(\cdot)$ has a decreasing jump, and this case is not covered by Theorem 2.1. Indeed, different forward characteristic solutions of

$$\dot{x} = -\mathrm{sgn}x \tag{2.21}$$

may impinge one on each other at finite time. Hence, the backward flow generated by (2.21) is ill-posed, and the hyperbolic solution transported along these characteristics cannot remain Lipschitz for $t > 0$.

Example 3. We consider the equation

$$\dot{x} = (T-x)^\alpha, \quad 0 < \alpha < 1, \tag{2.22}$$

which is 'solved' backwards (of course, the solution is not unique in this case), starting with the final time $T > 0$. Here, $a(x) = (T-x)^\alpha$ fails to satisfy the OSLC assumption (2.4), although the analogue one-sided Hölder condition holds. Consequently, the corresponding forward transport solution will be only Hölder continuous. In contrast to (2.22), let us consider the case

$$\dot{x} = a(x), \quad a(x) = -(T-x)^\alpha, \quad 0 < \alpha < 1. \tag{2.23}$$

Although $a(x)$ is only Hölder (but not Lipschitz) continuous, the OSLC assumption (2.4) is now fulfilled thanks to the judicious minus sign in (2.23), and the result of Theorem 2.1 is valid.

In the next section we study the error equation associated with the conservation law (1.1) and its viscous regularization in (1.2). The dual problem of such an error equation leads to a backward transport equation like (2.1). For future reference we therefore state

THEOREM 2.2. (The backward transport equation). *Consider the linear transport equation*

$$\frac{\partial}{\partial t}[\phi(x,t)] + a(x,t)\frac{\partial}{\partial x}[\phi(x,t)] = 0, \quad t \leq T, \tag{2.24}$$

with Lipschitz continuous data prescribed at $t = T$,

$$\phi(x, t = T) = \phi(x). \tag{2.25}$$

We assume that

I. (Uniform Boundedness). $a(x,t)$ *is uniformly bounded.*

II. (OSLC). $a(x,t)$ *satisfies the following One-Sided Lipschitz Condition.*

$$L^+[a(\cdot,t)] \equiv ess\sup_{x\neq y}\left(\frac{a(x,t)-a(y,t)}{x-y}\right)^+ \leq m(t), \quad m\epsilon L^1[t,T]. \tag{2.26}$$

Then for $t < T$, *there exists a unique Lipschitz continuous solution,* $\phi(x,t)$, *of (2.24), (2.25), such that the following estimate holds*

$$\|\phi(\cdot,t)\|_{W^{1,\infty}} \leq \|\phi(\cdot)\|_{W^{1,\infty}} \cdot e^{M(t)}, \quad M(t) \equiv \int_t^T m(\tau)d\tau, \quad t \leq T. \tag{2.27}$$

3. A PRIORI ESTIMATE ON MOMENTS OF THE ERROR

We return to the scalar, genuinely nonlinear (say, strictly convex) conservation law,

$$\frac{\partial}{\partial t}[u(x,t)] + \frac{\partial}{\partial x}[f(u(x,t))] = 0, \quad t \geq 0, \quad f'' \geq \alpha > 0. \tag{3.1}$$

It is well-known, [8], [15], that the unique entropy solution of (3.1) is characterized by Oleinik's E-condition [11]

$$\frac{u(x,t)-u(y,t)}{x-y} \leq \frac{1}{\alpha t}, \quad t > 0. \tag{3.2}$$

Later on we shall need a slightly stronger version of this E-condition – interesting for its own sake, which is the content of our next result.

THEOREM 3.1. *Consider the (possibly degenerate) parabolic regularization of (3.1),*

$$\frac{\partial}{\partial t}[u_\epsilon(x,t)] + \frac{\partial}{\partial x}[f(u_\epsilon(x,t))] = \varepsilon\frac{\partial^2}{\partial x^2}[Q(u_\epsilon(x,t))], \quad \varepsilon Q' \geq 0. \tag{3.3}$$

Then the following estimate holds

$$L^+[Q(u_\epsilon(\cdot,t))] \leq \frac{L^+[Q(u_\epsilon(\cdot,0))]}{1+\beta t L^+[Q(u_\epsilon(\cdot,0))]}, \quad \beta \equiv \min\frac{f''(\cdot)}{Q'(\cdot)} > 0. \tag{3.4}$$

PROOF. Let us first assume that Q' is strictly positive, so that the uniformly parabolic equation (3.3) admits a smooth solution. Multiplying (3.3) by $Q'(u_\varepsilon(x,t))$ we obtain

$$\frac{\partial}{\partial t}[Q(u_\varepsilon)] + a(u_\varepsilon)\frac{\partial}{\partial x}[Q(u_\varepsilon)] = \varepsilon Q'(u_\varepsilon)\frac{\partial^2}{\partial x^2}[Q(u_\varepsilon)], \quad a(\cdot) \equiv f'(\cdot). \tag{3.5}$$

Next we denote

$$w_\varepsilon(x,t) = \frac{\partial}{\partial x}[Q(u_\varepsilon(x,t))] \equiv Q'(u_\varepsilon)\frac{\partial}{\partial x}[u_\varepsilon(x,t)]. \tag{3.6}$$

Differentiation of (3.5) w.r.t. x yields

$$\frac{\partial w_\varepsilon}{\partial t} + [a(u_\varepsilon) - \varepsilon\frac{Q''(u_\varepsilon)}{Q'(u_\varepsilon)}w_\varepsilon]\frac{\partial w_\varepsilon}{\partial x} + \frac{a'(u_\varepsilon)}{Q'(u_\varepsilon)}w_\varepsilon^2 = \varepsilon Q'(u_\varepsilon)\frac{\partial^2 w_\varepsilon}{\partial x^2}. \tag{3.7}$$

By a standard regularization argument of the Heaviside function (which we omit), (3.7) implies for $w_\varepsilon^+ \equiv H(w_\varepsilon)w_\varepsilon$,

$$\frac{\partial w_\varepsilon^+}{\partial t} + [a(u_\varepsilon) - \varepsilon\frac{Q''(u_\varepsilon)}{Q'(u_\varepsilon)}w_\varepsilon]\frac{\partial w_\varepsilon^+}{\partial x} + \beta(w_\varepsilon^+)^2 \leq \varepsilon Q'(u_\varepsilon)\frac{\partial^2 w_\varepsilon^+}{\partial x^2}. \tag{3.8}$$

The maximum principle shows that (3.8) is dominated by Ricatti equation

$$\frac{d}{dt}[\sup_x w_\varepsilon^+(x,t)] + \beta[\sup_x w_\varepsilon^+(x,t)]^2 \leq 0,$$

which in turn leads to (3.9),

$$L^+[Q(u_\varepsilon(\cdot,t))] = \sup_x w_\varepsilon^+(x,t) \leq \frac{1}{\frac{1}{\sup_x w_\varepsilon^+(x,0)} + \beta t} = \frac{L^+[Q(u_\varepsilon(\cdot,0))]}{1+\beta t L^+[Q(u_\varepsilon(\cdot,0))]}. \tag{3.9}$$

Finally, we treat the possibly degenerate case where $Q' \geq 0$. As in [18], we introduce a further regularization where $Q(u_\varepsilon)$ is replaced by $Q^\delta(u_\varepsilon) = Q(u_\varepsilon) + \delta u_\varepsilon$, and (3.4) is then recovered by letting $\delta \downarrow 0$.

REMARKS.

1. Theorem 3.1 covers the convective porous medium equation where $Q(u_\varepsilon) = |u_\varepsilon|^\gamma u_\varepsilon$, $\gamma > 0$.

2. In view of the convexity assumption on f, we may set $Q(u_\varepsilon) = a(u_\varepsilon)$ and obtain

$$L^+[a(u_\varepsilon(\cdot,t))] \leq \frac{L^+[a(u_\varepsilon(\cdot,0))]}{1+tL^+[a(u_\varepsilon(\cdot,0))]}. \tag{3.10}$$

Since in this case (where $Q'(u_\varepsilon) \geq \alpha > 0$), $u_\varepsilon(x,t)$ converges strongly to the entropy solution, $u(x,t)$, of (3.1), the $\varepsilon \downarrow 0$ limit of (3.10) also gives us

$$L^+[a(u(\cdot,t))] \leq \frac{L^+[a(u(\cdot,0))]}{1+tL^+[a(u(\cdot,0))]}. \tag{3.11}$$

The a priori estimate (3.11), stated in somewhat weaker form, can be found in [5, Theorem 1]. It shows that the compact solution operator of the nonlinear problem (3.1) tends to 'linearize' the problem as $a_x(\cdot, t \uparrow \infty) = 0$. As noted in [5], the inequality (3.11) requires f to be merely C^1, and is sharper than both – Oleinik's result (3.2) as well as its generalization in [2, Proposition 1] which apply to C^2 fluxes.

3. Theorem 3.1 implies, in particular, that the positive variation of $Q(u_\varepsilon(\cdot, t))$ supported on any compact domain is bounded. Consequently, the total variation of $Q(u_\varepsilon(\cdot, t))$ over such domains is upper-bounded by

$$\begin{aligned}\|Q(u_\varepsilon(\cdot,t))\|_{BV} &\leq \text{Const}_0 \frac{L^+[Q(u_\varepsilon(\cdot,0))]}{1+\beta t L^+[Q(u_\varepsilon(\cdot,0))]} \leq \\ &\leq \text{Const}_0 \frac{\max Q'}{\min a'} \cdot \frac{L^+[a(u_\varepsilon(\cdot,0))]}{1+tL^+[a(u_\varepsilon(\cdot,0))]},\end{aligned} \tag{3.12}$$

where Const_0 equals twice the size of $\text{supp}Q(u_\varepsilon(\cdot, t))$ We note that (3.12) holds for BV initial data, $u_\varepsilon(\cdot, 0)$, over arbitrary domains (with a different coefficient Const_0).

Equipped with Theorem 3.1, we now turn to the main of this section – the local convergence rate of small viscosity solutions for uniformly parabolic equations

$$\frac{\partial}{\partial t}[u_\varepsilon(x,t)] + \frac{\partial}{\partial x}[f(u_\varepsilon(x,t))] = \varepsilon\frac{\partial^2}{\partial x^2}[Q(u_\varepsilon(x,t))], \quad t \geq 0, \quad Q' \geq q > 0. \tag{3.13}$$

The difference between u_ε and its entropy limit u,

$$e_\varepsilon(x,t) = u_\varepsilon(x,t) - u(x,t),$$

satisfies the error equation

$$\frac{\partial}{\partial t}[e_\varepsilon(x,t)] + \frac{\partial}{\partial x}[\overline{a}_\varepsilon(x,t)e_\varepsilon(x,t)] = \varepsilon\frac{\partial^2}{\partial x^2}[Q(u_\varepsilon(x,t))], \quad t \geq 0, \tag{3.14a}$$

where $\overline{a}_\varepsilon(x,t)$ denotes the mean-value

$$\overline{a}_\varepsilon(x,t) = \int_{\xi=0}^{1} a(\xi u_\varepsilon(x,t) + (1-\xi)u(x,t))d\xi. \tag{3.14b}$$

We recall that Theorem 3.1 applies to both $u_\varepsilon(x,t)$ and its entropy limit $u(x,t)$. In particular, in view of the strict convexity of Q, (3.4) implies

$$L^+[u_\varepsilon(\cdot,t)] \leq \frac{1}{q} \cdot \frac{L^+[Q(u_\varepsilon(\cdot,0))]}{1+\beta t L^+[Q(u_\varepsilon(\cdot,0))]}, \quad t \geq 0,$$

and in view of the strict convexity of a, (3.11) implies

$$L^+[u(\cdot,t)] \leq \frac{1}{\alpha} \cdot \frac{L^+[a(u(\cdot,0))]}{1+tL^+[a(u(\cdot,0))]}, \quad t \geq 0.$$

Inserting the last two inequalities into (3.14b) and using the convexity of $a(\cdot,t)$ once more, we find after little rearrangement the following.

PROPOSITION 3.2 (OSLC). *The averaged convective velocity $\overline{a}_\varepsilon(x,t)$ given in (3.14b) satisfies the OSLC*

$$L^+[\overline{a}_\varepsilon(\cdot,t)] \leq m(t), \quad m(t) = \frac{\eta L_0^+}{1+tL_0^+}. \tag{3.15}$$

REMARK. The constants $L_0^+ \leq \infty$ and $\eta \geq 1$ are given respectively by

$$L_0^+ = \max(L^+[a(u_\varepsilon(\cdot,0))], L^+[a(u(\cdot,0))]) \tag{3.16a}$$

$$\eta = \frac{\max a' \cdot \max Q'}{\min a' \cdot \min Q'}. \tag{3.16b}$$

Let us now form the dual problem of the error equation (3.14). This is given by the backward linear transport equation

$$\frac{\partial}{\partial t}[\phi_\varepsilon(x,t)] + \overline{a}_\varepsilon(x,t)\frac{\partial}{\partial x}[\phi_\varepsilon(x,t)] = 0, \quad t \leq T, \tag{3.17a}$$

with, say C^1 data, independent of ε prescribed at $t=T$,

$$\phi_\varepsilon(x,t=T) = \phi(x). \tag{3.17b}$$

Although $\overline{a}_\varepsilon(\cdot,t)$ may – and in the generic case, will be discontinuous, Proposition 3.2 tells us that Theorem 2.2 applies in this case; namely, by (2.27) we have

$$\|\phi_\varepsilon(\cdot,t)\|_{\mathrm{Lip}} \leq \left(\frac{1+TL_0^+}{1+tL_0^+}\right)^\eta \|\frac{d\phi}{dx}\|_{L^\infty}, \quad 0 \leq t \leq T. \tag{3.18}$$

We are now ready to proceed with our main result announced earlier.

We convolute the error equation (3.14a) against $\phi_\varepsilon(\cdot,t)$ over its compact support; we convolute the adjoint equation (3.17a) against $e_\varepsilon(\cdot,t)$; their sum results in

$$\frac{d}{dt}e_\varepsilon(\cdot,t) * \phi_\varepsilon(\cdot,t) = \varepsilon\frac{\partial^2}{\partial x^2}[Q(u_\varepsilon(\cdot,t))] * \phi_\varepsilon(\cdot,t). \tag{3.19}$$

Equation (3.19) governs the evolution of moments of the error. By (3.12) and (3.18), its RHS is upper-bounded by

$$
\begin{aligned}
&\|\varepsilon\frac{\partial^2}{\partial x^2}[Q(u_\varepsilon(\cdot,t))]*\phi_\varepsilon(\cdot,t)\|_{L^\infty} \le \varepsilon\|Q(u_\varepsilon(\cdot,t))\|_{BV}\|\phi_\varepsilon(\cdot,t)\|_{\mathrm{Lip}} \le \\
&\varepsilon\mathrm{Const}_0\cdot L_0^+\cdot\frac{\min a'}{\max Q'}\cdot\frac{\eta(1+TL_0^+)^\eta}{(1+tL_0^+)^{\eta+1}}\|\frac{d\phi}{dx}\|_{L^\infty}, \quad 0\le t\le T.
\end{aligned} \tag{3.20}
$$

Temporal integration of (3.19) together with (3.20) yields

THEOREM 3.3. *Let $u(x,t)$ and $u_\varepsilon(x,t)$ be the entropy solution and the corresponding viscosity solution of (3.1) and (3.13), respectively. Then there exist constants $K > 0$ and $\eta \ge 1$[1] such that for any C^1-function, $\phi(x)$, the following estimate holds*

$$
\|u_\varepsilon(\cdot,T)*\phi-u(\cdot,T)*\phi\|_{L^\infty} \le K(1+L_0^+T)^{\eta+1}\|\frac{d\phi}{dx}\|_{L^\infty}\cdot[\varepsilon+\|\int^x(u_\varepsilon(\cdot,0)-u(\cdot,0))\|_{L^1}]. \tag{3.21}
$$

The dependence of the error estimate (3.21) on the initial data, $u(\cdot,0)$ and $u_\varepsilon(\cdot,0)$, is reflected by the following two quantities:

1. The $W^{-1,1}$ size of the initial error. Of course, in order to obtain the desired $O(\varepsilon)$ convergence rate, we shall need a rather weak consistency assumption in this direction, requiring

$$
\|u_\varepsilon(\cdot,0)-u(\cdot,0)\|_{W^{-1,1}} \le \mathrm{Const.}\cdot\varepsilon. \tag{3.22}
$$

2. The one-sided Lipschitz size of the initial data, measured by

$$
L_0^+ = \max(L^+[a(u_\varepsilon(\cdot,0))], \quad L^+[a(u(\cdot,0))]).
$$

We restrict our attention to initial data for which L_0^+ is finite, i.e.,

$$
ess\sup_{x\neq y}\left(\frac{u_\varepsilon(x,0)-u_\varepsilon(y,0)}{x-y}\right)^+, ess\sup_{x\neq y}\left(\frac{u(x,0)-u(y,0)}{x-y}\right)^+ \le \mathrm{Const.} \tag{3.23}
$$

In other words, (3.23) assumes general initial data as long as they do not contain non-Lipschitzian increasing discontinuities; in particular, arbitrary C^1 initial data are permitted. It would be desirable to extend our result to arbitary BV initial data. In this context, the reader is referred to [4] for an almost optimal convergence result for the case of monotonically increasing initial data.

---

[1] The constants K and η may depend on the strictly positive extreme values of f'' and ϕ' (and K may depend on the size of suppϕ), but otherwise are independent of ε and t.

We have shown

THEOREM 3.4 (Uniform estimate on the moments). *Let $u(x,t)$ and $u_\varepsilon(x,t)$ be the entropy solution and the corresponding viscosity solution of (3.1) and (3.13) respectively. We assume that*

A. The initial viscosity data, $u_\varepsilon(\cdot,0)$, is consistent with the initial entropy data, $u(\cdot,0)$, in the sense that (3.22) holds.

B. The increasing part of the viscosity and entropy initial data is Lipschitz, i.e., (3.23) holds.

Then for any $T > 0$, there exists a constant $K = K(T)$ such that for all C^1-functions, $\phi(x)$, we have

$$\|u_\varepsilon(\cdot,T) * \phi - u(\cdot,T) * \phi\|_{L^\infty} \leq K \cdot \varepsilon \cdot \|\frac{d\phi}{dx}\|_{L^\infty}. \tag{3.24}$$

REMARK. Theorem 3.4 suggests $W^{-1,1}$ as a possible weak topology to study the convergence of u_ε. P. D. Lax [7, p. 191] used a similar setup to prove convergence (without a rate estimate) already in 1954.

The estimate on the moments of the error in Theorem 3.4 can be converted into an L^1 error estimate at the expense of 'losing' an additional factor of $\sqrt{\varepsilon}$. This results in the usual $O(\sqrt{\varepsilon})$ L^1-convergence rate in agreement with, [6], [12]. We close this section with

COROLLARY 3.5. (L^p-error estimate). *Assume that the conditions of Theorem 3.4 hold. Then for any $T > 0$ and $p \geq 1$ there exists a constant K (which depends on T and $\frac{p-1}{p}$, but otherwise is independent of ε) such that*

$$\|u_\varepsilon(\cdot,T) - u(\cdot,T)\|_{L^p} \leq K \cdot \varepsilon^{\frac{1}{2p}}, \quad p \geq 1. \tag{3.25}$$

PROOF. Let $\zeta(x)$ be a C_0^1 function with unit mass. For any $\phi \epsilon L^\infty$ we consider

$$e_\varepsilon(\cdot,T) * \phi = e_\varepsilon(\cdot,T) * \phi_\delta + e_\varepsilon(\cdot,T) * (\phi - \phi_\delta), \quad \phi_\delta \equiv \phi * \frac{1}{\delta}\zeta(\frac{\cdot}{\delta}). \tag{3.26}$$

By Theorem 3.4, the first term on the right does not exceed

$$\|e_\varepsilon(\cdot,T) * \phi_\delta\|_{L^\infty} \leq K \cdot \varepsilon \cdot \|\frac{d\phi_\delta}{dx}\|_{L^\infty} \leq \text{Const.}\frac{\varepsilon}{\delta} \cdot \|\phi\|_{L^\infty}. \tag{3.27}$$

By (3.12), $u_\varepsilon(\cdot,T)$, and likewise, $u(\cdot,T)$, have bounded variations, and hence the second term on the RHS of (3.26) is upper bounded by

$$
\begin{aligned}
e_\varepsilon(\cdot,T)*(\phi-\phi_\delta) &= \int_y e_\varepsilon(y,T)\phi(\cdot-y)dy- \\
&- \int_y e_\varepsilon(y,T)\int_z \phi(\cdot-y-z)\frac{1}{\delta}\zeta(\frac{z}{\delta})dzdy = \\
&= \int_z[\int_y[e_\varepsilon(y,T)-e_\varepsilon(y-z,T)]\phi(\cdot-y)dy]\frac{1}{\delta}\zeta(\frac{z}{\delta})dz \leq \\
&\leq \text{Const.}\delta\|\phi\|_{L^\infty}
\end{aligned}
\tag{3.28}
$$

Inserting (3.27) and (3.28) into (3.26) we obtain

$$
|\int_x e_\varepsilon(x,T)\phi(x)dx| \leq \text{Const.}(\frac{\varepsilon}{\delta}+\delta)\|\phi\|_{L^\infty}. \tag{3.29}
$$

Choosing the free parameter $\delta \sim \sqrt{\varepsilon}$, (3.29) with $\phi = e_\varepsilon^{p-1}(\cdot,T)$ yields

$$
\|e_\varepsilon(\cdot,T)\|_{L^p}^p \leq \text{Const.}\sqrt{\varepsilon}\|e_\varepsilon(\cdot,T)\|_{L^\infty}^{p-1}, \tag{3.30}
$$

and the result (3.25) follows.

4. LOCAL ERROR ESTIMATES

In the previous section we proved that the error between C^1-moments of u_ε and the corresponding C^1-moments of u is of order ε. In this section we show that given u_ε, we can use its moments in order to recover pointwise values of u (and any of its derivatives) with an error as close to ε as desired.

The idea of such recovery technique is not new. The reader is referred to Mock and Lax [9] who post-processed difference approximations in order to recover accurately the pointwise values of discontinuous solutions for linear hyperbolic equations. Their approach was later extended in [1] and [3] to include spectral and ψdospectral approximations for such equations. We shall give a bird's eye view of both approaches to the post-processing technique in the present context of nonlinear equations.

Assume that $u(\cdot,T)$ is smooth in some fixed θ-neighborhood of x, say

$$
u(\cdot,T)\epsilon C^p(x-\theta,x+\theta), \quad p\geq 1. \tag{4.1}
$$

Let $\phi(x)$ be a C_0^∞ function supported on the (-1,1) interval such that

$$
\int_{-1}^1 \phi(x)dx = 1, \quad \int_{-1}^1 x^k\phi(x) = 0, \quad k=1,2,\cdots,p-1. \tag{4.2}
$$

With $\phi_\delta(x) \equiv \frac{1}{\delta}\phi(\frac{x}{\delta})$, where $\delta\epsilon(0,\theta)$ is a free parameter to be chosen later, we have

$$(4.3) \qquad |(u(\cdot,T) * \phi_\delta)(x) - u(x,T)| \leq \frac{\delta^p}{p!}\|\phi\|_{L^1} \cdot \|u(\cdot,T)\|_{W^{p,\infty}(x-\theta,x+\theta)}.$$

The error bound in (4.3) can be made small in two ways: by decreasing δ – the size of suppϕ_δ, as was done in [9]; and by increasing p – so that further cancellation due to the oscillatory behavior of ϕ occurs, as was done in [3]. We shall the sketch the details of both ways.

By Theorem 3.4 we have

$$(4.4) \qquad |(u_\varepsilon(\cdot,T) * \phi_\delta)(x) - (u(\cdot,T) * \phi_\delta)(x)| \leq K \cdot \varepsilon \cdot \|\frac{d\phi_\delta}{dx}\|_{L^\infty} \leq K \cdot \frac{\varepsilon}{\delta^2}\|\frac{d\phi}{dx}\|_{L^\infty}$$

We now have two types of error estimates: mollification in (4.3) and approximation in (4.4). If we choose the free parameter δ as

$$(4.5) \qquad \delta = \theta\left(\frac{\varepsilon}{1+|u^{(p)}|_{\text{loc}}}\right)^{\frac{1}{p+2}}, \quad |u^{(p)}|_{\text{loc}} = \|u(\cdot,T)\|_{W^{p,\infty}(x-\theta,x+\theta)},$$

then the contribution of the two error terms is of the same order and (4.3), (4.4) yield

$$(4.6) \qquad |(u_\varepsilon(\cdot,T) * \phi_\delta)(x) - u(x,T)| \leq \text{Const}_p \cdot (1+|u^{(p)}|_{\text{loc}})^{\frac{2}{p+2}} \cdot \varepsilon^{\frac{p}{p+2}}.$$

Thus, the ϕ_δ-moments of $u_\varepsilon(\cdot,T)$ recover the pointwise values of $u(\cdot,T)$ with accuracy as close to ε as the local smoothness of $u(\cdot,T)$ permits.

The estimate (4.6) makes use of a dilated localizer $\phi(x)$ whose support lies in the interval (-1,1). If, instead, we choose the support of $\phi(x)$ to lie in, say (0,1), then (4.6) holds with $|u^{(p)}|_{\text{loc}}$ restricted to $[x+, x+\theta)$,

$$(4.6) \qquad |u^{(p)}|_{\text{loc}} = \|u(\cdot,T)\|_{W^{p,\infty}[x+,x+\theta)}.$$

Thus we are able to recover pointwise values of $u(\cdot,T)$ up to the discontinuity.

In a similar manner, we can recover any spatial derivative of the entropy solution, $\frac{\partial^s}{\partial x^s}u(\cdot,T)$. Indeed, Theorem 3.4 gives us

$$(4.7) \quad |(\frac{\partial^s}{\partial x^s}u_\varepsilon(\cdot,T)*\phi_\delta)(x) - (\frac{\partial^s}{\partial x^s}u(\cdot,T)*\phi_\delta)(x)| \leq K\cdot\varepsilon\cdot\|\frac{d^{s+1}\phi_\delta}{dx^{s+1}}\|_{L^\infty} \leq K\cdot\frac{\varepsilon}{\delta^{s+2}}\cdot\|\frac{d^{s+1}\phi}{dx^{s+1}}\|_{L^\infty},$$

Hence, by choosing

$$(4.8) \qquad \delta = \theta\left(\frac{\varepsilon}{1+|u^{(p+s)}|_{\text{loc}}}\right)^{\frac{1}{p+s+2}}, \quad |u^{(p+s)}|_{\text{loc}} = \|u(\cdot,T)\|_{W^{p+s,\infty}(x+\theta\text{supp}\phi)},$$

we obtain a generalization of (4.6) to higher derivatives, which we summarize in

THEOREM 4.1. *Assume that the conditions of Theorem 4.3 hold. Then if we choose $\phi_\delta(x)$ as described above (4.2), (4.8), the ϕ_δ-moments of $\frac{\partial^s}{\partial x^s}u_\varepsilon(\cdot,T)$ recover the pointwise values of $u(\cdot,T)$ and its derivatives, and the following estimate holds,*

$$|(\frac{\partial^s}{\partial x^s}u_\varepsilon(\cdot,T) * \phi_\delta)(x) - \frac{\partial^s}{\partial x^s}u(x,T)| \le \mathrm{Const}_p \cdot (1 + |u^{(p+s)}|_{\mathrm{loc}})^{\frac{s+2}{p+s+2}} \cdot \varepsilon^{\frac{p}{p+s+2}}, \tag{4.9}$$

Next, let us consider the generic case of smooth initial data, $u(x,0)\epsilon\mathcal{S}$. Then, the entropy solution of (3.1) (with C^∞ convex flux f) is piecewise smooth [13]. In this case, we are able to recover the pointwise values of $\frac{\partial^s}{\partial x^s}u(x,T)$ with error as close to ε as desired, if we take p large enough in (4.9).

We close this section with a brief description of the spectral post-processing technique [3] which enables the pointwise recovery of $u(\cdot,T)$ and its derivatives.

We restrict our attention to the case (4.1) where the symmetric interval $(x+\theta, x+\theta)$ is free of discontinuities of $u(\cdot,T)$. Let $\zeta(x)$ be a $C_0^\infty(-1,1)$ function, normalized such that

$$\zeta(x=0) = 1, \tag{4.10}$$

and we set

$$\phi_N(x) = \frac{1}{\theta}\zeta(\frac{x}{\theta})D_N(\frac{x}{\theta}), \quad D_N(\xi) \equiv \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})\xi}{\sin\frac{1}{2}\xi}. \tag{4.11}$$

We note that in this case the support of the regularization kernel, $\phi_N(x)$ is kept fixed. Instead, by increasing N, we obtain a highly oscillatory kernel whose monomial moments satisfy (4.2) modulo a spectrally small negligible error.

Standard error estimates for the truncated Fourier projection, S_N, give us [3, Section 3]

$$\begin{aligned} |(u(\cdot,T) * \phi_N)(x) - u(x,T)| &= |(I - S_N)[u(x-\theta\cdot,T)\zeta(\cdot) - u(x,T)](0)| \\ &\le \mathrm{Const}_p \cdot \frac{ln\ N}{N^p}\|u(\cdot,T)\|_{W^{p,\infty}(x-\theta,x+\theta)}. \end{aligned} \tag{4.12}$$

Applying Theorem (3.4) with $\phi = \phi_N$ we find

$$|(u_\varepsilon(\cdot,T) * \phi_N)(x) - (u(\cdot,T) * \phi_N)(x)| \le K \cdot \varepsilon \cdot \|\frac{d\phi_N}{dx}\|_{L^\infty} \le \mathrm{Const.}\varepsilon \cdot N^2. \tag{4.13}$$

Hence, by choosing

$$N \sim \left(\frac{|u^{(p)}|_{\mathrm{loc}}}{\varepsilon}\right)^{\frac{1}{p+2}}, \quad |u^{(p)}|_{\mathrm{loc}} = \|u(\cdot,T)\|_{W^{p,\infty}(x-\theta,x+\theta)} \tag{4.14}$$

we recover from (4.12), (4.13) the local error estimate we had in (4.6), namely we have

$$|(u_\varepsilon(\cdot,T) * \phi_N)(x) - u(x,T)| \le \mathrm{Const}_p \cdot |u^{(p)}|_{\mathrm{loc}}^{\frac{2}{p+2}} \cdot \varepsilon^{\frac{p}{p+2}}. \tag{4.15}$$

5. CONCLUDING REMARKS

The results of Section 4 hinge on the a priori estimate of the moments in Theorem 3.4, which, in turn, is based on the $W^{1,\infty}$-energy estimates for the OSLC satisfying linear transport equations studied in Section 2. In this section we provide still another derivation of Theorem 3.4 which amplify the direct linkage between the OSLC and Theorem 3.4. To demonstrate our point we will concentrate on the, say 2π, periodic case.

Let

$$E_\varepsilon(x,t) = \int^x u_\varepsilon(\xi,t)d\xi - \int^x u(\xi,t)d\xi \tag{5.1}$$

denotes the difference between the primitives of the small viscosity approximation, $U_\varepsilon(x,t) = \int^x u_\varepsilon(\xi,t)d\xi$, and that of the entropy solution, $U(x,t) = \int^x u(\xi,t)d\xi$. Integration of the error equation (3.14a) gives us

$$\frac{\partial}{\partial t}[E_\varepsilon(x,t)] + \overline{a}_\varepsilon(x,t)\frac{\partial}{\partial x}[E_\varepsilon(x,t)] = \varepsilon\frac{\partial}{\partial x}[Q(u_\varepsilon(x,t))]. \tag{5.2}$$

We shall now energy estimate (5.2) along the lines of our study of irreversible linear transport equations in Section 2, compare (2.16). Integrating (5.2) against $\operatorname{sgn}E_\varepsilon$, we obtain

$$\begin{aligned}&\frac{d}{dt}\|E_\varepsilon(\cdot,t)\|_{L^1} + \int_x \overline{a}_\varepsilon(x,t)\frac{\partial}{\partial x}[E_\varepsilon(x,t)]dx = \\ &= \varepsilon\int_x \operatorname{sgn}E_\varepsilon(x,t)\frac{\partial}{\partial x}[Q(u_\varepsilon(x,t))]dx.\end{aligned} \tag{5.3}$$

Integration by parts of the second term on the RHS of (5.3), together with the upper bound of the RHS by the BV estimate of $Q(u_\varepsilon(\cdot,t))$ in (3.12), yield

$$\frac{d}{dt}\|E_\varepsilon(\cdot,t)\|_{L^1} \le \int_x \frac{\partial}{\partial x}[\overline{a}_\varepsilon(x,t)]\cdot E_\varepsilon(x,t)dx + \text{Const.}\varepsilon. \tag{5.4}$$

Thanks to the judicious positive sign on the right, we may now use the OSLC (3.15) to obtain

$$\frac{d}{dt}\|E_\varepsilon(\cdot,t)\|_{L^1} \le m(t)\|E_\varepsilon(\cdot,t)\|_{L^1} + \text{Const.}\varepsilon. \tag{5.5}$$

By integration of the last inequality we conclude – in agreeement with Theorem 3.4, that

$$\|U_\varepsilon(\cdot,t) - U(\cdot,t)\|_{L^1} \le K\cdot\varepsilon, \quad 0\le t\le T. \tag{5.6}$$

Using this we can derive L^∞ (and consequently, L^p) estimate as follows. Let $E_\varepsilon(x,t)$ assumes the Fourier expansion

$$E_\varepsilon(x,t) = \sum_{|k|\le\infty} \hat{E}(k,t)e^{ikx}. \tag{5.7}$$

By (5.6) we have

$$|\hat{E}_\varepsilon(k,t) \leq \frac{1}{2\pi}\|E_\varepsilon(\cdot,t)\|_{L^1} \leq \frac{K}{2\pi}\cdot\varepsilon. \tag{5.8}$$

Moreover, since both $u_\varepsilon(\cdot,t) \equiv \frac{\partial}{\partial x}U_\varepsilon(\cdot,t)$ and $u(\cdot,t) \equiv \frac{\partial}{\partial x}U(\cdot,t)$ are uniformly BV (w.r.t. ε), their k-th Fourier coefficients decay like $\lesssim |k|^{-1}$, and hence

$$|\hat{E}_\varepsilon(k,t)| \leq \text{Const.}\frac{1}{|k|^2}, \quad |k| \geq 1. \tag{5.9}$$

Inserting (5.8), (5.9) into (5.7) we find

$$\|E_\varepsilon(\cdot,t)\|_{L^\infty} \leq \sum_{|k|\leq\frac{1}{\sqrt{\varepsilon}}}\frac{K}{2\pi}\cdot\varepsilon + \sum_{|k|>\frac{1}{\sqrt{\varepsilon}}}\text{Const.}\frac{1}{|k|^2} \leq \text{Const.}\sqrt{\varepsilon}. \tag{5.10}$$

Standard interpolation between the L^1 error estimate in (5.6) and the L^∞ estimate in (5.10) gives us

THEOREM 5.1. *Assume that the conditions of Theorem 3.4 hold. Then for any $T > 0$ there exists a constant $K = K(T)$ such that*

$$\|U_\varepsilon(\cdot,t) - U(\cdot,t)\|_{L^p} \geq K\cdot\varepsilon^{\frac{p+1}{2p}}, \quad p\geq 1, \quad 0\leq t\leq T. \tag{5.11}$$

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16. Abstract

Let $u(x,t)$ be the possibly discontinuous entropy solution of a nonlinear scalar conservation law with smooth initial data. Suppose $u_\varepsilon(x,t)$ is the solution of an approximate viscosity regularization, where $\varepsilon > 0$ is the small viscosity amplitude. We show that by post-processing the small viscosity approximation u_ε, we can recover pointwise values of u and its derivatives with an error as close to ε as desired.

The analysis relies on the adjoint problem of the forward error equation, which in this case amounts to a backward linear transport with discontinuous coefficients. The novelty of our approach is to use a (generalized) E-condition of the forward problem in order to deduce a $W^{1,\infty}$ energy estimate for the discontinuous backward transport equation; this, in turn, leads us to ε-uniform estimate on moments of the error $u_\varepsilon - u$.

Our approach does not "follow the characteristics" and, therefore, applies mutatis mutandis to other approximate solutions such as E-difference schemes.

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