NASA Contractor Report 181958
ICASE Report No. 89–71

# ICASE

# MODIFIED CHEBYSHEV PSEUDOSPECTRAL METHOD WITH $O(N^{-1})$ TIME STEP RESTRICTION

Dan Kosloff
Hillel Tal-Ezer

Contract No. NAS1–18605
December 1989

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, Virginia 23665–5225

Operated by the Universities Space Research Association



National Aeronautics and
Space Administration

**Langley Research Center**
Hampton, Virginia 23665-5225

# MODIFIED CHEBYSHEV PSEUDOSPECTRAL METHOD WITH $O\left(N^{-1}\right)$ TIME STEP RESTRICTION

Dan Kosloff
Department of Geophysical and Planetary Sciences
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University

Hillel Tal-Ezer[1]
Institute for Computer Applications in Science and Engineering
and
School of Mathematical Sciences
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University

## ABSTRACT

The extreme eigenvalues of the Chebyshev pseudospectral differentiation operator are $O(N^2)$ where $N$ is the number of grid points [4]. As a result of this, the allowable time step in an explicit time marching algorithm is $O(N^{-2})$ which, in many cases, is much below the time step dictated by the physics of the P.D.E. In this paper we introduce a new set of interpolating points such that the eigenvalues of the differentiation operator are $O(N)$ and the allowable time step is $O(N^{-1})$. The properties of the new algorithm are similar to those of the Fourier method but in addition it provides highly accurate solution for nonperiodic boundary value problems.

[1]Research was supported in part by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665. Additional support was provided by the Air Force Office of Scientific Research under Grant No. 85-0303.

## 1. Introduction

Consider the first order hyperbolic initial boundary value problem

$$u_t - u_x = 0 \qquad -1 < x < 1\ ,\ t \geq 0 \tag{1.1}$$

$$u(x,0) = u^0(x) \qquad -1 < x < 1 \tag{1.2}$$

$$u(1,t) = s(t) \qquad t \geq 0. \tag{1.3}$$

A standard pseudospectral method [5] for solving (1.1)-(1.3) is based on interpolating the solution at the extremal points

$$x_i = \cos\left(\frac{i\pi}{N}\right) \qquad i = 0, \cdots, N \tag{1.4}$$

of the Nth-order Chebyshev polynomial

$$T_N(x) = \cos(N \arccos(x)). \tag{1.5}$$

Using this method for space discretization and a standard explicit scheme (e.g., Runge-Kutta) for time discretization, one encounters a stability condition which has to satisfy [5]

$$\Delta t = O(N^{-2}). \tag{1.6}$$

This restriction is very stringent and forces the user to march in time steps which, in many cases, are much bellow the time step dictated by the physics of the problem. A way of overcoming this annoying phenomenon is to use implicit or semi-implicit time marching techniques. Since the pseudospectral differentiation matrix is dense, the resulting algorithm is highly time consuming. Therefore, we would like to find a way to advance explicitly in time with a less restrictive stability condition. Our research is aimed at this target.

Chebyshev points (1.4) are bunched near the boundaries with minimal spacing of $O(N^{-2})$. Since the pseudospectral method is global, there is no direct relation between the minimal spacing and the stability condition as in the finite-difference method [9]. Nevertheless, numerical experience and heuristic reasoning led us to 'blame' the super fine grid near the boundaries for the severe stability condition (1.6). When there are sharp gradients near the boundaries, the clustering of points is needed for resolution and the small time step can also be anticipated by physical reasonings. But when the high gradients are elsewhere, or if the solution is evenly smooth, there seem to be no justification in putting more points near the boundary. Thus, we are led to the conclusion that the numerical tool we are using (polynomial interpolation) is not appropriate in these cases.

In [2], Bayliss et al. describe a physical model with very sharp gradients. In order to overcome the numerical difficulties, they have designed an algorithm where the problem is transformed so as to minimize some functional. In our research, we use a similar transformation approach. The collocation points are chosen as

$$x_i = g(y_i; \alpha) \qquad -1 \le x_i \le 1 \qquad i = 0, \cdots, N, \qquad 0 \le \alpha < 1, \tag{1.7}$$

where

$$y_i = \cos\left(\frac{i\pi}{N}\right) \qquad i = 0, \cdots, N, \tag{1.8}$$

$\alpha$ is a parameter and $g(y;\alpha)$ is a 'stretching function'. By a proper choice of the parameter $\alpha$ we increase the minimal spacing near the boundaries such that

$$\Delta x_{min} = O\left(N^{-1}\right). \tag{1.9}$$

Consequently, we are able to advance in time with the favorable stability condition

$$\Delta t = O(N^{-1}). \tag{1.10}$$

Moreover, it is shown that, as $N \to \infty$ , one needs only two points per wavelength for resolution (as in the Fourier method ) and not $\pi$ points as in the Chebyshev case [5]. Thus, fewer points are needed to model the P.D.E. (a saving of almost 40% ). The transformation function is described in Section 2. In Section 3 we present the resolution analysis which also reveals the approximation subspace on which the solution is projected. In order to efficiently implement the algorithm, it is important to choose the appropriate parameter $\alpha$ and this subject is discussed in Section 4. In Section 5 we present a more general transformation which gives additional flexibility to the new interpolation method. The paper is concluded in Section 6 in which we present numerical results.

## 2. New Interpolation Points

Chebyshev pseudospectral solution of (1.1)-(1.3) is based on approximating the spatial derivative by differentiating analytically the interpolating polynomial. If $v$ is an $N$ dimensional vector which approximates $u(x)$ at the interpolation points (1.4) then the vector

$$v' = Dv \tag{2.1}$$

approximates $u'(x)$ at (1.4). $D$ is the spatial differentiation matrix which incorporates the boundary condition (1.3). The entries of $D$ are given in [6] ((2.1) can be accomplished by using FFTs requiring only O(NlogN) operations [6]). The matrix $D$ is very ill-conditioned, with eigenvalues scattered in the left side of the complex plane [4]. While most of the eigenvalues grow like $O(N)$, a few of them are $O(N^2)$ [4]. These extreme eigenvalues can be considered as the reason for the severe stability condition (1.6). We have to choose $\Delta t = O(N^{-2})$ so that all the eigenvalues of $\Delta t D$ will be included in the domain of stability of the time marching scheme.

Furthermore,

$$\Delta x_{min} = \min_i |x_{i+1} - x_i| = 1 - \cos(\pi/N) = O(N^{-2}). \tag{2.2}$$

This phenomenon of having domain of eigenvalues whose size is proportional to the reciprocal of the minimal spacing is typical to many differentiation matrices. Even in cases where this correspondence does not hold, as in the Legendre pseudospectral method [10] , we still have to choose time step dictated by the minimal spacing due to numerical instability whose origin is the ill-conditioning of the matrix which diagonalizes the differentiation matrix [12]. Thus, we would like to have a set of interpolating points with larger minimal spacing. We are going to attain this goal by mapping the Chebyshev points (1.8) to another set of points in $[-1, 1]$ such that the minimal spacing near the boundary is 'stretched'.

Let us consider the transformation

$$x = g(y; \alpha) = \frac{\arcsin(\alpha y)}{\arcsin(\alpha)} \qquad x, y \in [-1, 1]. \tag{2.3}$$

Computing the derivative at the grid points $x_i$

$$x_i = g(y_i; \alpha) \qquad 0 \le i \le N, \tag{2.4}$$

where

$$y_i = \cos\left(\frac{i\pi}{N}\right) \qquad 0 \le i \le N \tag{2.5}$$

is accomplished by making use of the chain rule. For given $f \in C^1[-1, 1]$ , we have

$$\frac{df}{dx} = \frac{1}{g'(y; \alpha)} \frac{df}{dy}. \tag{2.6}$$

Hence, we modify (2.1) to read

$$v' = ADv \tag{2.7}$$

where $A$ is a diagonal matrix

$$A_{ii} = \frac{1}{g'(y_i; \alpha)} \tag{2.8}$$

and

$$g'(y; \alpha) = \frac{\alpha}{\arcsin(\alpha)} \frac{1}{\sqrt{1 - (\alpha y)^2}}. \tag{2.9}$$

$v$ and $v'$ contains the approximated values of $u(x)$ and $u'(x)$ respectively at $x_i = g(y_i; \alpha)$. We have

**Lemma 2.1.** *If* $x_i$ ,$0 \le i \le N$, *satisfy (2.4) then the minimal spacing between the points is attained near the boundaries.*

**Proof:** Define

$$\theta = \arccos(y). \tag{2.10}$$

Using the mean value theorem

$$\begin{aligned} \Delta x_i &= x_{i+1} - x_i \\ &= \frac{dg}{d\theta}(\xi_i)\Delta\theta, \qquad \theta_i \le \xi_i \le \theta_{i+1}, \qquad 0 \le i \le N-1 \end{aligned} \tag{2.11}$$

while

$$\Delta\theta = \frac{\pi}{N}. \tag{2.12}$$

By (2.3) and (2.10) we have

$$\frac{dg}{d\theta} = \frac{\alpha}{\arcsin(\alpha)} h(\theta) \tag{2.13}$$

where

$$h(\theta) = -\frac{\sin(\theta)}{\sqrt{1 - (\alpha\cos(\theta))^2}} . \tag{2.14}$$

Since

$$h'(\theta) = \frac{\cos(\theta)\,(\alpha^2 - 1)}{\left(1 - (\alpha\cos(\theta))^2\right)^{\frac{3}{2}}} \tag{2.15}$$

is negative in $0 < \theta < \frac{\pi}{2}$ and positive in $\frac{\pi}{2} < \theta < \pi$ , $-h(\theta)$ attains its minima at $\theta = 0, \pi$ and the result follows.

Furthermore,

$$\Delta x_{min} = 1 - x_1 = 1 - \frac{\arcsin(\alpha\cos(\frac{\pi}{N}))}{\arcsin(\alpha)}. \tag{2.16}$$

It is easily verified that the R.H.S. of (2.16) is monotonically increasing with $\alpha$ , $\alpha \in (0,1)$ and

$$\lim_{\alpha\to 1} \Delta x_{min} = \frac{2}{N} \qquad (\textit{as in Fourier case}) \tag{2.17}$$

$$\lim_{\alpha\to 0} \Delta x_{min} = 1 - \cos\left(\frac{\pi}{N}\right) \qquad (\textit{as in Chebyshev case}). \tag{2.18}$$

**Lemma 2.2.** *If*

$$\alpha = 1 - \frac{c}{N^2} + O\left(N^{-3}\right) \tag{2.19}$$

*then*

$$\Delta x_{min} = \frac{2\pi}{\sqrt{\pi^2 + 2c} + \sqrt{2c}} \left(\frac{1}{N}\right) + O\left(N^{-2}\right) . \tag{2.20}$$

**Proof:** Define $z = \frac{1}{N}$ then

$$\Delta x_{min} = 1 - h(z) \tag{2.21}$$

where

$$h(z) = \frac{\arcsin(g_1(z))}{\arcsin(g_2(z))}, \tag{2.22}$$

$$g_1(z) = (1 - cz^2)\cos(\pi z) + O\left(z^3\right), \tag{2.23}$$

$$g_2(z) = (1 - cz^2) + O\left(z^3\right). \tag{2.24}$$

We have $h(0) = 1$ and

$$h'(0) = \frac{2}{\pi}\lim_{z\to 0}\left(\frac{g_1'}{\sqrt{1-g_1^2}} - \frac{g_2'}{\sqrt{1-g_2^2}}\right). \tag{2.25}$$

Using L'Hospital's rule it easily verified that

$$\begin{aligned} h'(0) &= -\frac{2}{\pi}\left(\sqrt{g_1''(0)} - \sqrt{g_2''(0)}\right) \\ &= \frac{2}{\pi}\left(\sqrt{\pi^2 + 2c} - \sqrt{2c}\right). \end{aligned} \tag{2.26}$$

and the result follows. Based on Lemma 2.2 we conjecture that the time restriction of the new interpolation method satisfies (1.10). Numerical results reported in section 6 assist this conjecture.

## 3. Resolution Analysis

Let

$$f_1(x) = \cos(r\pi x) \qquad -1 \le x \le 1, \tag{3.1}$$

$$f_2(x) = \sin(r\pi x) \qquad -1 \le x \le 1 \tag{3.2}$$

be functions whose derivative we want to approximate ( $r$ is a real number which indicates the wave number). Substituting (2.3) in (3.1) and (3.2) we have

$$f_1(x) = \tilde{f}_1(y) = \cos[m\arcsin(\alpha y)] \tag{3.3}$$

$$f_2(x) = \tilde{f}_2(y) = \sin[m\arcsin(\alpha y)] \tag{3.4}$$

where

$$m = \frac{r\pi}{\arcsin(\alpha)}. \tag{3.5}$$

In the appendix we show that

for $m$ even

$$\cos(m\psi) = (-1)^{\frac{m}{2}} T_m(\sin(\psi)), \tag{3.6}$$

for $m$ odd

$$\sin(m\psi) = (-1)^{\frac{m-1}{2}} T_m(\sin(\psi)). \tag{3.7}$$

Hence, using (3.3)-(3.5)

$$\tilde{f}_1(y) = (-1)^{\frac{m}{2}} T_m(\alpha y) \qquad (\ m\ even\ ) \tag{3.8}$$

and

$$\tilde{f}_2(y) = (-1)^{\frac{m-1}{2}} T_m(\alpha y) \qquad (\ m\ odd\ ). \tag{3.9}$$

$T_m(\alpha y)$ is a polynomial in $y$, therefore, interpolating at $N+1$ points, $m \leq N$, will result in the function itself. Hence, the new algorithm is exact for the following $N+1$ functions (N even)

$$1, \cos(2px), \cos(4px), \cdots, \cos(Npx), \sin(px), \sin(3px), \cdots, \sin((N-1)px) \tag{3.10}$$

where

$$p = \arcsin(\alpha). \tag{3.11}$$

The set of functions (3.10) span the approximation subspace. An elaborate discussion of this subspace will be given in future paper.

Let us clarify now what we mean by resolution. If the Chebyshev expansion of a function $h(y)$ is

$$h(y) = \sum_{k=0}^{\infty} \frac{1}{c_k} a_k T_k(y) \qquad c_0 = 2,\ c_k = 1\ for\ k \neq 0, \tag{3.12}$$

and there is $K$ such that $a_k$ decreases rapidly when $k$ increases beyond $K$ then we say that $h(y)$ is resolved by $K$ terms. Since Chebyshev expansion is qualitatively similar to interpolation in Chebyshev points (1.8), it is equivalent to saying that we need $K$ points in order to resolve $h(y)$ by interpolation.

We speculate that

**Conjecture:** *The function* $T_m(\alpha y)$ *is resolved by* $M+1$ *terms where*

$$M = [\alpha m]\ . \tag{3.13}$$

The reasoning for this conjecture goes as follows: let

$$T_m(\alpha y) = \sum_{k=0}^{m} \frac{1}{c_k} a_k^m T_k(y) \qquad c_0 = 2,\ c_k = 1\ for\ k \neq 0, \tag{3.14}$$

then

$$a_k^m = \frac{2}{\pi}\int_{-1}^{1} \frac{T_m(\alpha y)T_k(y)}{\sqrt{1-y^2}}dy. \tag{3.15}$$

Chebyshev polynomials satisfy the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{3.16}$$

Hence

$$a_k^m = \frac{2}{\pi}\int_{-1}^{1} \frac{\{2\alpha y T_{m-1}(\alpha y) - T_{m-2}(\alpha y)\}\, T_k(y)}{\sqrt{1-y^2}}dy, \qquad m \geq 2 \tag{3.17}$$

$$a_0^0 = 0, \qquad a_0^1 = 0, \qquad a_1^1 = \alpha. \tag{3.18}$$

By (3.16), $2yT_k(y) = T_{k+1}(y) + T_{k-1}(y)$ which imply the relation

$$a_k^m = c_k\alpha\left(a_{k-1}^{m-1} + a_{k+1}^{m-1}\right) - a_k^{m-2} \qquad k \geq 0\ ,\ m \geq 2, \qquad (a_{-1}^{m-1} = 0). \tag{3.19}$$

We have programmed (3.19) with initial values (3.18) and ran it for many values of $m$ and $\alpha$ and have always observed that when $k \leq M$ , $a_k^m$ are nondecreasing. Once $k$ is greater then $M$ , $a_k^m$ decreases very rapidly.

Therefore, using (3.5), the maximal wave number which can be resolved by the new method is

$$r_{max} = \frac{N \arcsin(\alpha)}{\pi\alpha} \tag{3.20}$$

while, since (3.10)

$$r_N = \frac{N \arcsin(\alpha)}{\pi} \tag{3.21}$$

is the largest wave number of mode resolved exactly by the new method.

Observe that

$$\lim_{\alpha\to 1} r_{max} = N/2 \qquad (as\ in\ Fourier\ case), \tag{3.22}$$

$$\lim_{\alpha\to 0} r_{max} = N/\pi \qquad (as\ in\ Chebyshev\ case). \tag{3.23}$$

Thus, by (3.22), asymptotically, two points per wavelength are needed for resolution.

We have shown that for $r$ satisfying (3.5) with $m$ even(odd), $\tilde{f}_1(y)\left(\tilde{f}_2(y)\right)$ is a polynomial in $y$. Let us discuss now the resolution of general trigonometric function

$$f(x) = \exp(ir\pi x). \tag{3.24}$$

We have

$$\tilde{f}(y) = f\left(g(y;\alpha)\right) = \exp[i\frac{r\pi}{\arcsin(\alpha)}\arcsin(\alpha y)]. \tag{3.25}$$

If

$$\frac{r\pi}{\arcsin(\alpha)} = k+\beta \quad , \quad 0 \le \beta \le 1 \, , k \ \textit{integer} \tag{3.26}$$

then

$$\tilde{f}(y) = \exp[ik\arcsin(\alpha y)]\exp[i\beta\arcsin(\alpha y)]. \tag{3.27}$$

Let us assume, without loss of generality, that $k$ is even then by Lemmas A.1, A.2

$$\tilde{f}(y) = (-1)^{\frac{k}{2}}\left(T_k(\alpha y) - i\sqrt{1-(\alpha y)^2}P_{k-1}(\alpha y)\right)\left(\sqrt{1-(\alpha y)^2}+i\alpha y\right)^{\beta}. \tag{3.28}$$

Resolution of $\tilde{f}(y)$ by interpolation is influenced by the degree $k$ of the polynomials involved and by the singularities at $\pm\frac{1}{\alpha}$. The degree of the interpolating polynomial has to be at least $\alpha k$ but the asymptotic rate of convergence depends only on the singularities. The relevant theory is presented below in greater generality.

Let $f(x)$ be an analytic function in $E \supset [-1,1]$. Since $g'(y;\alpha)$ (2.9) has singularities at $y = \pm 1/\alpha$, so does $\tilde{f}(y) = f(g(y;\alpha))$. Define

$$B = \left(-\frac{1}{\alpha}\, , \, \frac{1}{\alpha}\right) \tag{3.29}$$

and assume that $\alpha$ is close enough to $1$ such that

$$B \subset g^{-1}(E). \tag{3.30}$$

Thus, $\tilde{f}(y)$ is analytic in $B$. The rate of convergence of polynomial interpolation at Chebyshev points is based on the following theory from [13]: let $K$ be a bounded continuum in $C$ such that $K^c =$ the complement $K$ is simply connected in the extended plane and contains the point at infinity. For such $K$ there exist a conformal mapping $\Psi(w)$ which maps the complement of the unit disc onto $K^c$ [13]. Let $\Phi(z)$ be the inverse of $\Psi(w)$ and

$$B_t = \{z : \ |\Phi(z)| = t\} \qquad (\, t > 1) \tag{3.31}$$

denote the level curves in $K^c$.

**Theorem 2.1:** *Suppose $t > 1$ is the largest number such that $f(z)$ is analytic inside $B_t$. The interpolating polynomials $P_n(z)$ with interpolating points $z_i^n$ that are uniformly distributed on $K$ then satisfy*

$$\lim_{n\to\infty}\max_{z\in K}|f(z)-P_n(z)|^{\frac{1}{n}} = \frac{1}{t}. \tag{3.32}$$

Since Chebyshev points satisfy the definition of uniformly distributed points on $[-1,1]$ [13], the asymptotic rate of convergence can be computed by making use of this theorem. We choose $K=[-1,1]$, and the relevant conformal mapping is given by [8]

$$\Phi(y) = y \pm \sqrt{y^2-1}\ . \tag{3.33}$$

Assume now that if $\tilde{f}(y)$ has additional singularities in the complex plane then $\alpha$ is close enough to 1 so that the largest $t$ corresponds to the singular points $\pm\frac{1}{\alpha}$. Thus

$$t = \frac{1+\sqrt{1-\alpha^2}}{\alpha} \tag{3.34}$$

and the asymptotic rate of convergence is

$$\frac{1}{t} = \frac{1-\sqrt{1-\alpha^2}}{\alpha}. \tag{3.35}$$

Hence, by interpolating at $N+1$ points, the asymptotic accuracy is $c\varepsilon$ where

$$\varepsilon = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^N \tag{3.36}$$

and $c$ is constant which depends on $f$ but does not depend on $N$ or $y$.

## 4. On the Choice of the Parameter $\alpha$

For a predetermined degree $N$ , we would like to choose the appropriate parameter $\alpha$. We give below three constructive ways for choosing $\alpha$, based on different considerations.

Resolution considerations - Sometimes we have an idea on the maximal wave number ($r_{max}$) which we want to resolve. For instance, if there is a source term in our equations with known band of frequencies. In this case we will solve (3.20) for $\alpha$. Assuming that $\alpha$ is close to 1 we can simplify (3.20) and use instead

$$\alpha = \sin\left(\frac{\pi r_{max}}{N}\right) \quad , \quad r_{max} < \frac{N}{2}. \tag{4.1}$$

If

$$r_{max} = \frac{N}{2} - j \quad , \quad j \ll \frac{N}{2} \tag{4.2}$$

then

$$\alpha = \sin\left(\frac{\pi}{2} - \frac{j}{N}\pi\right) = \cos\left(\frac{j}{N}\pi\right), \tag{4.3}$$

$j$ being the number of modes which we 'give up' resolving. Expanding in Taylor series

$$\alpha = 1 - \frac{1}{2}j^2\left(\frac{\pi}{N}\right)^2 + \cdots \tag{4.4}$$

which satisfies (2.19) and by using (4.4) and (2.20) we get

$$\Delta x_{min} = \frac{2}{j+\sqrt{j^2+1}}\frac{1}{N} + O\left(N^{-2}\right). \tag{4.5}$$

*Remark 4.1* - Resolution analysis is closely related to maximal spacing analysis. By the sampling theorem we know that for any sampling interval $\Delta$, there is maximal mode $w_c$ called the Nyquist critical frequency and is given by $w_c = \frac{1}{\Delta}$. For Fourier method we have $\Delta x = \frac{2}{N}$ , hence the maximal mode which can be resolved is $\frac{N}{2}$. Similarly, in the Chebyshev case $\Delta x_{max} = \frac{\pi}{N}$ and the maximal mode is $\frac{N}{\pi}$ which is equivalent to stating that $\pi$ points per wavelength are needed for resolution. This result is given also in [5] based on expanding the trigonometric functions in Chebyshev polynomials. By Lemma 2.1 we get that for the transformed interpolating points, the maximal spacing is attained in the center of the interval. Therefore

$$\begin{aligned}\Delta x_{max} &= g\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{N}\right)\right] - g(0) \\ &= \frac{\arcsin[\alpha\sin(\frac{\pi}{N})]}{\arcsin(\alpha)}.\end{aligned} \tag{4.6}$$

Substituting (4.3) in (4.6) we get

$$\lim_{N\to\infty}\frac{1}{\Delta x_{max}} = \frac{N}{2} - j \tag{4.7}$$

as in (4.2).

<u>Accuracy considerations</u> - For given $\varepsilon$ and $N$, we can solve (3.36) for $\alpha$ and get an explicit expression

$$\alpha = \frac{2}{t+t^{-1}} \quad , \quad t = \varepsilon^{-\frac{1}{N}}. \tag{4.8}$$

To examine the minimal spacing dictated by this choice, we expand $\alpha$ in Taylor series

$$\alpha = 1 - \frac{1}{2}\ln^2(\varepsilon)\left(\frac{1}{N}\right)^2 + O\left(N^{-2}\right). \tag{4.9}$$

The expansion (4.9) is of the form (2.19), using it and (2.20) we get

$$\Delta x_{min} = \frac{2\pi}{\left(|\ln(\varepsilon)| + \sqrt{\pi^2+\ln^2(\varepsilon)}\right)}\frac{1}{N} + O(N^{-2}). \tag{4.10}$$

The agreement between $\varepsilon$ and the computed accuracy will depend on the constant $c$. Observe that there is a 'give and take' relation between resolution and accuracy. By decreasing $r_{max}, \alpha$ is decreased (4.1) and therefore $\varepsilon$ is getting smaller (3.36). Hence, by sacrificing the resolution of the high modes we get in return higher accuracy on the rest of the modes. See numerical results in Section 6.

Adaptive approach - We have described above two formulas which give explicit expressions for $\alpha$. One can also consider a third approach, an adaptive algorithm for computing $\alpha$. Observing (2.6) we can regard the method described in this paper as a 'preconditioning' one. For a given function with large gradients, we are looking for a parameter $\alpha$ such that after the transformation $\tilde{f}(y)$ will be a smooth function. One can consider the tail of the series of pseudospectral Chebyshev coefficients to measure the smoothness of $\tilde{f}(y)$. Since, by stability considerations we would like $\alpha$ to be as large as possible, the adaptive algorithm should find

$$\alpha_{max} = \max\left\{\alpha \;\Big|\; \frac{\sum_{i=N-k}^{N}|a_i(\alpha)|}{\sum_{i=0}^{N}|a_i(\alpha)|} < \varepsilon_0 \;,\; k \ll N\right\} \tag{4.11}$$

where $\varepsilon_0$ is a given tolerance and $a_i$ are the computed Chebyshev coefficients. When the adaptive approach is implemented in time dependent problems, the search for an optimal $\alpha$ should restart whenever the solution behavior has been changed significantly.

## 5. Non-Symmetric Transformation

The transformation function (2.3) is symmetric. The interpolating points (2.4) are distributed symmetrically around the origin. When there is a boundary layer on one side of the domain, we would like to have the flexibility of putting more points on this side. To this end we modify the transformation (2.3) and take

$$x = g(y;\alpha,\beta) = \frac{1}{a}\left(\tilde{g}(y;\alpha,\beta) - b\right), \tag{5.1}$$

where

$$\tilde{g}(y;\alpha,\beta) = \arcsin\left(\frac{2\alpha\beta y + \alpha - \beta}{\alpha+\beta}\right) \tag{5.2}$$

and

$$a = \frac{1}{2}\left\{\tilde{g}(1;\alpha,\beta) - \tilde{g}(-1;\alpha,\beta)\right\} \tag{5.3}$$

$$b = \frac{1}{2}\left\{\tilde{g}(1;\alpha,\beta) + \tilde{g}(-1;\alpha,\beta)\right\}. \tag{5.4}$$

For the derivative we have

$$\frac{1}{g'(y;\alpha,\beta)} = \frac{a}{\sqrt{\alpha\beta}}\sqrt{(1-\alpha y)(1+\beta y)}. \tag{5.5}$$

and the parameters $\alpha$ and $\beta$ control the distribution of interpolation points near $x = 1$ and $x = -1$ respectively. An elaborate discussion of this transformation will be given in a future paper.

## 6. Numerical Results

The following notations are used in this section:

$N =$ number of interpolating points,

$E =$ relative error of the derivative in the maximum norm,

$j =$ number of unresolved modes (4.2),

$y_i =$ Chebyshev interpolating points (2.5),

$z_i =$ check points,

$$z_i = g\left(\cos(y_{i+\frac{1}{2}};\alpha)\right), \qquad 0 \le i \le N-1. \tag{6.1}$$

In the first table we present the spectral radius $\rho$ of $AD$ (2.7) where $\alpha$ is given by (4.3). The spectral radius of the Chebyshev pseudospectral differentiation operator $D$ is given in the last column. Using the new method for time dependent problems we have observed that

$$\frac{\Delta t(new\ method)}{\Delta t(Chebyshev)} \approx \frac{\rho(D)}{\rho(AD)}. \tag{6.2}$$

Thus, from Table I we see that for $N = 128$ for example, the time step restriction of the new algorithm is almost 8 times larger then the one used in a standard Chebyshev discretization.

**Table I** *Spectral Radius*

| $j$ | $N$ | $\alpha$ | $\rho(AD)$ | $\rho(D)$ |
|---|---|---|---|---|
| 1 | 16 | 0.9808 | 18.927 | 23.560 |
| 1 | 32 | 0.9952 | 42.897 | 91.559 |
| 1 | 64 | 0.9987 | 92.286 | 363.779 |
| 1 | 128 | 0.9997 | 192.165 | 1452.706 |
| 2 | 16 | 0.9239 | 17.934 | 23.560 |
| 2 | 32 | 0.9808 | 41.061 | 91.559 |
| 2 | 64 | 0.9952 | 89.957 | 363.779 |
| 2 | 128 | 0.9988 | 189.454 | 1452.706 |
| 3 | 16 | 0.8312 | 18.624 | 23.560 |
| 3 | 32 | 0.9569 | 44.600 | 91.559 |
| 3 | 64 | 0.9892 | 97.846 | 363.779 |
| 3 | 128 | 0.9973 | 204.997 | 1452.706 |

The results given in Tables II and III demonstrate the resolution and accuracy properties of the new algorithm and clarify the 'give and take' relation between the two as mentioned in Section 4. We applied the new differentiation algorithm to the trigonometric functions

$$u_k(x) = \cos(k\pi x) \qquad 1 \le k \le 16, \tag{6.3}$$

and obtained approximations $v_k'$, $k = 1, \cdots, 16$. $\alpha$ is given by (4.3). Corresponding results for Chebyshev method are shown in the last column. In the last row we have printed $\varepsilon$ as given by (3.36). The resolution property of $\frac{N}{2} - j$ modes (4.2), (4.7) is clearly demonstrated in Tables II, III. As $j$ increases, so does the accuracy of the modes resolved. As shown in Section 3 the new method should be exact if

$$m = \frac{2kN}{N - 2j} . \tag{6.4}$$

is an even integer. This explains the high accuracy exhibited in relevant entries of Tables II and III. Comparing Table III to Table II we see that the accuracy by which the low modes are resolved is almost the same. The effect of increasing $N$ is in the number of modes resolved with accuracy imposed by the choice of $j$.

**Table II** $N = 32$

| $k$ | $E(j=1)$ | $E(j=2)$ | $E(j=3)$ | $E(j=4)$ | $E_{chb}$ |
|---|---|---|---|---|---|
| 1 | 9.35E−03 | 7.14E−04 | 3.78E−05 | 1.57E−06 | 1.57E−12 |
| 2 | 1.86E−02 | 1.35E−03 | 6.23E−05 | 1.77E−06 | 5.83E−13 |
| 3 | 2.84E−02 | 1.82E−03 | 5.84E−05 | 1.66E−06 | 3.29E−13 |
| 4 | 3.70E−02 | 2.02E−03 | 2.06E−05 | 2.97E−06 | 5.65E−11 |
| 5 | 4.76E−02 | 1.88E−03 | 5.14E−05 | 4.42E−06 | 4.41E−08 |
| 6 | 5.44E−02 | 1.25E−03 | 1.42E−04 | 3.90E−14 | 7.56E−06 |
| 7 | 6.29E−02 | 3.19E−14 | 2.10E−04 | 1.36E−05 | 4.14E−04 |
| 8 | 7.04E−02 | 2.04E−03 | 1.68E−04 | 2.93E−05 | 9.33E−03 |
| 9 | 7.76E−02 | 5.07E−03 | 1.54E−04 | 4.10E−14 | 9.34E−02 |
| 10 | 8.36E−02 | 9.26E−03 | 1.07E−03 | 2.46E−04 | 4.52E−01 |
| 11 | 8.58E−02 | 1.48E−03 | 3.12E−03 | 1.16E−03 | 8.94E−01 |
| 12 | 8.46E−02 | 2.10E−02 | 6.60E−03 | 3.01E−14 | 1.56E+00 |
| 13 | 7.82E−02 | 2.48E−02 | 3.52E−14 | 3.75E−01 | 1.72E+00 |
| 14 | 5.70E−02 | 4.07E−14 | 7.39E−01 | 1.32E+00 | 1.70E+00 |
| 15 | 3.87E−14 | 1.25E+00 | 1.79E+00 | 1.86E+00 | 1.34E+00 |
| 16 | 1.69E+00 | 1.92E+00 | 1.76E+00 | 1.79E+00 | 1.63E+00 |
| $\varepsilon$ | 4.32E−02 | 1.79E−03 | 6.99E−05 | 2.49E−06 | − |

**Table III** $N = 64$

| $k$ | $E(j=1)$ | $E(j=2)$ | $E(j=3)$ | $E(j=4)$ | $E_{chb}$ |
|---|---|---|---|---|---|
| 1 | 4.55E−03 | 3.53E−04 | 2.04E−05 | 1.04E−06 | 3.49E−11 |
| 3 | 1.36E−02 | 1.03E−04 | 5.53E−05 | 2.49E−06 | 7.16E−12 |
| 5 | 2.26E−02 | 1.60E−03 | 7.29E−05 | 2.26E−06 | 3.50E−12 |
| 7 | 3.16E−02 | 1.99E−03 | 6.35E−05 | 1.99E−13 | 1.85E−12 |
| 9 | 4.04E−02 | 2.13E−03 | 2.14E−05 | 3.53E−06 | 1.02E−12 |
| 11 | 4.90E−02 | 1.93E−03 | 5.26E−05 | 6.13E−06 | 2.15E−12 |
| 13 | 5.74E−02 | 1.26E−03 | 1.47E−04 | 4.12E−06 | 1.55E−08 |
| 15 | 6.52E−02 | 6.58E−03 | 2.32E−04 | 6.77E−06 | 1.48E−05 |
| 17 | 7.25E−02 | 1.99E−03 | 2.44E−04 | 2.76E−05 | 2.62E−03 |
| 19 | 7.89E−02 | 4.89E−03 | 6.26E−05 | 4.56E−05 | 9.76E−02 |
| 21 | 8.38E−02 | 8.91E−03 | 5.40E−04 | 5.81E−14 | 6.61E−01 |
| 23 | 8.66E−02 | 1.42E−03 | 1.96E−03 | 3.12E−04 | 1.78E+00 |
| 25 | 8.57E−02 | 2.07E−03 | 4.94E−03 | 1.53E−03 | 1.76E+00 |
| 27 | 7.85E−02 | 2.72E−03 | 1.01E−02 | 4.48E−03 | 1.86E+00 |
| 29 | 5.82E−02 | 2.46E−02 | 9.01E−14 | 7.12E−01 | 1.62E+00 |
| 31 | 1.48E−13 | 1.51E+00 | 2.03E+00 | 1.91E+00 | 1.77E+00 |
| $\varepsilon$ | 4.32E−02 | 1.84E−03 | 7.80E−05 | 3.21E−06 | − |

In Tables IV and V we present mesh refinement charts for the functions

$$f_1(x) = \frac{0.05}{x^2 + 0.05}\,, \tag{6.5}$$

$$f_2(x) = \frac{\exp(2x)}{2 + \cos(15x)} \tag{6.6}$$

respectively. In Table IV, $\alpha$ is given by (4.3) with $j = 3$. Observe the fast convergence up to $N = 64$ . The error is not decreasing beyond this points, since all the modes have been resolved to the accuracy enforced by the choice of $j$. In Table V, $\alpha$ is computed by (4.8) with $\varepsilon = 1.E - 05$ . The results for Chebyshev method are given in the last column.

**Table IV** *Mesh refinement chart,* $f_1(x) = \frac{0.05}{x^2+0.05}$

| $N$ | $\alpha$ | $E(j=3)$ | $E_{chb}$ |
|---|---|---|---|
| 16 | 0.83147 | 9.394E−02 | 1.777E−01 |
| 32 | 0.95694 | 1.019E−03 | 9.281E−03 |
| 64 | 0.98918 | 1.507E−06 | 1.486E−05 |
| 128 | 0.99729 | 1.794E−06 | 8.845E−11 |
| 256 | 0.99932 | 1.939E−06 | 9.923E−11 |

**Table V** *Mesh refinement chart,* $f_2(x) = \frac{\exp(2x)}{2+\cos(15x)}$

| $N$ | $\alpha$ | $E(\varepsilon = 1.E\text{-}05)$ | $E_{chb}$ |
|---|---|---|---|
| 16 | 0.80761 | 1.853E−01 | 1.717E−01 |
| 32 | 0.94208 | 9.007E−02 | 6.461E−02 |
| 64 | 0.98452 | 2.298E−03 | 9.032E−03 |
| 128 | 0.99603 | 1.968E−06 | 5.774E−05 |
| 256 | 0.99899 | 6.344E−06 | 1.669E−09 |

We have solved the model problem described in the introduction (1.1)-(1.3) and the results are reported in Table VI. The solution is computed at $t = 1$ using fourth order Runge-Kutta as time marching algorithm. The initial and boundary conditions are

$$u^0(x) = \left[x \exp^{-(x-1)^2} \cos(m\pi x) - (-1)^m\right]^4, \tag{6.7}$$

$$s(t) = 0 \tag{6.8}$$

respectively. The numerical solution is compared to the exact solution

$$u(x,t) = \begin{cases} 0 & x+t \geq 1 \\ \left[(x+t)\exp^{-(x+t-1)^2} \cos[m\pi(x+t)] - (-1)^m\right]^4 & x+t \leq 1\,. \end{cases} \tag{6.9}$$

The results presented in the table provide a comparison between the new method, where $\alpha$ is computed by (4.3) with $j = 1$, and standard Chebyshev method. *nsteps* is the number of time steps. For $m = 6$ we needed 91 points, in the Chebyshev case, in order to achieve the accuracy given in the last column. For stability we had to use 1300 time steps . Taking smaller $\Delta t$ would not decrease the error as shown in the second row. Using the new algorithm, we took only 65 points. 80 time steps were sufficient for stability. In order to get accuracy close to the one we had in the Chebyshev case, 200 time steps were needed. Reducing $\Delta t$ beyond this point would not reduce the error significantly as shown by the fifth row. In the next set of experiments we took $m = 12$. In both cases, we had to double the number of points in order to resolve the solution. The results are provided in the rest of the table.

**Table VI** *Time-dependent problem (1.1)-(1.3)*

| $m$ | $N$ | $\alpha$ | *nsteps* | $\max(\lvert E\rvert)/\max(\lvert u\rvert)$ |
|---|---|---|---|---|
| 6 | 91 | 0 | 1300 | 8.59E-03 |
| 6 | 91 | 0 | 2600 | 8.53E-03 |
| 6 | 65 | 0.9988 | 80 | 1.79E-01 |
| 6 | 65 | 0.9988 | 200 | 7.08E-03 |
| 6 | 65 | 0.9988 | 300 | 5.64E-03 |
| 12 | 181 | 0 | 5200 | 5.32E-03 |
| 12 | 129 | 0.9997 | 200 | 2.22E-01 |
| 12 | 129 | 0.9997 | 600 | 5.71E-03 |

**Lamb Problem**

The problem is of wave propagation in a uniform and isotropic elastic two dimensional halfspace subjected to a point source applied in the vicinity of the free surface. This problem is numerically challenging because of the presence of Rayleigh surface waves around the free surface the calculation of which requires an accurate representation of the boundary conditions.

Let $x$ and $y$ denote horizontal and vertical cartesian coordinates respectively and $t$ the time variable. The system of equations to be solved is

$$\frac{\partial V_x}{\partial t} = \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\right) + f_x \tag{6.9}$$

$$\frac{\partial V_y}{\partial t} = \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right) + f_y \tag{6.10}$$

$$\frac{\partial \sigma_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial V_x}{\partial x} + \lambda\frac{\partial V_y}{\partial y} \tag{6.11}$$

$$\frac{\partial \sigma_{yy}}{\partial t} = \lambda\frac{\partial V_x}{\partial x} + (\lambda + 2\mu)\frac{\partial V_y}{\partial y} \tag{6.12}$$

$$\frac{\partial \sigma_{xy}}{\partial t} = \mu\frac{\partial V_x}{\partial y} + \mu\frac{\partial V_y}{\partial x}. \tag{6.13}$$

$V_x$ and $V_y$ respectively denote the horizontal and vertical velocities, $\sigma_{xx}, \sigma_{yy}$ and $\sigma_{xy}$ are the stress components, $f_x$ and $f_y$ are the body forces, $\rho$ is the density and $\lambda$ and $\mu$ are Lamb's constants. The system is the same as the one used by Bayliss et. al. [1] for a fourth order finite difference scheme.

During the calculations, the variables $V_x, V_y, \sigma_{xx}, \sigma_{yy}$ and $\sigma_{xy}$ are advanced in time after specification of the body forces $f_x$ and $f_y$. In this work we choose to approximate the horizontal derivative by the Fourier method whereas for the vertical coordinate $y$ we choose the modified Chebyshev discretization as described in this paper, using the transformation function (5.1). The boundary conditions at $y = 0$ is $\sigma_{yy} = \sigma_{xy} = 0$ whereas for the bottom boundary $y = L$ we choose the condition that the incoming characteristics are zero [1]. In addition an absorbing strip was applied along the lower boundary and the sides of the grid to prevent reflections or wraparound from the boundaries [7]. For the present problem the material parameters had uniform values of $\rho = 1.2_{gr/cm^2}$, and $P$ and $S$ velocities of $V_p = \sqrt{\frac{\lambda+2\mu}{\rho}} = 2000_{m/sec}$ and $V_s = \sqrt{\frac{\mu}{\rho}} = 1155_{m/sec}$ respectively. For the body forces, $f_x = 0$ and $f_y = \delta(x - x_0)\delta(y - y_0)h(t)$, where $x_0 = 250_m$, $y_0 = 1.8_m$ and $h(t)$ was a band limited Ricker wavelet with highest frequency of 40Hz [11]. For the spatial discretization $\Delta x = \Delta y = 10_m$ and the grid modification parameters (5.1) are $\alpha = 0.729$ and $\beta = 0.620$. The solution was advanced in time to $1_{sec}$ by the fourth-order Runge-Kutta method using a time step of 0.002 seconds. This time step is approximately <u>seven times</u> larger then the maximum allowable time step for an ordinary Chebyshev discretization, and is approximately equal to the time step which would be used with a uniform Fourier grid from accuracy considerations (e.g. $\frac{Cdt}{dx} \approx 0.4$ ).

Figures 1–3 present a comparison between the numerical and analytical horizontal displacement time histories at points located at respective distances of $200_m, 500_m$ and $800_m$ from the source. A corresponding comparison of vertical displacements is presented in Figures 4–6. As the figures show the match between numerical and analytical solutions is virtually perfect.

# References


[1] A. Bayliss, K. E. Jordan, B. J. LeMesurier, and E. Turkel, "A fourth order accurate scheme for the computation of elastic waves," Bull. Seismol. Soc. Amer., Vol. 76, No. 4, pp. 1115-1132 (1986).

[2] A. Bayliss, D. Gottlieb, B. J. Matkowsky, and M. Minkoff, "An adaptive pseudospectral method for reaction diffusion problems," ICASE Report No. 87-67, NASA Langley Research Center, Hampton, VA 23665 (1987).

[3] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin-Heidelberg-New York (1987).

[4] M. Dubiner, "Asymptotic analysis of spectral methods," J. Sci. Comput., Vol. 2, No. 1 (1987).

[5] D. Gottlieb and S. Orszag, "Numerical analysis of spectral methods: Theory and applications," CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM Publisher, Philadelphia, PA (1977).

[6] D. Gottlieb and E. Turkel, "Spectral methods for time dependent partial differential equations," ICASE Report No. 83-54, NASA Langley Research Center, Hampton, VA 23665 (1983).

[7] R. Kosloff and D. Kosloff, "Absorbing boundaries for wave propogation problems," J. Comput. Phys., Vol. 63, pp. 363-376 (1986).

[8] A. I. Markushevich, *Theory of Functions of a Complex Variable*, Chelsea, New York (1977).

[9] A. Solomonoff and E. Turkel, "Global properties of pseudospectral methods," J. Comput. Phys., Vol. 81, pp. 239-276 (1989).

[10] H. Tal-Ezer, "A pseudospectral Legendre method for hyperbolic equation with an improved stability condition," J. Comput. Phys., Vol. 67, pp. 145-172 (1986).

[11] H. Tal-Ezer, D. Kosloff, and Z. Koren, "An accurate scheme for seismic forward modeling," Geophysical Prospecting, Vol. 35, No. 5, pp. 479-490 (1987).

[12] L. N. Trefethen and M. R. Trummer, "An instability phenomenon in spectral methods," SIAM J. Numer. Anal., Vol. 24, pp. 1008-1023 (1987).

[13] J. L. Walsh, "Interpolation and approximation by rational functions in the complex domain," American Mathematical Society, Providence, Rhode Island, 1956.

## APPENDIX

**Lemma A.1:** *For* $m$ *even*

$$\cos(m\psi) = (-1)^{\frac{m}{2}} T_m(\sin(\psi)). \tag{A.1}$$

*For* $m$ *odd*

$$\sin(m\psi) = (-1)^{\frac{m-1}{2}} T_m(\sin(\psi)). \tag{A.2}$$

**Proof:** Chebyshev polynomials satisfy the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{A.3}$$

Using basic trigonometric identities, the reccurence relation and induction we get

$$\begin{aligned}
\cos((m+2)\psi) &= 2[2\cos^2(\psi) - 1]\cos(m\psi) - \cos[(m-2)\psi] \\
&= (-1)^{\frac{m}{2}+1}\left\{2[2\sin^2(\psi) - 1]T_m(\sin(\psi)) - T_{m-2}(\sin(\psi))\right\} \\
&= (-1)^{\frac{m}{2}+1} T_{m+2}(\sin(\psi)).
\end{aligned} \tag{A.4}$$

We will use now (A.4) to show (A.2).

$$\begin{aligned}
\sin((m+2)\psi) &= \cos(2\psi)\sin(m\psi) + \cos(m\psi)\sin(2\psi) \\
&= \left(1 - 2\sin^2\psi\right)(-1)^{\frac{m-1}{2}} T_m(\sin(\psi)) + \sin(\psi)\left\{\cos((m+1)\psi) + \cos((m-1)\psi)\right\} \\
&= (-1)^{\frac{m+1}{2}}\left\{(2\sin^2\psi - 1)T_m(\sin(\psi)) + \sin(\psi)\left[T_{m+1}(\sin(\psi)) - T_{m-1}(\sin(\psi))\right]\right\} \\
&= (-1)^{\frac{m+1}{2}}\left\{2\left(2\sin^2(\psi) - 1\right)T_m(\sin(\psi)) - T_{m-2}(\sin(\psi))\right\} \\
&= (-1)^{\frac{m+1}{2}} T_{m+2}.
\end{aligned} \tag{A.5}$$

**Lemma A.2:** *For* $m$ *odd*

$$\cos(m\psi) = \cos(\psi)P_{m-1}(\sin(\psi)) \tag{A.6}$$

*and for* $m$ *even*

$$\sin(m\psi) = \cos(\psi)Q_{m-1}(\sin(\psi)) \tag{A.7}$$

where $P_{m-1}, Q_{m-1}$ are polynomials of degree $m-1$.

Lemma A.2 is easily verified by using trigonometric identities, induction and the results of Lemma A.1.

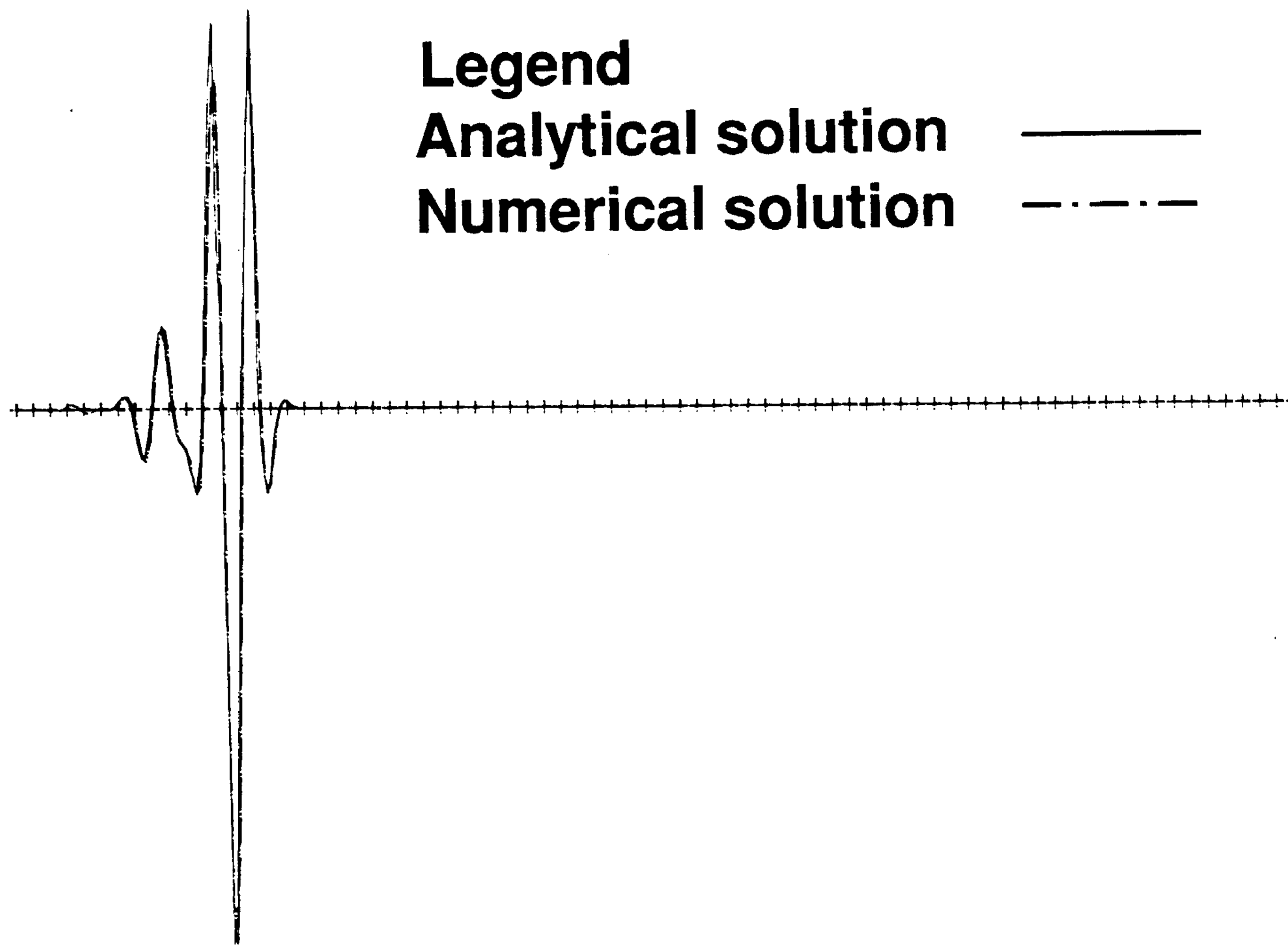


Figure 1. Comparison between the numerical and analytical horizontal displacement time history at distance of $200_m$ from the source.

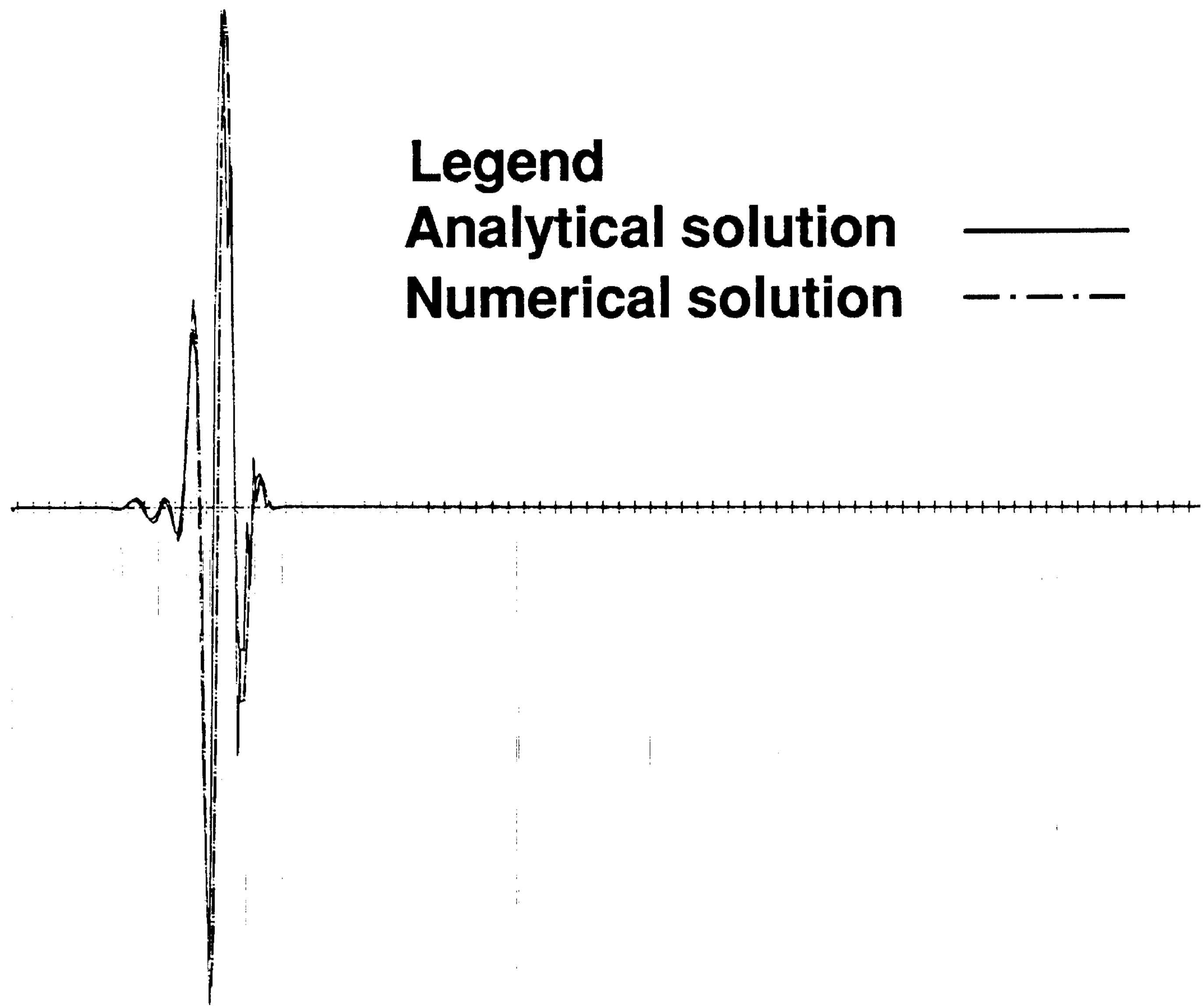


Figure 2. Comparison between the numerical and analytical horizontal displacement time history at distance of $500_m$ from the source.

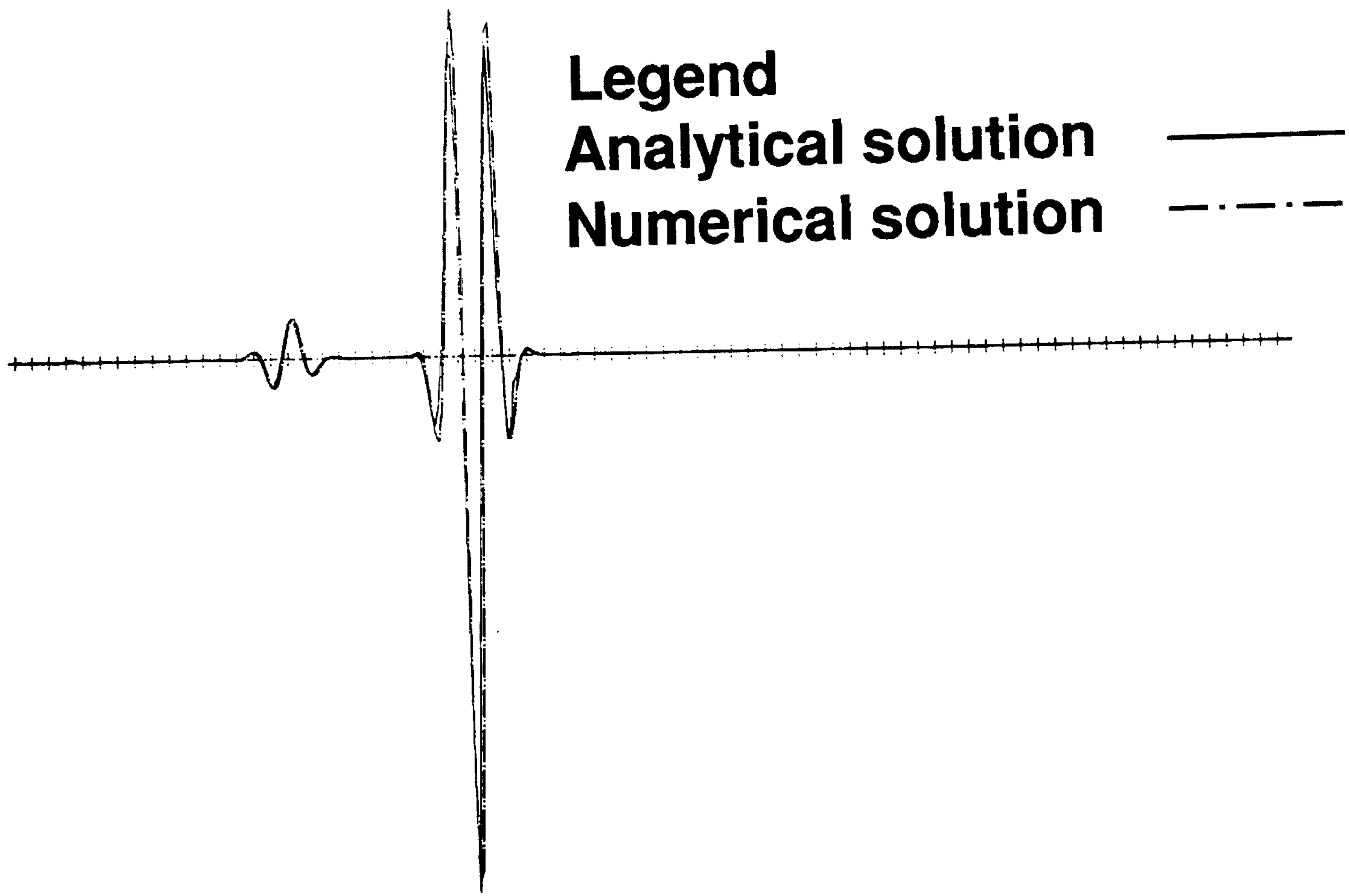


Figure 3. Comparison between the numerical and analytical horizontal displacement time history at distance of $800_m$ from the source.

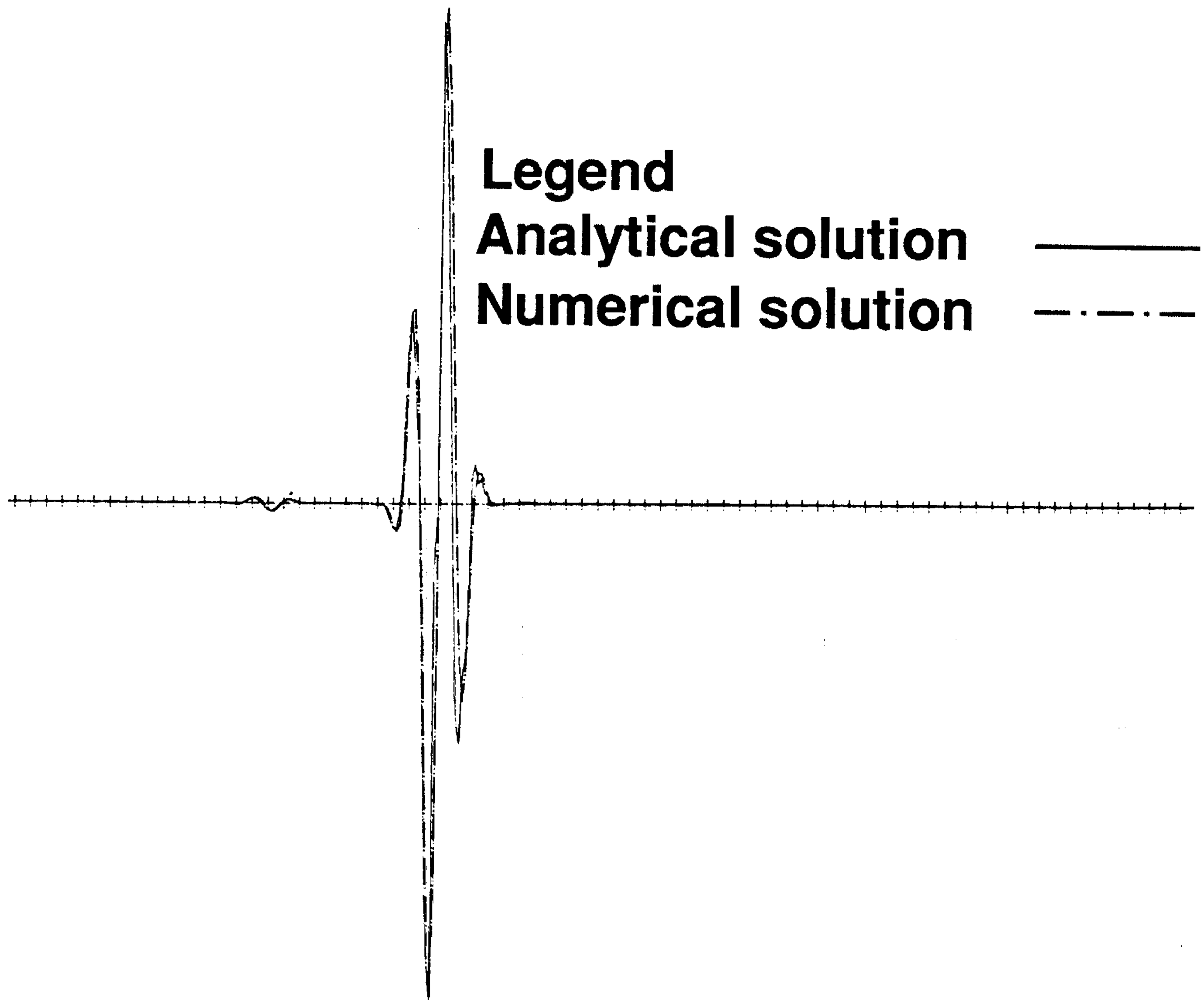


Figure 4. Comparison between the numerical and analytical vertical displacement time history at distance of $200_m$ from the source.

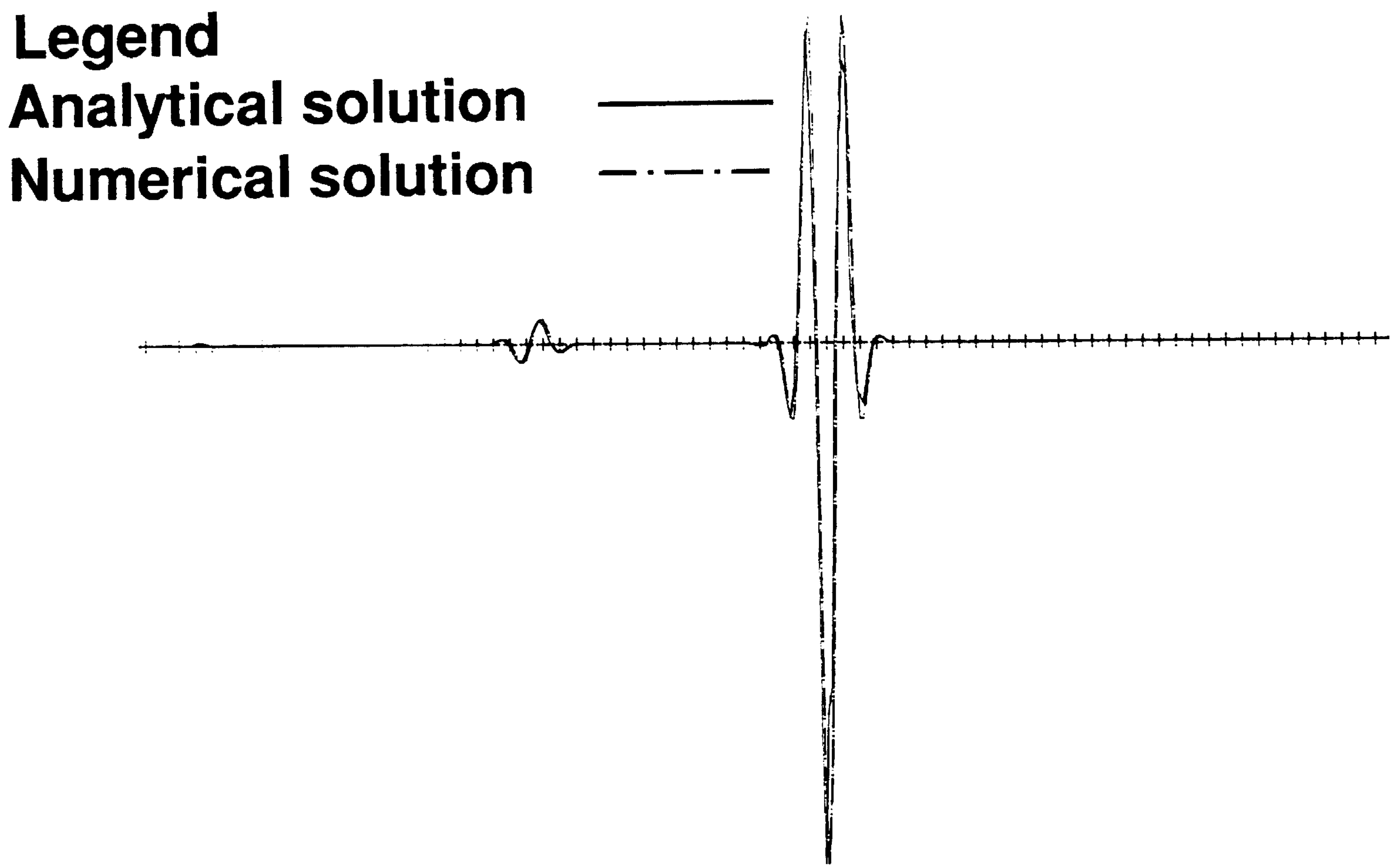


Figure 5. Comparison between the numerical and analytical vertical displacement time history at distance of $500_m$ from the source.

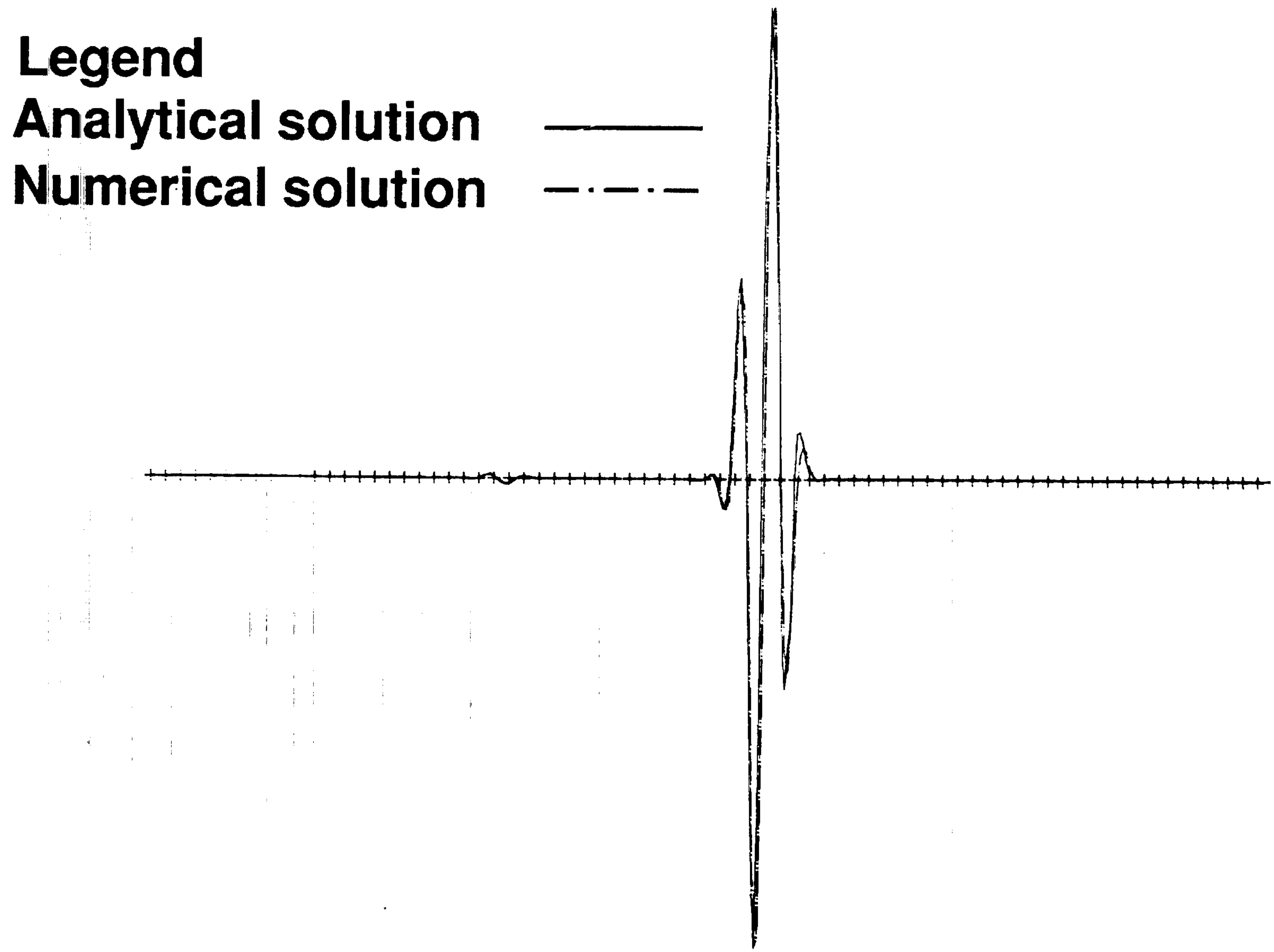


Figure 6. Comparison between the numerical and analytical vertical displacement time history at distance of $800_m$ from the source.

**NASA** National Aeronautics and Space Administration

# Report Documentation Page

| 1. Report No. | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|
| NASA CR-181958<br>ICASE Report No. 89-71 | | |

| 4. Title and Subtitle | 5. Report Date |
|---|---|
| MODIFIED CHEBYSHEV PSEUDOSPECTRAL METHOD WITH $O(N^{-1})$ TIME STEP RESTRICTION | December 1989 |
| | 6. Performing Organization Code |
| 7. Author(s)<br>Dan Kosloff<br>Hillel Tal-Ezer | 8. Performing Organization Report No.<br>89-71 |
| | 10. Work Unit No.<br>505-90-21-01 |
| 9. Performing Organization Name and Address<br>Institute for Computer Applications in Science and Engineering<br>Mail Stop 132C, NASA Langley Research Center<br>Hampton, VA 23665-5225 | 11. Contract or Grant No.<br>NAS1-18605 |
| 12. Sponsoring Agency Name and Address<br>National Aeronautics and Space Administration<br>Langley Research Center<br>Hampton, VA 23665-5225 | 13. Type of Report and Period Covered<br>Contractor Report |
| | 14. Sponsoring Agency Code |

15. Supplementary Notes

Langley Technical Monitor:
Richard W. Barnwell

Submitted to Journal of Computational Physics

Final Report

16. Abstract

The extreme eigenvalues of the Chebyshev pseudospectral differentiation operator are $O(N^2)$ where N is the number of grid points (4). As a result of this, the allowable time step in an explicit time marching algorithm is $O(N^{-2})$ which, in many cases, is much below the time step dictated by the physics of the P.D.E. In this paper we introduce a new set of interpolating points such that the eigenvalues of the differenteation operator are O(N) and the allowable time step is $O(N^{-1})$. The properties of the new algorithm are similar to those of the Fourier method but in addition it provides highly accurate solution for non-periodic boundary value problems.

| 17. Key Words (Suggested by Author(s)) | 18. Distribution Statement |
|---|---|
| Chebyshev, pseudospectral method, interpolation, differentiation | 64 - Numerical Analysis<br>71 - Acoustics<br><br>Unclassified - Unlimited |

| 19. Security Classif. (of this report) | 20. Security Classif. (of this page) | 21. No. of pages | 22. Price |
|---|---|---|---|
| Unclassified | Unclassified | 28 | A03 |

NASA FORM 1626 OCT 86