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LOAD-SHORTENING BEHAVIOR OF AN INITIALLY CURVED ECCENTRICALLY LOADED COLUMN

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SUMMARY

To explore the feasibility of using buckled columns to provide a soft support system for simulating a free-free boundary condition in dynamic testing, the nonlinear load-shortening behavior of initially imperfect, eccentrically loaded slender columns is analyzed. Load-shortening curves are obtained for various combinations of load eccentricity and uniform initial curvature and are compared, for reference purposes, with the limiting case of the classical elastica. Results for numerous combinations of initial curvature and load eccentricity show that, over a wide range of shortening, an axially loaded slender column exhibits load-deflection compliance which is of the same order as that of a straight but otherwise identical cantilever beam under lateral tip loading.

LIST OF SYMBOLS

A	area of cross section
C	constant of integration
E	Young's modulus
I	cross-sectional moment of inertia
L	length of column
M	moment
P	applied vertical force
R	radius of curvature
a	moment arm for applied vertical force
k	force-stiffness parameter
p	parameter
q	normalized vertical force
r	parameter
s	arc-length coordinate

t	integration variable
x	vertical coordinate
y	horizontal coordinate
γ	column tip rotation
Δ	column shortening
θ	column cross-section rotation
κ	column curvature

Subscripts 0 and e denote initial state and Euler buckling value, respectively.

INTRODUCTION

In many structural dynamics tests it is desirable to minimize the interaction between a structure and its support system. One frequently used scheme involves suspending the structure on long, flexible cords when, for example, free vibration frequencies are of interest. Such an arrangement is most satisfactory when the suspension cords are long enough to minimize side loads and the natural frequencies of the suspension system are separated sufficiently from those of the test article.

Another possible approach is to mount the structure on compression springs, if the springs can be made to have sufficiently low stiffness while supporting the weight of the test article. One device which might be able to satisfy the combined flexibility and load-bearing requirements, and simultaneously avoid the need for spacious overhead support capability, is the buckled column. After buckling, the column continues to support its buckling load, while over a considerable deflection range the slope of its load-shortening curve (i.e., its tangent stiffness) may be low enough to satisfy test requirements.

To investigate the feasibility of such an approach, the shortening due to the large lateral deflections of an eccentrically loaded, slender elastic column with initial curvature is examined. Both the initial curvature and the load eccentricity serve to preclude buckling instability which would be troublesome in a support system. For reference, the load-shortening curves for various combinations of initial imperfection and load eccentricity are compared with results from reference 1 for the classical elastica problem.

ANALYSIS

Governing Equations

A sketch of a cantilevered, initially curved, inextensional column, before and during eccentric loading at the free end, is shown in figure 1. For simplicity, the initial imperfection shape is assumed to be a circular arc of radius R_0. Hence, the initial tip rotation γ_0 is given by $\gamma_0 = \frac{L}{R_0}$,where L is the length of the column. The vertical force P ia applied at a normal distance a from the column centroid, and so as to cause tip rotation of the same sense as the initial curvature. During deformation, the moment arm of length a is assumed to rotate with the end of the column. Equating the internal moment at a distance s measured along the column centerline from the tip, to the moment due to the applied force P yields

$$-EI(\kappa-\kappa_0)=P(y+a\cos\gamma) \qquad (1)$$

where the initial curvature is $\kappa_0 = \frac{1}{R_0}$, γ is the angle of tip rotation relative to the vertical, y is the lateral displacement relative to the tip, κ is the total curvature, and EI is the column bending stiffness.

In terms of θ, the angle of rotation of the tangent at s, $\kappa = \frac{d\theta}{ds}$. Thus, equation (1) can be written as

$$\frac{d\theta}{ds} = -\frac{1}{R_o} - k^2(y + a\cos\gamma) \qquad (2)$$

where $k^2 = \frac{P}{EI}$. The boundary conditions are

$$\theta(0) = \gamma \text{ (and } y = 0) \qquad (3)$$

and

$$\theta(L) = 0 \qquad (4)$$

Differentiation of equation (2) gives

$$\frac{d^2\theta}{ds^2} = -k^2\frac{dy}{ds}$$

or

$$\frac{d^2\theta}{ds^2} = - k^2 \sin\theta \tag{5}$$

Multiplication by $\frac{d\theta}{ds}$ and integration yields

$$\left(\frac{d\theta}{ds}\right)^2 = 2k^2 \cos\theta + C \tag{6}$$

Applying boundary condition (3) and using equation (2) in equation (5) gives

$$C = \left(\frac{1}{R_o} + k^2 a \cos\gamma\right)^2 - 2k^2 \cos\gamma$$

Thus,

$$\frac{d\theta}{ds} = -\sqrt{\left(\frac{1}{R_o} + k^2 a \cos\gamma\right)^2 - 2k^2\left(\cos\gamma - \cos\theta\right)} \tag{7}$$

where the negative sign is consistent with the fact that $\theta(s)$ is a decreasing function of s.

Employing the identity $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, integrating over the length of the column, and using boundary condition (4) yields

$$L = \int_0^{\gamma} \frac{d\theta}{\sqrt{\left(\frac{1}{R_0} + k^2 a \cos\gamma\right)^2 + 4k^2\left(\sin^2\frac{\gamma}{2} - \sin^2\frac{\theta}{2}\right)}} \tag{8}$$

With the definition $p = \sin\frac{\gamma}{2}$, and the change of variable $pt = \sin\frac{\theta}{2}$, equation (8) becomes

$$L = \frac{2p}{\sqrt{\left(\frac{1}{R_o} + k^2 a \cos\gamma\right)^2 + 4k^2 p^2}} \int_0^1 \frac{dt}{\sqrt{1 - p^2 t^2}\sqrt{1 - r^2 t^2}} \tag{9}$$

where $r = \sqrt{\dfrac{\pi^2 p^2 q}{\left(\gamma_0 + \dfrac{\pi^2 e\, q \cos\gamma}{4L}\right)^2 + \pi^2 p^2 q}}$. With the Euler buckling force for a perfect column denoted by $P_e = \dfrac{\pi^2 EI}{4L^2}$, then $k^2 = \dfrac{\pi^2}{4L^2} q$, where $q = \dfrac{P}{P_e}$ is the normalized applied force. Finally, equation (9) can be written as:

$$1 = \frac{2r}{\pi\sqrt{q}} \int_0^1 \frac{dt}{\sqrt{1 - p^2 t^2}\sqrt{1 - r^2 t^2}} \tag{10}$$

which is the governing relation between the applied force and the initial and final tip rotations.

To obtain the relation between applied force and the column shortening, which is defined herein as the vertical tip displacement in excess of that due to the initial curvature, an expression for vertical displacement is needed. (Note that in the present analysis the axial compressive compliance of the column has been neglected.) Since $\frac{dx}{ds} = \cos\theta$, where x is measured downward from the tip of the column, then

$dx = \cos\theta ds = \cos\theta \frac{ds}{d\theta} d\theta$, so that use of equation (7) gives

$$dx = -\frac{\cos\theta\, d\theta}{\sqrt{\left(\frac{1}{R_o} + k^2 a \cos\gamma\right)^2 + 4k^2\left(\sin^2\frac{\gamma}{2} - \sin^2\frac{\theta}{2}\right)}} \tag{11}$$

With the definitions and change of variable employed earlier, integration of equation (11) over the length of the column yields

$$x_t = \frac{4rL}{\pi\sqrt{q}} \int_0^1 \frac{\sqrt{1 - r^2 t^2}}{\sqrt{1 - p^2 t^2}} dt - L \tag{12}$$

where x_t is the height of the deformed column, and equation (10) has been used to simplify the result. The column shortening due solely to initial curvature is $L - L\frac{\sin\gamma_o}{\gamma_o}$; hence, the net shortening due to the vertical tip force is $\Delta = L - x_t - \left(L - L\frac{\sin\gamma_o}{\gamma_o}\right)$ or

$$\frac{\Delta}{L} = 1 + \frac{\sin\gamma_o}{\gamma_o} - \frac{4r}{\pi\sqrt{q}}\int_0^1 \frac{\sqrt{1 - p^2t^2}}{\sqrt{1 - r^2t^2}}dt \tag{13}$$

To obtain the relation between applied force and net column shortening for specified values of initial curvature and force eccentricity, equations (10) and (13) must be solved simultaneously.

Computations

A numerical approach was used in solving equations (10) and (13). Equation (10) was rewritten as

$$f(\gamma,\gamma_o,a,q) = \frac{2r}{\pi\sqrt{q}}\int_0^1 \frac{dt}{\sqrt{1 - p^2t^2}\sqrt{1 - r^2t^2}} - 1 = 0 \tag{14}$$

which is the form required by routines for solving nonlinear equations. Before this equation could be solved, however, a numerical problem in evaluating the integrals in equations (13) and (14) required resolution. When r is very close to one, both integrals are nearly singular, giving rise to numerical integration inaccuracy. To circumvent this problem a change of variable was made, and asymptotic expansions were developed to evaluate the integrals on the subinterval $0.99 \le t \le 1.0$. The integrals and their replacements are

$$I_1 = \int_0^1 \frac{dt}{\sqrt{1 - p^2t^2}\sqrt{1 - r^2t^2}} = \frac{1}{r}(J_{11} + J_{12})$$

where

$$J_{11} = \int_{\sqrt{1-r^2}}^{T} \frac{du}{\sqrt{1 - u^2}\sqrt{1 - \frac{p^2}{r^2}(1 - u^2)}}$$

$$J_{12}=\frac{1}{\sqrt{2}}\left[2\sqrt{1-T}+\frac{2}{3}\left(\frac{1}{4}+\frac{p^2}{r^2}\right)\sqrt{(1-T)^3}+\frac{2}{5}\left(\frac{3}{32}-\frac{1}{4}\frac{p^2}{r^2}+\frac{3}{2}\frac{p^4}{r^4}\right)\sqrt{(1-T)^5}\right.$$
$$\left.+\frac{2}{7}\left(\frac{5}{128}-\frac{1}{32}\frac{p^2}{r^2}-\frac{9}{8}\frac{p^4}{r^4}+\frac{5}{2}\frac{p^6}{r^6}\right)\sqrt{(1-T)^7}\right]$$

and

$$I_2=\int_0^1\frac{\sqrt{1-p^2t^2}}{\sqrt{1-r^2t^2}}dt=\frac{1}{r}(J_{21}+J_{22})$$

where

$$J_{21}=\int_{\sqrt{1-r^2}}^{T}\frac{\sqrt{1-\frac{p^2}{r^2}(1-u^2)}}{\sqrt{1-u^2}}du$$

$$J_{22}=\frac{1}{\sqrt{2}}\left[2\sqrt{1-T}+\frac{2}{3}\left(\frac{1}{4}-\frac{p^2}{r^2}\right)\sqrt{(1-T)^3}+\frac{2}{5}\left(\frac{3}{32}+\frac{1}{4}\frac{p^2}{r^2}-\frac{1}{2}\frac{p^4}{r^4}\right)\sqrt{(1-T)^5}\right.$$
$$\left.+\frac{2}{7}\left(\frac{5}{128}+\frac{1}{32}\frac{p^2}{r^2}+\frac{3}{8}\frac{p^4}{r^4}-\frac{1}{8}\frac{p^6}{r^6}\right)\sqrt{(1-T)^7}\right]$$

and where, for the present computations, $T = 0.99$.

For prescribed values of a, γ_0 and γ, equation (14) can be solved for the normalized force q. The integrals J_{11} and J_{21} were evaluated using a standard mathematical library routine based on Simpson's rule. Equation (14) was solved by use of an available mathematical library routine which first bounds the root and then uses interpolation to obtain a converged value. An error criterion for f of 10^{-4} was used in equation (14). After equation (14) was solved, $\frac{\Delta}{L}$ was calculated from equation (13), again using the routine based on Simpson's rule. A complete load-shortening curve was generated by incrementing γ and repeating the solution process for numerous prescribed values of a and γ_0.

RESULTS AND DISCUSSION

The results are presented in figure 2 as plots of $q(\frac{\Delta}{L})$ for various combinations of initial curvature and load eccentricity. For reference, the elastica solution, which the present solution approaches from below as load eccentricity and initial curvature vanish, is also shown in each figure.

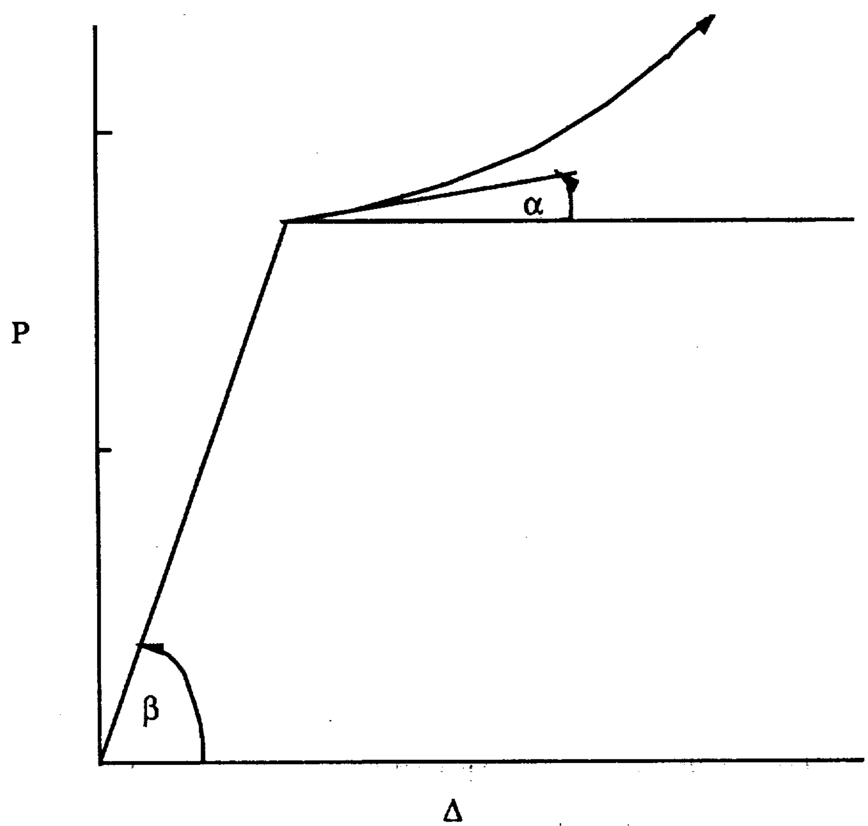
Figure 2(a) shows that for zero load eccentricity, the present results are barely distinguishable from the elastica solution when initial curvature is very small ($\gamma_0 = 0.0001$). As initial curvature is increased, the deviation from the elastica solution becomes more pronounced, particularly at small values of column shortening. However, for $\frac{\Delta}{L}$ greater than about 0.05, all of the load-shortening curves are essentially parallel to the elastica curve, indicating that the tangent stiffness of all the "buckled" columns is about equal to that of the elastica.

For progressively larger dimensionless load eccentricity, $\frac{a}{L}$, the relative importance of initial curvature is seen to diminish. In figure 2(f), for example, note the small range in the results for initial curvature values from 0.0001 to 0.03. Hence, a value of 0.05 for dimensionless load eccentricity would ensure smooth nonlinear load-shortening behavior, with little chance of troublesome column instability. However, the slope of the curve is somewhat greater than that of the elastica in the $\frac{\Delta}{L}$-range of interest, say, 0.1 to 0.3, indicating a tangent stiffness which is above the minimum achievable.

Figures 2(b) through 2(e) are for values of load eccentricity in the range $0.002 \le \frac{a}{L} \le 0.02$. As expected, all of these results lie in the envelope defined by the curves for load eccentricity, $\frac{a}{L}$, of zero and 0.05, and show in the $\frac{\Delta}{L}$-range of interest a gradual but discernable trend toward higher slope (thus, higher tangent stiffness) with increasing load eccentricity. Thus, the desire for the lowest practicable tangent stiffness would require that load eccentricity be kept reasonably small.

The results also show that the effects of initial curvature and load eccentricity are essentially the same. Thus, the relatively soft "postbuckling" behavior of the nearly perfect slender column can be obtained by imposing reasonable values of either initial curvature or load eccentricity. In most situations it would appear that, of these two perturbing influences, load eccentricity would be easier to quantify and control.

An appreciation of the magnitude of the tangent stiffness of the bent column can be gained by comparing the slope of the load-shortening curve for bending with that of the prebuckling load-shortening curve for the straight column under axial compression. For illustrative purposes, the two curves are combined in the sketch below to represent a continuous prebuckling and postbuckling load-shortening curve for a perfect column. The slope of the postbuckling curve is exaggerated for easier visualization.



Note that $\tan\alpha = \left(\frac{dP}{d\Delta}\right)_b = \frac{\pi^2}{8}\left(\frac{\rho}{L}\right)^2 \tan\beta << \left(\frac{dP}{d\Delta}\right)_c = \tan\beta.$

From the elastica solution (ref. 1), it can be shown (See Appendix) that the initial postbuckling slope of the load-shortening curve is, in dimensionless terms, $\frac{dq}{d\left(\frac{\Delta}{L}\right)} = \frac{1}{2}$, which can be rewritten as $\left(\frac{dP}{d\Delta}\right)_b = \frac{\pi^2 EI}{8L^3}$, where the subscript b denotes bending. (As can be seen in figure 2, this is a close approximation to the present results for a large range of column shortening.) The corresponding expression for prebuckling axial compression is $\left(\frac{dP}{d\Delta}\right)_c = \frac{EA}{L}$, where the c denotes compression. The ratio of the two tangent stiffness measures is

$$\frac{\left(\frac{dP}{d\Delta}\right)_b}{\left(\frac{dP}{d\Delta}\right)_c} = \frac{\pi^2 I}{8L^2 A} = \frac{\pi^2}{8}\left(\frac{\rho}{L}\right)^2,$$

where ρ is the radius of gyration of the column cross section. For slender columns, this ratio is very small. However, to put this result in perspective, note that the initial postbuckling slope is of the same order of magnitude as the slope of the load-deflection curve for the tip-loaded straight cantilever beam of length L, i.e.,

$$\frac{dP}{d\Delta} = \frac{EI}{3L^3}$$

This means, of course, that a buckled-column support system would simulate free-free specimen support conditions about as well as would a system employing cantilever beams. Hence, although the slope of the load-shortening curve for the buckled column is very small in comparison with the perfect column's prebuckling slope, it is not negligible, particularly when compared with the initial postbuckling slope of the plot of load as a function of lateral tip deflection, which is zero.

CONCLUDING REMARKS

In this paper, the load-shortening behavior of an eccentrically loaded, initially imperfect slender column has been analyzed. The results indicate that such columns can be designed to behave as relatively soft springs over a wide range of column shortening, for reasonably broad ranges of initial curvature and load eccentricity. Both initial curvature and load eccentricity are seen to promote similar nonlinear behavior. Because load eccentricity is probably easier to impose with accuracy, it is likely to be the preferred parameter to specify in order to produce and control the soft "postbuckling" behavior of the slender column. If the tangent stiffness of a "buckled" column is low enough to satisfy test requirements, then a specimen support system that incorporates eccentrically loaded slender columns may present a useful alternative to a system requiring very long suspension cords. It should be noted, however, that the tangent stiffness of the deformed column is of the same order as that of an initially straight, tip-loaded cantilever beam and, thus, may not be low enough to satisfy some test requirements.

REFERENCES

1. Timoshenko, Stephen P. and Gere, James M.: Theory of Elastic Stability, second edition, McGraw-Hill Book Company, New York, 1961.
2. Abramowitz, Milton; and Stegun, Irene, Eds.: Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series 55, 1968

APPENDIX

The Initial Postbuckling Slope of a Perfect Inextensional Column

From reference 1, the two equations governing the load-shortening behavior of the elastica are, in dimensionless terms,

$$\frac{\Delta}{L} = \frac{2}{kL}E(p) - 1 \tag{A1}$$

and

$$kL = K(p) \tag{A2}$$

where K(p) and E(p) are complete elliptic integrals of the first and second kind, respectively, $p=\sin\frac{\alpha}{2}$, and $k=\frac{\pi}{2}\sqrt{q}$. Letting $\lambda = \frac{\Delta}{L}$, (A1) and (A2) become

$$1+\lambda = \frac{2E(p)}{K(p)} \tag{A3}$$

and

$$\frac{\pi}{2}\sqrt{q} = K(p) \tag{A4}$$

Differentiating both equations with respect to λ,

$$\frac{dp}{d\lambda} = \frac{K^2}{2\left(E\frac{dK}{dp} - K\frac{dE}{dp}\right)} \tag{A5}$$

and

$$\frac{dq}{d\lambda} = \frac{4\sqrt{q}}{\pi}\frac{dK}{dp}\frac{dp}{d\lambda} \tag{A6}$$

Inserting (A5) into (A6) gives

$$\frac{dq}{d\lambda} = \frac{2\sqrt{q}}{\pi}\frac{K^2\frac{dK}{dp}}{E\frac{dK}{dp} - K\frac{dE}{dp}} \tag{A7}$$

For load only slightly above the buckling load (q=1), $p=\sin\frac{\alpha}{2}$ is very small.

Hence, K(p) and E(p) have the valid power series

$$K(p) = \frac{\pi}{2}\left[1 + \frac{p^2}{4} + \frac{9}{64}p^4 + \ldots\right] \tag{A8}$$

and

$$E(p) = \frac{\pi}{2}\left(1 - \frac{p^2}{4} - \frac{3}{64}p^4 + \ldots\right) \tag{A9}$$

Substitution of (A8) and (A9) into (A7) yields

$$\frac{dq}{d\lambda} = \frac{\sqrt{q}}{2}\left(1 + O\left(p^2\right)\right)$$

As $q \to 1+$, $p \to 0$; therefore, $\dfrac{dq}{d\left(\dfrac{\Delta}{L}\right)} \to \dfrac{1}{2}$ as the buckling load is approached from above.

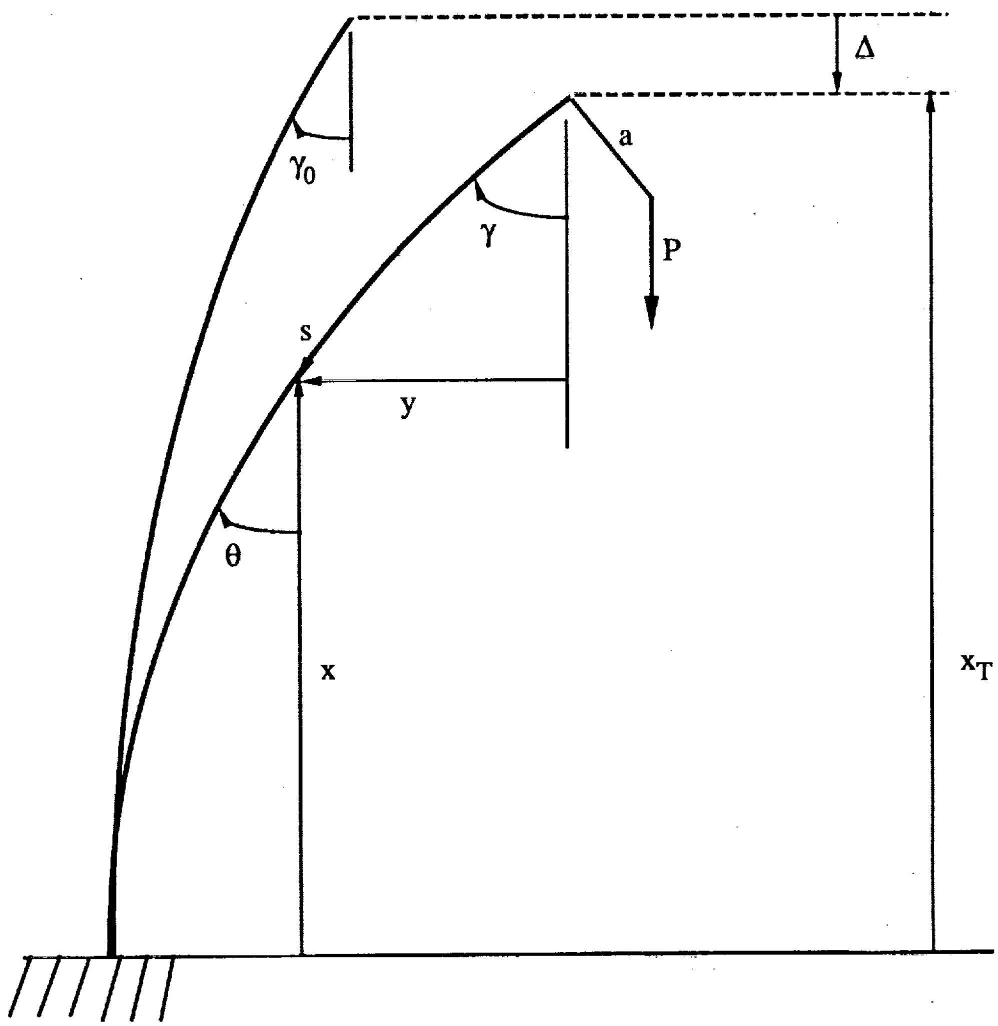
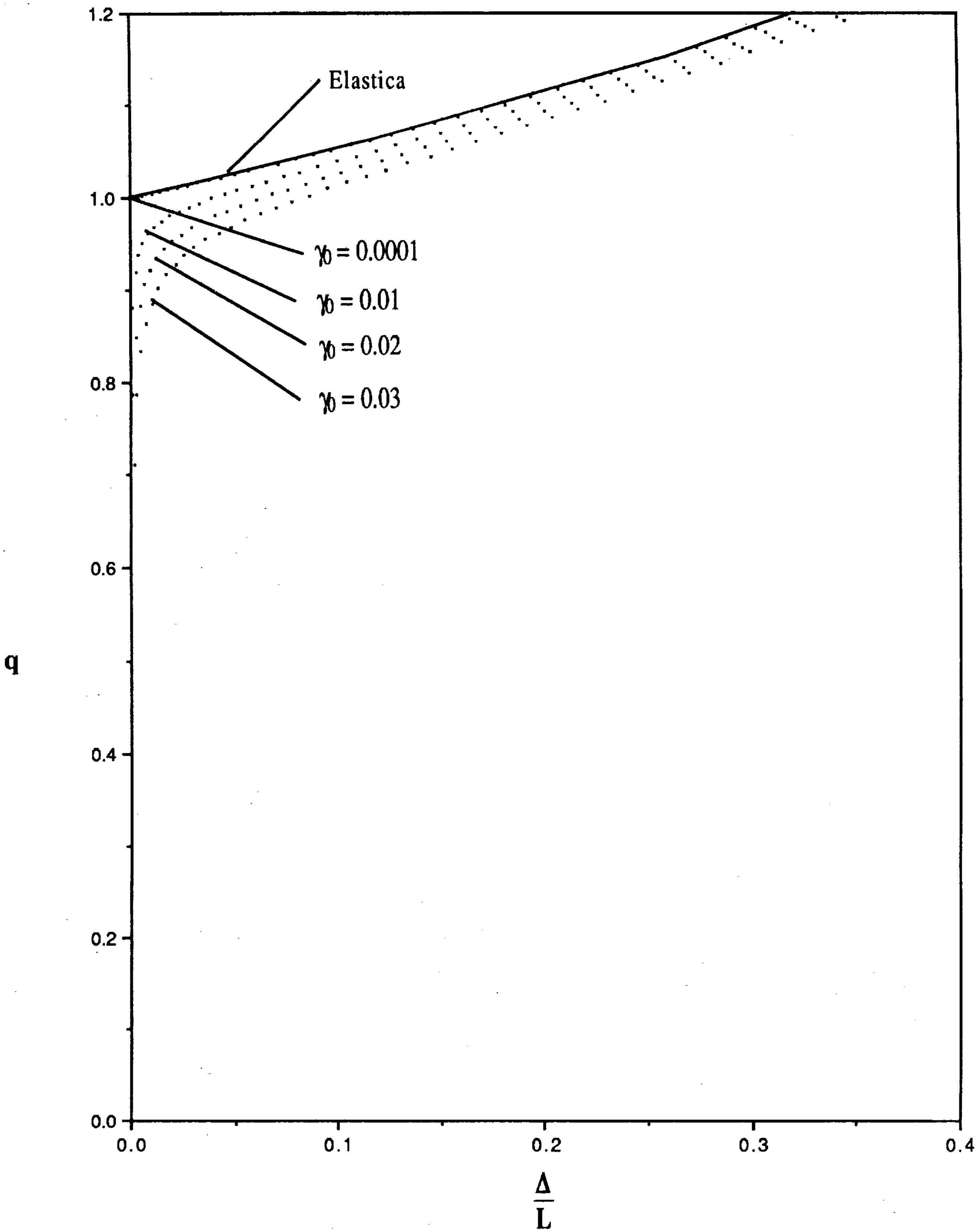
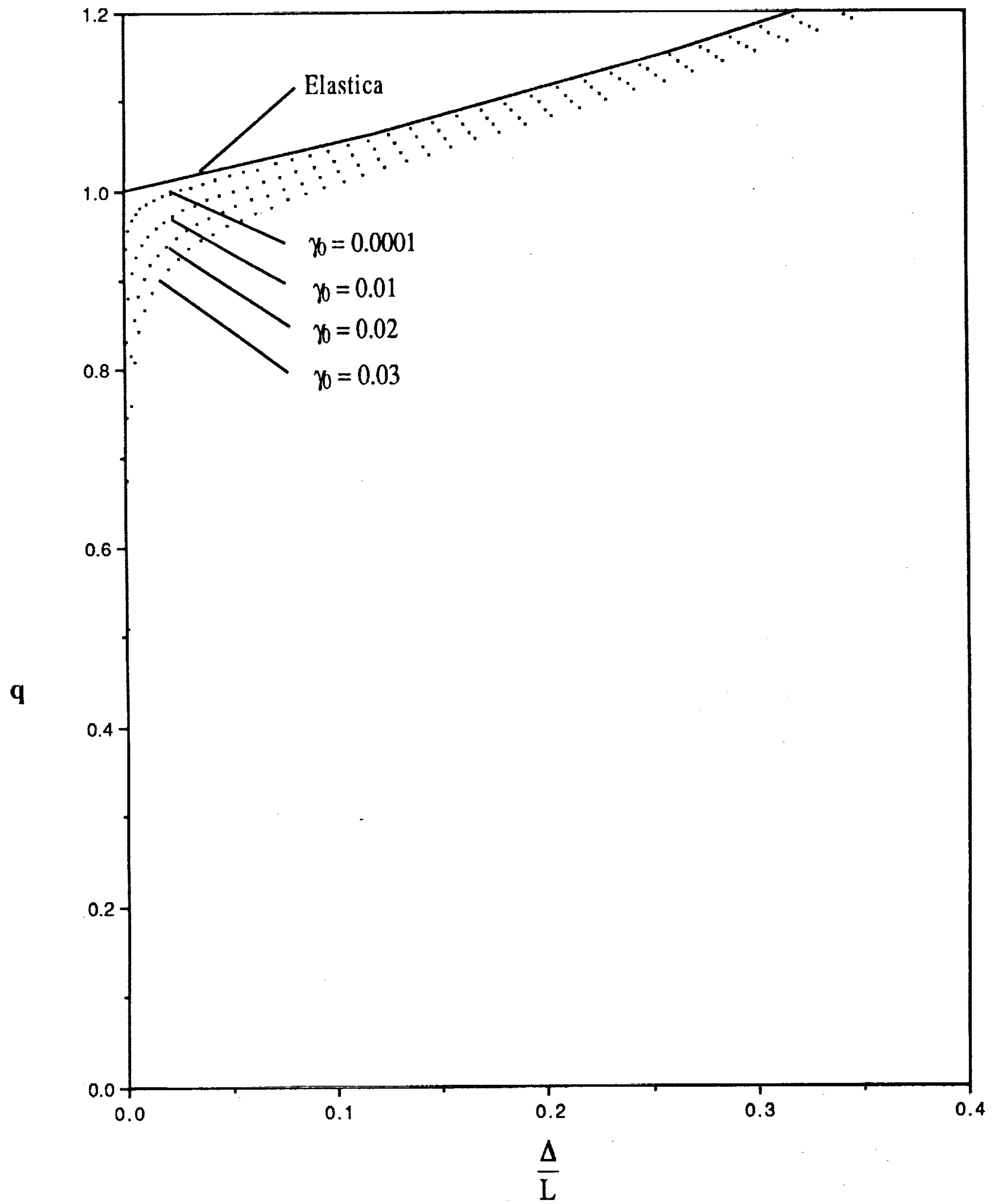


Figure 1. Coordinate system and nomenclature.



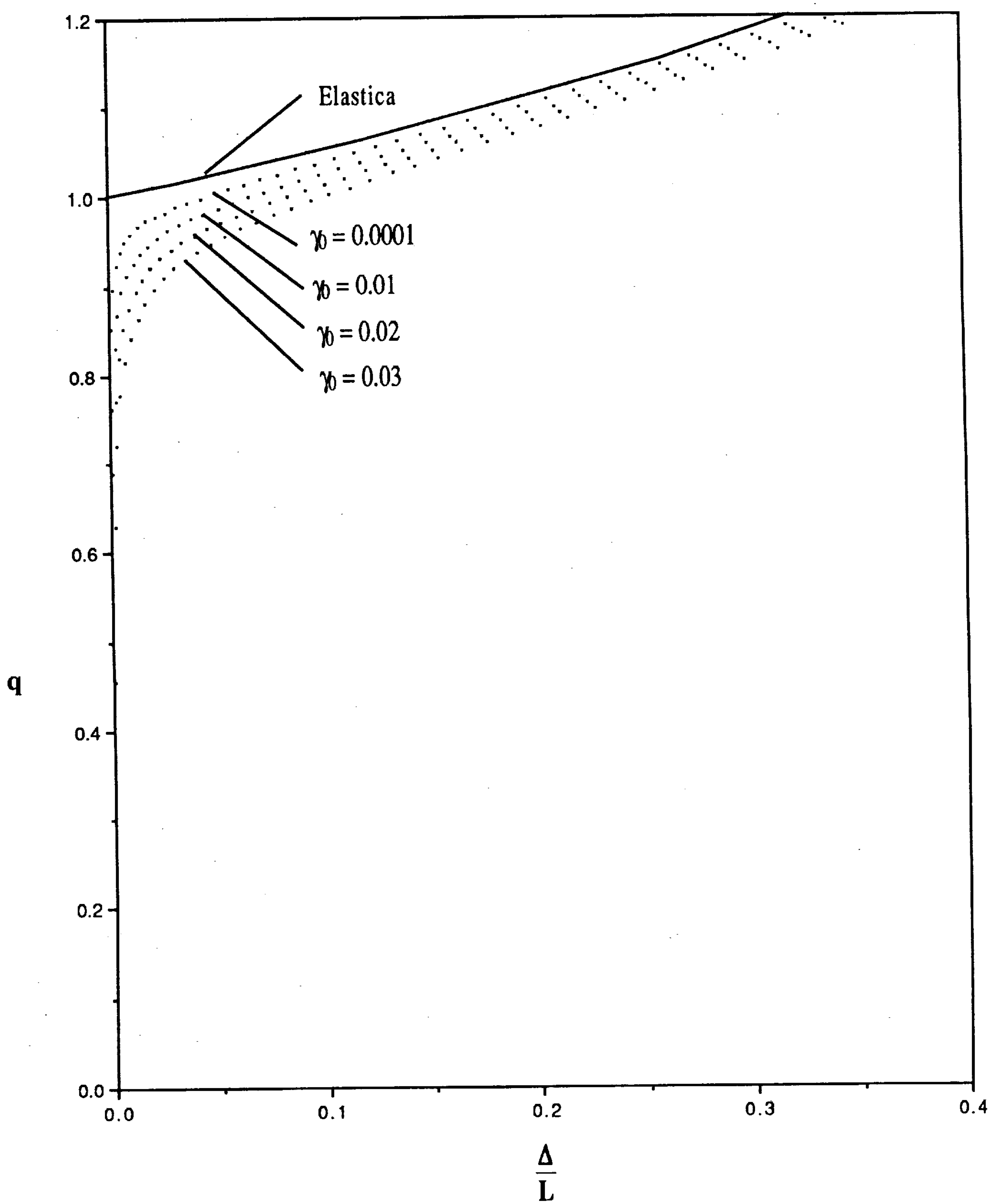
(a) $\frac{a}{L} = 0$

Figure 2.- Load-shortening curves for an initially curved, eccentrically loaded column.



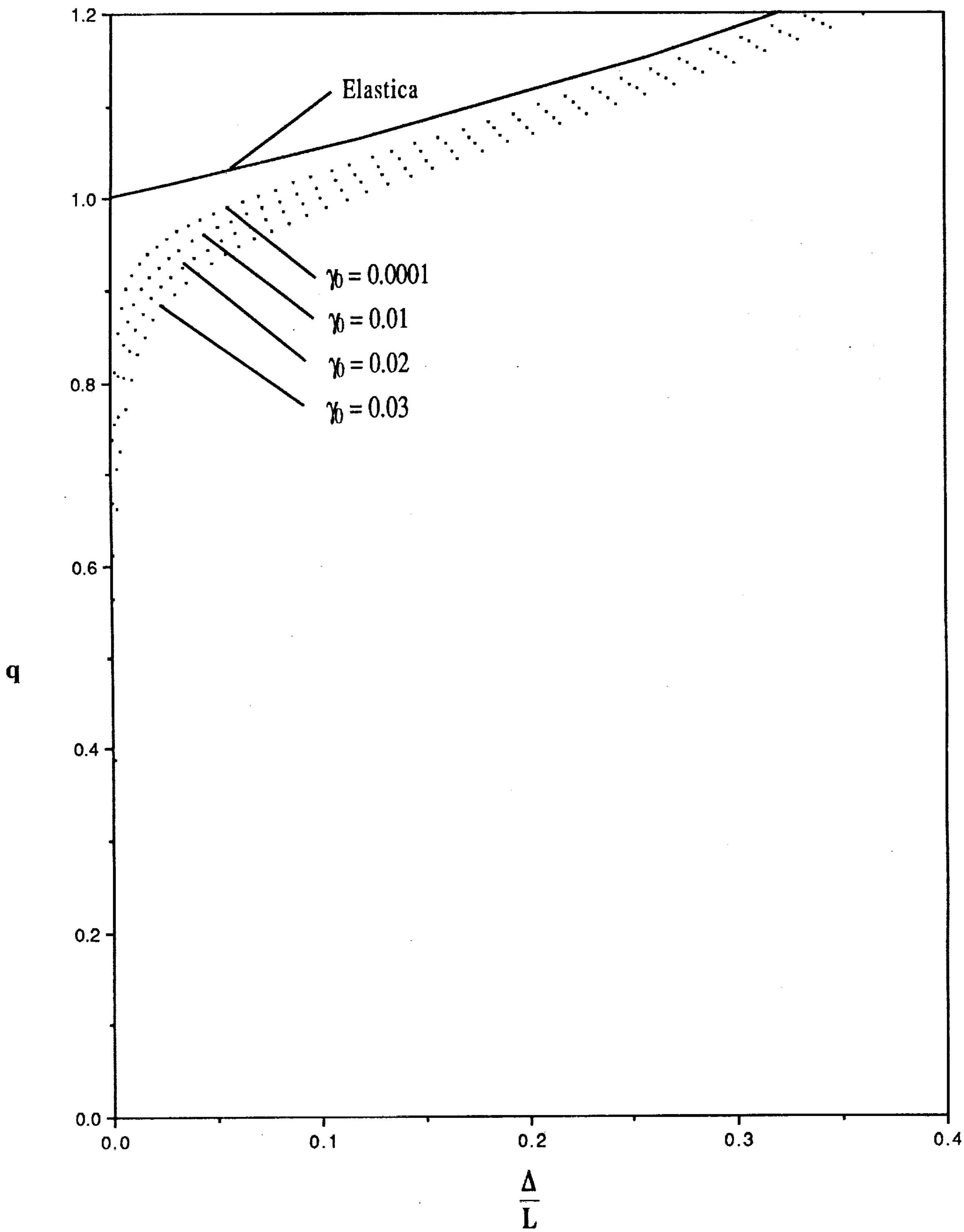
(b) $\frac{a}{L} = 0.002$

Figure 2.- Continued.



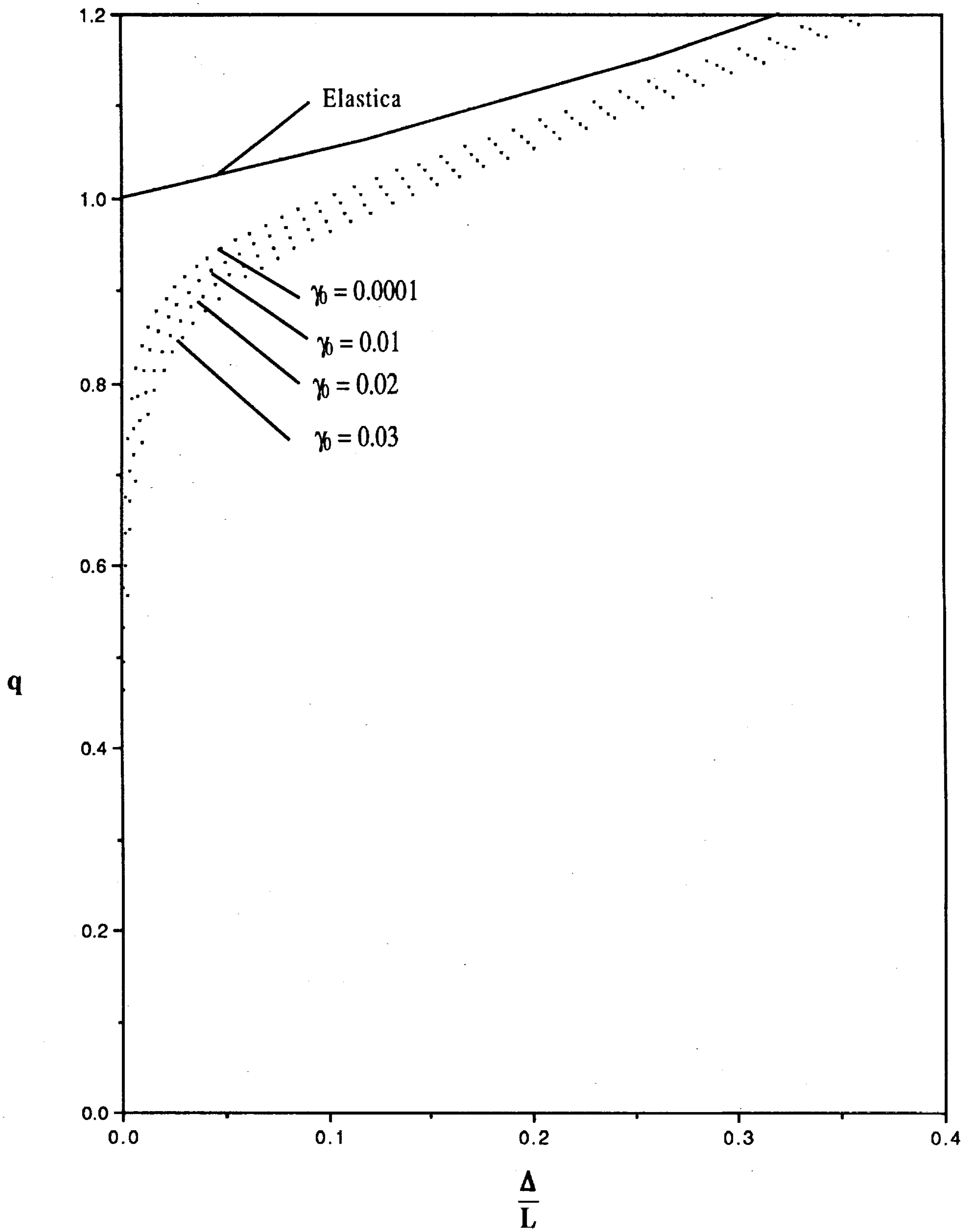
(c) $\frac{a}{L} = 0.005$

Figure 2.- Continued.



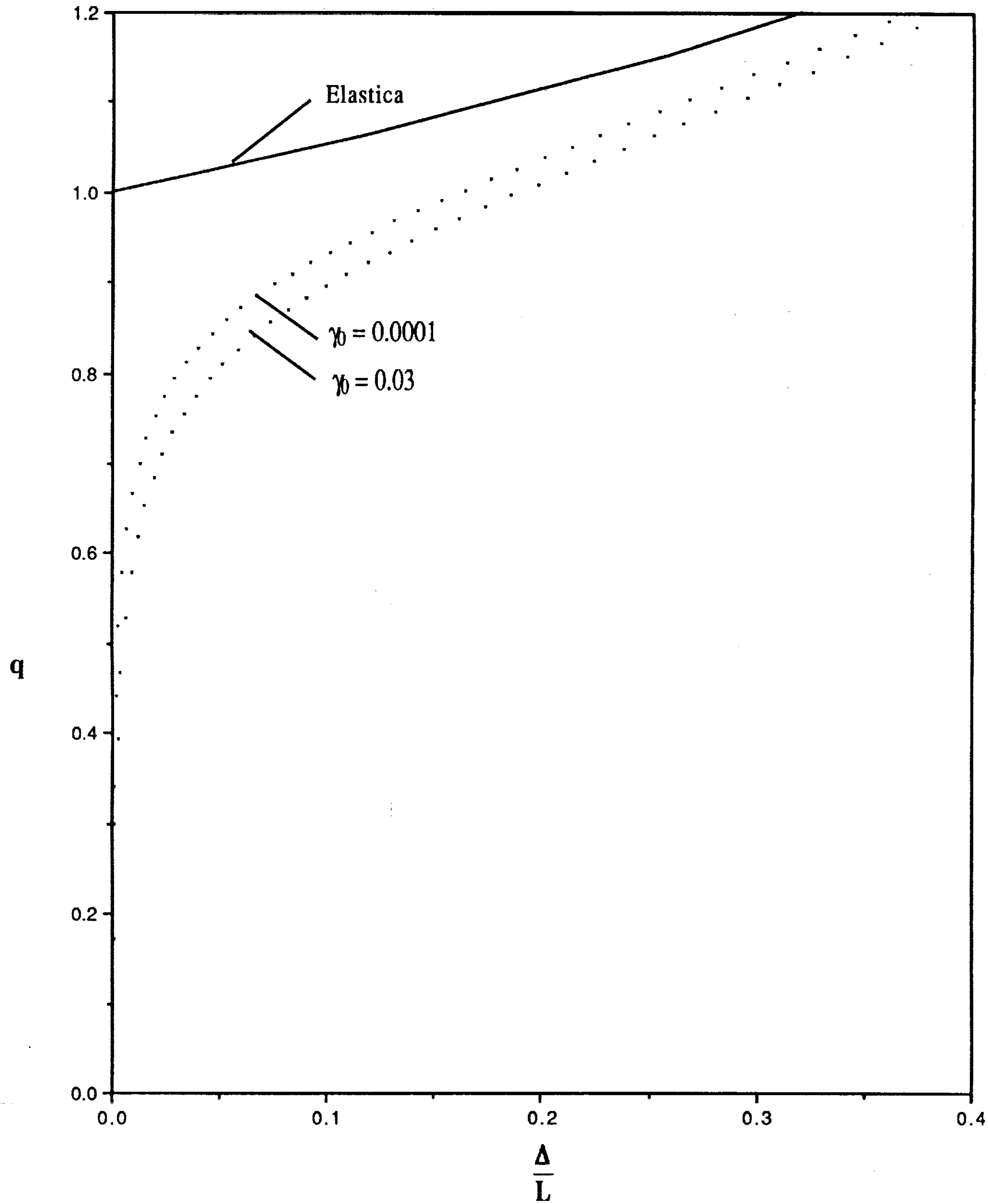
(d) $\frac{a}{L} = 0.01$

Figure 2.- Continued.



(e) $\frac{a}{L} = 0.02$

Figure 2.- Continued.



(f) $\frac{a}{L} = 0.05$

Figure 2.- Concluded.

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16. Abstract

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