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COMPATIBILITY CONDITIONS OF STRUCTURAL MECHANICS FOR FINITE ELEMENT ANALYSIS

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SUMMARY

The equilibrium equations and the compatibility conditions are fundamental to the analyses of structures. However, anyone who undertakes even a cursory generic study of the compatibility conditions can discover, with little effort, that historically this facet of structural mechanics had not been adequately researched by the profession. Now the compatibility conditions (CC's) have been researched and are understood to a great extent. For finite element discretizations, the CC's are banded and can be divided into three distinct categories: (1) the interface CC's, (2) the cluster or field CC's, and (3) the external CC's. The generation of CC's requires the separating of a local region, then writing the deformation displacement relation (ddr) for the region, and finally, the eliminating of the displacements from the ddr. The procedure to generate all three types of CC's is presented and illustrated through examples of finite element models. The uniqueness of the CC's thus generated is shown.

The utilization of CC's has resulted in the novel integrated force method (IFM). The solution that is obtained by the IFM converges with a significantly fewer number of elements, compared to the stiffness and the hybrid methods.

INTRODUCTION

In the analyses of structures both the conditions of equilibrium and of compatibility come into play, except for the trivial statically determinate case. However the conditions of equilibrium are the most familiar to structural analysts, perhaps because of a concern about the internal forces required for design and the wide acceptance of equilibrium as a universal and natural concept. In contrast, the concept of compatibility is much less familiar to structural analysts. The compatibility conditions (CCS's) were not known until mathematicians formulated them about a century ago (ref. 1), long after the equilibrium equations (EE's) had been derived. Even so, the early forms of the compatibility conditions were developed mainly for the manual analyses of simple structures and were based on the concept of redundant structural elements. The notion of cutting the redundant members, which leads to the conditions of compatibility that have to be restored (henceforth referred to as classical

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compatibility conditions (CCC's)), was formulated in the precomputer era. Formulating CCC's for large-scale computations proved inconvenient and inefficient, so they have almost disappeared from current structural engineering practice. The general, and mathematically rigorous, compatibility conditions (analogous to St. Venant's strain formulation in elasticity (ref. 1)) have been formulated for finite element analysis. These finite element CC's are referred to as the global compatibility conditions (GCC's) or simply CC's. The GCC's are banded, and they are amenable to computer automation. (In stiffness analyses, linkage and continuity of displacements at element interface are popularly referred to as the compatibility conditions; in a strict sense, such constraints are continuity conditions.)

The utilization of compatibility conditions has resulted in the integrated force method (IFM) (refs. 2 to 16). In the IFM all the internal forces are treated as the primary unknown, and the system equilibrium equations are coupled to the global compatibility conditions in a fashion paralleling approaches in continuum mechanics, such as the Beltrami-Michell formulation in elasticity (ref. 17).

The compatibility conditions, in basic form, have been introduced and compared with the CCC's in references 2 to 4. The purposes of this report are to (1) describe the physical aspects of the compatibility conditions, including the interface CC's of finite element models; (2) demonstrate the generation of GCC's from their local counterparts; and (3) illustrate the benefits that accrue from the use of the global compatibility conditions in finite element analysis.

The subject matter of this report is presented in the subsequent five sections. In the second section the governing equations of the IFM are presented. In the third section we demonstrate the procedure to generate the compatibility conditions, and these concepts are illustrated in the fourth section. In the fifth section, comparison of results obtained by the IFM, the stiffness method, and the hybrid methods are presented, and the conclusions are given in the sixth section.

EQUATIONS OF THE INTEGRATED FORCE METHOD

In the integrated force method, a discretized structure for the purpose of analysis is designated as structure (n,m) where (1) structure denotes the types of structure (truss, frame, plate, shell discretized by finite elements, their combinations, etc.) and (2) n and m are force and displacement degrees of freedom fof and dof, respectively. The structure (n,m) has m equilibrium equations and $r = n - m$ compatibility conditions. The m EE's, $[B]\{F\} = \{P\}$, and the r CC's, $[C][G]\{F\} = \{\delta R\}$, are coupled to obtain the governing equations of the IFM as follows:

$$\left[\frac{[B]}{[C][G]}\right]\{F\} = \left\{\frac{P}{\delta R}\right\} \qquad \text{or} \qquad [S]\{F\} = \{P\}^* \tag{1}$$

where

[B] $m \times n$ equilibrium matrix
[C] $r \times n$ compatibility matrix

[G] $n \times n$ concatenated flexibility matrix (containing material properties) and it links deformations $\{\beta\}$ to forces $\{F\}$ as $\{\beta\} = [G]\{F\}$

{P} m-component load vector

{δR} r-component effective initial deformation vector such that $\{\delta R\} = -[C]\{\beta_0\}$ where $\{\beta_0\}$ is the n-component initial deformation vector

[S] $n \times n$ IFM governing matrix

The matrices [B], [C], [G], and [S] are banded, and they have full row ranks of m, r, n, and n, respectively.

The solution of equation (1) yields the n-forces {F}. The m-displacements {X} are obtained from the forces {F} by back substitution (ref. 9) as

$$\{X\} = [J]\{[G]\{F\} + \{\beta_0\}\} \tag{2}$$

where [J] is the $m \times n$ deformation coefficient matrix defined as $[J] = m$ rows of $[[S]^{-1}]^T$.

Equations (1) and (2) represent the two key relations of the IFM for finite element analysis that are needed to calculate forces and displacements, respectively.

GENERATION OF THE COMPATIBILITY CONDITIONS

The compatibility conditions and the associated coefficient matrix [C] are obtained from St. Venant's strain formulation in elasticity (ref. 1) as an extension to discrete structural mechanics. The strain formulation is illustrated through the plane stress elasticity problem. The strain displacement relations (SDR) of the problem are

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \tag{3a}$$

Since in the SDR three strains (ε_x, ε_y, γ_{xy}) are expressed as functions of two displacements (u,v), the SDR contains one compatibility condition, which can be obtained by eliminating the two displacements from the three SDR as

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = 0 \tag{3b}$$

The two steps that are necessary to generate CC's from St. Venant's strain formulation are (1) establish the strain displacement relations given by equation (3a) and (2) eliminate displacements from the SDR to obtain the compatibility condition given by equation (3b).

In the mechanics of discrete structures, the equivalent of SDR are the deformation displacement relations (DDR). (Deformations of discrete analysis $\{\beta\}$ are analogous to strains $\{\varepsilon\}$ of elasticity.) The DDR can be assembled directly or obtained on an energy basis by utilizing the well known equality relation of internal strain energy and external work, which can be written in the case of a discrete structure (n,m) as

$$\frac{1}{2}\{F\}^T\{\beta\} = \frac{1}{2}\{X\}^T\{P\} \tag{4a}$$

Equation (4a) can be rewritten by eliminating loads {P} in favor of internal forces {F} by using the EE ([B]{F} = {P}) to obtain

$$\frac{1}{2}\{X\}^T[B]\{F\} = \frac{1}{2}\{F\}^T\{\beta\} \qquad \text{or} \qquad \frac{1}{2}\{F\}^T\{[B]^T\{X\} - \{\beta\}\} = 0 \tag{4b}$$

Since the force vector {F} is not a null set, we finally obtain the following relation between member deformations and nodal displacements:

$$\{\beta\} = [B]^T\{X\} \tag{5}$$

The expression given by equation (5) represents the global deformation displacement relation applicable to any finite element model whose system equilibrium equations can be symbolized as [B]{F} = {P}. In the DDR, n-deformations $\{\beta\}$ are expressed in terms of m-displacements {X}; thus, there are $r = n - m$ constraints on deformations, which represent the r compatibility conditions of the structure (n,m). The r CC's, in terms of total deformations, can be obtained by the elimination of m-displacements from the n DDR, and in matrix notation the CC's can be written as

$$[C]\{\beta\} = \{0\} \tag{6a}$$

The CC's, in terms of elastic deformations $\{\beta\}_e$ that are given by equation (6b), are obtained from equation (6a) and from the definition of the total deformation $\{\beta\}$, which is composed of the initial deformations $\{\beta\}_0$ and the elastic deformations $\{\beta\}_e$ as $\{\beta\} = \{\beta\}_e + \{\beta\}_0$. Thus

$$[C]\{\beta\}_e = \{\delta R\} \tag{6b}$$

where

$\{\delta R\} = -[C]\{\beta_0\}$
[C] $r \times n$ global compatibility matrix

The efficient generation of the CC matrix [C] is the subject matter of this report. The matrix [C] is rectangular and banded, and it has full row rank r. The CC's are kinematic relationships that are independent of sizing design parameters (such as area of bars, moments of inertia of beams, etc.), material properties, and external loads. The compatibility conditions depend on the initial deformation in the structure. For numerical efficiency, directly eliminating displacements from the DDR to obtain CC's is not recommended for large-scale computations. Instead, the global compatibility matrix [C] is efficiently generated by utilizing such physical features of the compatibility conditions of finite element models as bandwidths, the determinacy of the grid points, and so on.

PROCEDURE TO GENERATE GLOBALCOMPATIBILITY CONDITIONS

To generate the CC's, (1) separate a local region from the structural model on the basis of interface, cluster, or external bandwidth considerations as explained later in this section; (2) establish the local deformation displacement relations (ddr) for the local region, and eliminate the displacements from the ddr to obtain the CC's for the region under consideration; and (3) repeat steps (1) and (2) until all r CC's of the structure (n,m) are generated. These steps are elaborated in the section Illustrative Examples. The order of generation of the CC's is immaterial; however, we recommend generating the interface CC's first, since these are most numerous, followed by the cluster CC's, and finally, the external CC's.

Bandwidths of the Compatibility Conditions

The CC's of discretized structures are banded. On the basis of bandwidth considerations, the CC's are divided into three distinct categories: (1) interface compatibility conditions, (2) cluster or field compatibility conditions, and (3) external compatibility conditions. By assuming the example of a finite element model as shown in figure 1, the three types of compatibility conditions can be illustrated as in figure 2.

Interface Compatibility Conditions

Numerous interfaces internal to the structure are created in the discretization processes. The interface is the common boundary shared by two or more elements. In the model shown in figure 1, the common boundary along nodes 1 and 7 is the interface between elements 1 and 2, the boundary connecting nodes 12 and 17 is the interface between elements 13 and 14, and so on. The interface between elements 1 and 2 is shown in figure 2(a). The deformations of elements 1 and 2 must be compatible along the common boundary defined by nodes 1 and 7, which gives rise to interface compatibility conditions. The number of CC's at the interface depends on the element types (such as membrane, flexure or solid tetrahedron, etc.) and element numbers. The maximum bandwidth of the interface compatibility condition can be calculated as

$$MBW_{icc} = \sum_{j=1}^{JT} (fof_{ej}) \tag{7a}$$

where

MBW_{icc} maximum bandwidth of the interface compatibility conditions
JT total number of elements present at the interface
fof_{ej} force degrees of freedom of the element j present at the interface

The bandwidth MBW_{icc} represents the maximum bandwidths of the interface compatibility conditions written either in terms of forces {F}, as in [C][G]{F} = {0} (here we are referring to the bandwidth of the product matrix [C][G], or in terms of deformations {β}, as in [C]{β} = {0} (here the bandwidth is that of the compatibility matrix [C]). The actual bandwidth of the compatibility matrix [C] is smaller than its maximum bandwidth. The interface compatibility conditions of discrete analysis are analogous to the boundary compatibility conditions in elasticity (ref. 8). The interface CC's are the most numerous in any finite element model. These are generated by writing the deformation displacement relation for the local region (such as shown in fig. 2(a) for the interface defined by nodes 1 and 7) and then eliminating the displacements from the local ddr as explained in the section Illustrative Examples. For the interface shown in figure 2(a), there are two elements (i.e., JT = 2). Let us assume that both are membrane elements; the fof of the triangular element fof_t is 3 and that of the quadrilateral element, fof_q is 5. Then the MBW_{icc} calculated from equation (7a) is 8.

Cluster Compatibility Conditions

Consider any element in the model shown in figure 1, such as 19. Element 19 along with its eight neighboring elements are shown in figure 2(b). The deformations of element 19 must be compatible with its neighboring elements (14 to 16 and 18 to 23). The CC's of the cluster of elements are referred to as the cluster compatibility conditions, which essentially represent the field CC's of St. Venant's strain formulation. The maximum bandwidth of the cluster CC's can be calculated as

$$MBW_{ccc} = \sum_{j=1}^{JTC} (fof_{ej}) \qquad (7b)$$

where

MBW_{ccc} maximum bandwidth of the cluster compatibility conditions
JTC total number of elements present in the cluster

Assuming that quadrilateral elements have $fof_q = 5$ and triangular elements have $fof_t = 3$, the MBW_{cc} of the cluster shown in figure 2(b) can be calculated from equation (7b) as 41.

In a finite element idealization, the number of cluster compatibility conditions are fewer than the number of interface CC's. Generating cluster compatibility conditions requires the establishment of the ddr for the local cluster (such as shown in fig. 2(b) for element 19) and then elimination of the displacements from the local ddr as explained in the section Illustrative Examples.

External Compatibility Conditions

Reactions are induced at the nodes where displacements are restrained. If such restraints are sufficient only for the kinematic stability of the structure, then the structure is externally determinate, and it has no external compatibility conditions. If, however, the restraints on the boundary exceed the number of rigid body motions of the structure, then the structure is externally indeterminate. The degree of external indeterminacy can be calculated as follows:

$$R_{ext} = N_x - N_f \qquad (7c)$$

where

R_{ext} number of external indeterminacy
N_x number of displacement components suppressed on the boundary
N_f number of boundary conditions required only for the kinematic stability of the structure

Let us assume that the finite element model shown in figure 1 represents a membrane structure. Then its external indeterminacy $R_{ext} = 7 - 3 = 4$, since the number of actual boundary restraints N_x is 7 and the kinematic stability requirement N_f is 3.

To calculate the bandwidth of the external compatibility conditions, separate the local region connecting any two boundary nodes. Let the number of elements between the two nodes be represented by JTE, then the bandwidth of the external CC's is given by

$$MBW_{ecc} = \sum_{j=1}^{JTE} (fof_{ej}) \tag{7d}$$

where

MBW_{ecc} maximum bandwidth of the external compatibility conditions

Assuming as before that the quadrilateral elements have $fof_q = 5$ and the triangular elements have $fof_t = 3$, the bandwidth of the external CC's shown in figure 2(c) can be calculated from equation (7d) as 8.

If the boundary represents a determinate boundary, then no boundary CC's will be generated. The boundary CC's are obtained by eliminating the displacements from the deformation displacement relations written for the local boundary segment (e.g., for the model shown in fig. 1, the segment containing nodes 1 and 3 and elements 2 and 3, also shown in fig. 2(c)) by following the procedure explained in Illustrative Examples.

The interface, cluster, and external CC's represent the local CC's. All three categories of local CC's are concatenated to form the system or the global CC's of the structure (n,m). The sum $r = r_{icc} + r_{ccc} + r_{ecc}$ of the number of interface CC's, cluster CC's, and external CC's is equal to $r = n - m$ of the model. The values of r_{icc}, r_{ccc}, and r_{ecc} can be calculated for discrete models; however it is not necessary to determine their values before generating the CC's.

ILLUSTRATIVE EXAMPLES

Examples of a few structures that are idealized by triangular membrane elements and bar elements are presented to illustrate the generation of global compatibility conditions from the local conditions such as interface CC's, cluster CC's, and external CC's. In the examples, triangular elements given by Przemieniecki (ref. 18) and standard bar elements, which are adequate to illustrate the compatibility concepts yet simple enough for closed-form presentation, are chosen. The elements are shown in figure 3. The membrane element has three force unknowns, F_{1e}, F_{2e}, and F_{3e}; its six displacement degrees of freedom are X_{1e}, X_{2e}, . . . , X_{6e} (fig. 3(a)). The bar element has one force and four displacement unknowns (fig. 3(b)). For the membrane element, the 6 × 3 equilibrium matrix $[B]_e$ and its symmetrical flexibility matrix $[G]_e$ of dimension 3 × 3 are obtained in closed form (ref. 18) as

$$[B]_e = \begin{bmatrix} -\ell_{12} & 0.0 & \ell_{31} \\ -m_{12} & 0.0 & m_{31} \\ \ell_{12} & -\ell_{23} & 0.0 \\ m_{12} & -m_{23} & 0.0 \\ 0.0 & \ell_{23} & -\ell_{31} \\ 0.0 & m_{23} & -m_{31} \end{bmatrix} \tag{8a}$$

where ℓ_{ij} and m_{ij} denote the direction cosines of the direction vector defined by the edge ij; and

$$[G]_e = \left(\frac{2}{Et}\right) \times \begin{bmatrix} \dfrac{\sin\theta_3}{\sin\theta_1 \sin\theta_2} & \cos\theta_2 \cot\theta_2 - \nu\sin\theta_2 & \cos\theta_1\cot\theta_1 - \nu\sin\theta_1 \\ \cos\theta_2\cot\theta_2 - \nu\sin\theta_2 & \dfrac{\sin\theta_1}{\sin\theta_2\sin\theta_3} & \cos\theta_3\cot\theta_3 - \nu\sin\theta_3 \\ \cos\theta_2\cot\theta_2 - \nu\sin\theta_2 & \cos\theta_3\cot\theta_3 - \nu\sin\theta_3 & \dfrac{\sin\theta_2}{\sin\theta_3\sin\theta_1} \end{bmatrix} \tag{8b}$$

where

E Young's modulus
ν Poisson's ratio
t membrane thickness

The angles θ_i are defined in figure 3(a).

Even though the 6 × 6 elemental stiffness matrix for the membrane can be generated in closed form, its explicit form is too complicated for presentation here.

Example I - The Membrane

Generation of the interface compatibility conditions is illustrated through the example of a membrane shown in figure 4(a). The membrane is made of steel, with Young's modulus E = 30 ksi and Poisson's ratio $\nu = 0.3$;

dimensions a and b are 100 in. and thickness $t = 1$ in. The membrane is subjected to concentrated loads. The example is also solved herein by the IFM, and the stiffness solution in symbolic form is included for comparison. The membrane is idealized by two triangular elements. The discretization has six force degrees of freedom, which are the concatenation of the two element force unknowns such that

$$\{F\}^T = \langle F_1 = F_{1e1}, F_2 = F_{2e1}, \ldots, F_6 = F_{3e2} \rangle \tag{9a}$$

where the subscript iej indicates the i^{th} force of element j.

A corresponding deformation component β_k is associated with each force component F_k. The six-component deformation vector of the membrane is

$$\{\beta\}^T = \langle \beta_1 = \beta_{1e1}, \beta_2 = \beta_{2e1}, \ldots, \beta_6 = \beta_{3e2} \rangle \tag{9b}$$

The five system displacement degrees of freedom shown in figure 4(a) are represented by $\{X\}$ as

$$\{X\}^T = \langle X_1, X_2, \ldots, X_5 \rangle \tag{10}$$

The membrane is designated "membrane (6,5)" since it has six force and five displacement unknowns. Membrane (6,5) has five EE's and one CC (i.e., $r = (n - m) = (6-5) = 1$).

Equilibrium equations of the membrane (6,5). - The five system EE's of membrane (6,5), in terms of the six forces, are obtained from the elemental equilibrium matrices (refer to eq. (8a)) by following finite element assembly technique as

$$\begin{bmatrix} 1.0 & 0.0 & 0.7071 & 0.7071 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.7071 & -0.7071 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.7071 & 0.7071 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} 50.00 \\ 100.00 \\ 50.00 \\ 50.00 \\ 100.00 \end{Bmatrix} \tag{11}$$

Since the system equilibrium matrix [B] has the dimension 5 × 6, the five EE's (eq. (11)) cannot be solved for the six forces; one CC is required to augment the five EE's to a solvable set of six equations in six unknowns.

Compatibility condition for membrane (6,5) or the interface compatibility conditions. - Membrane (6,5) has one CC; therefore its local CC and global CC are represented by the same equation. The first step in the generation of the CC is to establish the deformation displacement relation for the membrane. The 6 × 5 global DDR is obtained from equations (5) and (11) as

$$\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{Bmatrix} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.707 & -0.7071 & 0.7071 & 0.0 & 0.0 \\ 0.707 & -0.7071 & 0.7071 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{Bmatrix} \tag{12}$$

Even though the single CC for this simple problem can be obtained by direct elimination of the five displacements from the six ddr given by equation (12), this procedure is not recommended because it can become numerically expensive for large structures. The concept of node determinacy, which greatly enhances computational efficiency in the generation of CC's, is presented next.

The Node Determinacy Condition

Node determinacy for general application is presented first; then it is specialized for membrane (6,5). Forces in determinate structures can be obtained from EE's alone; such determinate forces do not participate in the CC's. The concept of determinate structure is extended to the nodes or grid points of a finite element model, and to enhance computational efficiency, determinate nodes or grid points are identified and eliminated at intermediate stages of the generation of CC's.

Take any node of a finite element model, for example the node i. Let K_i represent the number of force components present in the EE's written for node i. Let L_i represent the displacement degrees of freedom of the node i, which also is the number of EE's that can be written at that node. The indeterminacy of node i, designated (NR_i), is defined as

$$NR_i = K_i - L_i \tag{13}$$

If $NR_i = 0$, then node i is designated determinate. Forces present at a determinate node i, referred to as determinate forces, do not participate in the compatibility conditions since such forces can be determined from the nodal equilibrium equations alone. Consequently, for determinate node i, K_i forces along with L_i EE's, which correspond to $L_i = K_i$ displacements, can be dropped simultaneously from the equilibrium matrix [B] to obtain the reduced equilibrium matrix $[B]^{(r1)}$ without affecting the CC's in any manner. Dropping of forces and displacements is also equivalent to the elimination of appropriate columns and rows of the deformation displacement relations. The reduced deformation displacement relation (designated DDR_{r1}) that is obtained after imposing the node determinacy condition has the following form:

$$DDR_{r1} \Rightarrow \{\beta\}^{(r1)} = [B]^{(r1)^T} \{X\}^{(r1)} \tag{14}$$

In equation (14), matrix $[B]^{(r1)T}$ has a dimension of $\{(n - K_i) \cdot (m - K_i)\}$. The deformation vector $\{\beta\}^{(r1)}$ has the dimension $n - K_i$, and displacements

$\{X\}^{(r1)}$ are of dimension $m - K_i$. As expected, the number of CC's contained in the DDR_{r1} given by equation (14) is $r = n - m$ or $r = \{(n - K_i) - (m - K_i\}$, since no CC has yet been generated. The node determinacy condition has reduced the number of deformation displacement relations from n to $n - K_i$; however, the number of compatibility conditions remains the same. The motivation behind dropping determinate variables at the intermediate stage of the generation of the CC is to enhance node determinacy at as many grid points as possible.

For the example of membrane (6,5), observe that the last two of its EE's, given by equation (11), have two determinate forces, F_5 and F_6, and that these correspond to the two displacements X_4 and X_5 at node 4; therefore $K_i = 2$ and $L_i = 2$. The node $i = 4$ is determinate since $NR_i = K_i - L_i = 0$. The reduced DDR_{r1} that is obtained by taking into consideration the determinacy for node 4 has the following explicit form:

$$\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{Bmatrix} = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.707 & -0.707 & 0.707 \\ 0.707 & -0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \tag{15}$$

The number of CC's given by the DDR_{r1} (eq. (15)) still remains one ($r = m - n = 1$), since no CC has yet been generated. The local structure that is obtained after the elimination of the determinate node 4 is shown in figure 4(b). Since node 1 is fixed and node 4 has been dropped, the local ddr_{r1} given by equation (15) corresponds to the DDR of nodes 2 and 3, which represents the interface between elements 1 and 2. The interface DDR has four deformations (β_1, β_2, β_3, and β_4) expressed in terms of three displacements (X_1, X_2, and X_3). The elimination of the three displacements from the four DDR given by equation (15) yields the only CC associated with the interface of the two membrane elements 1 and 2 along their common boundary defined by nodes 2 and 3). It has the following explicit form:

$$(\beta_3 - \beta_4) = 0 \tag{16}$$

The CC given by equation (16) represents the deformation balance condition along the interface of adjoining elements. Such CC's are referred to as the interface CC's. For the membrane model, each interface has one interface CC. However the number of CC's at an interface of any discrete model will depend on the type and number of elements that are connected to the interface. For example, the flexure problem given in the section Benefits Derived from the Compatibility Conditions has three CC's at each interface. The interface CC's represent the majority, though not all of the CC's of a finite element idealization. Generating interface CC's requires that the ddr for the local interface be established and then that the displacements be eliminated from it.

The CC, $[C]\{\beta\} = \{0\}$, for membrane (6,5), which is given by equation (16), has to be expressed in forces, $[C][G]\{F\} = \{0\}$, so that it can be coupled to the EE's (eq. (11)) to obtain the IFM governing equations, $[S]\{F\} = \{P\}^*$. Deformations $\{\beta\}$ are transformed into forces $\{F\}$ by the constitutive relation $\{\beta\} = [G]\{F\}$. Here, the 6×6 matrix $[G]$ is the block diagonal concatenation of element matrices $\{G_e\}$ (see eq. (8b)) for elements 1 and 2. The interface CC ($\beta_3 - \beta_4 = 0$) in terms of forces has the following form:

$$\begin{bmatrix} 0.5 & 0.5 & 1.0 & -1.0 & -0.5 & -0.5 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \{0\} \tag{17}$$

For membrane (6,5) the maximum bandwidth of the CC (MBW_{icc}) is 6. The actual bandwidth of the compatibility matrix [C] ($BW_{actual\ C}$) is 2. The actual CC bandwidth of the composite matrix [C][G] ($BW_{actual\ CG}$) is 6.

The integrated force method solution for membrane (6,5). - For completeness, the solution of membrane (6,5) by the IFM is presented. The IFM governing equation $[S]\{F\} = \{P\}^*$ is obtained by coupling the EE's (eq. (11)) to the CC (eq. (17)) as

$$\begin{bmatrix} 1.0 & 0.0 & 0.7071 & 0.7071 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.7071 & -0.7071 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.7071 & 0.7071 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ \hline 0.5 & 0.5 & 1.0 & -1.0 & -0.5 & -0.5 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} 50.00 \\ 100.00 \\ 50.00 \\ 50.00 \\ 100.00 \\ \hline 0 \end{Bmatrix} \tag{18}$$

The solution to equation (18) yields the six forces. The five displacements are obtained from the forces by back substitution in equation (2) as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} 200.00 \\ 200.00 \\ -168.60 \\ -43.58 \\ 50.00 \\ 100.00 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{Bmatrix} = \begin{Bmatrix} 0.152 \\ -0.126 \\ 0.152 \\ 0.0260 \\ 0.152 \end{Bmatrix} \tag{19}$$

<u>Analysis of the membrane by the stiffness method</u>. - For comparison, the membrane is also analyzed by the stiffness method. The stiffness equations are well known but complicated; therefore, the analysis is carried out in symbolic form. To establish the parallelism between the IFM and the stiffness method, a slightly different procedure than normal is followed; the purpose will become evident in the process of the solution. For the membrane, a displacement vector $\{X_c\}$ of dimension 12 that represents the concatenation of the 2 elemental displacement degrees of freedom (see fig. 4(c)) is defined as:

$$\{X_c\}^T = \langle X_{c1} = X_{1e1}, X_{c2} = X_{2e1}, \ldots, X_{c12} = X_{6e2} \rangle \tag{20}$$

where the subscript iej represents the i^{th} displacement component for the j^{th} element.

Notice the similarities between the displacement vector $\{X_c\}$ of equation (20) and the force vector $\{F\}$ given by equation (9a). These vectors represent the concatenation of the elemental displacements and the elemental force degrees of freedom, respectively. The equilibrium matrix in the stiffness method, in terms of concatenated displacement vector $\{X_c\}$, is obtained by following assembly techniques as

$$\Big[[K_1] : [K_2]\Big]\{X_c\} = \{P\} \tag{21}$$

The stiffness matrix $[K_1]$ has the dimension 3 × 6. Its three rows represent contributions to the EE's at node 2 along X_1, and at node 3 along X_2 and X_3 (see fig. 4(a)). Likewise, the matrix $[K_2]$ has the dimension 5 × 6, which contributes to the equilibrium at node 2 along X_1, node 3 along X_2 and X_3, and node 4 along X_4 and X_5. Equation (21), which represents 5 EE's in terms of 12 unknown displacements cannot be solved for the 12 variables. Seven displacement continuity conditions are required to augment the EE's given by equation (21) to arrive at a solvable 12 × 12 system. The seven displacement continuity conditions of the membrane are as follows:

$$\left.\begin{aligned} X_{1e1} &= 0 \\ X_{2e1} &= 0 \\ X_{3e1} &= X_{3e2} \\ X_{4e1} &= X_{4e2} \\ X_{5e1} &= X_{1e2} \\ X_{6e1} &= 0 \\ X_{2e2} &= 0 \end{aligned}\right\} \tag{22a}$$

The seven displacement continuity conditions can be represented by a single matrix equation

$$[CTY]\{X\} = \{0\} \tag{22b}$$

where the matrix [CTY] is the displacement continuity matrix of the dimension 7 × 12.

The displacement continuity conditions that are given by equation (22) and the EE's in terms of displacements that are given by equation (21) are coupled to obtain the 12 × 12 equation system (eq. (23)) from which the 12 concatenated displacements X_c can be obtained:

$$\begin{bmatrix} [K_1 : K_2] \\ \text{-------} \\ [CTY] \end{bmatrix} \{X_c\} = \begin{Bmatrix} P \\ \text{---} \\ 0.0 \end{Bmatrix} \tag{23}$$

From the displacements, the internal forces can be calculated by back substitutions.

From the structures of IFM equations (eqs. (1) and (18)) and from the equation of the stiffness method (eq. (23)), we observe the following: (1) In the IFM, the EE's, written in terms of forces, are augmented by the CC's, also in terms of forces; (2) in the stiffness method the EE's are expressed in terms of displacements and these EE's are augmented by the displacement continuity conditions; and (3) the IFM equations (eq. (18)) are fewer in number, and also sparser than the stiffness equations (eq. (23)). Details about equation sparsity and the computations required to generate the solution by the IFM and the stiffness method are given in reference 14. The IFM satisfies both the EE's and the CC's simultaneously, whereas the stiffness method is based on the EE's and displacement continuity conditions.

Example II - The Two-Bay Membrane

The generation of both the interface and the cluster CC's are illustrated by using the example of a two-bay membrane shown in figure 5. The membrane is discretized by eight triangular elements, and it has eight nodes. The model, designated "membrane (24,15)," has $n = 24$ force unknowns and $m = 15$ displacement unknowns. Membrane (24,15) has $m = 15$ EE's and $r = n - m = 9$ CC's. Because of the increase in complexity, the algebra for the example is presented in symbolic form. The system equilibrium matrix $[B_m]$ of dimension 15 × 24 for membrane (24,15) is assembled by following standard techniques. The global deformation displacement relations (GDDR) that correspond to the equilibrium matrix $[B_m]$ can be symbolized as

$$\text{GDDR} \Rightarrow \{B\} = [B_m]^T\{X\} \tag{24}$$

The GDDR (given by eq. (24)) contains 9 CC's, since its 24 deformations are expressed in terms of 15 displacements. The model shown in figure 5 has eight interelement boundaries, defined by nodes 1 and 5, and nodes 5 and 8, and so on. From example I, we know that each interface has one CC; therefore, the eight interfaces yield eight CC's, which can be generated by following both parts of step I, explained in the following paragraphs. Steps I(a), I(b), II, and III, are parts of the general procedure to generate the CC's. For clarity, the steps are explained by using the example of membrane (24,15) as an illustration.

Step I(a) - Local structure and interface compatibility conditions. - Consider any interface - for example, the interface along nodes 1 and 5 (between elements 1 and 2) for membrane (24,15). Separate the interface and the elements as shown in figure 6(a). The local structure shown therein is statically unstable; therefore impose adequate numbers of restraints to make the local structure kinematically stable. The restraints do not influence the CC's, and they can be imposed at any of the nodes of the local structure. The structure requires 2 restraints, which are imposed at node 5. The stable local structure (6,5) has one interface CC, designated as CC_{11}. The deformation displacement relation for the local structure, designated ddr_{L1}, is extracted from the global GDDR. The local ddr_{L1} consists of six deformations (β_1, β_2, . . . , β_6) expressed in terms of five displacements (X_1, X_4, X_5, X_6, and X_7). The single compatibility condition in ddr_{L1}, is generated by following the procedure given for example I, and it turns out to be $\beta_3 - \beta_4 = 0$.

Step I(b) - Update the global deformation displacement relations. - The number of GDDR of the structure (n,m) is reduced after the generation of each CC. The reduced GDDR, which is designated $GDDR_{r1}$, has $m_1 = m$ rows and $n_1 = n - n_{c1}$ columns, where n_{c1} represents the number of CC's generated in step I(a). For the example in step I(a), n_{c1} is 1 since only one CC was generated in Step I(a). The row and column dimensions of $GDDR_{r1}$ are $n_1 = (24 - 1) = 23$ rows and $m_1 = 15$ columns. The $GDDR_{r1}$ contains eight CC's since only one out of $r = 9$ CC's of structure (24,15) has been generated. The $GDDR_{r1}$ is obtained by dropping one deformation displacement relation from the GDDR. Any deformation that has participated in the CC generated in step I(a) can be dropped. For the example, deformation β_3 or β_4 can be dropped.

Step II - Local structure and its interface compatibility conditions. - The local structure consisting of elements 2 and 3 is separated next, and its interface CC's along nodes 2 and 6 are generated by following steps I(a) and I(b). Steps I and II are repeated until all the interface CC's have been generated. For this problem there are eight interface CC's at the end of whose generation the $GDDR_{r8}$ will have 16 rows and 15 columns containing 1 CC.

Steps I and II are sufficient to generate all the interface CC's, which are the most numerous CC's in all finite element models. Since the interface CC depends on the few elements that are common to the interface, the computation time required for the generation of such CC's after the equilibrium matrix has been established is insignificant compared to the total solution time.

Step III - Cluster compatibility conditions. - In a finite element model, a cluster is defined as a series of adjoining elements. The cluster compatibility conditions represent constraints on the deformations of the elements that belong to the cluster. A cluster can be generated for any element. For membrane (24,15), take element (4). The cluster for this element, shown in figure 6(b), consists of four elements, 2 to 4 and 8, and six nodes, 1 to 3, 5, 6, and 9. Let us designate its deformation displacement relation as ddr_{CL1}. The cluster is stable, so there is no need to impose any of the restraints indicated in step I(a). If the cluster was unstable, then it would be necessary to impose the additional constraints indicated in step I(a). The ddr_{CL1} of the cluster contains nine deformations, β_4, β_6, β_9 to β_{12}, and β_{22} to β_{24} (note that β_5, β_7, and β_8 have been eliminated during the generation of the interface CC), which are expressed in terms of nine displacements, X_1 to X_3,

X_6 to X_9, X_{14}, and X_{15}. The cluster, which has nine deformations expressed in nine displacements is determinate and contains no additional CC's.

The second cluster that is defined for element 2 is shown in figure 6(c). The ddr_{CL2} of the second cluster contains 7 elements and 14 deformations (since 7 deformations have been dropped during the generations of the 7 interface CC's belonging to the cluster), expressed in terms of 13 displacements. The cluster contains one CC, which is obtained by following step I(a):

$$\{0.447(\beta_6 - \beta_2 + \beta_8 - \beta_{20}) - \beta_{16} - \beta_{12} + 0.894(\beta_1 - \beta_5 + \beta_{17} - \beta_{23})\} = 0 \tag{25}$$

Step I(b) should be exercised to reduce the $GDDR_{C12}$. The $GDDR_{C12}$ has 13 deformations expressed in terms of 13 displacements, and it contains no CC's, thereby indicating that all 9 CC's (8 interface CC's and 1 cluster CC) have been generated for membrane (24,15).

Example 3 - The Stiffened Membrane

The generation of external CC's and interface CC's when different element types are present in the discretization is illustrated by taking the example of the stiffened membrane shown in figure 7(a). This membrane is discretized by 8 membrane elements and 16 bar elements. Nodes 1 and 3 of the membrane are fully restrained. The fof of the membrane ($n = 40$) consists of the 24 membrane forces and 16 bar forces. The dof of the membrane is $m = 14$. The membrane is designated "membrane (40,14);" it has 14 EE's and $r = (40 - 14) = 36$ CC's. The global deformation displacement relation for the structure, designated as $GDDR_{sm}$ and consisting of 40 deformations expressed in terms of 14 displacements, is assembled by following standard techniques.

Interface compatibility conditions of membrane (40,14). - The first interface, which is defined by nodes 1 and 5, and associated membrane elements 1 and 2, and bars 9 to 12 and 16, is considered. One boundary constraint, the displacement at node 2, is imposed for its overall stability as shown in figure 7(b). This local structure has $n_{L1} = 11$ fof consisting of six membrane forces and five bar forces, and $m_{L1} = 5$ dof. The local structure, shown in figure 7(b), is designated as SML_{L1} (11,5); its six ($r_{L1} = 6$) interface compatibility conditions, which can be generated by following step I, are:

$$(\beta_3 - \beta_4) = 0 \tag{26}$$

$$\left.\begin{aligned} \beta_1 - \beta_9 &= 0 \\ \beta_2 - \beta_{16} &= 0 \\ \beta_3 - \beta_{10} &= 0 \\ \beta_5 - \beta_{12} &= 0 \\ \beta_6 - \beta_{11} &= 0 \end{aligned}\right\} \tag{27}$$

The CC given by equation (26) represents the membrane-to-membrane interface compatibility condition, and those given by equation (27) represent the five membrane-to-bar interface CC's.

Steps I and II are repeated for interfaces 4 and 5, 4 and 8, 5 and 8, 5 and 9, 5 and 6, 2 and 6, and 2 and 5 to generate, respectively, 3, 3, 3, 3, 2, 3, and 1 additional local interface CC's. After the generation of the 24 interface compatibility conditions, the reduced deformation displacement relation has 2 CC's and consists of 16 deformations that are expressed in terms of 14 displacements.

Cluster and external compatibility conditions of membrane (40,14). - Step III is evoked and yields two compatibility conditions: one is a cluster CC identical to the CCC of membrane (24,15) as given by equation (25), and the other $\beta_4 + \beta_{10} = 0$, which represents a constraint on the deformations between boundary nodes 1 to 3, is the external compatibility condition of the membrane. Generation of the external CC is further explained with the example of a bridge truss.

External compatibility conditions of a bridge truss. - A bridge truss supported at two nodes that are far apart (see fig. 8(a)) illustrates the generation of external CC's. The bridge truss being analyzed can be designated as truss (26,20); it has 20 EE's and 6 CC's, and its global GDDR has 26 deformations expressed in terms of 20 displacements. Skeletal structures such as trusses and frames do not have any interface CC's; their CC's can be either cluster or external ones.

Cluster compatibility conditions of the truss (26,20). - The cluster for element 1, consisting of six bars, is shown in figure 8(b). The unstable cluster is made stable by imposing a constraint at node 3. The cluster, designated "bay (6,5)," has one CC, which is obtained by following step I. At this time any one bar, for example bar 1 that corresponds to deformation β_1, is dropped; the resulting bay (5,5) is shown in figure 8(c). Steps I and II are repeated until all five cluster CC's are generated and the structure shown in figure 8(d) is obtained.

External compatibility conditions. - The reduced structure shown in figure 8(d) has 1 CC since its ddr consists of 21 bar deformations expressed in terms of 20 displacements. The single CC is obtained by first imposing a node determinacy condition that reduces the ddr to six deformations expressed in five displacements and then by eliminating the displacements. The external CC ($\beta_2 + \beta_{10} + \beta_{15} + \beta_{20} + \beta_{25} = 0$) represents a homogeneous constraint on all the deformations between boundary nodes 1 and 12, as shown in figure 8(e). The bandwidths of the external CC's for restrained nodes that are far apart, typically encountered in long span bridges, can be large from physical considerations, since the deformations between fully constrained node of a truss that are far apart have to be zero $\left(\sum_{i=1}^{5} \beta_{j(i)} = 0 \right)$. The larger bandwidths of few external CC's do not impose any major problem because in the IFM the solution process is carried out by using sparse matrix techniques. Quite often such external CC's can be trivially generated by mere inspection.

Uniqueness of Compatibility Matrix

The compatibility conditions are homogeneous equations; therefore the CC's can be multiplied by any nonsingular matrix, for example $[R_u]$, to obtain the feasible compatibility matrix $[C_u]$ given by equation (28). The feasible matrix $[C_u]$ is a linear combination of the rows of matrix [C]:

$$[R_u][C]\{\beta\} = \{0\} \qquad \text{or} \qquad [C_u]\{\beta\} = \{0\} \tag{28}$$

The procedure presented in this paper generates the matrix [C] and not $[C_u]$. This can be proved by observation. Take the example of the two membrane interface CC's (CC_1 and CC_2) defined as:

$$CC_1 \Rightarrow \beta_{k1} - \beta_{k2} = 0 \tag{29a}$$

$$CC_2 \Rightarrow \beta_{k3} - \beta_{k4} = 0 \tag{29b}$$

A linear combination of CC_1 and CC_2 yields a feasible compatibility condition CC_3 as

$$CC_3 \Rightarrow \beta_{k1} - \beta_{k2} + \beta_{k3} - \beta_{k4} = 0 \tag{29c}$$

Notice that in equation (29c), deformations of CC_1 such as β_{k1} and β_{k2} and deformations of CC_2 such as β_{k3} and β_{k4} are present. However, after the generation of CC_1, one of the two deformations β_{k1} or β_{k2} must be dropped (step I(b)); therefore their combination cannot occur in subsequent CC's. In other words, after CC_1 has been generated, the feasible compatibility condition CC_3, which includes CC_1 cannot be obtained. Dropping a deformation that has participated in the CC, immediately after its generation, avoids the possibility of its further participation in any other CC. The process generates the matrix [C] and not its combinations such as the usable CC matrix $[C_u]$. Therefore the unique [C] matrix is generated.

Product of Compatibility and Equilibrium Matrices - a Null Matrix

The product of the CC matrix [C] and [EE] matrix [B] is a null matrix ($[C][B]^T = [0]$). This can be verified by direct substitution of the DDR in the CC's as

$$[C]\{\beta\} = \{0\} \tag{30a}$$

and

$$\{\beta\} = [B]^T\{X\} \tag{30b}$$

Next, eliminate deformations in favor of displacements between equations (30a) and (30b) to obtain

$$[B]^T[C] = [C]^T[B] = [0] \tag{30c}$$

The null product property of the two fundamental EE and CC operators implies that errors in equilibrium equations can propagate to the compatibility conditions and vice versa.

BENEFITS DERIVED FROM COMPATIBILITY CONDITIONS

The accuracy of solutions obtained by the integrated force method, the stiffness method, or the hybrid method is of paramount importance, since all are approximate formulations. All methods attempt to satisfy the equilibrium equations written in terms of forces or displacements. However, explicit compatibility conditions, in a strict sense, are imposed only in the integrated force method. Any improved solution accuracy, of the integrated force method over the other formulations should be a consequence of the explicit presence of the global compatibility conditions in the IFM. Based on the theory of the integrated force method, a Generalized Integrated Force Technique (GIFT) computer code has been developed. To illustrate the solution accuracy in finite element calculations, a plate bending problem is solved by the GIFT code and other standard analyzers such as MSC/NASTRAN (ref. 19), ASKA stiffness codes (ref. 20), a mixed formulation MHOST (ref. 21), and Chang's hybrid method (ref. 22).

The plate parameters considered (see fig. 9) are the following: size of the plate; $a = b = 40$ in. (101.6 cm); aspect ratio a/b, varied between 1 and 2; thickness of plate, $h = 0.2$ in. (5.08 mm); Young's modulus, $E = 30\ 000$ ksi (21 091.81 kg/mm^2); Poisson's ratio, $\nu = 0.3$; and magnitude of concentrated load at the center, $P = 500$ lb (226.795 kg). Both simply supported and clamped boundary conditions are considered.

To compare solution accuracy, the problem is solved by using two types of elements: a four-node rectangular element and a three-node triangular element. The IFM elements assume three forces (such as a shear force and two bending moments) and three displacements (a transverse displacement and two slopes) per node, as depicted in figure 9. A cubic polynomial with 12 constants is used to approximate the transverse displacement in the element field. Normal moments M_x and M_y are assumed to have linear distributions, and the twisting moment M_{xy} is constant in the element domain (ref. 14).

The elements of the general purpose programs NASTRAN, ASKA, and MHOST are specialized to generate only the flexure solution. The elements used are the following: (1) QUAD4 (both ASKA and MSC/NASTRAN have QUAD4 elements), (2) TRIB3 (triangular element of ASKA program), (3) TRIA3 (triangular element of MSC/NASTRAN), and (4) TUBA3 of ASKA code (a higher order triangular element with six dof per grid point). For the first three elements, which are well known in the literature and popular in practice, the bending response of the elements represents three dof per node.

The hybrid elements have more unknowns, for example, for flexural response. Chang's program has the equivalent of seven unknowns at the nodes, whereas the mixed formulation MHOST has more unknowns per grid point. The IFM elements and the stiffness elements (such as QUAD4 of MSC/NASTRAN, QUAD4 of ASKA, TRIB3 of ASKA, and TRIA3 of MSC/NASTRAN) are equivalent with respect to their nodal degrees of freedom. The hybrid and TUBA3 elements are higher order elements than those of the IFM.

In the stiffness method, nodal stress parameters that are calculated from displacements by back substitution are discontinuous and ambiguous at grid points (ref. 23); therefore calculation of forces at the nodes are routinely avoided in the stiffness method. In this situation the noncontroversial nodal displacement is used in the comparison. Remember however that in the IFM, forces are the primary variables from which the secondary displacement unknowns are obtained by back substitution. The central deflection of the plate w_c, given by Timoshenko (ref. 24), is 0.2036 in. (5.715 mm).

MacNeal (ref. 25) introduced a grading scheme for the evaluation of finite elements as follows:

A <2-percent error
B 2- to 10-percent error
C 10- to 20-percent error
D 20- to 50-percent error
F >50-percent error

The results obtained by all four formulations were graded by using MacNeal's scheme and are presented in tables I to III. Results obtained by the IFM and the stiffness formulations are also presented graphically in figures 10 and 11. The IFM results for simply supported and clamped boundary conditions for different aspect ratios are presented in table IV. From the numerical results of the plate flexure problem presented in tables I to IV and figures 10 and 11, we observe the following:

(1) For the IFM rectangular element, convergence occurs for the first model, which consists of four elements. If symmetry is taken into consideration, then convergence occurs for a single-element idealization. Both stiffness (MSC/NASTRAN and ASKA) and hybrid methods (MHOST and Chang's) converge slowly. To achieve an A grade, MSC/NASTRAN QUAD4 element idealization requires 36 elements, whereas ASKA QUAD4 secures only a B grade, even for 100 elements. The hybrid method of Chang secures an A grade for 64 elements, whereas MHOST secures a B grade for the same level of discretization.

(2) For the IFM triangular elements (see fig. 11), the result is discernible from an analytical solution for the first model, which has four elements, but even so, the result displays engigeering accuracy. The next model, with eight elements, converges to the analytical solution and also achieves an A grade. None of the stiffness elements, such as TRIA3 of MSC/NASTRAN and TRIB3 and TUBA3 of ASKA, could secure a grade of A, even for models with fine discretization.

(3) The IFM result for a simply supported boundary follows the pattern of a clamped boundary; namely, it secures a grade of A for the first model, which has four elements. The IFM element retains an A grade for aspect ratios up to 1.6, but for the ratio 2.0, a total of eight elements, which corresponds to a 2 × 4 mesh, is required to secure an A grade. Other examples more or less follow the pattern depicted in tables I to IV. Overall the IFM convergence rate is very fast whereas both the stiffness and hybrid methods converge slowly or struggle to do so.

CONCLUSIONS

1. The structural mechanics profession recognizes that both equilibrium equations (EE's) and compatibility conditions (CC's) are essential for stress analysis. However, the compatibility conditions in typical finite element calculations were promoted via such concepts as cutting and closing the gaps, or

displacement matching at nodes (deflection or slope should have unique values) and so on. Although such concepts are somewhat related to the CC's of finite element models, they do not represent the true CC's that are analogous to the strain formulation of St. Venant. The true compatibility conditions of finite element analysis have been understood to a great extent, though we do not claim that the understanding is total. Attempts should be made by the profession to understand the CC's in totality, rather than to avoid them because they are mathematically formidable or analytically more difficult than the familiar equilibrium equations.

2. In finite element analysis, the system equilibrium equations in terms of forces or displacements can be assembled from element matrices. The question is, can such an assembly technique be developed for the compatibility conditions also? The generation of compatibility conditions is not equivalent to the direct-assembly technique of the finite element analysis, even though there is a close resemblance in that the global compatibility conditions are assembled from their local counterparts such as interface CC's, cluster CC's, and external CC's. We do not yet know a direct assembly scheme, but such a possibility has not altogether been ruled out. We do, however, believe that the compatibility generation scheme given in this paper is rather elegant, since element characteristics, connectivities, and such, which are already contained in the equilibrium equations (and consequently in the deformation displacement relation because it is the raw ingredient of CC's) are referred to only once.

3. The compatibility conditions are unique. The computation time required to generate the CC's is a small fraction of the total solution time.

4. The quality of the solution in approximate methods is dependent on the extent to which equilibrium equations and compatibility conditions are satisfied. Since the integrated force method (IFM) satisfies both the EE's and the CC's simultaneously, the solution via the IFM is accurate, as expected. The stress parameters obtained by the stiffness and hybrid methods do not satisfy the EE's even at the cardinal grid points; therefore solution quality by such methods is prone to be poor in comparison to IFM results.

5. Since all the finite element formulations are approximate in nature, we recommend generating solutions via the integrated force method, and by the stiffness method and then comparing them, rather than qualifing the results obtained by any one formulation by successive mesh refinements.

APPENDIX - SYMBOLS

a,b	dimensions of rectangular element
[B]	equilibrium matrix of dimension $m \times n$
$[B]_e$	element of equilibrium matrix
$[B]_m$	system equilibrium matrix
[C]	compatibility matrix of dimension $r \times n$
CC	compatibility condition
CCC	classical compatibility condition
[CTY]	displacement continuity matrix
ddr	local deformation displacement relation
dof	displacement degrees of freedom
E	Young's modulus
EE	equilibrium equations
{F}	force vector of dimension n
fof	force degrees of freedom
[G]	flexibility matrix of dimension $n \times n$
$[G]_e$	element flexibility matrix
GCC	global compatibility conditions
GDDR	global deformation displacement relations
IFM	integrated force method
[J]	deformation coefficient matrix of dimension $m \times n$
JT	total number of elements at interface
JTC	total number of elements in cluster
JTE	total number of elements between two boundary nodes
[K]	stiffness matrix of dimension $m \times m$
ℓ_{ij},m_{ij}	direction cosines along edge ij
MBW	maximum bandwidth
M_x,M_y,M_{xy}	plate bending moments

m	number of displacement variables
NR_i	indeterminancy of node i
n	number of force variables
$\{P\}$	load vector of dimension m
r	number of compatibility condition; $(n - m)$
$[S]$	IFM governing matrix of dimension $n \times n$
SDR	strain displacement relations
t	membrane thickness
u,v	membrane displacement components
w_c	central deflection of plate
$\{X\}$	displacement vector of dimension m
$\{X_c\}$	concatenated displacement vector
$\{\beta\}$	deformation vector of dimension n
β_k	k^{th} deformation component
$\gamma_{xy},\varepsilon_x,\varepsilon_y$	strain components
ν	Poisson's ratio

Subscripts:

ccc	cluster compatibility conditions
ej	element j
ecc,ext	external compatibility conditions
icc	interface compatibility conditions
q	quadrilateral
t	triangular
0	initial

Superscripts:

r	number of compatibility conditions
rl	reduced
T	transpose

REFERENCES

1. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity. Dover, New York, 1944.

2. Patnaik, S.N.; and Joseph, K.T.: Generation of the Compatibility Matrix in the Integrated Force Method. Comput. Methods Appl. Mech. Eng., vol. 55, 1986, pp. 239-257.

3. Patnaik, S.N.; and Joseph, K.T.: Compatibility Conditions from Deformation Displacement Relations. AIAA J., vol. 23, 1985, pp. 1291 - 1293.

4. Nagabhusanam, J.; and Patnaik, S.N.: General Purpose Programme to Generate Compatibility Matrix in the Integrated Force Method. AIAA J., to be published.

5. Patnaik, S.N.: An Integrated Force Method for Discrete Analysis. Int. J. Numer. Methods Eng., vol. 6, no. 7, 1973, pp. 237 - 251.

6. Patnaik, S.N. and Yadagiri, S.: Design for Frequency by the Integrated Force Method in Structural Vibration Analysis. Comput. Methods Appl. Mech. Eng., vol. 16, Nov. 1978, pp. 213 - 230.

7. Patnaik, S.N.; and Yadagiri, S.: Frequency Analysis of Structures by the Integrated Force Method. J. Sound Vib., vol. 83, July 8, 1982), pp. 93 - 109.

8. Patnaik, S.N.: Variational Energy Formulation for the Integrated Force Method. AIAA J., vol. 24, Jan. 1986, pp. 129 - 137.

9. Patnaik, S.N.: Integrated Force Method Versus the Standard Force Method. Comput. Struct., vol. 22, no. 2, 1986, pp. 151 - 163.

10. Patnaik, S.N.: Behavior of Trusses with Stress and Displacement Constraints. Comput. Struct., vol. 22, no. 4, 1986, pp. 619 - 623.

11. Patnaik, S.N.; and Gallagher, R.H.: Gradients of Behavior Constraints and Reanalysis Via the Integrated Force Method for Structural Analysis. Int. J. Numer. Methods Eng., vol. 23, Dec. 1986, pp. 2205-2212.

12. Patnaik, S.N.; and Nagraj, M.S.: Analysis of Continuum by the Integrated Force Method. Comput. Struct., vol. 26, no. 6, 1987), pp. 889-905.

13. Vijayakumar, K.; Krishna Murty, A.V.; and Patnaik, S.N.: Basis for the Analysis of Solid Continuum by the Integrated Force Method. AIAA J., vol. 26, May 1988, pp. 626-629.

14. Patnaik, S.N.; Berke, L.; and Gallagher, R.H.: Integrated Force Method Versus Displacement Method For Finite Element Analysis. NASA TP-2937, to be published.

15. Patnaik, S.N.: Analytical Initial Design for Optimization by the Integrated Force Method. Comput. Struct., to be published.

16. Patnaik, S.N.; and Satish, H.: Analysis of Continuum using the Boundary Compatibility Conditions of Integrated Force Method. Comput. Struct., to be published.

17. Sokolnikoff, I.S.: Mathematical Theory of Elasticity. McGraw Hill, 1956.

18. Przemieniecki, J.S.: Theory of Matrix Structural Analysis. McGraw Hill, 1968.

19. Merrard, A.V., ed: MSC/NASTRAN. The MacNeal-Schwendler Corp., Los Angeles, CA, 1981.

20. Argyris, J.H.: ASKA - Automatic System for Kinematic Analysis. Nucl. Eng. Des., vol. 10, no. 4, Aug. 1969, pp. 441-455.

21. Nakazawa, S.: MHOST Version 4.2 Volume 1: Users Manual. (MARC Analysis Research Corp.; NASA Contract: NAS3-23697) NASA CR-182235-VOL-1, 1989.

22. Saleeb, A.F.; and Chang, Y.T.: An efficient Quadrilateral Element for Plate Bending Analysis. Int. J. Numer. Methods Eng., vol. 24, June 1987, pp. 1123 - 1155.

23. Stress Recovery at Grid Points. MSC/NASTRAN Application Manual, McNeal-Schwendler Corp., Los Angeles, CA, 1982, Section 2.2.

24. Timoshenko, S.: Theory of Plates and Shells. McGraw Hill, 1959.

TABLE I. - REPORT CARD FOR RECTANGULAR ELEMENT WITH CLAMPED BOUNDARY CONDITION

Number of elements for full plate (mesh)	IFM	MSC/NASTRAN QUAD4	ASKA QUAD4
4 (2×2)	A	F	F
16 (4×4)	A	B	B
36 (6×6)	A	A	↓
64 (8×8)	-	A	↓
100 (10×10)	-	-	↓

TABLE II. - REPORT CARD FOR INTEGRATED FORCE METHOD WITH RECTANGULAR ELEMENT

Number of elements for full plate (mesh)	Aspect ratio	Clamped boundary	Simply supported boundary
4 (2×2)	1.00	A	A
16 (4×4)	1.00	A	A
4 (2×2)	1.20	A	-
4 (2×2)	1.40	A	-
4 (2×2)	1.60	A	-
4 (2×2)	1.80	B	-
4 (2×2)	2.00	B	-
8 (2×4)	2.00	A	-

TABLE III. - REPORT CARD FOR RECTANGULAR ELEMENT

Number of elements for full plact (mesh)	IFM	Mixed method MHOST	Hybrid method Chang (ref. 22)
4 (2×2)	A	F	F
16 (4×4)	A	C	C
64 (8×8)	A	B	A

TABLE IV. - REPORT CARD FOR TRIANGULAR ELEMENT

Number of elements for full plate	IFM	MSC/NASTRAN TRIA3	ASKA TRIB3	ASKA TUBA3
4	B	F	-	F
8	A	D	F	F
16	A	C	-	-
32	-	B	C	D
128	-	-	B	B

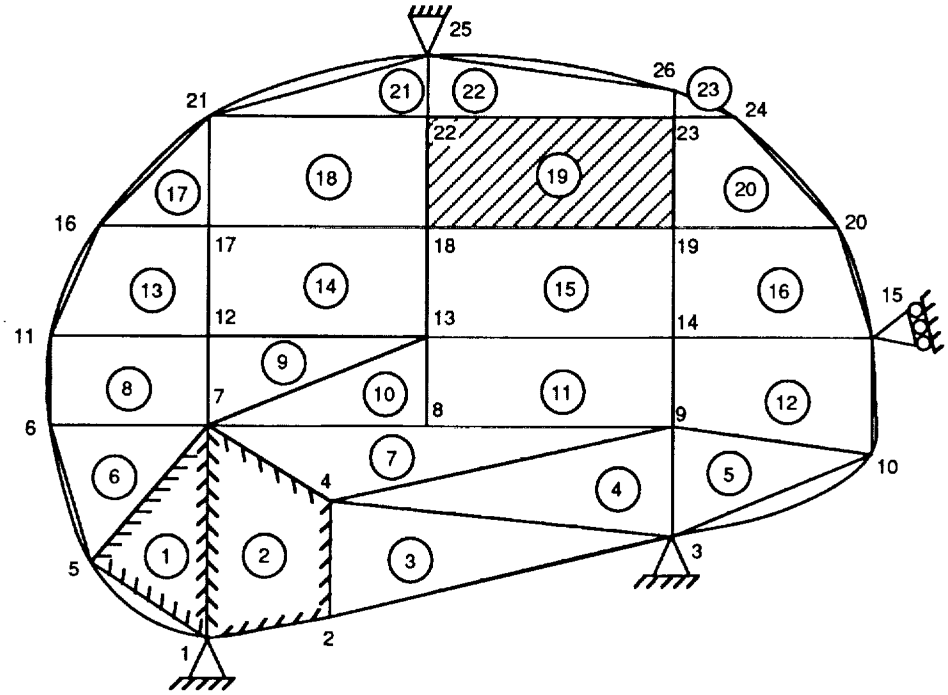
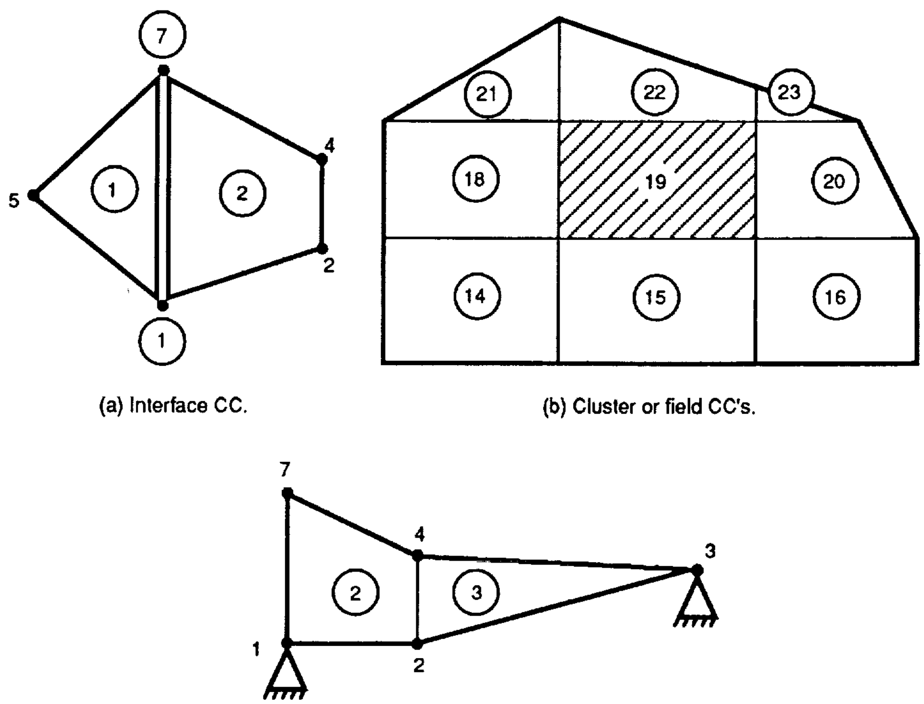


Figure 1. - Finite element model.

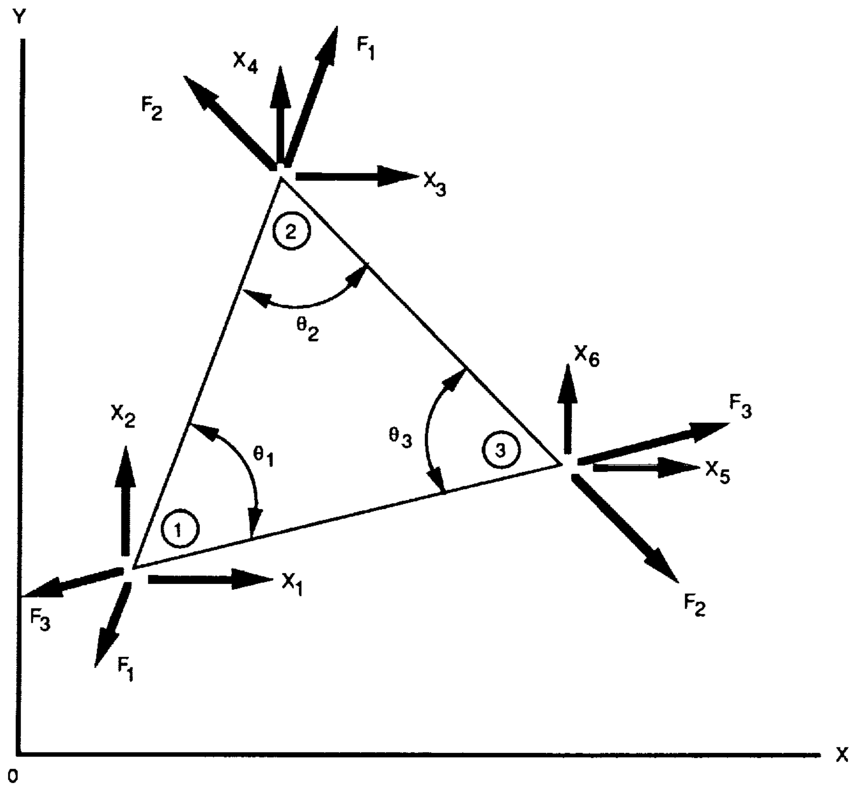


(a) Interface CC.

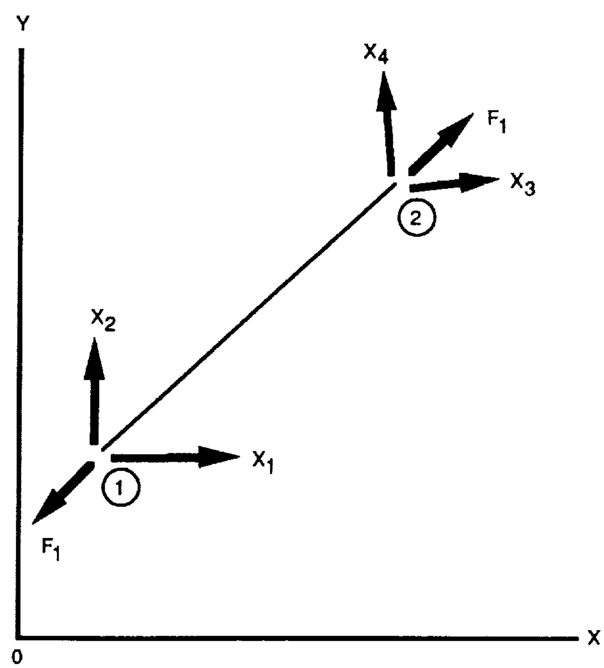
(b) Cluster or field CC's.

(c) External CC's.

Figure 2. - Bandwidth of compatibility conditions (CC's).

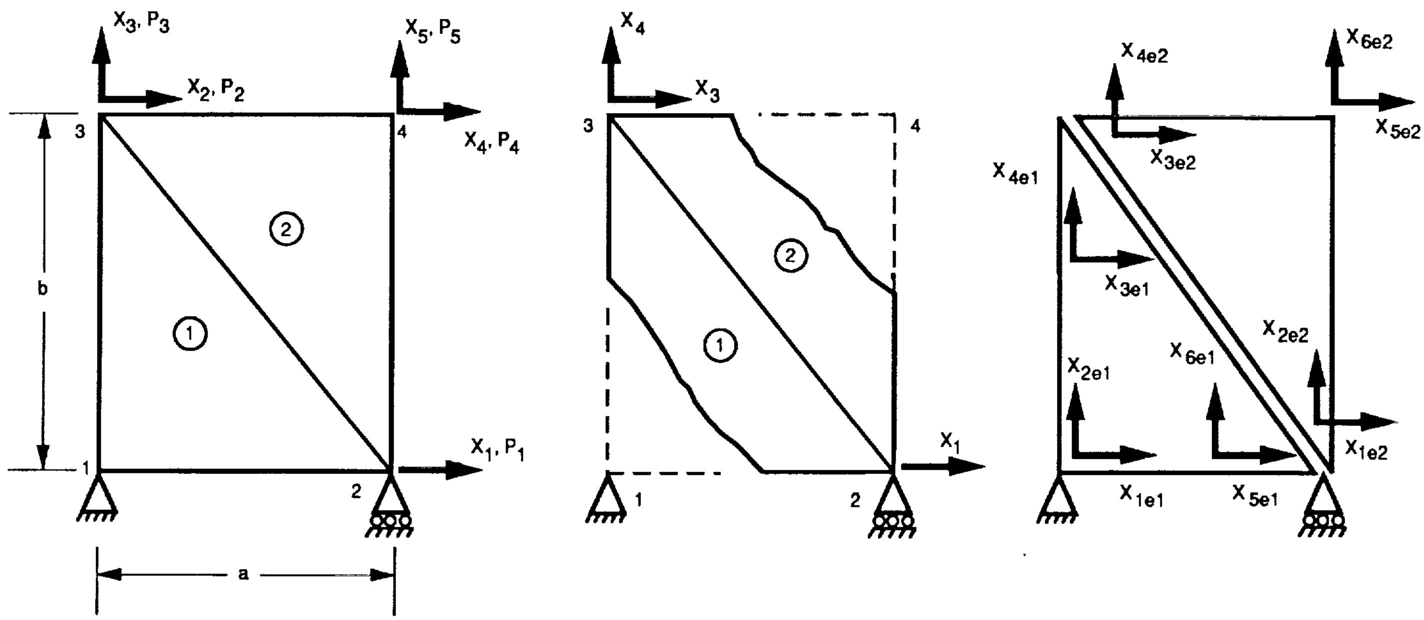


(a) Membrane element; F_1, F_2, and F_3 are force degrees of freedom (fof) and X_1, X_2, . . ., X_6 are displacement degrees of freedom (dof).



(b) Bar element; F_1 is fof and X_1, X_2, X_3, and X_4 are dof.

Figure 3. - Membrane and bar elements.



(a) Membrane.

(b) Interface nodes 2 and 3.

(c) Conatenated model.

Figure 4. - Analysis of membrane.

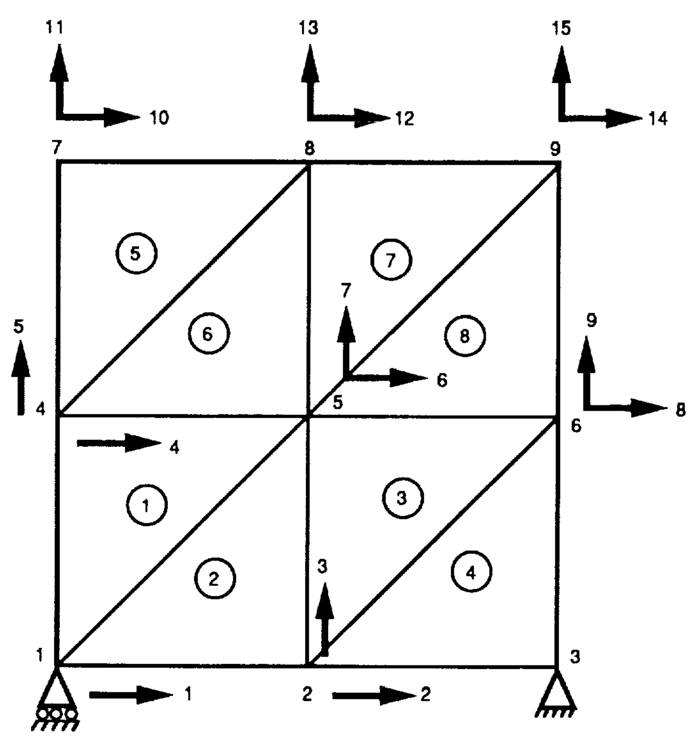
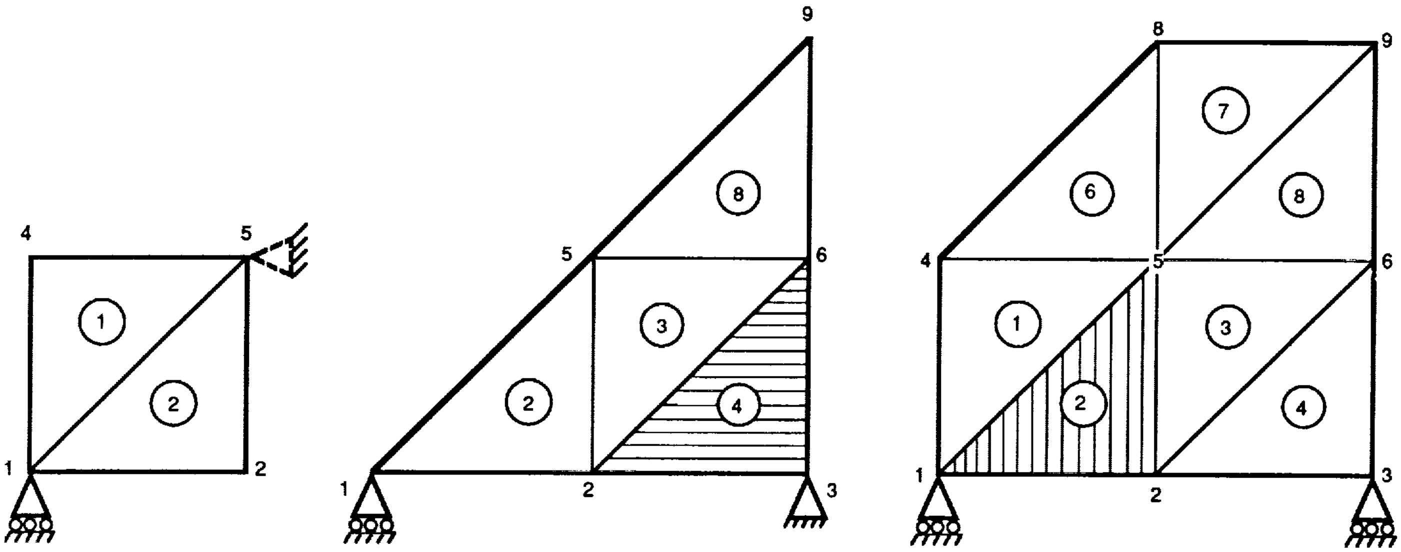


Figure 5. - Two-bay membrane.

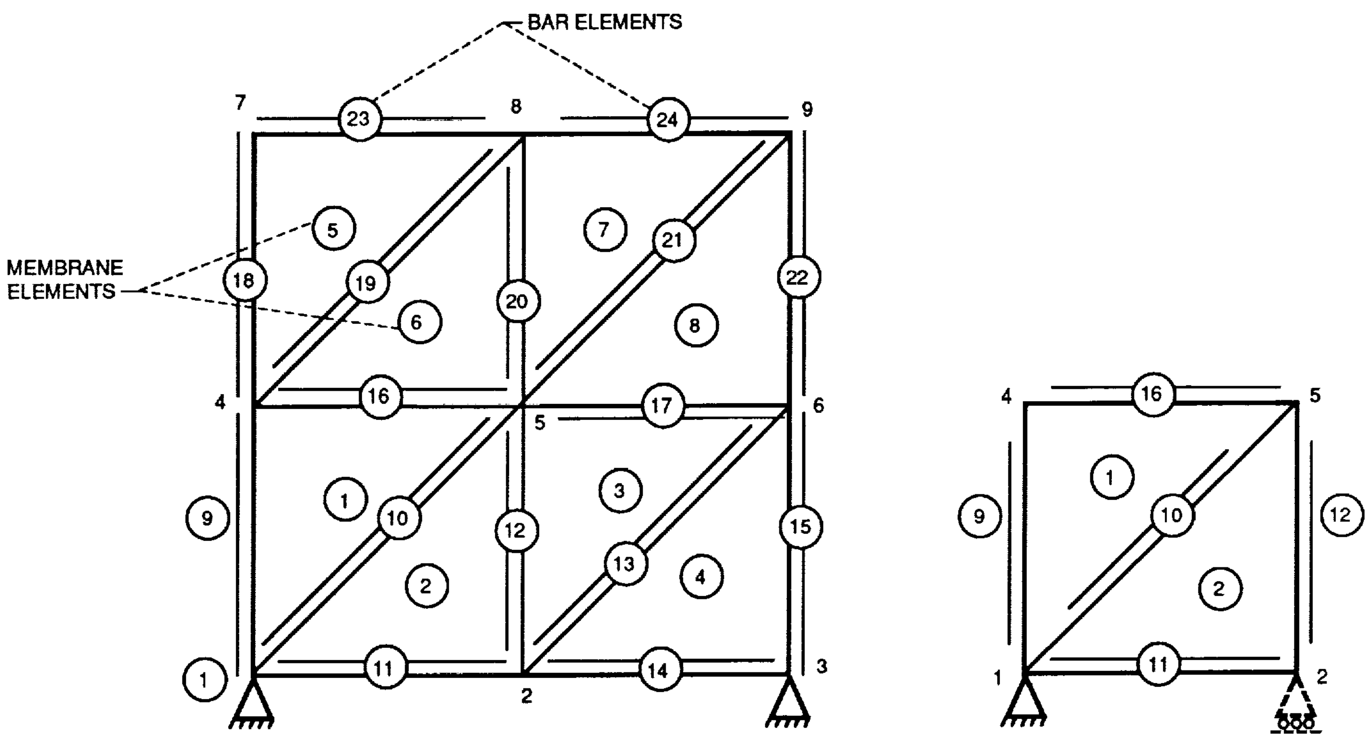


(a) Interface CC.

(b) Determinate cluster.

(c) Indeterminate cluster.

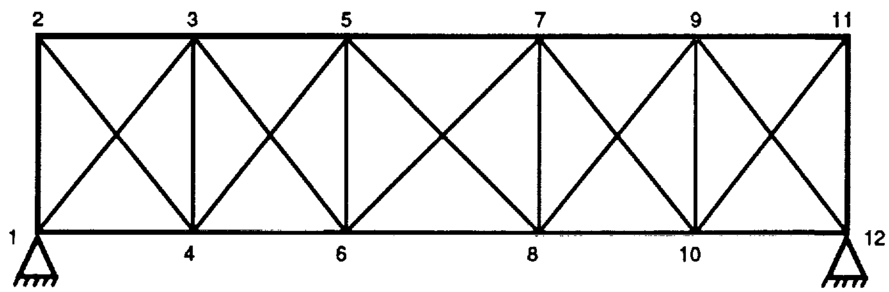
Figure 6. - Compatibility conditions for two-bay membrane.



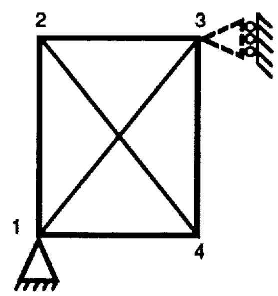
(a) Model with 8 membrane elements and 16 bar elements.

(b) Interface CC.

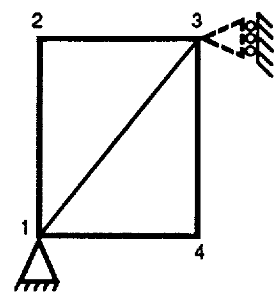
Figure 7. - Composite (membrane and bar) structure.



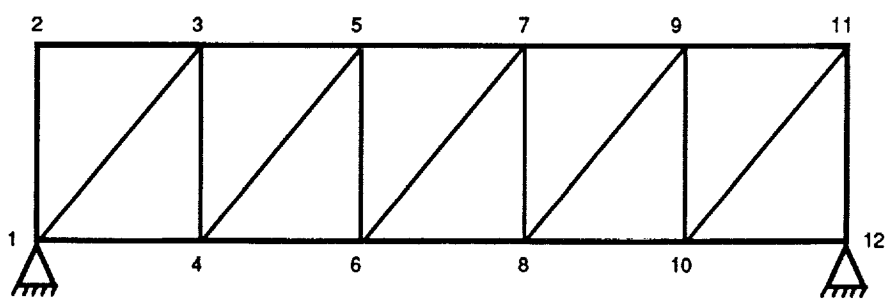
(a) Bridge truss.



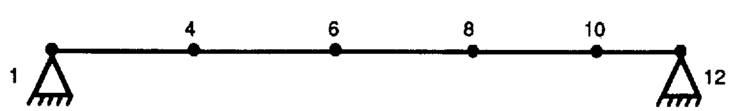
(b) Indeterminate cluster.



(c) Determinate cluster.

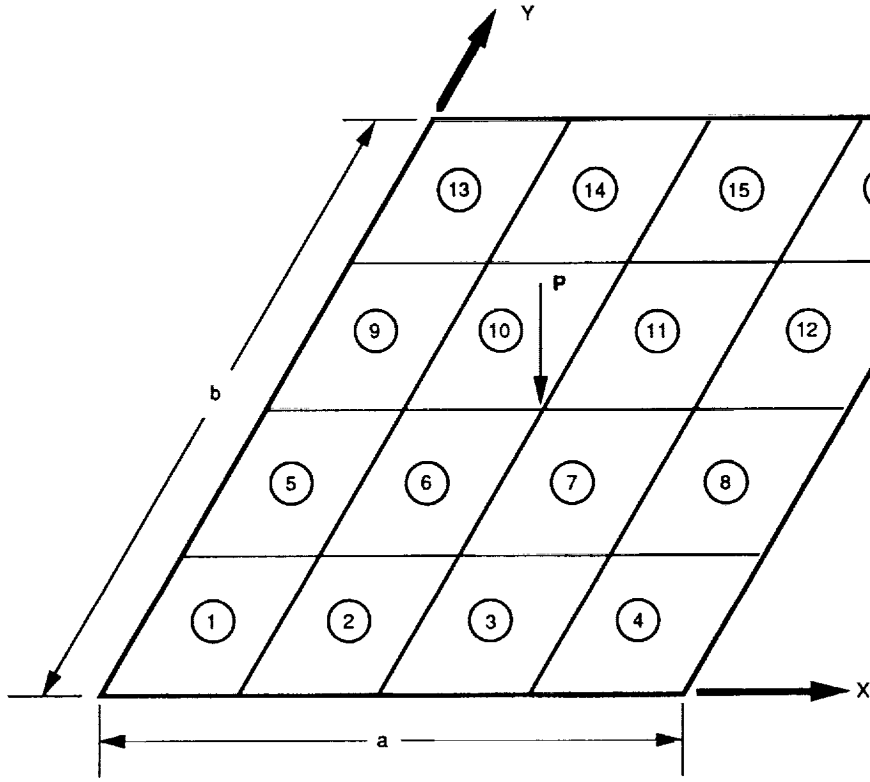


(d) Reduced truss structure.

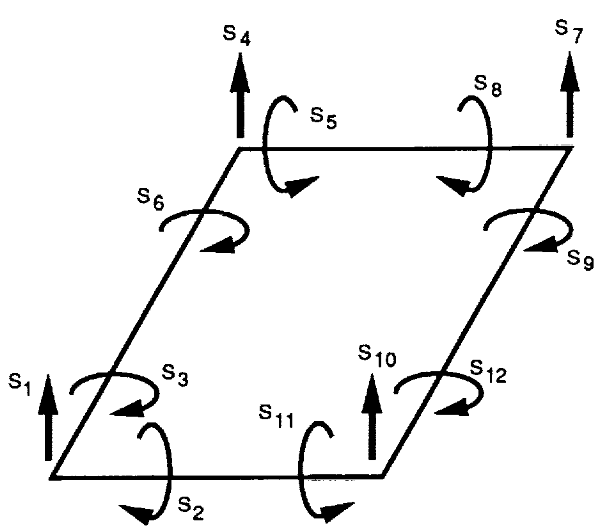


(e) External CC's.

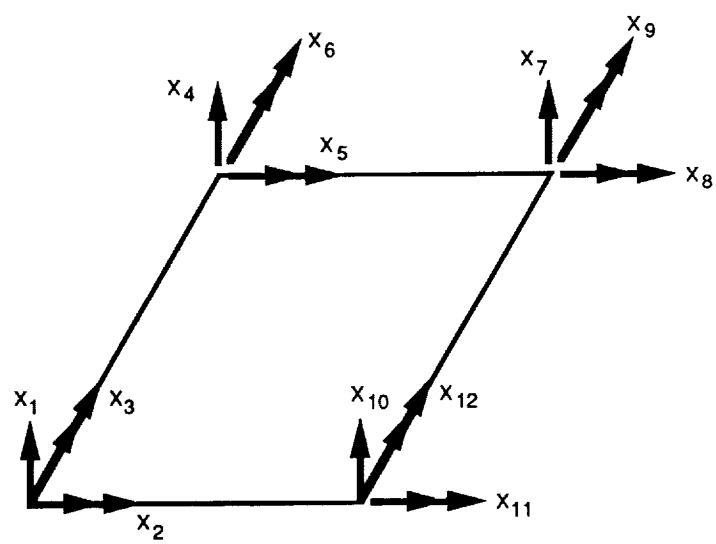
Figure 8. - Compatibility conditions of a bridge truss.



(a) Model with 16 elements.



(b) Elemental forces.



(c) Elemental displacements.

Figure 9. - Rectangular plate in flexture.

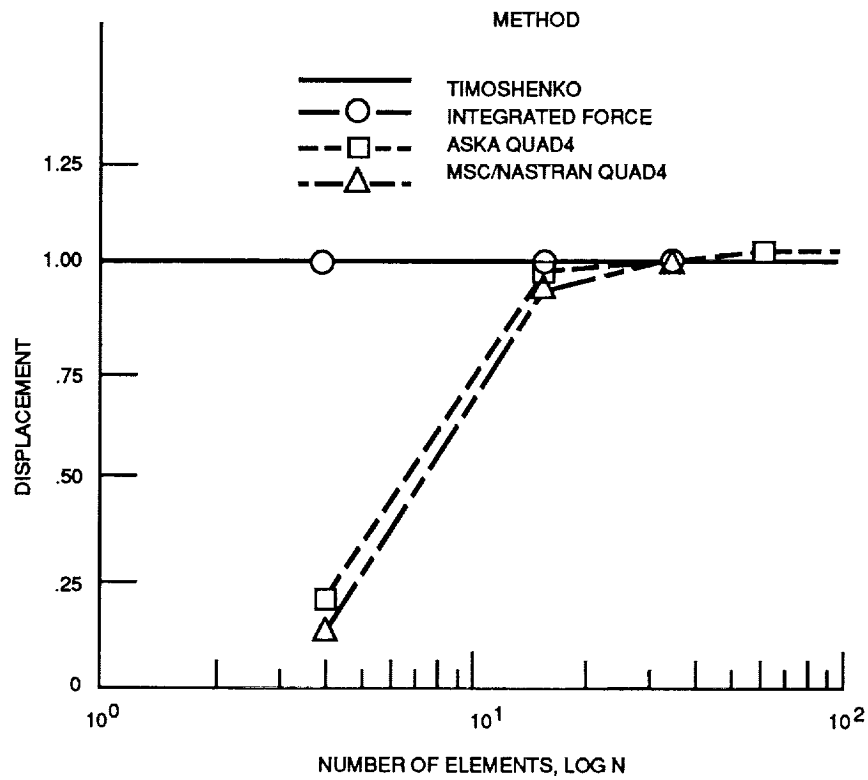


Figure 10. - Rate of convergence for rectangular elements.

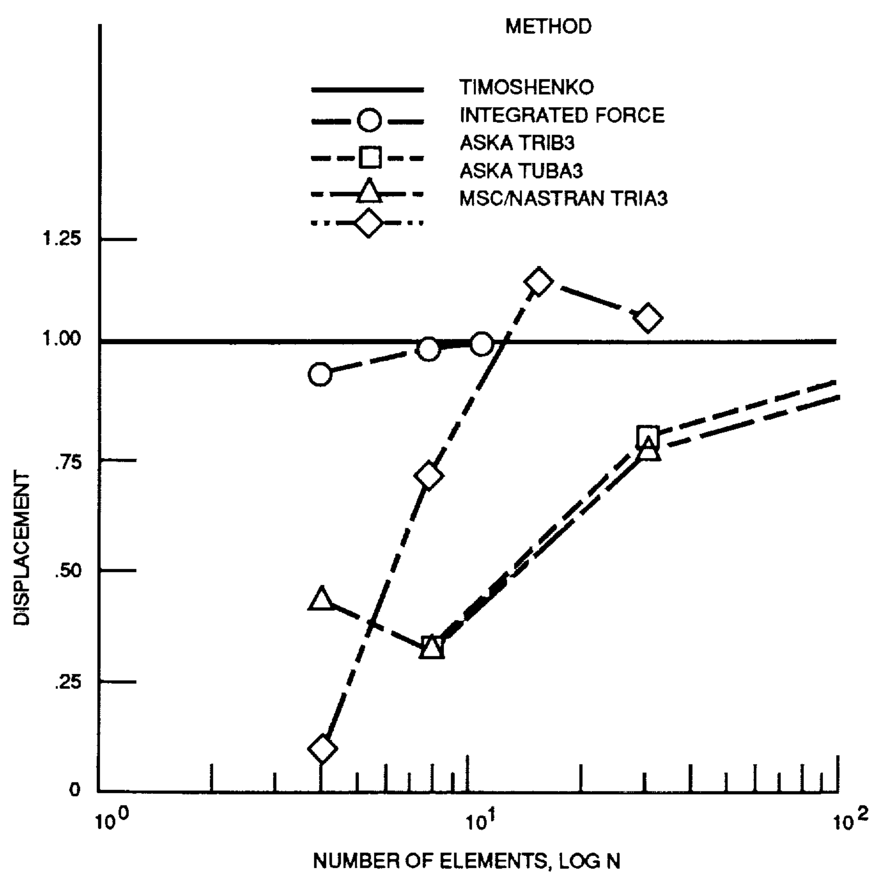


Figure 11. - Rate of convergence for triangular elements.

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15. Supplementary Notes

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16. Abstract

The equilibrium equations and the compatibility conditions are fundamental to the analyses of structures. However, anyone who undertakes even a cursory generic study of the compatibility conditions can discover, with little effort, that historically this facet of structural mechanics had not been adequately researched by the profession. Now the compatibility conditions (CC's) have been researched and are understood to a great extent. For finite element discretizations, the CC's are banded and can be divided into three distinct categories: (1) the interface CC's, (2) the cluster or field CC's, and (3) the external CC's. The generation of CC's requires the separating of a local region, then writing the deformation displacement relation (ddr) for the region, and finally, the eliminating of the displacements from the ddr. The procedure to generate all three types of CC's is presented and illustrated through examples of finite element models. The uniqueness of the CC's thus generated is shown.

17. Key Words (Suggested by Author(s)): Finite elements; Integrated force method; Compatibility conditions; Interface; Cluster; External compatibility conditions; Deformation displacement relations; Strain formulation	18. Distribution Statement: Unclassified – Unlimited; Subject Category 39

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