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DYNAMICS OF COLUMN STABILITY WITH PARTIAL END RESTRAINTS

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ABSTRACT

The purpose of this investigation is to conduct a theoretical study of the dynamic behavior of columns with partial end restraints subjected to an axial dead load and a pulsating load. The governing differential equation is solved using a lumped impulse recurrence formula relative to time and coupled with a finite difference discretization along the member length. A computer program was developed to determine the first two critical frequencies as a function of end stiffness. Results obtained for a pinned ended column case compares very well with an exact analytical solution. Also, the natural frequency and buckling load equations are derived for equal and unequal end restraints.

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1. INTRODUCTION

1.1 Preliminary Remarks

A column is one of the most basic elements of a structural system. Although a number of studies have investigated the response of a columnm, the effects of partial end restraints, especially under dynamic loading, has not been well defined in the literature. There are three basic conditions of end restraints; 1) pinned, 2) fixed, and 3) partially restrained. The majority of analyses for a column considers the end restraint as either pinned or fixed; yet, most of the as built connections are neither pinned or fixed; they are partially restrained to some degree. Therefore, the concept of partial restraint is of a great practical interest. Another fairly undefined condition is that of a pulsating load on a column, such as the supports for an unbalanced rotating piece of machinery. Such a pulsating force under the right conditions can result in the instability of a column well below the Euler Buckling load. For pinned-end condition, the pulsating load case can be found in ref. 1, but no references to the case with partial end restraints exists. Therefore, the case of a column with paritial end restraints and a pulsating load is of practical interest, and it is the subject of this investigation.

1.2 Literature Review

A literature review of the topic indicates a limited amount of research on the general idea of a pulsating load and no research was found on the idea of partial end restraints with a pulsating load.

References 1 and 5 give the solution of a pinned-pinned column with a pulsating load $P_o + S_o \cos \Omega t$. The solution is in the form of a graph of the instability regions as a function of ω_o, Ω, P_o, S_o, and P_e. Reference 5 gives the instability boundary equations and a supporting table of values. A graph of the instability regions is shown on figure 3.

The case of a column with both ends fixed was discussed by F. Weidenhammer, Ingr.-Arch., vol. 19, page 162, 1951 (in German). Several stability problems under pulsating loads was found in the book by B. B. Bolotin, "Dynamic Stability of Elastic Systems," Moscow, 1956, (Russian).

1.3 Problem Statement

Figure 1 shows a schematic diagram of a column of length L that is bent about its weak axis with rotational end stiffnesses K_1 and K_2. The modulus of elasticity, E, and the moment of inertia, I, are constant through the length of the beam; and the axial load is represented by $P_o + S_o \cos \Omega t$.

The problem is to find the critical frequencies, Ωi, such that instability occurs before the buckling load is reached, and the critical frequencies should be in general terms of the parameters K, E, I, L, Ω, and ω.

1.4 Objectives and Scope

The objectives of this study are;

1. Solve the differential equation using finite difference techniques.
2. Verify the solution against known bench mark cases.
3. Identify the first two critical frequencies in general terms.
4. Determine the natural frequencies and buckling load of a column with equal and unequal end restraints.

1.5 Assumptions

1. The column has a uniform cross section and is subjected to bending only about its' weak axis
2. The material stress-strain relation is linear elastic
3. The loads pass through the centroid of the beam resulting in no eccentric loading
4. No local instability is considered
5. The period of the pulsating force is very large in comparison with the longitudinal natural period of the column

2. THEORETICAL DEVELOPMENT

2.1 Governing Equations

The governing differential equation of lateral vibrations for a column with damping and a pulsating load, $P_o + S_o \cos \Omega t$, (see fig 1) is:

$$EI \frac{\partial^4 w}{\partial x^4} + (P_o + S_o \cos \Omega t) \frac{\partial^2 w}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} = 0 \tag{1}$$

The boundary conditions which include the effects of partial restraints are;

$$EI \frac{\partial^2 w(0)}{\partial x^2} = K_1 \frac{\partial w(0)}{\partial x} \tag{2}$$

$$EI \frac{\partial^2 w(L)}{\partial x^2} = - K_2 \frac{\partial w(L)}{\partial x} \tag{3}$$

$$w(0) = w(L) = 0 \tag{4}$$

Equation (1) is associated with pure buckling. This implies that there is no lateral displacement prior to buckling. That would result in a highly complicated eigenvalue problem which is beyond the scope of this investigation. To circumvent this problem, an initial imperfection will be applied to the column. This will give a continuous displacement response of the column to the load. The exact solution would give discrete regions of instability; whereas with an initial imperfection, the response would asymptotically approach the theoretical regions of instability. Since all as built columns have some degree of imperfection, this assumption is valid. For the subject investigation, the initial imperfection will be of the form;

$$\bar{w} = \delta \sin \frac{\pi x}{L} \tag{5}$$

where δ is the maximum initial displacement at the center of the column. This quantity remains variable so that the smaller δ is made the closer the response gets to the exact solution. With initial imperfections, equation (1) becomes;

$$EI \frac{\partial^4 w}{\partial x^4} + P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial x^2}\right) + \rho \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} = 0 \tag{6}$$

If equation (6) is rearranged such that known quantities are on the right hand side, we arrive at;

$$EI \frac{\partial^4 w}{\partial x^4} + P \frac{\partial^2 w}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} = - P \frac{\partial^2 \tilde{w}}{\partial x^2} \tag{7}$$

2.2 Non-Dimensional Equations

To generalize the results of this analysis, equation (7) will be put in non-dimensional form. To non-dimensionalize equation (7), the following variables are introduced.

$$\bar{w} = \frac{w\,L}{A} \qquad \bar{\tilde{w}} = \frac{\tilde{w}\,L}{A}$$

$$\bar{x} = \frac{x}{L} \qquad \bar{t} = \frac{t}{T_o}$$

Substitutiting the above parameters into equation (7) and rearranging, we get;15

$$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + \frac{P\,L^2}{EI} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\rho L^4}{EIT_o^2} \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{C\,L^4}{EIT_o} \frac{\partial \bar{w}}{\partial \bar{t}} = - \frac{P\,L^2}{EI} \frac{\partial^2 \bar{\tilde{w}}}{\partial \bar{x}^2} \tag{8}$$

To simplify equation (8), the following variables are defined;

$$Z_1 = \frac{PL^2}{EI} = (P_o + S_o \cos \Omega t)\, \frac{L^2}{EI}$$

$$Z_2 = \frac{\rho L^4}{EIT_o^{\,2}}$$

$$Z_3 = \frac{C\ L^4}{EIT_o}$$

Therefore equation (8) becomes;

$$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + Z_2 \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + Z_3 \frac{\partial \bar{w}}{\partial \bar{t}} = - Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \qquad (9)$$

The non-dimensional boundary conditions are;

$$\bar{w}(0) = \bar{w}(1) = 0 \qquad (10)$$

$$\frac{\partial^2 \bar{w}(0)}{\partial \bar{x}^2} = \frac{K_1 L}{EI} \frac{\partial \bar{w}(0)}{\partial \bar{x}} \qquad (11)$$

$$\frac{\partial^2 \bar{w}(1)}{\partial \bar{x}^2} = - \frac{K_2 L}{EI} \frac{\partial \bar{w}(1)}{\partial \bar{x}} \qquad (12)$$

2.3 Finite Difference Formulation

There are two basic steps in the solution of eq. (9):

1. First, a column of length L is divided into descrete sections of length h; and a central difference equation is written for each node to formulate the geometric stiffness matrix

$$[K]\{\bar{w}\} = \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \tag{13}$$

Where $\{\bar{w}\}$ is a vector of nodal displacements.

The central difference equation's used are (ref. 3);

$$\frac{\partial w}{\partial x} = \frac{1}{2h} (- w_{i-1} + w_{i+1}) \tag{14}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{h^2} (w_{i-1} - 2w_i + w_{i+1}) \tag{15}$$

$$\frac{\partial^4 w}{\partial x^4} = \frac{1}{h^4} (w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}) \tag{16}$$

2. Next, a lumped impulse recurrence formula is used to step the response through a succesion of discrete time intervals. The equations used are (ref. 4);

$$w^{(1)} = \frac{1}{2} \ddot{w}^{(0)} (\Delta t)^2 \tag{17}$$

$$w^{(s+1)} = 2w^{(s)} - w^{(s-1)} + \ddot{w}^{(s)} (\Delta t)^2 \tag{18}$$

$$\dot{w}^{(s)} = \frac{w^{(s)} - w^{(s-1)}}{\Delta t} + \ddot{w}^{(s)} \frac{\Delta t}{2} \tag{19}$$

In the above equations, the superscript indicates the time step. The first equation gives the first time step and the second gives each subsequent time step according to the two preceeding steps. From this proceedure, the column's deflection vs. time can be plotted. If the deflection constantly increases with time, then the column is unstable.

2.3.1 Geometric Stiffness Matrix

The stiffness matrix equation (13), can be formulated for a column of length L divided into 12 sections with 11 unknown displacement nodes (fig. 2). Equation (13) can be written for all eleven nodes. When writing the equation at nodes 10 and 11, the ficticious nodes a and b must be expressed in terms of the other nodes. To do this, the boundary conditions, equations (10) and (11) are written at the support nodes. The resulting matrix (11 by 11) is a function of the parameters E, I, K_1, K_2, L, P_o, S_o, Ω, and t.

To solve for node a, equations (14) and (15) are substituted into equation (11) and rearranged to arrive at;

$$w_a = w_{11} Q_1$$

$$\text{where} \quad Q_1 = \frac{\left(\frac{K_1 L\bar{h}}{2EI} - 1\right)}{\left(\frac{K_1 L\bar{h}}{2EI} + 1\right)}$$

For node b, equations (14) and (15) are substituted into equation (12), resulting in;

$$w_b = w_{10} Q_2$$

$$\text{where} \quad Q_2 = \frac{\left(\frac{K_2 L\bar{h}}{2EI} - 1\right)}{\left(\frac{K_2 L\bar{h}}{2EI} + 1\right)}$$

Note that the non-dimensional element length, $\bar{h}$ is defined as;

$$\bar{h} = \frac{h}{L} = \frac{L}{12L} = \frac{1}{12}$$

Now equations (15) and (16) can substituted into equation (13) at all eleven nodes, resulting in the following nodal equations;

1) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_1 - 4\bar{w}_2 - 4\bar{w}_3 + \bar{w}_4 + \bar{w}_5)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_1 + \bar{w}_2 + \bar{w}_3)$$

2) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_2 - 4\bar{w}_4 - 4\bar{w}_1 + \bar{w}_3 + \bar{w}_6)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_2 + \bar{w}_4 + \bar{w}_1)$$

3) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_3 - 4\bar{w}_1 - 4\bar{w}_5 + \bar{w}_2 + \bar{w}_7)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_3 + \bar{w}_1 + \bar{w}_5)$$

4) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_4 - 4\bar{w}_6 - 4\bar{w}_2 + \bar{w}_8 + \bar{w}_1)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_4 + \bar{w}_6 + \bar{w}_2)$$

5) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_5 - 4\bar{w}_3 - 4\bar{w}_7 + \bar{w}_1 + \bar{w}_9)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_5 + \bar{w}_3 + \bar{w}_7)$$

6) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_6 - 4\bar{w}_8 - 4\bar{w}_4 + \bar{w}_{10} + \bar{w}_2)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_6 + \bar{w}_8 + \bar{w}_4)$$

7) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_7 - 4\bar{w}_5 - 4\bar{w}_9 + \bar{w}_3 + \bar{w}_{11})$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_7 + \bar{w}_5 + \bar{w}_9)$$

8) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_8 - 4\bar{w}_{10} - 4\bar{w}_6 + \bar{w}_4)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{w}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_8 + \bar{w}_{10} + \bar{w}_6)$$

9) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_9 - 4\bar{w}_7 - 4\bar{w}_{11} + \bar{w}_5)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_9 + \bar{w}_7 + \bar{w}_{11})$$

10) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{\bar{h}^4} (6\bar{w}_{10} - 4\bar{w}_8 + \bar{w}_6 + \bar{w}_{10} Q_2)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_{10} + \bar{w}_8)$$

11) $$\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} = \frac{1}{h^4} (6\bar{w}_{11} - 4\bar{w}_9 + \bar{w}_7 + \bar{w}_{11} Q_1)$$

$$Z_1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = \frac{Z_1}{\bar{h}^2} (-2\bar{w}_{11} + \bar{w}_9)$$

The following variables are defined;

$$\alpha = \frac{6}{\bar{h}^4} - \frac{2PL^2}{\bar{h}^2 EI}$$

$$\beta = \frac{PL^2}{\bar{h}^2 EI} - \frac{4}{\bar{h}^2}$$

$$\gamma = \frac{1}{\bar{h}^4}$$

$$\eta_1 = \frac{(6 + Q_1)}{\bar{h}^4} - \frac{2PL^2}{\bar{h}^2 EI}$$

$$\eta_2 = \frac{(6 + Q_2)}{\bar{h}^4} - \frac{2PL^2}{\bar{h}^2 EI}$$

Using the above variables and the eleven nodal equations, the following stiffness matrix is formed.

$$[K]\,\{\bar{w}\} = \begin{bmatrix}
\alpha & \beta & \beta & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \alpha & \gamma & \beta & 0 & \gamma & 0 & 0 & 0 & 0 & 0 \\
 & & \alpha & 0 & \beta & 0 & \gamma & 0 & 0 & 0 & 0 \\
 & & & \alpha & 0 & \beta & 0 & \gamma & 0 & 0 & 0 \\
 & & & & \alpha & 0 & \beta & 0 & \gamma & 0 & 0 \\
 & & & & & \alpha & 0 & \beta & 0 & \gamma & 0 \\
 & & & & & & \alpha & 0 & \beta & 0 & \gamma \\
 & & \text{SYMMETRIC} & & & & & \alpha & 0 & \beta & 0 \\
 & & & & & & & & \alpha & 0 & \beta \\
 & & & & & & & & & \eta_2 & 0 \\
 & & & & & & & & & & \eta_1
\end{bmatrix}
\begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \\ \bar{w}_4 \\ \bar{w}_5 \\ \bar{w}_5 \\ \bar{w}_7 \\ \bar{w}_8 \\ \bar{w}_9 \\ \bar{w}_{10} \\ \bar{w}_{11} \end{bmatrix} \tag{20}$$

2.3.2 Initial Imperfections

The initial imperfection, equation (5), and its second, derivative in non-dimensional form are;

$$\bar{\tilde{w}} = \frac{\delta L}{A} \sin \pi\bar{x}$$

$$\frac{\partial^2\bar{\tilde{w}}}{\partial\bar{x}^2} = -\frac{\delta L\pi^2}{A} \sin \pi\bar{x}$$

Now the vector $\left\{\bar{\tilde{w}}_{xx}\right\}$ is defined as the non-dimensional initial imperfection curvatures;

$$\left\{\bar{\tilde{w}}_{xx}\right\} = -\frac{\delta L\pi^2}{A}\begin{bmatrix} 1 \\ \sin 7\pi/12 \\ \sin 7\pi/12 \\ \sin 8\pi/12 \\ \sin 8\pi/12 \\ \sin 9\pi/12 \\ \sin 9\pi/12 \\ \sin 10\pi/12 \\ \sin 10\pi/12 \\ \sin 11\pi/12 \\ \sin 11\pi/12 \end{bmatrix} \qquad (21)$$

2.3.3 Lumped Impulse Recurrence Formula

To apply the recurrence formulas equations (17), (18) and (19), the differential equation (9) is put into vector form. Using equations (20) and (21) and defining the following vectors,

$$\left\{\ddot{\bar{w}}\right\} = \frac{\partial^2\bar{w}}{\partial\bar{t}^2}$$

$$\left\{\dot{\bar{w}}\right\} = \frac{\partial \bar{w}}{\partial \bar{t}}$$

we arrived at the vector matrix form of equation (9) as;

$$[K] \left\{\bar{w}\right\} + Z_2 \left\{\ddot{\bar{w}}\right\} + Z_3 \left\{\dot{\bar{w}}\right\} = -Z_1 \left\{\bar{\tilde{w}}_{xx}\right\} \tag{22}$$

The non-dimensional time used in this formulation is;

$$\bar{t} = \frac{t}{T_o}$$

$$\Delta\bar{t} = \frac{\bar{t}_i - \bar{t}_{i-1}}{T_o} = \frac{\Delta t}{T_o}$$

For the first time step, the acceleration vector is found from equation (21) to be;

$$\left\{\ddot{\bar{w}}\right\}^{(s)} = -\frac{1}{Z_2}\left[[K]\left\{\bar{w}\right\}^{(s)} + Z_3\left\{\dot{\bar{w}}\right\}^{(s)} + Z_1^{(s)}\left\{\bar{\tilde{w}}_{xx}\right\}\right] \tag{23}$$

If the above equation is substituted into equation (17), accounting for the condition that all terms are zero except the initial imperfection curvature at time $t = 0$, we get;

$$\left\{w\right\}^{(1)} = -\frac{Z_1}{2Z_2}\left\{\bar{w}_{xx}\right\}(\Delta\bar{t})^2 \tag{24}$$

Equation (24) is the first interval of the time sequence. For the rest of the intervals, we first substitute equation (19) into equation (23) and get;

$$\left\{\ddot{\bar{w}}\right\}^{(s)} = -\frac{1}{Z_2}\left[[K]\left\{\bar{w}\right\}^{(s)} + Z_1{}^{(s)}\left\{\bar{w}_{xx}\right\} + Z_3\left[\frac{\left\{\bar{w}\right\}^{(s)} - \left\{\bar{w}\right\}^{(s-1)}}{\Delta\bar{t}} + \left\{\ddot{\bar{w}}\right\}^{(s)}\frac{\Delta\bar{t}}{2}\right]\right]$$

Then after solving the above equation explicity for the acceleration vector and substituting it into equation (18), we get;

$$\left\{\ddot{\bar{w}}\right\}^{(s)} = \frac{[K]\left\{\bar{w}\right\}^{(s)} + Z_1{}^{(s)}\left\{\bar{w}_{xx}\right\} + \frac{Z_3}{\Delta\bar{t}}\left[\left\{\bar{w}\right\}^{(s)} - \left\{\bar{w}\right\}^{(s-1)}\right]}{\left(Z_2 + \frac{Z_3\,\Delta\bar{t}}{2}\right)}$$

$$\left\{\bar{w}\right\}^{(s+1)} = 2\left\{\bar{w}\right\}^{(s)} - \left\{\bar{w}\right\}^{(s-1)} - \left[\frac{[K]\left\{\bar{w}\right\}^{(s)} + Z_1{}^{(s)}\left\{\bar{w}\right\}^{(s)} + \frac{Z_3}{\Delta\bar{t}}\left[\left\{\bar{w}\right\}^{(s)} - \left\{\bar{w}\right\}^{(s-1)}\right]}{\left(Z_2 + \frac{Z_3\Delta\bar{t}}{2}\right)}\right]\Delta t^2 \tag{25}$$

Equations (24) and (25) are the equations needed to step the response of the column through time.

3. RESULTS

3.1 Comparative Results

The case of a column with pinned ends and a pulsating force $P_o + S_o \cos \Omega t$ has been well documented, ref (1) and (5). The solution of the differential equation (1) with no damping takes the form;

$$w = A\, f(t) \sin \frac{\Omega x}{L}$$

First, the above equation is substituted into the differential equation (1). Then the following variables are defined;

$$\omega_o^2 = \frac{\pi^4 EI}{\rho L^4} \qquad \tau = \Omega t$$

$$P_e = \frac{\pi^2 EI}{L^2} \qquad p = \frac{P_o}{P_e} \qquad s = \frac{S_o}{P_e}$$

Using the above variables, the transformed differential equation has the form;

$$\frac{d^2 f(\tau)}{d\tau^2} + (a + b \cos \tau)\ f(\tau) = 0$$

$$\text{where: } a = \frac{\omega_o^2}{\Omega^2}(1-p)$$

$$b = -\frac{\omega_o^2}{\Omega^2} s$$

The above differential equation is known as the Mathieu equation, and the character of the solution depends on the values of a and b. For a particular set of values of a and b, the solution is either stable or unstable. The unstable conditions are indicated by vibration which grow with time. Figure (3) is a plot of a vs b with the shaded region representing a stable condition and the unshaded areas indicating the regions of instability.

The computer program can now be compared to the pinned-pinned case as given by the Mathieu equation. The program was run using fixed values of a and Ω with $K_1 = K_2 = 0$ for the pinned-pinned case. Then S_o was increased by increments of 1 lb until the response became unstable. Six cases were evaluated for the condition of P_o=0 and ten cases with $P = 1/2\ P_{cr}$. The variables used in the program are listed below;

$E = 30 \times 10^6$ psi

$I = .0021\ in^4$

$L = 144$ in

$A = .0859\ in^2$

$\rho = 6.3 \times 10^{-5}$

$T_o = .41749$

The results are shown in table (1). The exact value of a corresponding to the value of b is from Figure (3) and a table of values from Reference (5). The comparison between the approximate solution and the exact solution is very good.

3.2 Critical Frequencies

The most valued result from this investigation would be to have an envelope similar to that of figure (3) as a function of K, L, E, I, Ω, ω, ρ, P_o, and S_o. This would require the generation of hundreds of data points and a search for the correct relationship between the variables involved. An attempt was made to use an analogous axis of figure 3 with the exception that ω_o became ω as a function of end stiffness and axial load. The results showed that there was not a direct correlation between the two, and that there was a shift along the a axis as the end stiffness increased. As a result, the idea of a total envelope was abandoned and only that of a more practical interest was pursued.

There are several practical considerations which would narrow the areas of practical interest. From Figure 3, there are stable regions such that $P_o + S_o \geq P_{cr}$; however, any practical design code would tend to limit the maximum load $P_o + S_o$ to be less than P_{cr}. This will limit our concern to the lower half of the diagonal a = b.

In addition the areas of critical interest are those areas in which very low values of S_o cause instability, and the first few critical frequencies cause such instability. Therefore, the first few frequencies will be determined for a range of end stiffnesses such that $P_o + S_o \leq P_{cr}$.

The results of this analysis, to find the critical frequencies, do not include the effects of damping or unequal end restraints. In the program, the parameter C was set equal to zero and K_1 was always equal to K_2. To locate the critical frequencies the program was run repeatedly, starting with K=0 and its known critical frequencies and then K was increased and its subsequent frequencies were found.

The first two critical frequencies were easily determined for $S_o \geq 5\% P_{cr}$. The remaining critical frequencies were more difficult to find; and they are not included in this report. Table 2 shows the first two critical frequencies for KL/EI varying from 0 to 2286 with $P_o = 0$. At KL/EI equal to 2000, P_{cr}/P_e is 3.992 which has a .2% dfference with that of the fixed-fixed case. Consequently with KL/EI = 2000, the ends are essentially fixed. It was found that as P_o increased, the relationship of Ω/ω remained constant when ω is the natural frequency as a function of end stiffness K and axial load P_o. It was also found that there was a finite relationship between KL/EI and Ω/ω. Figures 5 and 6 are graphs of Ω_1/ω vs KL/EI and Ω_2/ω vs KL/EI, respectively. From these two graphs, the first two critical frequencies can be determined if K, L, E, I, and ω are known. The graphs start at Ω_1 equal to $2\omega_o$ and Ω_2 equal to ω_o with K=0 which is consistent with the pinned-pinned case. At KL/EI = 2000 Ω_1 is equal to 1.91ω and Ω_2 is equal to $.955\omega$. Changing the variables did little to effect the general behavior of the response at the critical frequencies. A typical response of Ω_1 and Ω_2 is on figures 7 and 8 respectively. The center plot on figures 7 and 8 is the critical frequency while the upper and lower plots show its stability at slightly above and below the critical frequency. Figure 7 shows a double beat corresponding to the critical frequency at about twice the natural frequency; and at the critical frequency, the double beat fades as it becomes unstable.

Appendix A formulates the natural frequency and buckling load for a column with equal and unequal end restraints. Figures 9 and 10 are graphs of the dimensionless natural period vs KL/EI for a range of axial loads. From the graph, either the natural period or the stiffness can be determined if the other is known. With the natural period and the end stiffness known, the first two critical frequencies can be determined from figures 5 and 6. In addition P_{cr}/P_e vs KL/EI is graphed on figures 11 and 12 with specific values of the graph in table 3. Therefore, if the end stiffness or the natural frequency can be estimated or determined experimentally, the first two critical frequencies can be found by using figures 5, 6, 9 and 10, and the static buckling load can be determined from figures 11 and 12 or table 3.

4. CONCLUSIONS AND FUTURE RESEARCH

4.1 Conclusions

The following conclusions can be made from this report:

1. A numerical analysis program is written (Appendix B) such that all input quantities are variable and the response of deflection vs time can be plotted.
2. The program gives results which compare very well for the pinned-pinned case (table 1).
3. The natural frequencies and buckling load formulas of a column with equal and unequal end restraints have been developed (Appendix A).
4. The dimensionless natural period vs KL/EI has been plotted for a range of axial loads (figures 9 and 10).
5. The first two critical frequencies have been determined (figures 5 and 6) as a function of K, L, E, I, and ω .
6. If either the end stiffness K or the natural period T_o is known, the other can be evaluated from figures 9 and 10. With the end stiffness and natural period known, the first two critical frequencies can be found from figures 5 and 6.

4.2 Future Research

Although the solution procedure presented in this report did include damping and unequal end restraints, no studies have been conducted concerning the effects of these two variables. Therefore, future research can be performed to investigate the effect of damping and unequal end restraints on the critical frequencies. Future work may also include determining all the critical frequencies as only two have been determined in this report. Also, the effects of material plasticity on the critical frequencies could be the subject of future investigations. In addition, testing could be conducted to verify the results herein.

NOMENCLATURE

A	area
C	damping coefficient
E	modulus of elasticity
f	natural frequency
h	finite difference element length
h	dimensionless finite difference element length
I	moment of inertia
K	geometric stiffness matrix
K_1	rotational stiffness at end 1
K_2	rotational stiffness at end 2
L	length of column
P	$P_o + S_o \cos \Omega t$
P_{cr}	critical buckling load
P_e	Euler buckling load
P_o	dead load
S_o	pulsating load
T_o	natural period
t	time
$\bar{t}$	dimensionless time
w	deflection
$\tilde{w}$	initial imperfection displacement
$\bar{w}$	dimensionless deflection
$\bar{\tilde{w}}$	dimensionless initial imperfection
$\dot{w}$	velocity
$\bar{\dot{w}}$	dimensionless velocity
w_c	centerline deflection excluding initial imperfection

$\ddot{w}$ acceleration

$\ddot{\bar{w}}$ dimensionless acceleration

X axial distance

$\bar{X}$ dimensionless axial distance

ρ density per unit length

δ maximum centerline initial imperfection

ω natural frequency with effects of end stiffness K and axial load P_o

Ω_1 first critical frequency

Ω_2 second critical frequency

Ω forcing function

REFERENCES

1. Timoshenko, S.P.; and Gere, J.M.: Theory of Elastic Stability, 2nd Edition, pp. 158-160, McGraw-Hill Book Company, New York, 1961

2. Razzaq, Z.; Voland, R.T.; Bush, H.G.; and Mikulas, M.M., Jr.: Stability, Vibration, and Passive Damping of Partially Restrained Imperfect Columns. NASA Technical Memorandum 85697, October , 1983.

3. James, M.L.; Smith, G.M.; and Wolford, J.C.: Applied Numerical Methods for Digital Computation, 2nd Edition, p. 346, Harper & Row Publishers, New York, 1977

4. Biggs, J.M.: Introduction to Structural Dynamics pp. 4-7, McGraw-Hill Book Company, New York, 1964

5. Lubkin, S.; and Stoker, J.J.: Stability of Columns and Strings Under Periodically Varying Forces. Quarterly Applied Mathematics, Volume 1, No. 3, p. 215, 1943

TABLES

Table 1. Comparison of Exact vs Computer Approximation for for a Pinned-Pinned Colum

P=0 Pcr=29.986 ω_o =15.05

b	Ω	Scr	a	a
Computer Results				Exact
.229	41.74	53	.13	.129
0	30.1	0	.25	.25
1.8	17.378	72	.75	.743
0	15.05	0	1.0	1.0
1.5	9.518	18	2.5	2.48
3.73	8.044	32	3.5	3.43

P=½ Pcr= 14.993

.225	29.515	26	.13	.131
0	21.284	0	.25	.25
.567	15.05	17	.5	.488
0	10.642	0	1.0	1.0
1.30	8.689	13	1.5	1.43
2.93	7.525	22	2.0	1.95
3.75	7.095	25	2.25	2.22
2.6	6.144	13	3.0	2.93
9.6	5.321	36	4.0	3.92
4.0	4.759	12	5.0	4.81

Table 2. First Two Natural Frequencies as a Function of KL/EI

K	Ω_1	Ω_2	ω	KL/EI	Ω_1/ω	Ω_2/ω
0	30.1	15.05	15.05	0	2	1
500	35.6	17.7	17.99	1.143	1.979	.984
1000	39.7	19.7	20.14	2.286	1.971	.978
2000	45.2	22.4	23.11	4.571	1.956	.970
5000	53.1	26.4	27.52	11.43	1.930	.960
10000	58.2	29.0	30.32	22.86	1.920	.957
20000	61.8	30.8	32.25	45.71	1.916	.955
52500	64.6	32.2	33.72	120	1.915	.955
87500	65.3	32.6	34.12	200	1.914	.955
1×10^6	66.3	33.1	34.7	2286	1.910	.955

Variables used; $E = 30 \times 10^6$ psi
$I = .0021$ in^4
$L = 144$ in
$L/r = 920$

Table 3. KL/EI vs P_{cr}/P_e for Equal End Stiffness

KL/EI	P_{cr}/P_e
.0	1.0000
.1	1.0402
.2	1.0797
.3	1.1185
.4	1.1567
.5	1.1942
.6	1.2310
.7	1.2671
.8	1.3026
.9	1.3374
1.0	1.3715
1.1	1.4050
1.2	1.4378
1.3	1.4701
1.4	1.5017
1.5	1.5327
1.6	1.5631
1.7	1.5929
1.8	1.6221
1.9	1.6508
2.0	1.6789
2.1	1.7065
2.2	1.7335
2.3	1.7600
2.4	1.7860
2.5	1.8115
2.6	1.8365
2.7	1.8611
2.8	1.8851
2.9	1.9088
3.0	1.9319
3.1	1.9547
3.2	1.9770
3.3	1.9988
3.4	2.0203
3.5	2.0414
3.6	2.0621
3.7	2.0824
3.8	2.1024
3.9	2.1219
4.0	2.1412
4.1	2.1600
4.2	2.1786
4.3	2.1968
4.4	2.2147
4.5	2.2323
4.6	2.2495
4.7	2.2665
4.8	2.2832
4.9	2.2996
5.0	2.3157
5.1	2.3315
5.2	2.3470
5.3	2.3623
5.4	2.3774
5.5	2.3922
5.6	2.4067
5.7	2.4210
5.8	2.4351
5.9	2.4490
6.0	2.4626
6.1	2.4760
6.2	2.4892
6.3	2.5022
6.4	2.5149
6.5	2.5275
6.6	2.5399
6.7	2.5521
6.8	2.5641
6.9	2.5759
7.0	2.5875
7.1	2.5990
7.2	2.6103
7.3	2.6214
7.4	2.6324
7.5	2.6432
7.6	2.6538
7.7	2.6643
7.8	2.6746
7.9	2.6848
8.0	2.6948
8.1	2.7047
8.2	2.7144
8.3	2.7241
8.4	2.7335
8.5	2.7429
8.6	2.7521
8.7	2.7612
8.8	2.7701
8.9	2.7790
9.0	2.7877
9.1	2.7963
9.2	2.8048
9.3	2.8132
9.4	2.8214
9.5	2.8296

Table 3. continued

KL/EI	P_{cr}/P_e
9.6	2.8376
9.7	2.8456
9.8	2.8534
9.9	2.8612
10.0	2.8688
10.1	2.8764
10.2	2.8838
10.3	2.8912
10.4	2.8984
10.5	2.9056
10.6	2.9127
10.7	2.9197
10.8	2.9266
10.9	2.9334
11.0	2.9402
11.1	2.9468
11.2	2.9534
11.3	2.9599
11.4	2.9663
11.5	2.9727
11.6	2.9790
11.7	2.9852
11.8	2.9913
11.9	2.9974
12.0	3.0033
12.1	3.0093
12.2	3.0151
12.3	3.0209
12.4	3.0266
12.5	3.0323
12.6	3.0379
12.7	3.0434
12.8	3.0489
12.9	3.0543
13.0	3.0597
13.1	3.0650
13.2	3.0702
13.3	3.0754
13.4	3.0805
13.5	3.0856
13.6	3.0906
13.7	3.0955
13.8	3.1005
13.9	3.1053
14.0	3.1101
14.1	3.1149
14.2	3.1196
14.3	3.1243

KL/EI	$P_{cr}P/_e$
14.4	3.1289
14.5	3.1334
14.6	3.1380
14.7	3.1424
14.8	3.1469
14.9	3.1512
15.0	3.1556
15.1	3.1599
15.2	3.1641
15.3	3.1684
15.4	3.1725
15.5	3.1767
15.6	3.1808
15.7	3.1848
15.8	3.1888
15.9	3.1928
16.0	3.1967
16.1	3.2006
16.2	3.2045
16.3	3.2083
16.4	3.2121
16.5	3.2159
16.6	3.2196
16.7	3.2233
16.8	3.2269
16.9	3.2306
17.0	3.2342
17.1	3.2377
17.2	3.2412
17.3	3.2447
17.4	3.2482
17.5	3.2516
17.6	3.2550
17.7	3.2584
17.8	3.2617
17.9	3.2650
18.0	3.2683
18.1	3.2716
18.2	3.2748
18.3	3.2780
18.4	3.2812
18.5	3.2843
18.6	3.2874
18.7	3.2905
18.8	3.2936
18.9	3.2966
19.0	3.2996

Table 3. continued

KL/EI	P_{cr}/P_e	KL/EI	P_{cr}/P_e
20.0	3.3284	68.0	3.7761
21.0	3.3549	69.0	3.7792
22.0	3.3795	70.0	3.7822
23.0	3.4023	71.0	3.7851
24.0	3.4235	72.0	3.7880
25.0	3.4432	73.0	3.7907
26.0	3.4617	74.0	3.7934
27.0	3.4790	75.0	3.7960
28.0	3.4952	76.0	3.7986
29.0	3.5105	77.0	3.8011
30.0	3.5249	78.0	3.8035
31.0	3.5384	79.0	3.8059
32.0	3.5512	80.0	3.8082
33.0	3.5634	81.0	3.8105
34.0	3.5749	82.0	3.8127
35.0	3.5858	83.0	3.8149
36.0	3.5961	84.0	3.8170
37.0	3.6060	85.0	3.8191
38.0	3.6154	86.0	3.8211
39.0	3.6244	87.0	3.8231
40.0	3.6329	88.0	3.8250
41.0	3.6411	89.0	3.8269
42.0	3.6489	90.0	3.8287
43.0	3.6564	91.0	3.8305
44.0	3.6636	92.0	3.8323
45.0	3.6705	93.0	3.8341
46.0	3.6771	94.0	3.8358
47.0	3.6835	95.0	3.8374
48.0	3.6896	96.0	3.8391
49.0	3.6954	97.0	3.8407
50.0	3.7011	98.0	3.8422
51.0	3.7066	99.0	3.8438
52.0	3.7118	100.0	3.8453
53.0	3.7169	101.0	3.8468
54.0	3.7218	102.0	3.8482
55.0	3.7265	103.0	3.8496
56.0	3.7311	104.0	3.8510
57.0	3.7355	105.0	3.8524
58.0	3.7398	106.0	3.8537
59.0	3.7440	107.0	3.8551
60.0	3.7480	108.0	3.8564
61.0	3.7519	109.0	3.8576
62.0	3.7557	110.0	3.8589
63.0	3.7593	111.0	3.8601
64.0	3.7629	112.0	3.8613
65.0	3.7663	113.0	3.8625
66.0	3.7697	114.0	3.8637
67.0	3.7730	115.0	3.8649

Table 3. continued

KL/EI	P_{cr}/P_e
116.0	3.8660
117.0	3.8671
118.0	3.8682
119.0	3.8693
120.0	3.8703
121.0	3.8714
122.0	3.8724
123.0	3.8734
124.0	3.8744
125.0	3.8754
126.0	3.8763
127.0	3.8773
128.0	3.8782
129.0	3.8791
130.0	3.8800
131.0	3.8809
132.0	3.8818
133.0	3.8827
134.0	3.8835
135.0	3.8844
136.0	3.8852
137.0	3.8860
138.0	3.8868
139.0	3.8876
140.0	3.8884
141.0	3.8892
142.0	3.8899
143.0	3.8907
144.0	3.8914
145.0	3.8922
146.0	3.8929
147.0	3.8936
148.0	3.8943
149.0	3.8950
150.0	3.8957
151.0	3.8963
152.0	3.8970
153.0	3.8977
154.0	3.8983
155.0	3.8990
156.0	3.8996
157.0	3.9002
158.0	3.9008
159.0	3.9015
160.0	3.9021
161.0	3.9026
162.0	3.9032
163.0	3.9038

KL/EI	P_{cr}/P_e
164.0	3.9044
165.0	3.9050
166.0	3.9055
167.0	3.9061
168.0	3.9066
169.0	3.9072
170.0	3.9077
171.0	3.9082
172.0	3.9088
173.0	3.9093
174.0	3.9098
175.0	3.9103
176.0	3.9108
177.0	3.9113
178.0	3.9118
179.0	3.9123
180.0	3.9127
181.0	3.9132
182.0	3.9137
183.0	3.9141
184.0	3.9146
185.0	3.9150
186.0	3.9155
187.0	3.9159
188.0	3.9164
189.0	3.9168
190.0	3.9172
191.0	3.9177
192.0	3.9181
193.0	3.9185
194.0	3.9189
195.0	3.9193
196.0	3.9197
197.0	3.9201
198.0	3.9205
199.0	3.9209
200.0	3.9213
201.0	3.9217
202.0	3.9221
203.0	3.9225
204.0	3.9228
205.0	3.9232
206.0	3.9236
207.0	3.9239
208.0	3.9243
209.0	3.9246
210.0	3.9250

Table 3. continued

KL/EI	P_{cr}/P_e
220.0	3.9284
230.0	3.9314
240.0	3.9342
250.0	3.9368
260.0	3.9392
270.0	3.9415
280.0	3.9435
290.0	3.9455
300.0	3.9472
310.0	3.9489
320.0	3.9505
330.0	3.9520
340.0	3.9534
350.0	3.9547
360.0	3.9560
370.0	3.9571
380.0	3.9583
390.0	3.9593
400.0	3.9603
410.0	3.9613
420.0	3.9622
430.0	3.9631
440.0	3.9639
450.0	3.9647
460.0	3.9655
470.0	3.9662
480.0	3.9669
490.0	3.9676
500.0	3.9682
510.0	3.9688
520.0	3.9694
530.0	3.9700
540.0	3.9705
550.0	3.9711
560.0	3.9716
570.0	3.9721
580.0	3.9726
590.0	3.9730
600.0	3.9735
610.0	3.9739
620.0	3.9743
630.0	3.9747
640.0	3.9751
650.0	3.9755
660.0	3.9759
670.0	3.9762
680.0	3.9766
690.0	3.9769

KL/EI	P_{cr}/P_e
700.0	3.9772
710.0	3.9776
720.0	3.9779
730.0	3.9782
740.0	3.9785
750.0	3.9788
760.0	3.9790
770.0	3.9793
780.0	3.9796
790.0	3.9798
800.0	3.9801
810.0	3.9803
820.0	3.9806
830.0	3.9808
840.0	3.9810
850.0	3.9812
860.0	3.9815
870.0	3.9817
880.0	3.9819
890.0	3.9821
900.0	3.9823
910.0	3.9825
920.0	3.9827
930.0	3.9829
940.0	3.9830
950.0	3.9832
960.0	3.9834
970.0	3.9836
980.0	3.9837
990.0	3.9839
1000.0	3.9841
1010.0	3.9842
1020.0	3.9844
1030.0	3.9845
1040.0	3.9847
1050.0	3.9848
1060.0	3.9850
1070.0	3.9851
1080.0	3.9852
1090.0	3.9854
1100.0	3.9855
1110.0	3.9856
1120.0	3.9858
1130.0	3.9859
1140.0	3.9860
1150.0	3.9861
1160.0	3.9862
1170.0	3.9864

Table 3. continued

KL/EI	P_{cr}/P_e
1180.0	3.9865
1190.0	3.9866
1200.0	3.9867
1210.0	3.9868
1220.0	3.9869
1230.0	3.9870
1240.0	3.9871
1250.0	3.9872
1260.0	3.9873
1270.0	3.9874
1280.0	3.9875
1290.0	3.9876
1300.0	3.9877
1310.0	3.9878
1320.0	3.9879
1330.0	3.9880
1340.0	3.9881
1350.0	3.9882
1360.0	3.9883
1370.0	3.9883
1380.0	3.9884
1390.0	3.9885
1400.0	3.9886
1410.0	3.9887
1420.0	3.9888
1430.0	3.9888
1440.0	3.9889
1450.0	3.9890
1460.0	3.9891
1470.0	3.9891
1480.0	3.9892
1490.0	3.9893
1500.0	3.9894
1510.0	3.9894
1520.0	3.9895
1530.0	3.9896
1540.0	3.9896
1550.0	3.9897
1560.0	3.9898
1570.0	3.9898
1580.0	3.9899
1590.0	3.9900
1600.0	3.9900
1610.0	3.9901
1620.0	3.9901
1630.0	3.9902
1640.0	3.9903
1650.0	3.9903

KL/EI	P_{cr}/P_e
1660.0	3.9904
1670.0	3.9904
1680.0	3.9905
1690.0	3.9906
1700.0	3.9906
1710.0	3.9907
1720.0	3.9907
1730.0	3.9908
1740.0	3.9908
1750.0	3.9909
1760.0	3.9909
1770.0	3.9910
1780.0	3.9910
1790.0	3.9911
1800.0	3.9911
1810.0	3.9912
1820.0	3.9912
1830.0	3.9913
1840.0	3.9913
1850.0	3.9914
1860.0	3.9914
1870.0	3.9915
1880.0	3.9915
1890.0	3.9915
1900.0	3.9916
1910.0	3.9916
1920.0	3.9917
1930.0	3.9917
1940.0	3.9918
1950.0	3.9918
1960.0	3.9919
1970.0	3.9919
1980.0	3.9919
1990.0	3.9920
2000.0	3.9920
2010.0	3.9921
2020.0	3.9921
2030.0	3.9921
2040.0	3.9922
2050.0	3.9922
2060.0	3.9922
2070.0	3.9923
2080.0	3.9923
2090.0	3.9924
2100.0	3.9924
2110.0	3.9924
2120.0	3.9925
2130.0	3.9925

Table 4 Dimensionless Natural Period vs KL/EI $(P/P_e=0)$

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
.0	.6366
.1	.6242
.2	.6126
.3	.6018
.4	.5917
.5	.5822
.6	.5732
.7	.5648
.8	.5569
.9	.5494
1.0	.5423
1.1	.5355
1.2	.5291
1.3	.5230
1.4	.5172
1.5	.5117
1.6	.5064
1.7	.5014
1.8	.4966
1.9	.4919
2.0	.4875
2.1	.4833
2.2	.4792
2.3	.4753
2.4	.4715
2.5	.4679
2.6	.4644
2.7	.4610
2.8	.4577
2.9	.4546
3.0	.4516
3.1	.4486
3.2	.4458
3.3	.4431
3.4	.4404
3.5	.4378
3.6	.4353
3.7	.4329
3.8	.4306
3.9	.4283
4.0	.4261
4.1	.4240
4.2	.4219
4.3	.4199
4.4	.4179
4.5	.4160
4.6	.4141
4.7	.4123

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
4.8	.4105
4.9	.4088
5.0	.4071
5.1	.4055
5.2	.4039
5.3	.4023
5.4	.4008
5.5	.3993
5.6	.3979
5.7	.3964
5.8	.3950
5.9	.3937
6.0	.3924
6.1	.3911
6.2	.3898
6.3	.3886
6.4	.3874
6.5	.3862
6.6	.3850
6.7	.3839
6.8	.3828
6.9	.3817
7.0	.3806
7.1	.3796
7.2	.3785
7.3	.3775
7.4	.3766
7.5	.3756
7.6	.3746
7.7	.3737
7.8	.3728
7.9	.3719
8.0	.3710
8.1	.3701
8.2	.3693
8.3	.3685
8.4	.3676
8.5	.3668
8.6	.3661
8.7	.3653
8.8	.3645
8.9	.3638
9.0	.3630
9.1	.3623
9.2	.3616
9.3	.3609
9.4	.3602
9.5	.3595

Table 4. Continued ($P/P_e=0$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
9.6	.3588
9.7	.3582
9.8	.3575
9.9	.3569
10.0	.3563
10.1	.3556
10.2	.3550
10.3	.3544
10.4	.3538
10.5	.3533
10.6	.3527
10.7	.3521
10.8	.3516
10.9	.3510
11.0	.3505
11.1	.3499
11.2	.3494
11.3	.3489
11.4	.3484
11.5	.3479
11.6	.3474
11.7	.3469
11.8	.3464
11.9	.3459
12.0	.3455
12.1	.3450
12.2	.3445
12.3	.3441
12.4	.3436
12.5	.3432
12.6	.3428
12.7	.3423
12.8	.3419
12.9	.3415
13.0	.3411
13.1	.3407
13.2	.3403
13.3	.3399
13.4	.3395
13.5	.3391
13.6	.3387
13.7	.3383
13.8	.3380
13.9	.3376
14.0	.3372
14.1	.3369
14.2	.3365
14.3	.3361
14.4	.3358
14.5	.3355
14.6	.3351
14.7	.3348
14.8	.3344
14.9	.3341
15.0	.3338
15.1	.3335
15.2	.3331
15.3	.3328
15.4	.3325
15.5	.3322
15.6	.3319
15.7	.3316
15.8	.3313
15.9	.3310
16.0	.3307
16.1	.3304
16.2	.3301
16.3	.3299
16.4	.3296
16.5	.3293
16.6	.3290
16.7	.3288
16.8	.3285
16.9	.3282
17.0	.3280
17.1	.3277
17.2	.3274
17.3	.3272
17.4	.3269
17.5	.3267
17.6	.3264
17.7	.3262
17.8	.3259
17.9	.3257
18.0	.3255
18.1	.3252
18.2	.3250
18.3	.3248
18.4	.3245
18.5	.3243
18.6	.3241
18.7	.3238
18.8	.3236
18.9	.3234
19.0	.3232
19.1	.3230

Table 4. Continued ($P/P_e=0$)

KL/EI	$\frac{T_0}{L^2}\sqrt{\frac{EI}{\rho}}$
19.2	.3228
19.3	.3225
19.4	.3223
19.5	.3221
19.6	.3219
19.7	.3217
19.8	.3215
19.9	.3213
20.0	.3211
21.0	.3192
22.0	.3175
23.0	.3159
24.0	.3144
25.0	.3130
26.0	.3117
27.0	.3105
28.0	.3094
29.0	.3083
30.0	.3073
31.0	.3064
32.0	.3055
33.0	.3047
34.0	.3039
35.0	.3031
36.0	.3024
37.0	.3018
38.0	.3011
39.0	.3005
40.0	.2999
41.0	.2994
42.0	.2989
43.0	.2983
44.0	.2979
45.0	.2974
46.0	.2970
47.0	.2965
48.0	.2961
49.0	.2957
50.0	.2953
51.0	.2950
52.0	.2946
53.0	.2943
54.0	.2940
55.0	.2936
56.0	.2933
57.0	.2930
58.0	.2928
59.0	.2925

KL/EI	$\frac{T_0}{L^2}\sqrt{\frac{EI}{\rho}}$
60.0	.2922
61.0	.2920
62.0	.2917
63.0	.2915
64.0	.2912
65.0	.2910
66.0	.2908
67.0	.2906
68.0	.2904
69.0	.2901
70.0	.2899
71.0	.2898
72.0	.2896
73.0	.2894
74.0	.2892
75.0	.2890
76.0	.2889
77.0	.2887
78.0	.2885
79.0	.2884
80.0	.2882
81.0	.2881
82.0	.2879
83.0	.2878
84.0	.2876
85.0	.2875
86.0	.2874
87.0	.2872
88.0	.2871
89.0	.2870
90.0	.2869
91.0	.2868
92.0	.2866
93.0	.2865
94.0	.2864
95.0	.2863
96.0	.2862
97.0	.2861
98.0	.2860
99.0	.2859
100.0	.2858
101.0	.2857
102.0	.2856
103.0	.2855
104.0	.2854
105.0	.2853
106.0	.2852
107.0	.2851

Table 4. Continued ($P/P_e=0$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
108.0	.2851
109.0	.2850
110.0	.2849
111.0	.2848
112.0	.2847
113.0	.2846
114.0	.2846
115.0	.2845
116.0	.2844
117.0	.2844
118.0	.2843
119.0	.2842
120.0	.2841
121.0	.2841
122.0	.2840
123.0	.2839
124.0	.2839
125.0	.2838
126.0	.2837
127.0	.2837
128.0	.2836
129.0	.2836
130.0	.2835
131.0	.2834
132.0	.2834
133.0	.2833
134.0	.2833
135.0	.2832
136.0	.2832
137.0	.2831
138.0	.2831
139.0	.2830
140.0	.2830
141.0	.2829
142.0	.2829
143.0	.2828
144.0	.2828
145.0	.2827
146.0	.2827
147.0	.2826
148.0	.2826
149.0	.2825
150.0	.2825
151.0	.2824
152.0	.2824
153.0	.2823
154.0	.2823
155.0	.2823

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
156.0	.2822
157.0	.2822
158.0	.2821
159.0	.2821
160.0	.2821
161.0	.2820
162.0	.2820
163.0	.2819
164.0	.2819
165.0	.2819
166.0	.2818
167.0	.2818
168.0	.2818
169.0	.2817
170.0	.2817
171.0	.2817
172.0	.2816
173.0	.2816
174.0	.2816
175.0	.2815
176.0	.2815
177.0	.2815
178.0	.2814
179.0	.2814
180.0	.2814
181.0	.2813
182.0	.2813
183.0	.2813
184.0	.2812
185.0	.2812
186.0	.2812
187.0	.2811
188.0	.2811
189.0	.2811
190.0	.2811
191.0	.2810
192.0	.2810
193.0	.2810
194.0	.2810
195.0	.2809
196.0	.2809
197.0	.2809
198.0	.2808
199.0	.2808
200.0	.2808
201.0	.2808
202.0	.2807
203.0	.2807

Table 4. Continued ($P/P_e = .5$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
108.0	.3056
109.0	.3055
110.0	.3054
111.0	.3053
112.0	.3052
113.0	.3051
114.0	.3050
115.0	.3049
116.0	.3048
117.0	.3047
118.0	.3046
119.0	.3046
120.0	.3045
121.0	.3044
122.0	.3043
123.0	.3043
124.0	.3042
125.0	.3041
126.0	.3040
127.0	.3040
128.0	.3039
129.0	.3038
130.0	.3037
131.0	.3037
132.0	.3036
133.0	.3035
134.0	.3035
135.0	.3034
136.0	.3034
137.0	.3033
138.0	.3032
139.0	.3032
140.0	.3031
141.0	.3031
142.0	.3030
143.0	.3029
144.0	.3029
145.0	.3028
146.0	.3028
147.0	.3027
148.0	.3027
149.0	.3026
150.0	.3026
151.0	.3025
152.0	.3025
153.0	.3024
154.0	.3024
155.0	.3023

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
156.0	.3023
157.0	.3022
158.0	.3022
159.0	.3021
160.0	.3021
161.0	.3020
162.0	.3020
163.0	.3019
164.0	.3019
165.0	.3019
166.0	.3018
167.0	.3018
168.0	.3017
169.0	.3017
170.0	.3016
171.0	.3016
172.0	.3016
173.0	.3015
174.0	.3015
175.0	.3015
176.0	.3014
177.0	.3014
178.0	.3013
179.0	.3013
180.0	.3013
181.0	.3012
182.0	.3012
183.0	.3012
184.0	.3011
185.0	.3011
186.0	.3011
187.0	.3010
188.0	.3010
189.0	.3010
190.0	.3009
191.0	.3009
192.0	.3009
193.0	.3008
194.0	.3008
195.0	.3008
196.0	.3007
197.0	.3007
198.0	.3007
199.0	.3006
200.0	.3006
201.0	.3006
202.0	.3006
203.0	.3005

Table 4. Continued (P/P_e=.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
19.2	.3503
19.3	.3501
19.4	.3498
19.5	.3496
19.6	.3493
19.7	.3491
19.8	.3488
19.9	.3486
20.0	.3483
21.0	.3460
22.0	.3439
23.0	.3420
24.0	.3402
25.0	.3385
26.0	.3370
27.0	.3355
28.0	.3342
29.0	.3329
30.0	.3317
31.0	.3306
32.0	.3296
33.0	.3286
34.0	.3277
35.0	.3268
36.0	.3259
37.0	.3251
38.0	.3244
39.0	.3237
40.0	.3230
41.0	.3223
42.0	.3217
43.0	.3211
44.0	.3205
45.0	.3200
46.0	.3195
47.0	.3190
48.0	.3185
49.0	.3180
50.0	.3176
51.0	.3171
52.0	.3167
53.0	.3163
54.0	.3160
55.0	.3156
56.0	.3152
57.0	.3149
58.0	.3145
59.0	.3142

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
60.0	.3139
61.0	.3136
62.0	.3133
63.0	.3130
64.0	.3127
65.0	.3125
66.0	.3122
67.0	.3120
68.0	.3117
69.0	.3115
70.0	.3113
71.0	.3110
72.0	.3108
73.0	.3106
74.0	.3104
75.0	.3102
76.0	.3100
77.0	.3098
78.0	.3096
79.0	.3094
80.0	.3092
81.0	.3091
82.0	.3089
83.0	.3087
84.0	.3086
85.0	.3084
86.0	.3083
87.0	.3081
88.0	.3080
89.0	.3078
90.0	.3077
91.0	.3075
92.0	.3074
93.0	.3073
94.0	.3071
95.0	.3070
96.0	.3069
97.0	.3068
98.0	.3066
99.0	.3065
100.0	.3064
101.0	.3063
102.0	.3062
103.0	.3061
104.0	.3060
105.0	.3059
106.0	.3058
107.0	.3056

Table 4. Continued (P/P_e=.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
9.6	.3954
9.7	.3945
9.8	.3937
9.9	.3929
10.0	.3921
10.1	.3913
10.2	.3905
10.3	.3897
10.4	.3890
10.5	.3882
10.6	.3875
10.7	.3868
10.8	.3861
10.9	.3854
11.0	.3847
11.1	.3840
11.2	.3834
11.3	.3827
11.4	.3821
11.5	.3814
11.6	.3808
11.7	.3802
11.8	.3796
11.9	.3790
12.0	.3784
12.1	.3778
12.2	.3772
12.3	.3767
12.4	.3761
12.5	.3756
12.6	.3750
12.7	.3745
12.8	.3740
12.9	.3734
13.0	.3729
13.1	.3724
13.2	.3719
13.3	.3714
13.4	.3709
13.5	.3704
13.6	.3700
13.7	.3695
13.8	.3690
13.9	.3686
14.0	.3681
14.1	.3677
14.2	.3672
14.3	.3668

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
14.4	.3663
14.5	.3659
14.6	.3655
14.7	.3651
14.8	.3647
14.9	.3643
15.0	.3639
15.1	.3635
15.2	.3631
15.3	.3627
15.4	.3623
15.5	.3619
15.6	.3615
15.7	.3612
15.8	.3608
15.9	.3604
16.0	.3601
16.1	.3597
16.2	.3594
16.3	.3590
16.4	.3587
16.5	.3583
16.6	.3580
16.7	.3577
16.8	.3573
16.9	.3570
17.0	.3567
17.1	.3564
17.2	.3560
17.3	.3557
17.4	.3554
17.5	.3551
17.6	.3548
17.7	.3545
17.8	.3542
17.9	.3539
18.0	.3536
18.1	.3533
18.2	.3531
18.3	.3528
18.4	.3525
18.5	.3522
18.6	.3519
18.7	.3517
18.8	.3514
18.9	.3511
19.0	.3509
19.1	.3506

Table 4. Continued (P/P_e=.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$	KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
.0	.9003	4.8	.4645
.1	.8661	4.9	.4621
.2	.8360	5.0	.4598
.3	.8092	5.1	.4575
.4	.7852	5.2	.4553
.5	.7636	5.3	.4531
.6	.7439	5.4	.4510
.7	.7259	5.5	.4490
.8	.7095	5.6	.4470
.9	.6943	5.7	.4450
1.0	.6803	5.8	.4432
1.1	.6673	5.9	.4413
1.2	.6552	6.0	.4395
1.3	.6439	6.1	.4378
1.4	.6333	6.2	.4361
1.5	.6234	6.3	.4344
1.6	.6141	6.4	.4328
1.7	.6053	6.5	.4312
1.8	.5970	6.6	.4296
1.9	.5892	6.7	.4281
2.0	.5818	6.8	.4266
2.1	.5748	6.9	.4252
2.2	.5681	7.0	.4238
2.3	.5617	7.1	.4224
2.4	.5556	7.2	.4210
2.5	.5499	7.3	.4197
2.6	.5443	7.4	.4184
2.7	.5391	7.5	.4171
2.8	.5340	7.6	.4159
2.9	.5292	7.7	.4146
3.0	.5245	7.8	.4134
3.1	.5201	7.9	.4123
3.2	.5158	8.0	.4111
3.3	.5117	8.1	.4100
3.4	.5077	8.2	.4089
3.5	.5039	8.3	.4078
3.6	.5002	8.4	.4067
3.7	.4966	8.5	.4057
3.8	.4932	8.6	.4046
3.9	.4899	8.7	.4036
4.0	.4867	8.8	.4027
4.1	.4836	8.9	.4017
4.2	.4806	9.0	.4007
4.3	.4777	9.1	.3998
4.4	.4749	9.2	.3989
4.5	.4722	9.3	.3980
4.6	.4696	9.4	.3971
4.7	.4670	9.5	.3962

Table 4. Continued ($P/P_e=1.0$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$	KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
.1	3.1754	4.9	.5438
.2	2.2546	5.0	.5401
.3	1.8485	5.1	.5365
.4	1.6075	5.2	.5331
.5	1.4437	5.3	.5298
.6	1.3233	5.4	.5265
.7	1.2302	5.5	.5234
.8	1.1555	5.6	.5204
.9	1.0938	5.7	.5174
1.0	1.0419	5.8	.5146
1.1	.9975	5.9	.5118
1.2	.9589	6.0	.5091
1.3	.9250	6.1	.5065
1.4	.8949	6.2	.5040
1.5	.8680	6.3	.5015
1.6	.8438	6.4	.4991
1.7	.8218	6.5	.4968
1.8	.8018	6.6	.4945
1.9	.7835	6.7	.4923
2.0	.7667	6.8	.4901
2.1	.7511	6.9	.4880
2.2	.7367	7.0	.4859
2.3	.7232	7.1	.4839
2.4	.7107	7.2	.4820
2.5	.6990	7.3	.4800
2.6	.6880	7.4	.4782
2.7	.6777	7.5	.4764
2.8	.6680	7.6	.4746
2.9	.6588	7.7	.4728
3.0	.6502	7.8	.4711
3.1	.6420	7.9	.4695
3.2	.6342	8.0	.4678
3.3	.6268	8.1	.4662
3.4	.6197	8.2	.4647
3.5	.6130	8.3	.4632
3.6	.6066	8.4	.4617
3.7	.6005	8.5	.4602
3.8	.5946	8.6	.4588
3.9	.5890	8.7	.4574
4.0	.5837	8.8	.4560
4.1	.5785	8.9	.4546
4.2	.5736	9.0	.4533
4.3	.5688	9.1	.4520
4.4	.5643	9.2	.4508
4.5	.5599	9.3	.4495
4.6	.5556	9.4	.4483
4.7	.5515	9.5	.4471
4.8	.5476	9.6	.4459

Table 4. Continued (P/P_e=1.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
9.7	.4448
9.8	.4436
9.9	.4425
10.0	.4414
10.1	.4403
10.2	.4393
10.3	.4382
10.4	.4372
10.5	.4362
10.6	.4352
10.7	.4343
10.8	.4333
10.9	.4324
11.0	.4314
11.1	.4305
11.2	.4296
11.3	.4288
11.4	.4279
11.5	.4271
11.6	.4262
11.7	.4254
11.8	.4246
11.9	.4238
12.0	.4230
12.1	.4222
12.2	.4215
12.3	.4207
12.4	.4200
12.5	.4192
12.6	.4185
12.7	.4178
12.8	.4171
12.9	.4164
13.0	.4157
13.1	.4150
13.2	.4144
13.3	.4137
13.4	.4131
13.5	.4125
13.6	.4118
13.7	.4112
13.8	.4106
13.9	.4100
14.0	.4094
14.1	.4088
14.2	.4082
14.3	.4077
14.4	.4071

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
14.5	.4065
14.6	.4060
14.7	.4054
14.8	.4049
14.9	.4044
15.0	.4039
15.1	.4033
15.2	.4028
15.3	.4023
15.4	.4018
15.5	.4013
15.6	.4008
15.7	.4004
15.8	.3999
15.9	.3994
16.0	.3990
16.1	.3985
16.2	.3980
16.3	.3976
16.4	.3971
16.5	.3967
16.6	.3963
16.7	.3958
16.8	.3954
16.9	.3950
17.0	.3946
17.1	.3942
17.2	.3938
17.3	.3934
17.4	.3930
17.5	.3926
17.6	.3922
17.7	.3918
17.8	.3914
17.9	.3910
18.0	.3907
18.1	.3903
18.2	.3899
18.3	.3896
18.4	.3892
18.5	.3889
18.6	.3885
18.7	.3882
18.8	.3878
18.9	.3875
19.0	.3871
19.1	.3868
19.2	.3865

Table 4. Continued ($P/P_e=1.0$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
19.3	.3861
19.4	.3858
19.5	.3855
19.6	.3852
19.7	.3849
19.8	.3845
19.9	.3842
20.0	.3839
20.0	.3839
21.0	.3810
22.0	.3783
23.0	.3759
24.0	.3736
25.0	.3715
26.0	.3696
27.0	.3678
28.0	.3661
29.0	.3646
30.0	.3631
31.0	.3617
32.0	.3604
33.0	.3592
34.0	.3581
35.0	.3570
36.0	.3559
37.0	.3550
38.0	.3540
39.0	.3532
40.0	.3523
41.0	.3515
42.0	.3508
43.0	.3500
44.0	.3493
45.0	.3487
46.0	.3480
47.0	.3474
48.0	.3468
49.0	.3463
50.0	.3457
51.0	.3452
52.0	.3447
53.0	.3442
54.0	.3438
55.0	.3433
56.0	.3429
57.0	.3424
58.0	.3420
59.0	.3417

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
60.0	.3413
61.0	.3409
62.0	.3406
63.0	.3402
64.0	.3399
65.0	.3395
66.0	.3392
67.0	.3389
68.0	.3386
69.0	.3383
70.0	.3381
71.0	.3378
72.0	.3375
73.0	.3373
74.0	.3370
75.0	.3368
76.0	.3365
77.0	.3363
78.0	.3361
79.0	.3359
80.0	.3356
81.0	.3354
82.0	.3352
83.0	.3350
84.0	.3348
85.0	.3346
86.0	.3345
87.0	.3343
88.0	.3341
89.0	.3339
90.0	.3338
91.0	.3336
92.0	.3334
93.0	.3333
94.0	.3331
95.0	.3330
96.0	.3328
97.0	.3327
98.0	.3325
99.0	.3324
100.0	.3322
101.0	.3321
102.0	.3320
103.0	.3318
104.0	.3317
105.0	.3316
106.0	.3315
107.0	.3313

Table 4. Continued (P/P_e=1.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
108.0	.3312
109.0	.3311
110.0	.3310
111.0	.3309
112.0	.3308
113.0	.3307
114.0	.3305
115.0	.3304
116.0	.3303
117.0	.3302
118.0	.3301
119.0	.3300
120.0	.3299
121.0	.3298
122.0	.3298
123.0	.3297
124.0	.3296
125.0	.3295
126.0	.3294
127.0	.3293
128.0	.3292
129.0	.3291
130.0	.3291
131.0	.3290
132.0	.3289
133.0	.3288
134.0	.3287
135.0	.3287
136.0	.3286
137.0	.3285
138.0	.3284
139.0	.3284
140.0	.3283
141.0	.3282
142.0	.3282
143.0	.3281
144.0	.3280
145.0	.3280
146.0	.3279
147.0	.3278
148.0	.3278
149.0	.3277
150.0	.3276
151.0	.3276
152.0	.3275
153.0	.3275
154.0	.3274
155.0	.3273
156.0	.3273
157.0	.3272
158.0	.3272
159.0	.3271
160.0	.3271
161.0	.3270
162.0	.3270
163.0	.3269
164.0	.3269
165.0	.3268
166.0	.3267
167.0	.3267
168.0	.3267
169.0	.3266
170.0	.3266
171.0	.3265
172.0	.3265
173.0	.3264
174.0	.3264
175.0	.3263
176.0	.3263
177.0	.3262
178.0	.3262
179.0	.3261
180.0	.3261
181.0	.3261
182.0	.3260
183.0	.3260
184.0	.3259
185.0	.3259
186.0	.3258
187.0	.3258
188.0	.3258
189.0	.3257
190.0	.3257
191.0	.3257
192.0	.3256
193.0	.3256
194.0	.3255
195.0	.3255
196.0	.3255
197.0	.3254
198.0	.3254
199.0	.3254
200.0	.3253
201.0	.3253
202.0	.3253
203.0	.3252

Table 4. Continued ($P/P_e=1.5$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
1.5	3.5051
1.6	2.5213
1.7	2.0764
1.8	1.8099
1.9	1.6278
2.0	1.4935
2.1	1.3894
2.2	1.3057
2.3	1.2365
2.4	1.1782
2.5	1.1282
2.6	1.0848
2.7	1.0466
2.8	1.0127
2.9	.9824
3.0	.9550
3.1	.9302
3.2	.9076
3.3	.8869
3.4	.8678
3.5	.8502
3.6	.8338
3.7	.8186
3.8	.8044
3.9	.7911
4.0	.7787
4.1	.7670
4.2	.7559
4.3	.7455
4.4	.7356
4.5	.7263
4.6	.7174
4.7	.7089
4.8	.7009
4.9	.6933
5.0	.6859
5.1	.6790
5.2	.6723
5.3	.6659
5.4	.6597
5.5	.6538
5.6	.6482
5.7	.6427
5.8	.6375
5.9	.6325
6.0	.6276
6.1	.6229
6.2	.6184
6.3	.6140
6.4	.6098
6.5	.6057
6.6	.6017
6.7	.5979
6.8	.5942
6.9	.5906
7.0	.5871
7.1	.5837
7.2	.5804
7.3	.5772
7.4	.5741
7.5	.5711
7.6	.5682
7.7	.5653
7.8	.5625
7.9	.5598
8.0	.5572
8.1	.5546
8.2	.5521
8.3	.5497
8.4	.5473
8.5	.5450
8.6	.5427
8.7	.5405
8.8	.5383
8.9	.5362
9.0	.5341
9.1	.5321
9.2	.5301
9.3	.5282
9.4	.5263
9.5	.5245
9.6	.5226
9.7	.5209
9.8	.5191
9.9	.5174
10.0	.5158
10.1	.5141
10.2	.5125
10.3	.5110
10.4	.5094
10.5	.5079
10.6	.5064
10.7	.5050
10.8	.5035
10.9	.5021
11.0	.5008

Table 4. Continued (P/P_e=1.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
11.1	.4994
11.2	.4981
11.3	.4968
11.4	.4955
11.5	.4943
11.6	.4930
11.7	.4918
11.8	.4906
11.9	.4894
12.0	.4883
12.1	.4872
12.2	.4860
12.3	.4849
12.4	.4839
12.5	.4828
12.6	.4818
12.7	.4807
12.8	.4797
12.9	.4787
13.0	.4777
13.1	.4768
13.2	.4758
13.3	.4749
13.4	.4739
13.5	.4730
13.6	.4721
13.7	.4713
13.8	.4704
13.9	.4695
14.0	.4687
14.1	.4678
14.2	.4670
14.3	.4662
14.4	.4654
14.5	.4646
14.6	.4638
14.7	.4631
14.8	.4623
14.9	.4616
15.0	.4608
15.1	.4601
15.2	.4594
15.3	.4587
15.4	.4580
15.5	.4573
15.6	.4566
15.7	.4559
15.8	.4553

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
15.9	.4546
16.0	.4539
16.1	.4533
16.2	.4527
16.3	.4520
16.4	.4514
16.5	.4508
16.6	.4502
16.7	.4496
16.8	.4490
16.9	.4484
17.0	.4479
17.1	.4473
17.2	.4467
17.3	.4462
17.4	.4456
17.5	.4451
17.6	.4446
17.7	.4440
17.8	.4435
17.9	.4430
18.0	.4425
18.1	.4420
18.2	.4415
18.3	.4410
18.4	.4405
18.5	.4400
18.6	.4395
18.7	.4390
18.8	.4385
18.9	.4381
19.0	.4376
19.1	.4372
19.2	.4367
19.3	.4363
19.4	.4358
19.5	.4354
19.6	.4349
19.7	.4345
19.8	.4341
19.9	.4337
20.0	.4333
20.1	.4328
20.2	.4324
20.3	.4320
20.4	.4316
20.5	.4312
20.6	.4308

Table 4. Continued ($P/P_e=1.5$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$	KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
20.7	.4304	62.0	.3764
20.8	.4301	63.0	.3760
20.9	.4297	64.0	.3755
21.0	.4293	65.0	.3751
21.1	.4289	66.0	.3747
21.2	.4286	67.0	.3744
21.3	.4282	68.0	.3740
21.4	.4278	69.0	.3736
22.0	.4257	70.0	.3733
23.0	.4224	71.0	.3729
24.0	.4194	72.0	.3726
25.0	.4166	73.0	.3723
26.0	.4140	74.0	.3719
27.0	.4117	75.0	.3716
28.0	.4094	76.0	.3713
29.0	.4074	77.0	.3710
30.0	.4055	78.0	.3708
31.0	.4037	79.0	.3705
32.0	.4020	80.0	.3702
33.0	.4004	81.0	.3700
34.0	.3989	82.0	.3697
35.0	.3975	83.0	.3694
36.0	.3961	84.0	.3692
37.0	.3949	85.0	.3690
38.0	.3937	86.0	.3687
39.0	.3925	87.0	.3685
40.0	.3914	88.0	.3683
41.0	.3904	89.0	.3681
42.0	.3894	90.0	.3678
43.0	.3885	91.0	.3676
44.0	.3876	92.0	.3674
45.0	.3867	93.0	.3672
46.0	.3859	94.0	.3670
47.0	.3851	95.0	.3668
48.0	.3844	96.0	.3667
49.0	.3837	97.0	.3665
50.0	.3830	98.0	.3663
51.0	.3823	99.0	.3661
52.0	.3817	100.0	.3659
53.0	.3811	101.0	.3658
54.0	.3805	102.0	.3656
55.0	.3799	103.0	.3654
56.0	.3793	104.0	.3653
57.0	.3788	105.0	.3651
58.0	.3783	106.0	.3650
59.0	.3778	107.0	.3648
60.0	.3773	108.0	.3647
61.0	.3769	109.0	.3645

Table 4. Continued (P/P_e=1.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
110.0	.3644
111.0	.3642
112.0	.3641
113.0	.3640
114.0	.3638
115.0	.3637
116.0	.3636
117.0	.3634
118.0	.3633
119.0	.3632
120.0	.3631
121.0	.3630
122.0	.3628
123.0	.3627
124.0	.3626
125.0	.3625
126.0	.3624
127.0	.3623
128.0	.3622
129.0	.3621
130.0	.3620
131.0	.3619
132.0	.3618
133.0	.3617
134.0	.3616
135.0	.3615
136.0	.3614
137.0	.3613
138.0	.3612
139.0	.3611
140.0	.3610
141.0	.3609
142.0	.3609
143.0	.3608
144.0	.3607
145.0	.3606
146.0	.3605
147.0	.3604
148.0	.3604
149.0	.3603
150.0	.3602
151.0	.3601
152.0	.3601
153.0	.3600
154.0	.3599
155.0	.3598
156.0	.3598
157.0	.3597

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
158.0	.3596
159.0	.3596
160.0	.3595
161.0	.3594
162.0	.3594
163.0	.3593
164.0	.3592
165.0	.3592
166.0	.3591
167.0	.3590
168.0	.3590
169.0	.3589
170.0	.3589
171.0	.3588
172.0	.3587
173.0	.3587
174.0	.3586
175.0	.3586
176.0	.3585
177.0	.3585
178.0	.3584
179.0	.3584
180.0	.3583
181.0	.3582
182.0	.3582
183.0	.3581
184.0	.3581
185.0	.3580
186.0	.3580
187.0	.3579
188.0	.3579
189.0	.3578
190.0	.3578
191.0	.3577
192.0	.3577
193.0	.3577
194.0	.3576
195.0	.3576
196.0	.3575
197.0	.3575
198.0	.3574
199.0	.3574
200.0	.3573
201.0	.3573
202.0	.3573
203.0	.3572
204.0	.3572
205.0	.3571

Table 4. Continued ($P/P_e = 2.0$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
3.5	3.0741
3.6	2.5086
3.7	2.1762
3.8	1.9514
3.9	1.7867
4.0	1.6595
4.1	1.5575
4.2	1.4735
4.3	1.4027
4.4	1.3422
4.5	1.2895
4.6	1.2433
4.7	1.2023
4.8	1.1657
4.9	1.1326
5.0	1.1026
5.1	1.0753
5.2	1.0503
5.3	1.0272
5.4	1.0059
5.5	.9862
5.6	.9678
5.7	.9506
5.8	.9346
5.9	.9195
6.0	.9053
6.1	.8919
6.2	.8793
6.3	.8674
6.4	.8561
6.5	.8453
6.6	.8351
6.7	.8254
6.8	.8161
6.9	.8072
7.0	.7987
7.1	.7906
7.2	.7829
7.3	.7754
7.4	.7683
7.5	.7614
7.6	.7548
7.7	.7484
7.8	.7423
7.9	.7364
8.0	.7307
8.1	.7252
8.2	.7198

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
8.3	.7147
8.4	.7097
8.5	.7049
8.6	.7002
8.7	.6957
8.8	.6913
8.9	.6871
9.0	.6829
9.1	.6789
9.2	.6750
9.3	.6712
9.4	.6675
9.5	.6639
9.6	.6605
9.7	.6571
9.8	.6537
9.9	.6505
10.0	.6474
10.1	.6443
10.2	.6413
10.3	.6384
10.4	.6356
10.5	.6328
10.6	.6301
10.7	.6274
10.8	.6248
10.9	.6223
11.0	.6198
11.1	.6174
11.2	.6150
11.3	.6127
11.4	.6104
11.5	.6082
11.6	.6060
11.7	.6039
11.8	.6018
11.9	.5997
12.0	.5977
12.1	.5957
12.2	.5938
12.3	.5919
12.4	.5900
12.5	.5882
12.6	.5864
12.7	.5847
12.8	.5829
12.9	.5812
13.0	.5796

Table 4. Continued (P/P_e=2.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$	KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
13.1	.5780	17.9	.5232
13.2	.5763	18.0	.5224
13.3	.5748	18.1	.5217
13.4	.5732	18.2	.5209
13.5	.5717	18.3	.5201
13.6	.5702	18.4	.5193
13.7	.5687	18.5	.5186
13.8	.5673	18.6	.5179
13.9	.5658	18.7	.5171
14.0	.5644	18.8	.5164
14.1	.5631	18.9	.5157
14.2	.5617	19.0	.5150
14.3	.5604	19.1	.5143
14.4	.5591	19.2	.5136
14.5	.5578	19.3	.5129
14.6	.5565	19.4	.5122
14.7	.5552	19.5	.5115
14.8	.5540	19.6	.5109
14.9	.5528	19.7	.5102
15.0	.5516	19.8	.5096
15.1	.5504	19.9	.5089
15.2	.5492	20.0	.5083
15.3	.5481	20.1	.5077
15.4	.5470	20.2	.5070
15.5	.5459	20.3	.5064
15.6	.5448	20.4	.5058
15.7	.5437	20.5	.5052
15.8	.5426	20.6	.5046
15.9	.5416	20.7	.5040
16.0	.5405	20.8	.5034
16.1	.5395	20.9	.5029
16.2	.5385	21.0	.5023
16.3	.5375	21.1	.5017
16.4	.5365	21.2	.5012
16.5	.5355	21.3	.5006
16.6	.5346	21.4	.5001
16.7	.5336	21.5	.4995
16.8	.5327	21.6	.4990
16.9	.5318	21.7	.4985
17.0	.5309	21.8	.4979
17.1	.5300	21.9	.4974
17.2	.5291	22.0	.4969
17.3	.5282	22.1	.4964
17.4	.5274	22.2	.4959
17.5	.5265	22.3	.4954
17.6	.5257	22.4	.4949
17.7	.5249	22.5	.4944
17.8	.5241	22.6	.4939

Table 4. Continued (P/P_e=2.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
22.7	.4934
22.8	.4929
22.9	.4925
23.0	.4920
23.1	.4915
23.2	.4911
23.3	.4906
23.4	.4901
24.0	.4875
25.0	.4834
26.0	.4797
27.0	.4762
28.0	.4730
29.0	.4700
30.0	.4672
31.0	.4646
32.0	.4622
33.0	.4600
34.0	.4578
35.0	.4558
36.0	.4540
37.0	.4522
38.0	.4505
39.0	.4489
40.0	.4474
41.0	.4459
42.0	.4446
43.0	.4433
44.0	.4420
45.0	.4408
46.0	.4397
47.0	.4386
48.0	.4376
49.0	.4366
50.0	.4356
51.0	.4347
52.0	.4339
53.0	.4330
54.0	.4322
55.0	.4314
56.0	.4307
57.0	.4299
58.0	.4292
59.0	.4286
60.0	.4279
61.0	.4273
62.0	.4267
63.0	.4261

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
64.0	.4255
65.0	.4249
66.0	.4244
67.0	.4239
68.0	.4234
69.0	.4229
70.0	.4224
71.0	.4219
72.0	.4215
73.0	.4210
74.0	.4206
75.0	.4202
76.0	.4198
77.0	.4194
78.0	.4190
79.0	.4186
80.0	.4183
81.0	.4179
82.0	.4176
83.0	.4172
84.0	.4169
85.0	.4166
86.0	.4163
87.0	.4160
88.0	.4157
89.0	.4154
90.0	.4151
91.0	.4148
92.0	.4145
93.0	.4143
94.0	.4140
95.0	.4138
96.0	.4135
97.0	.4133
98.0	.4130
99.0	.4128
100.0	.4125
101.0	.4123
102.0	.4121
103.0	.4119
104.0	.4117
105.0	.4115
106.0	.4113
107.0	.4110
108.0	.4109
109.0	.4107
110.0	.4105
111.0	.4103

Table 4. Continued (P/P_e=2.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
112.0	.4101
113.0	.4099
114.0	.4097
115.0	.4096
116.0	.4094
117.0	.4092
118.0	.4091
119.0	.4089
120.0	.4087
121.0	.4086
122.0	.4084
123.0	.4083
124.0	.4081
125.0	.4080
126.0	.4078
127.0	.4077
128.0	.4075
129.0	.4074
130.0	.4073
131.0	.4071
132.0	.4070
133.0	.4069
134.0	.4068
135.0	.4066
136.0	.4065
137.0	.4064
138.0	.4063
139.0	.4061
140.0	.4060
141.0	.4059
142.0	.4058
143.0	.4057
144.0	.4056
145.0	.4055
146.0	.4054
147.0	.4052
148.0	.4051
149.0	.4050
150.0	.4049
151.0	.4048
152.0	.4047
153.0	.4046
154.0	.4045
155.0	.4045
156.0	.4044
157.0	.4043
158.0	.4042
159.0	.4041

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
160.0	.4040
161.0	.4039
162.0	.4038
163.0	.4037
164.0	.4036
165.0	.4036
166.0	.4035
167.0	.4034
168.0	.4033
169.0	.4032
170.0	.4032
171.0	.4031
172.0	.4030
173.0	.4029
174.0	.4028
175.0	.4028
176.0	.4027
177.0	.4026
178.0	.4026
179.0	.4025
180.0	.4024
181.0	.4023
182.0	.4023
183.0	.4022
184.0	.4021
185.0	.4021
186.0	.4020
187.0	.4019
188.0	.4019
189.0	.4018
190.0	.4017
191.0	.4017
192.0	.4016
193.0	.4016
194.0	.4015
195.0	.4014
196.0	.4014
197.0	.4013
198.0	.4013
199.0	.4012
200.0	.4012
201.0	.4011
202.0	.4010
203.0	.4010
204.0	.4009
205.0	.4009
206.0	.4008
207.0	.4008

Table 4. Continued (P/P_e=2.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
6.5	3.7009
6.6	3.0719
6.7	2.6870
6.8	2.4211
6.9	2.2235
7.0	2.0693
7.1	1.9448
7.2	1.8416
7.3	1.7543
7.4	1.6792
7.5	1.6138
7.6	1.5562
7.7	1.5050
7.8	1.4591
7.9	1.4176
8.0	1.3799
8.1	1.3455
8.2	1.3139
8.3	1.2848
8.4	1.2578
8.5	1.2328
8.6	1.2095
8.7	1.1877
8.8	1.1672
8.9	1.1481
9.0	1.1300
9.1	1.1130
9.2	1.0968
9.3	1.0816
9.4	1.0671
9.5	1.0534
9.6	1.0403
9.7	1.0278
9.8	1.0159
9.9	1.0045
10.0	.9936
10.1	.9832
10.2	.9732
10.3	.9636
10.4	.9544
10.5	.9455
10.6	.9370
10.7	.9288
10.8	.9208
10.9	.9132
11.0	.9058
11.1	.8987
11.2	.8918

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
11.3	.8851
11.4	.8787
11.5	.8724
11.6	.8664
11.7	.8605
11.8	.8548
11.9	.8492
12.0	.8439
12.1	.8386
12.2	.8336
12.3	.8286
12.4	.8238
12.5	.8192
12.6	.8146
12.7	.8102
12.8	.8058
12.9	.8016
13.0	.7975
13.1	.7935
13.2	.7896
13.3	.7858
13.4	.7820
13.5	.7784
13.6	.7748
13.7	.7714
13.8	.7680
13.9	.7646
14.0	.7614
14.1	.7582
14.2	.7551
14.3	.7520
14.4	.7490
14.5	.7461
14.6	.7432
14.7	.7404
14.8	.7377
14.9	.7350
15.0	.7323
15.1	.7297
15.2	.7272
15.3	.7247
15.4	.7222
15.5	.7198
15.6	.7174
15.7	.7151
15.8	.7128
15.9	.7106
16.0	.7084

Table 4. Continued (P/P_e=2.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
16.1	.7062
16.2	.7041
16.3	.7020
16.4	.7000
16.5	.6979
16.6	.6960
16.7	.6940
16.8	.6921
16.9	.6902
17.0	.6883
17.1	.6865
17.2	.6847
17.3	.6829
17.4	.6812
17.5	.6795
17.6	.6778
17.7	.6761
17.8	.6745
17.9	.6728
18.0	.6713
18.1	.6697
18.2	.6681
18.3	.6666
18.4	.6651
18.5	.6636
18.6	.6622
18.7	.6607
18.8	.6593
18.9	.6579
19.0	.6565
19.1	.6551
19.2	.6538
19.3	.6525
19.4	.6512
19.5	.6499
19.6	.6486
19.7	.6473
19.8	.6461
19.9	.6449
20.0	.6437
20.1	.6425
20.2	.6413
20.3	.6401
20.4	.6390
20.5	.6378
20.6	.6367
20.7	.6356
20.8	.6345
20.9	.6334
21.0	.6323
21.1	.6313
21.2	.6302
21.3	.6292
21.4	.6282
21.5	.6272
21.6	.6262
21.7	.6252
21.8	.6242
21.9	.6233
22.0	.6223
22.1	.6214
22.2	.6204
22.3	.6195
22.4	.6186
22.5	.6177
22.6	.6168
22.7	.6159
22.8	.6151
22.9	.6142
23.0	.6133
23.1	.6125
23.2	.6117
23.3	.6108
23.4	.6100
23.5	.6092
23.6	.6084
23.7	.6076
23.8	.6068
23.9	.6060
24.0	.6053
24.1	.6045
24.2	.6038
24.3	.6030
24.4	.6023
24.5	.6015
24.6	.6008
24.7	.6001
24.8	.5994
24.9	.5987
25.0	.5980
25.1	.5973
25.2	.5966
25.3	.5959
25.4	.5953
25.5	.5946
25.6	.5939

Table 4. Continued (P/P_e=2.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
25.7	.5933
25.8	.5926
25.9	.5920
26.0	.5913
26.1	.5907
26.2	.5901
26.3	.5895
26.4	.5889
27.0	.5853
28.0	.5797
29.0	.5746
30.0	.5699
31.0	.5656
32.0	.5615
33.0	.5577
34.0	.5542
35.0	.5509
36.0	.5478
37.0	.5449
38.0	.5421
39.0	.5395
40.0	.5371
41.0	.5348
42.0	.5326
43.0	.5305
44.0	.5285
45.0	.5267
46.0	.5249
47.0	.5231
48.0	.5215
49.0	.5199
50.0	.5185
51.0	.5170
52.0	.5156
53.0	.5143
54.0	.5131
55.0	.5118
56.0	.5107
57.0	.5096
58.0	.5085
59.0	.5074
60.0	.5064
61.0	.5054
62.0	.5045
63.0	.5036
64.0	.5027
65.0	.5019
66.0	.5010

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
67.0	.5002
68.0	.4995
69.0	.4987
70.0	.4980
71.0	.4973
72.0	.4966
73.0	.4959
74.0	.4953
75.0	.4947
76.0	.4940
77.0	.4934
78.0	.4929
79.0	.4923
80.0	.4918
81.0	.4912
82.0	.4907
83.0	.4902
84.0	.4897
85.0	.4892
86.0	.4887
87.0	.4883
88.0	.4878
89.0	.4874
90.0	.4870
91.0	.4865
92.0	.4861
93.0	.4857
94.0	.4853
95.0	.4850
96.0	.4846
97.0	.4842
98.0	.4839
99.0	.4835
100.0	.4832
101.0	.4828
102.0	.4825
103.0	.4822
104.0	.4819
105.0	.4815
106.0	.4812
107.0	.4809
108.0	.4806
109.0	.4804
110.0	.4801
111.0	.4798
112.0	.4795
113.0	.4793
114.0	.4790

Table 4. Continued (P/P_e=2.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
115.0	.4787
116.0	.4785
117.0	.4782
118.0	.4780
119.0	.4778
120.0	.4775
121.0	.4773
122.0	.4771
123.0	.4768
124.0	.4766
125.0	.4764
126.0	.4762
127.0	.4760
128.0	.4758
129.0	.4756
130.0	.4754
131.0	.4752
132.0	.4750
133.0	.4748
134.0	.4746
135.0	.4744
136.0	.4742
137.0	.4740
138.0	.4739
139.0	.4737
140.0	.4735
141.0	.4734
142.0	.4732
143.0	.4730
144.0	.4729
145.0	.4727
146.0	.4725
147.0	.4724
148.0	.4722
149.0	.4721
150.0	.4719
151.0	.4718
152.0	.4716
153.0	.4715
154.0	.4714
155.0	.4712
156.0	.4711
157.0	.4709
158.0	.4708
159.0	.4707
160.0	.4705
161.0	.4704
162.0	.4703

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
163.0	.4702
164.0	.4700
165.0	.4699
166.0	.4698
167.0	.4697
168.0	.4696
169.0	.4694
170.0	.4693
171.0	.4692
172.0	.4691
173.0	.4690
174.0	.4689
175.0	.4688
176.0	.4687
177.0	.4686
178.0	.4684
179.0	.4683
180.0	.4682
181.0	.4681
182.0	.4680
183.0	.4679
184.0	.4678
185.0	.4677
186.0	.4676
187.0	.4676
188.0	.4675
189.0	.4674
190.0	.4673
191.0	.4672
192.0	.4671
193.0	.4670
194.0	.4669
195.0	.4668
196.0	.4667
197.0	.4667
198.0	.4666
199.0	.4665
200.0	.4664
201.0	.4663
202.0	.4662
203.0	.4662
204.0	.4661
205.0	.4660
206.0	.4659
207.0	.4658
208.0	.4658
209.0	.4657
210.0	.4656

Table 4. Continued (P/P_e=3.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
12.5	3.3257
12.6	3.0693
12.7	2.8661
12.8	2.7000
12.9	2.5611
13.0	2.4427
13.1	2.3402
13.2	2.2505
13.3	2.1711
13.4	2.1001
13.5	2.0363
13.6	1.9785
13.7	1.9258
13.8	1.8775
13.9	1.8331
14.0	1.7921
14.1	1.7541
14.2	1.7186
14.3	1.6856
14.4	1.6546
14.5	1.6256
14.6	1.5983
14.7	1.5725
14.8	1.5481
14.9	1.5251
15.0	1.5032
15.1	1.4825
15.2	1.4627
15.3	1.4439
15.4	1.4259
15.5	1.4087
15.6	1.3923
15.7	1.3766
15.8	1.3615
15.9	1.3470
16.0	1.3331
16.1	1.3197
16.2	1.3069
16.3	1.2945
16.4	1.2825
16.5	1.2710
16.6	1.2598
16.7	1.2491
16.8	1.2386
16.9	1.2286
17.0	1.2188
17.1	1.2094
17.2	1.2002

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
17.3	1.1913
17.4	1.1827
17.5	1.1743
17.6	1.1662
17.7	1.1583
17.8	1.1506
17.9	1.1432
18.0	1.1359
18.1	1.1288
18.2	1.1219
18.3	1.1152
18.4	1.1086
18.5	1.1022
18.6	1.0960
18.7	1.0899
18.8	1.0840
18.9	1.0782
19.0	1.0725
19.1	1.0670
19.2	1.0616
19.3	1.0563
19.4	1.0511
19.5	1.0460
19.6	1.0411
19.7	1.0362
19.8	1.0315
19.9	1.0269
20.0	1.0223
20.1	1.0178
20.2	1.0135
20.3	1.0092
20.4	1.0050
20.5	1.0009
20.6	.9968
20.7	.9929
20.8	.9890
20.9	.9852
21.0	.9814
21.1	.9777
21.2	.9741
21.3	.9706
21.4	.9671
21.5	.9637
21.6	.9603
21.7	.9570
21.8	.9537
21.9	.9505
22.0	.9474

Table 4. Continued (P/P_e=3.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$	KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
22.1	.9443	26.9	.8383
22.2	.9412	27.0	.8368
22.3	.9382	27.1	.8352
22.4	.9353	27.2	.8336
22.5	.9324	27.3	.8321
22.6	.9295	27.4	.8306
22.7	.9267	27.5	.8291
22.8	.9240	27.6	.8276
22.9	.9212	27.7	.8262
23.0	.9186	27.8	.8247
23.1	.9159	27.9	.8233
23.2	.9133	28.0	.8219
23.3	.9108	28.1	.8205
23.4	.9082	28.2	.8191
23.5	.9057	28.3	.8177
23.6	.9033	28.4	.8163
23.7	.9009	28.5	.8150
23.8	.8985	28.6	.8137
23.9	.8961	28.7	.8123
24.0	.8938	28.8	.8110
24.1	.8915	28.9	.8098
24.2	.8893	29.0	.8085
24.3	.8871	29.1	.8072
24.4	.8849	29.2	.8060
24.5	.8827	29.3	.8047
24.6	.8806	29.4	.8035
24.7	.8785	29.5	.8023
24.8	.8764	29.6	.8011
24.9	.8743	29.7	.7999
25.0	.8723	29.8	.7987
25.1	.8703	29.9	.7975
25.2	.8684	30.0	.7964
25.3	.8664	30.1	.7952
25.4	.8645	30.2	.7941
25.5	.8626	30.3	.7930
25.6	.8607	30.4	.7919
25.7	.8589	30.5	.7908
25.8	.8570	30.6	.7897
25.9	.8552	30.7	.7886
26.0	.8535	30.8	.7875
26.1	.8517	30.9	.7865
26.2	.8500	31.0	.7854
26.3	.8482	31.1	.7844
26.4	.8465	31.2	.7833
26.5	.8449	31.3	.7823
26.6	.8432	31.4	.7813
26.7	.8416	31.5	.7803
26.8	.8399	31.6	.7793

Table 4. Continued (P/P_e=3.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
31.7	.7783
31.8	.7773
31.9	.7764
32.0	.7754
32.1	.7745
32.2	.7735
32.3	.7726
32.4	.7716
33.0	.7662
34.0	.7578
35.0	.7500
36.0	.7428
37.0	.7361
38.0	.7299
39.0	.7240
40.0	.7186
41.0	.7135
42.0	.7087
43.0	.7041
44.0	.6999
45.0	.6958
46.0	.6920
47.0	.6884
48.0	.6850
49.0	.6817
50.0	.6786
51.0	.6756
52.0	.6728
53.0	.6701
54.0	.6675
55.0	.6650
56.0	.6627
57.0	.6604
58.0	.6582
59.0	.6561
60.0	.6541
61.0	.6522
62.0	.6503
63.0	.6485
64.0	.6468
65.0	.6451
66.0	.6435
67.0	.6420
68.0	.6404
69.0	.6390
70.0	.6376
71.0	.6362
72.0	.6349
73.0	.6336
74.0	.6324
75.0	.6312
76.0	.6300
77.0	.6289
78.0	.6278
79.0	.6267
80.0	.6256
81.0	.6246
82.0	.6236
83.0	.6227
84.0	.6217
85.0	.6208
86.0	.6199
87.0	.6191
88.0	.6182
89.0	.6174
90.0	.6166
91.0	.6158
92.0	.6151
93.0	.6143
94.0	.6136
95.0	.6129
96.0	.6122
97.0	.6115
98.0	.6108
99.0	.6102
100.0	.6095
101.0	.6089
102.0	.6083
103.0	.6077
104.0	.6071
105.0	.6066
106.0	.6060
107.0	.6054
108.0	.6049
109.0	.6044
110.0	.6039
111.0	.6034
112.0	.6029
113.0	.6024
114.0	.6019
115.0	.6014
116.0	.6010
117.0	.6005
118.0	.6001
119.0	.5996
120.0	.5992

Table 4. Continued (P/P_e=3.0)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$	KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
121.0	.5988	169.0	.5847
122.0	.5984	170.0	.5845
123.0	.5979	171.0	.5843
124.0	.5975	172.0	.5841
125.0	.5972	173.0	.5839
126.0	.5968	174.0	.5837
127.0	.5964	175.0	.5835
128.0	.5960	176.0	.5833
129.0	.5956	177.0	.5831
130.0	.5953	178.0	.5829
131.0	.5949	179.0	.5827
132.0	.5946	180.0	.5825
133.0	.5942	181.0	.5824
134.0	.5939	182.0	.5822
135.0	.5936	183.0	.5820
136.0	.5932	184.0	.5818
137.0	.5929	185.0	.5817
138.0	.5926	186.0	.5815
139.0	.5923	187.0	.5813
140.0	.5920	188.0	.5812
141.0	.5917	189.0	.5810
142.0	.5914	190.0	.5808
143.0	.5911	191.0	.5807
144.0	.5908	192.0	.5805
145.0	.5905	193.0	.5804
146.0	.5902	194.0	.5802
147.0	.5899	195.0	.5800
148.0	.5896	196.0	.5799
149.0	.5894	197.0	.5797
150.0	.5891	198.0	.5796
151.0	.5888	199.0	.5794
152.0	.5886	200.0	.5793
153.0	.5883	201.0	.5792
154.0	.5881	202.0	.5790
155.0	.5878	203.0	.5789
156.0	.5876	204.0	.5787
157.0	.5873	205.0	.5786
158.0	.5871	206.0	.5785
159.0	.5869	207.0	.5783
160.0	.5866	208.0	.5782
161.0	.5864	209.0	.5781
162.0	.5862	210.0	.5779
163.0	.5860	211.0	.5778
164.0	.5857	212.0	.5777
165.0	.5855	213.0	.5775
166.0	.5853	214.0	.5774
167.0	.5851	215.0	.5773
168.0	.5849	216.0	.5772

Table 4. Continued (P/P_e=3.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$	KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
30.0	3.6597	34.8	1.9853
30.1	3.5609	34.9	1.9725
30.2	3.4702	35.0	1.9601
30.3	3.3866	35.1	1.9479
30.4	3.3091	35.2	1.9360
30.5	3.2370	35.3	1.9243
30.6	3.1698	35.4	1.9130
30.7	3.1070	35.5	1.9018
30.8	3.0481	35.6	1.8910
30.9	2.9926	35.7	1.8803
31.0	2.9404	35.8	1.8699
31.1	2.8910	35.9	1.8597
31.2	2.8443	36.0	1.8497
31.3	2.8000	36.1	1.8399
31.4	2.7580	36.2	1.8303
31.5	2.7179	36.3	1.8209
31.6	2.6798	36.4	1.8117
31.7	2.6434	36.5	1.8027
31.8	2.6086	36.6	1.7938
31.9	2.5753	36.7	1.7851
32.0	2.5435	36.8	1.7766
32.1	2.5129	36.9	1.7682
32.2	2.4836	37.0	1.7600
32.3	2.4554	37.1	1.7520
32.4	2.4283	37.2	1.7440
32.5	2.4021	37.3	1.7363
32.6	2.3770	37.4	1.7286
32.7	2.3527	37.5	1.7211
32.8	2.3293	37.6	1.7137
32.9	2.3067	37.7	1.7065
33.0	2.2848	37.8	1.6994
33.1	2.2636	37.9	1.6923
33.2	2.2431	38.0	1.6855
33.3	2.2233	38.1	1.6787
33.4	2.2041	38.2	1.6720
33.5	2.1854	38.3	1.6654
33.6	2.1673	38.4	1.6590
33.7	2.1498	38.5	1.6526
33.8	2.1327	38.6	1.6464
33.9	2.1161	38.7	1.6402
34.0	2.1000	38.8	1.6342
34.1	2.0843	38.9	1.6282
34.2	2.0690	39.0	1.6223
34.3	2.0542	39.1	1.6165
34.4	2.0397	39.2	1.6108
34.5	2.0256	39.3	1.6052
34.6	2.0118	39.4	1.5997
34.7	1.9984	39.5	1.5942

Table 4. Continued (P/P_e=3.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
39.6	1.5888
39.7	1.5835
39.8	1.5783
39.9	1.5731
40.0	1.5680
40.1	1.5630
40.2	1.5581
40.3	1.5532
40.4	1.5484
40.5	1.5436
40.6	1.5390
40.7	1.5343
40.8	1.5298
40.9	1.5253
41.0	1.5208
41.1	1.5164
41.2	1.5121
41.3	1.5078
41.4	1.5036
41.5	1.4994
41.6	1.4953
41.7	1.4912
41.8	1.4872
41.9	1.4832
42.0	1.4793
42.1	1.4754
42.2	1.4716
42.3	1.4678
42.4	1.4641
42.5	1.4604
42.6	1.4567
42.7	1.4531
42.8	1.4495
42.9	1.4460
43.0	1.4425
43.1	1.4390
43.2	1.4356
43.3	1.4322
43.4	1.4289
43.5	1.4256
43.6	1.4223
43.7	1.4191
43.8	1.4159
43.9	1.4127
44.0	1.4096
44.1	1.4065
44.2	1.4034
44.3	1.4004

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
44.4	1.3973
44.5	1.3944
44.6	1.3914
44.7	1.3885
44.8	1.3856
44.9	1.3828
45.0	1.3799
45.1	1.3771
45.2	1.3744
45.3	1.3716
45.4	1.3689
45.5	1.3662
45.6	1.3635
45.7	1.3609
45.8	1.3583
45.9	1.3557
46.0	1.3531
46.1	1.3506
46.2	1.3481
46.3	1.3456
46.4	1.3431
46.5	1.3406
46.6	1.3382
46.7	1.3358
46.8	1.3334
46.9	1.3311
47.0	1.3287
47.1	1.3264
47.2	1.3241
47.3	1.3218
47.4	1.3196
47.5	1.3173
47.6	1.3151
47.7	1.3129
47.8	1.3107
47.9	1.3085
48.0	1.3064
48.1	1.3043
48.2	1.3022
48.3	1.3001
48.4	1.2980
48.5	1.2959
48.6	1.2939
48.7	1.2919
48.8	1.2899
48.9	1.2879
49.0	1.2859
49.1	1.2840

Table 4. Continued (P/P_e=3.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
49.2	1.2820
49.3	1.2801
49.4	1.2782
49.5	1.2763
49.6	1.2744
49.7	1.2725
49.8	1.2707
49.9	1.2688
50.0	1.2670
51.0	1.2496
52.0	1.2333
53.0	1.2182
54.0	1.2042
55.0	1.1910
56.0	1.1786
57.0	1.1670
58.0	1.1561
59.0	1.1458
60.0	1.1360
61.0	1.1268
62.0	1.1180
63.0	1.1097
64.0	1.1018
65.0	1.0943
66.0	1.0871
67.0	1.0803
68.0	1.0737
69.0	1.0675
70.0	1.0615
71.0	1.0557
72.0	1.0502
73.0	1.0449
74.0	1.0399
75.0	1.0350
76.0	1.0303
77.0	1.0257
78.0	1.0214
79.0	1.0172
80.0	1.0131
81.0	1.0092
82.0	1.0054
83.0	1.0017
84.0	.9981
85.0	.9947
86.0	.9914
87.0	.9881
88.0	.9850
89.0	.9820

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
90.0	.9790
91.0	.9762
92.0	.9734
93.0	.9707
94.0	.9681
95.0	.9655
96.0	.9630
97.0	.9606
98.0	.9583
99.0	.9560
100.0	.9537
101.0	.9515
102.0	.9494
103.0	.9473
104.0	.9453
105.0	.9434
106.0	.9414
107.0	.9395
108.0	.9377
109.0	.9359
110.0	.9342
111.0	.9324
112.0	.9308
113.0	.9291
114.0	.9275
115.0	.9259
116.0	.9244
117.0	.9229
118.0	.9214
119.0	.9200
120.0	.9186
121.0	.9172
122.0	.9158
123.0	.9145
124.0	.9132
125.0	.9119
126.0	.9106
127.0	.9094
128.0	.9082
129.0	.9070
130.0	.9059
131.0	.9047
132.0	.9036
133.0	.9025
134.0	.9014
135.0	.9003
136.0	.8993
137.0	.8983

Table 4. Continued (P/P_e=3.5)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
138.0	.8973
139.0	.8963
140.0	.8953
141.0	.8943
142.0	.8934
143.0	.8924
144.0	.8915
145.0	.8906
146.0	.8897
147.0	.8889
148.0	.8880
149.0	.8872
150.0	.8863
151.0	.8855
152.0	.8847
153.0	.8839
154.0	.8832
155.0	.8824
156.0	.8816
157.0	.8809
158.0	.8801
159.0	.8794
160.0	.8787
161.0	.8780
162.0	.8773
163.0	.8766
164.0	.8759
165.0	.8753
166.0	.8746
167.0	.8740
168.0	.8733
169.0	.8727
170.0	.8721
171.0	.8715
172.0	.8709
173.0	.8703
174.0	.8697
175.0	.8691
176.0	.8685
177.0	.8680
178.0	.8674
179.0	.8668
180.0	.8663
181.0	.8658
182.0	.8652
183.0	.8647
184.0	.8642
185.0	.8637
186.0	.8632
187.0	.8627
188.0	.8622
189.0	.8617
190.0	.8612
191.0	.8607
192.0	.8602
193.0	.8598
194.0	.8593
195.0	.8589
196.0	.8584
197.0	.8580
198.0	.8575
199.0	.8571
200.0	.8567
201.0	.8562
202.0	.8558
203.0	.8554
204.0	.8550
205.0	.8546
206.0	.8542
207.0	.8538
208.0	.8534
209.0	.8530
210.0	.8526
211.0	.8522
212.0	.8518
213.0	.8515
214.0	.8511
215.0	.8507
216.0	.8504
217.0	.8500
218.0	.8496
219.0	.8493
220.0	.8489
221.0	.8486
222.0	.8483
223.0	.8479
224.0	.8476
225.0	.8473
226.0	.8469
227.0	.8466
228.0	.8463
229.0	.8460
230.0	.8456
231.0	.8453
232.0	.8450
233.0	.8447

Table 4. Continued (P/P_e=3.75)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
70.0	3.1424
70.1	3.1280
70.2	3.1138
70.3	3.0999
70.4	3.0861
70.5	3.0726
70.6	3.0593
70.7	3.0463
70.8	3.0334
70.9	3.0207
71.0	3.0082
71.1	2.9958
71.2	2.9837
71.3	2.9717
71.4	2.9600
71.5	2.9483
71.6	2.9369
71.7	2.9256
71.8	2.9144
71.9	2.9035
72.0	2.8926
72.1	2.8819
72.2	2.8714
72.3	2.8610
72.4	2.8507
72.5	2.8406
72.6	2.8306
72.7	2.8207
72.8	2.8110
72.9	2.8014
73.0	2.7919
73.1	2.7825
73.2	2.7733
73.3	2.7641
73.4	2.7551
73.5	2.7462
73.6	2.7374
73.7	2.7286
73.8	2.7200
73.9	2.7115
74.0	2.7031
74.1	2.6948
74.2	2.6866
74.3	2.6785
74.4	2.6705
74.5	2.6626
74.6	2.6547
74.7	2.6470
74.8	2.6393
74.9	2.6317
75.0	2.6242
75.1	2.6168
75.2	2.6095
75.3	2.6022
75.4	2.5950
75.5	2.5879
75.6	2.5809
75.7	2.5740
75.8	2.5671
75.9	2.5603
76.0	2.5535
76.1	2.5468
76.2	2.5402
76.3	2.5337
76.4	2.5272
76.5	2.5208
76.6	2.5145
76.7	2.5082
76.8	2.5020
76.9	2.4958
77.0	2.4897
77.1	2.4837
77.2	2.4777
77.3	2.4718
77.4	2.4659
77.5	2.4601
77.6	2.4543
77.7	2.4486
77.8	2.4430
77.9	2.4374
78.0	2.4318
78.1	2.4263
78.2	2.4209
78.3	2.4155
78.4	2.4101
78.5	2.4048
78.6	2.3996
78.7	2.3944
78.8	2.3892
78.9	2.3841
79.0	2.3790
79.1	2.3740
79.2	2.3690
79.3	2.3641
79.4	2.3592
79.5	2.3543

Table 4. Continued (P/P_e=3.75)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
79.6	2.3495
79.7	2.3447
79.8	2.3400
79.9	2.3353
80.0	2.3306
80.1	2.3260
80.2	2.3214
80.3	2.3168
80.4	2.3123
80.5	2.3079
80.6	2.3034
80.7	2.2990
80.8	2.2947
80.9	2.2903
81.0	2.2860
81.1	2.2818
81.2	2.2775
81.3	2.2733
81.4	2.2692
81.5	2.2650
81.6	2.2609
81.7	2.2569
81.8	2.2528
81.9	2.2488
82.0	2.2448
82.1	2.2409
82.2	2.2370
82.3	2.2331
82.4	2.2292
82.5	2.2254
82.6	2.2216
82.7	2.2178
82.8	2.2141
82.9	2.2103
83.0	2.2066
83.1	2.2030
83.2	2.1993
83.3	2.1957
83.4	2.1921
83.5	2.1886
83.6	2.1850
83.7	2.1815
83.8	2.1780
83.9	2.1746
84.0	2.1711
84.1	2.1677
84.2	2.1643
84.3	2.1609

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
84.4	2.1576
84.5	2.1543
84.6	2.1510
84.7	2.1477
84.8	2.1444
84.9	2.1412
85.0	2.1380
85.1	2.1348
85.2	2.1316
85.3	2.1285
85.4	2.1253
85.5	2.1222
85.6	2.1191
85.7	2.1160
85.8	2.1130
85.9	2.1100
86.0	2.1070
86.1	2.1040
86.2	2.1010
86.3	2.0980
86.4	2.0951
86.5	2.0922
86.6	2.0893
86.7	2.0864
86.8	2.0835
86.9	2.0807
87.0	2.0779
87.1	2.0751
87.2	2.0723
87.3	2.0695
87.4	2.0667
87.5	2.0640
87.6	2.0613
87.7	2.0586
87.8	2.0559
87.9	2.0532
88.0	2.0505
88.1	2.0479
88.2	2.0453
88.3	2.0427
88.4	2.0401
88.5	2.0375
88.6	2.0349
88.7	2.0324
88.8	2.0298
88.9	2.0273
89.0	2.0248
89.1	2.0223

Table 4. Continued ($P/P_e=3.75$)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
89.2	2.0198
89.3	2.0174
89.4	2.0149
89.5	2.0125
89.6	2.0101
89.7	2.0076
89.8	2.0053
89.9	2.0029
90.0	2.0005
91.0	1.9775
92.0	1.9558
93.0	1.9351
94.0	1.9155
95.0	1.8969
96.0	1.8791
97.0	1.8621
98.0	1.8459
99.0	1.8304
100.0	1.8156
101.0	1.8014
102.0	1.7878
103.0	1.7747
104.0	1.7621
105.0	1.7500
106.0	1.7384
107.0	1.7272
108.0	1.7164
109.0	1.7059
110.0	1.6959
111.0	1.6861
112.0	1.6767
113.0	1.6677
114.0	1.6589
115.0	1.6503
116.0	1.6421
117.0	1.6341
118.0	1.6263
119.0	1.6188
120.0	1.6115
121.0	1.6044
122.0	1.5975
123.0	1.5908
124.0	1.5843
125.0	1.5779
126.0	1.5717
127.0	1.5657
128.0	1.5599
129.0	1.5542
130.0	1.5486
131.0	1.5432
132.0	1.5379
133.0	1.5327
134.0	1.5277
135.0	1.5227
136.0	1.5179
137.0	1.5132
138.0	1.5087
139.0	1.5042
140.0	1.4998
141.0	1.4955
142.0	1.4913
143.0	1.4872
144.0	1.4832
145.0	1.4793
146.0	1.4754
147.0	1.4716
148.0	1.4679
149.0	1.4643
150.0	1.4608
151.0	1.4573
152.0	1.4539
153.0	1.4506
154.0	1.4473
155.0	1.4441
156.0	1.4409
157.0	1.4378
158.0	1.4348
159.0	1.4318
160.0	1.4288
161.0	1.4260
162.0	1.4231
163.0	1.4204
164.0	1.4176
165.0	1.4149
166.0	1.4123
167.0	1.4097
168.0	1.4072
169.0	1.4047
170.0	1.4022
171.0	1.3998
172.0	1.3974
173.0	1.3950
174.0	1.3927
175.0	1.3905
176.0	1.3882
177.0	1.3860

Table 4. Continued (P/P_e=3.75)

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
226.0	1.3085
227.0	1.3074
228.0	1.3063
229.0	1.3051
230.0	1.3040
231.0	1.3030
232.0	1.3019
233.0	1.3008
234.0	1.2998
235.0	1.2987
236.0	1.2977
237.0	1.2967
238.0	1.2956
239.0	1.2946
240.0	1.2936
241.0	1.2927
242.0	1.2917
243.0	1.2907
244.0	1.2898
245.0	1.2888
246.0	1.2879
247.0	1.2870
248.0	1.2861
249.0	1.2852
250.0	1.2843
251.0	1.2834
252.0	1.2825
253.0	1.2816
254.0	1.2808
255.0	1.2799
256.0	1.2791
257.0	1.2783
258.0	1.2774
259.0	1.2766
260.0	1.2758
261.0	1.2750
262.0	1.2742
263.0	1.2734
264.0	1.2726
265.0	1.2718
266.0	1.2711
267.0	1.2703
268.0	1.2696
269.0	1.2688
270.0	1.2681
271.0	1.2673
272.0	1.2666
273.0	1.2659

KL/EI	$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}}$
178.0	1.3839
179.0	1.3817
180.0	1.3796
181.0	1.3776
182.0	1.3755
183.0	1.3735
184.0	1.3715
185.0	1.3696
186.0	1.3677
187.0	1.3658
188.0	1.3639
189.0	1.3621
190.0	1.3602
191.0	1.3585
192.0	1.3567
193.0	1.3550
194.0	1.3532
195.0	1.3515
196.0	1.3499
197.0	1.3482
198.0	1.3466
199.0	1.3450
200.0	1.3434
201.0	1.3419
202.0	1.3403
203.0	1.3388
204.0	1.3373
205.0	1.3358
206.0	1.3344
207.0	1.3329
208.0	1.3315
209.0	1.3301
210.0	1.3287
211.0	1.3273
212.0	1.3259
213.0	1.3246
214.0	1.3233
215.0	1.3220
216.0	1.3207
217.0	1.3194
218.0	1.3181
219.0	1.3169
220.0	1.3156
221.0	1.3144
222.0	1.3132
223.0	1.3120
224.0	1.3108
225.0	1.3097

FIGURES

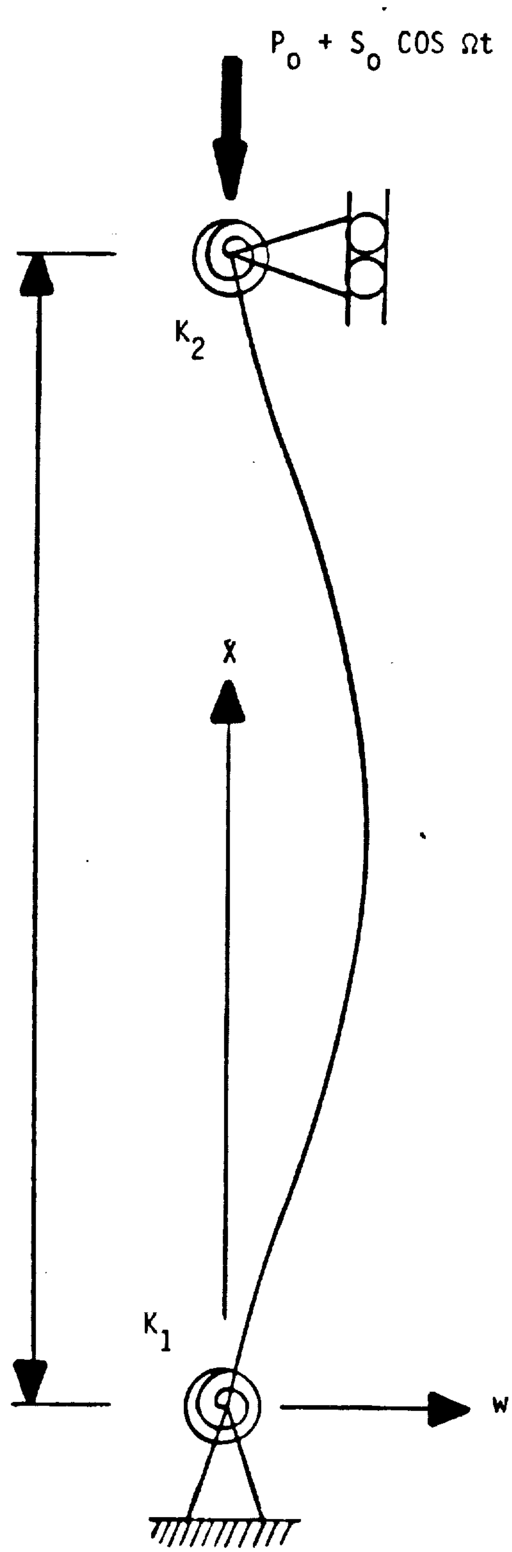


Figure 1. Problem Definition

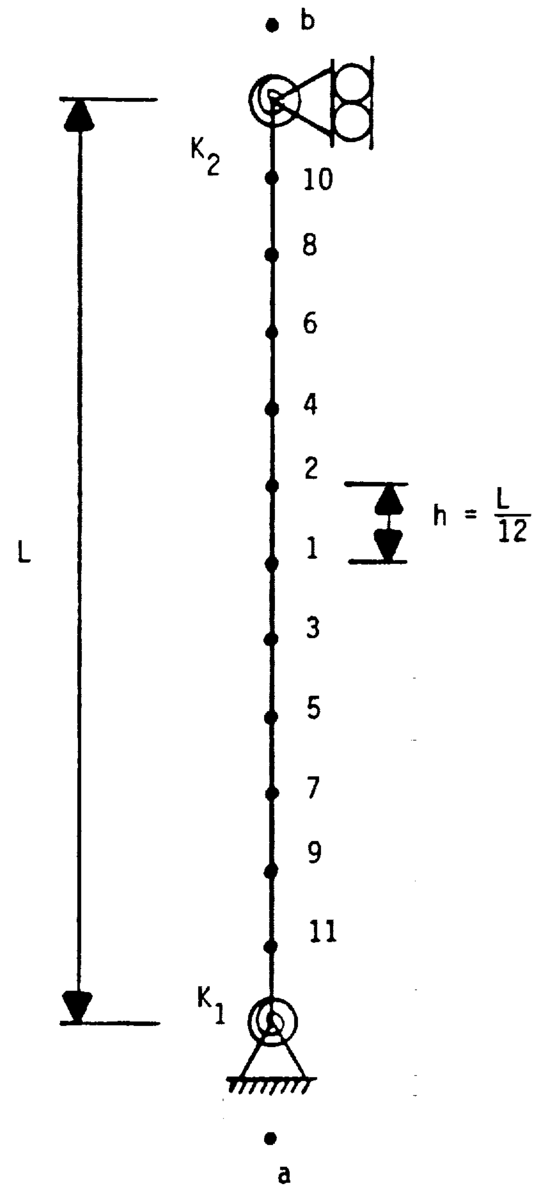


Figure 2. Finite Difference Node Definition

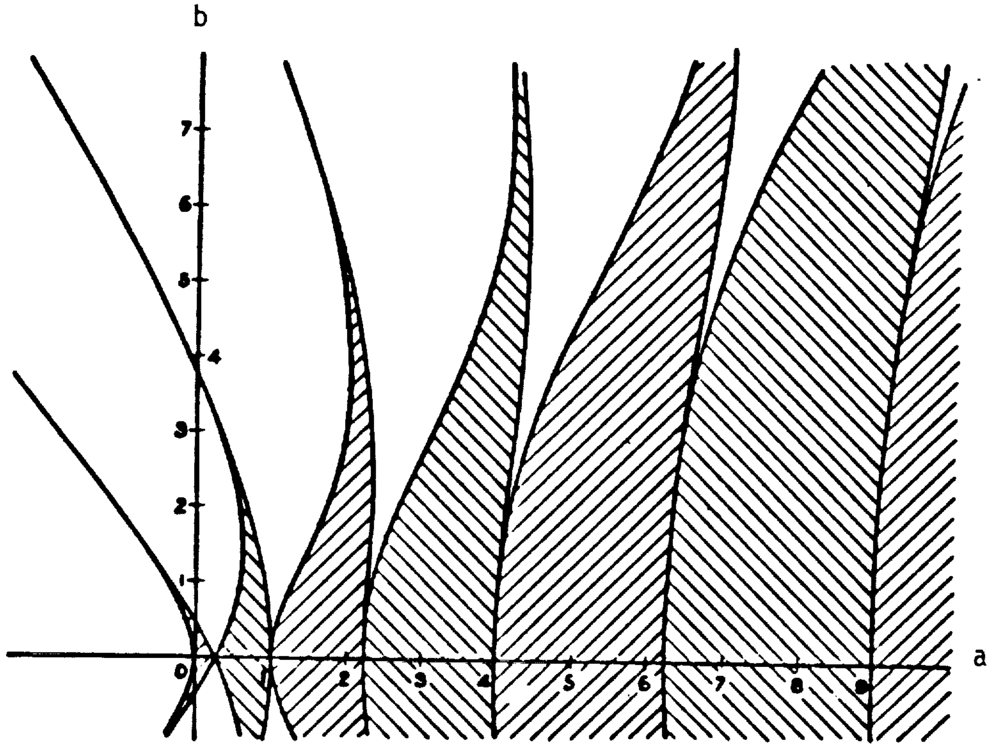


Figure 3. Plot of Instability Regions for a Pinned-Pinned Column (Ref. 1 and 5)

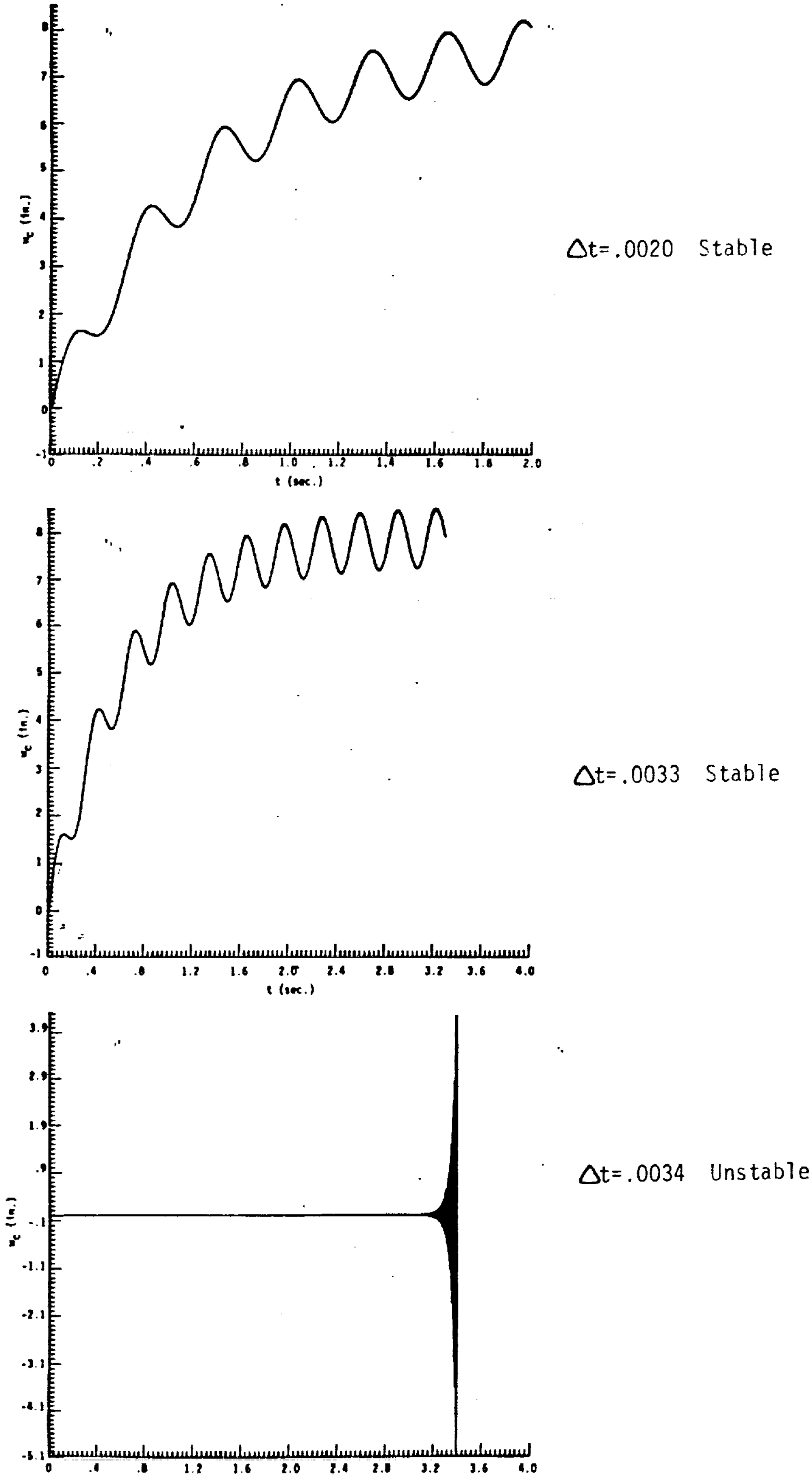


Figure 4. Effect on the Time Increment (Δt) on the Numerical Stability of the Program

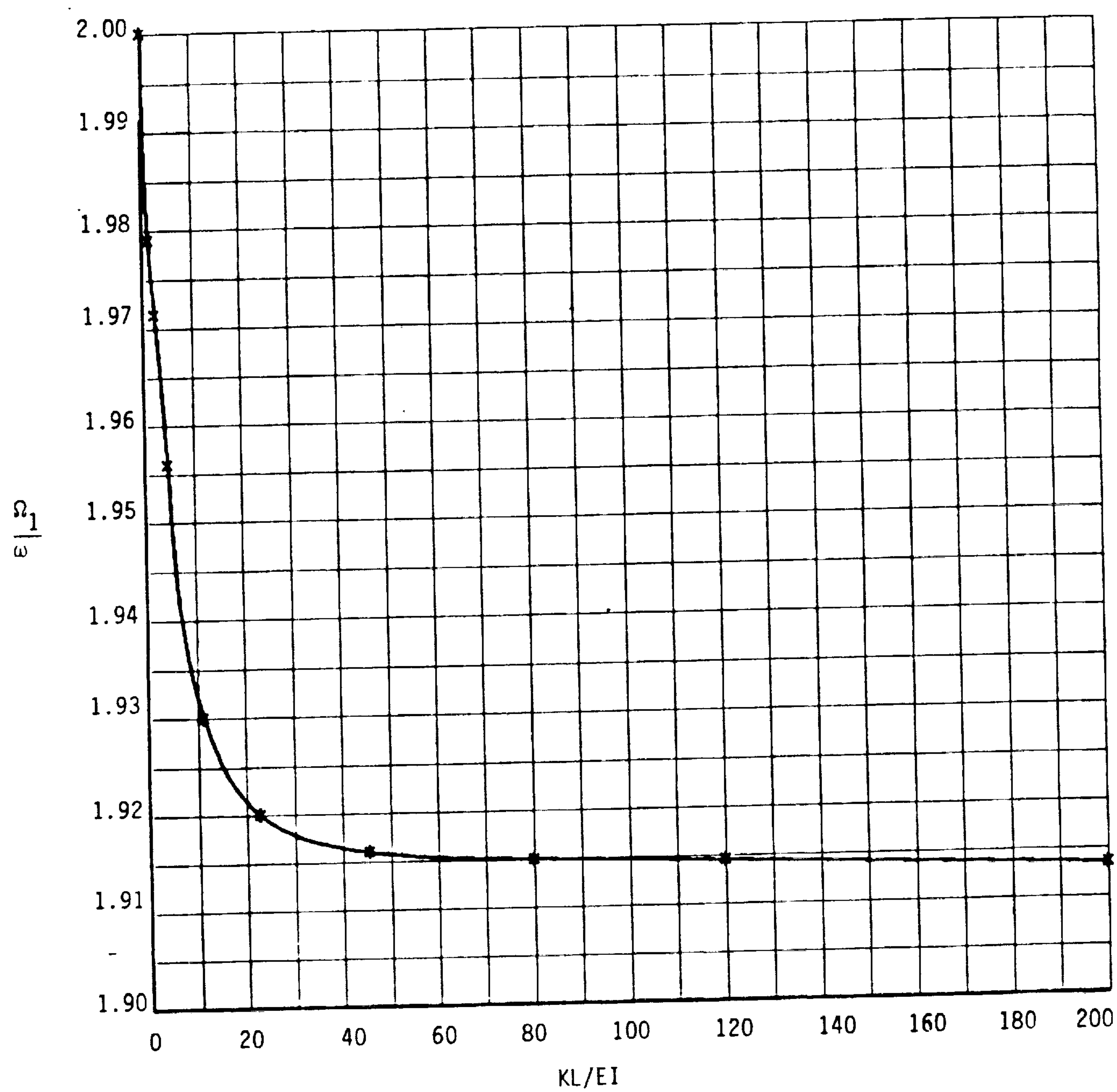


Figure 5. Plot of Ω_1/ω vs KL/EI for Equal End Stiffness

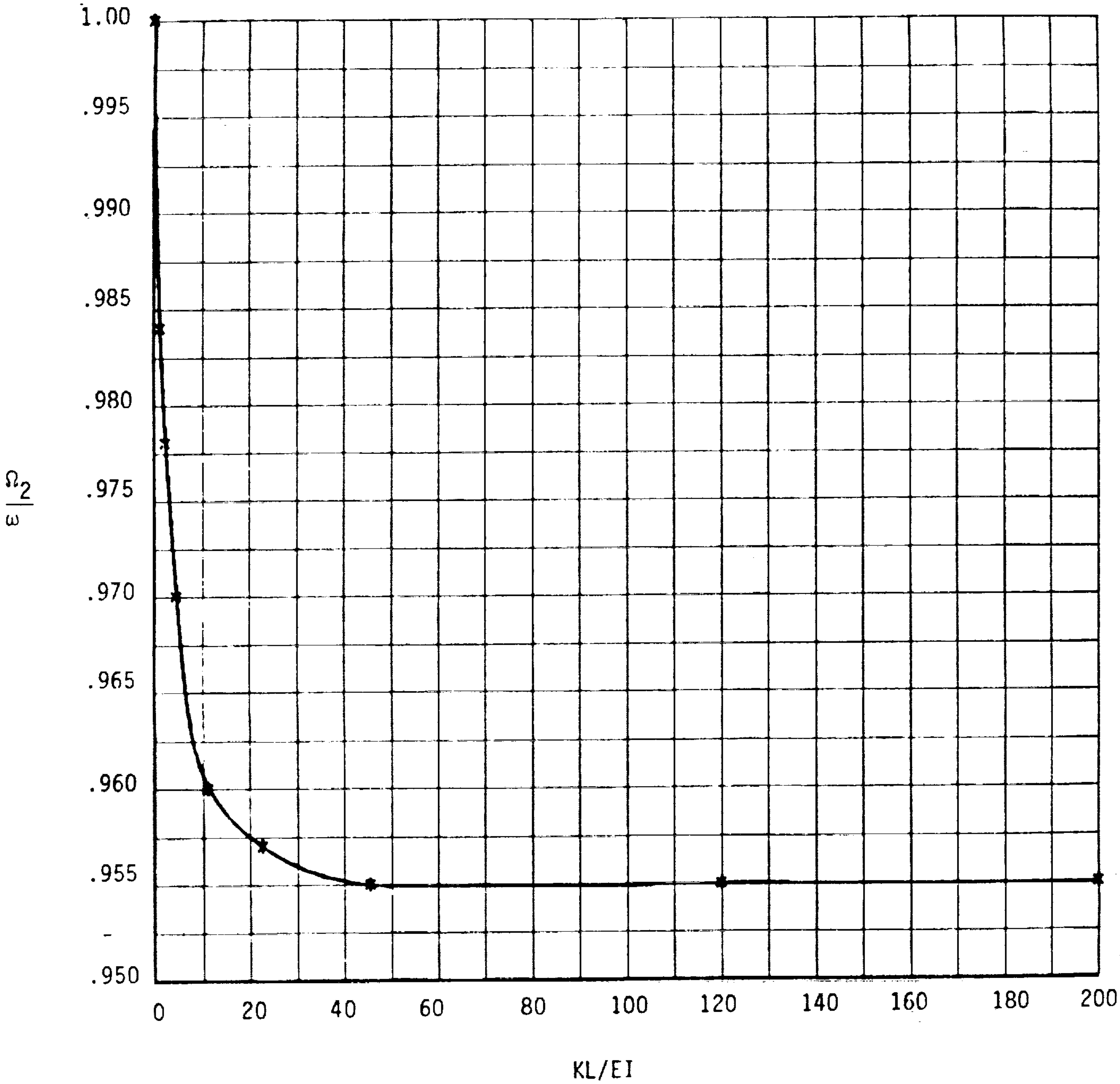
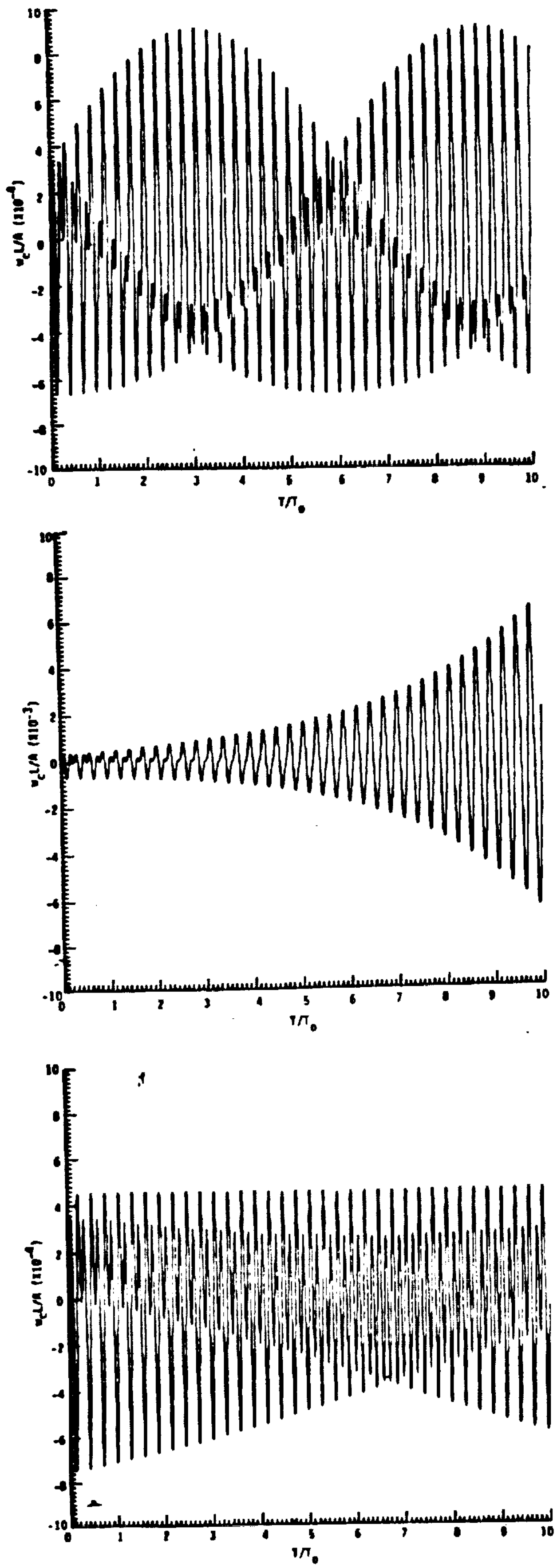


Figure 6. Plot of Ω_2/ω vs KL/EI for Equal End Stiffness



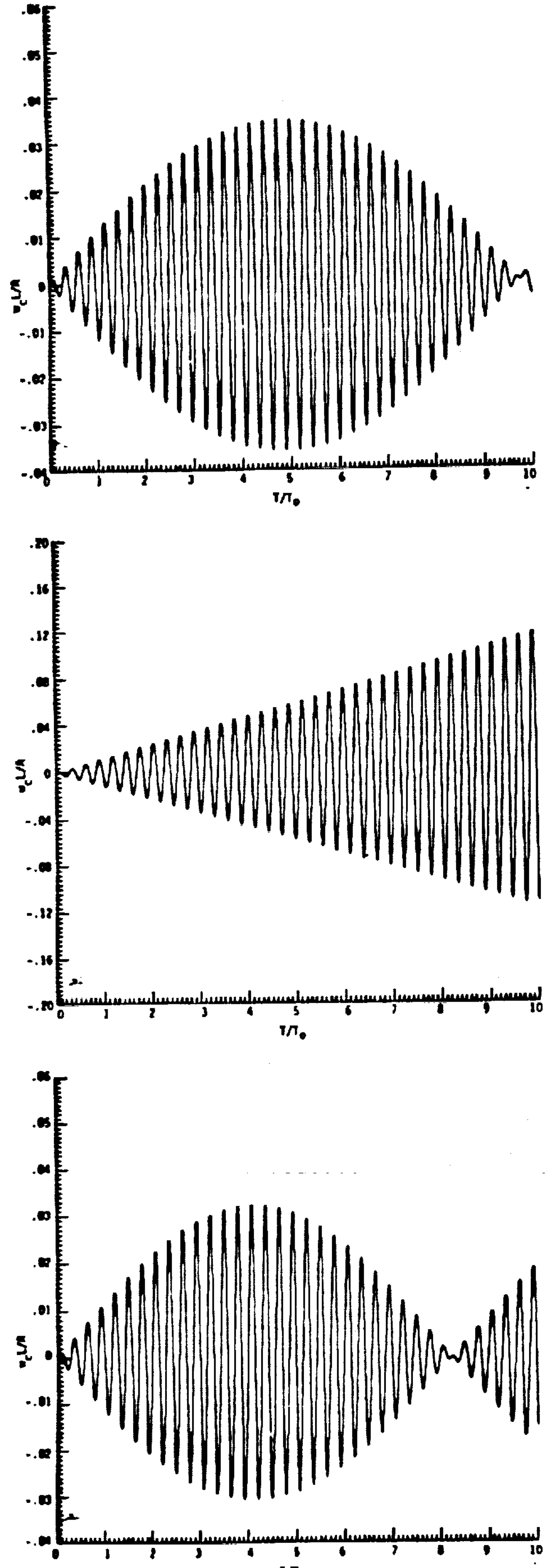
Ω = 2ω

(Critical Frequency)

Ω = 1.956ω

Ω = 1.912ω

Figure 7. Typical Deflection - Time Response at the First Critical Frequency (K=2000, S=4)



$\Omega = \omega$

(Critical Frequency)

$\Omega = .97\omega$

$\Omega = .94\omega$

Figure 8. Typical Deflection - Time Response at the Second Critical Frequency (K=2000, S=4)

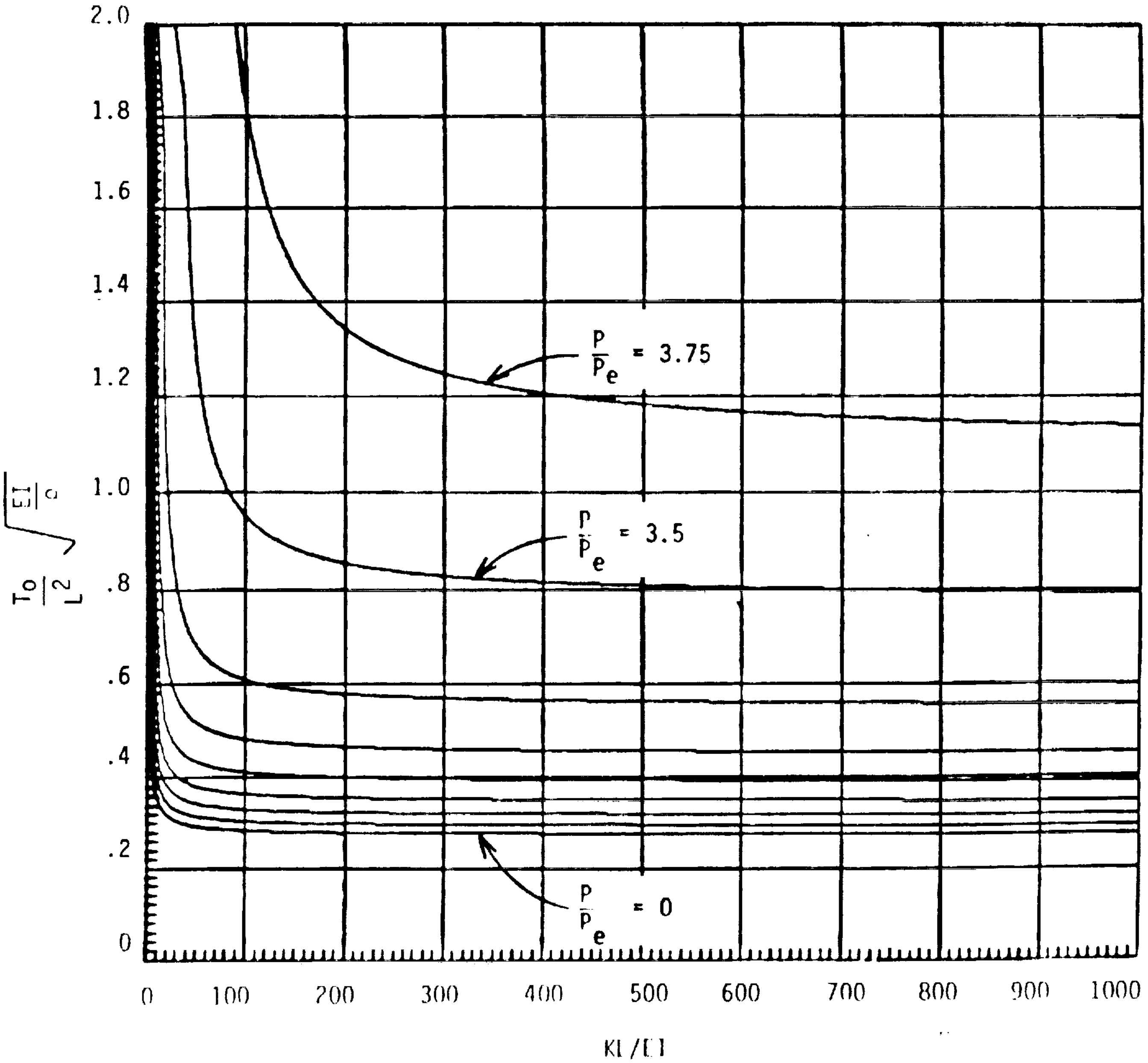


Figure 9. Graph of Equation (h) for Certain Values of P/P_e with KL/EI Varying From 0 to 1000. P/P_e Varies From 0 to 3.5 in .5 Increments With the Final One Equal to 3.75.

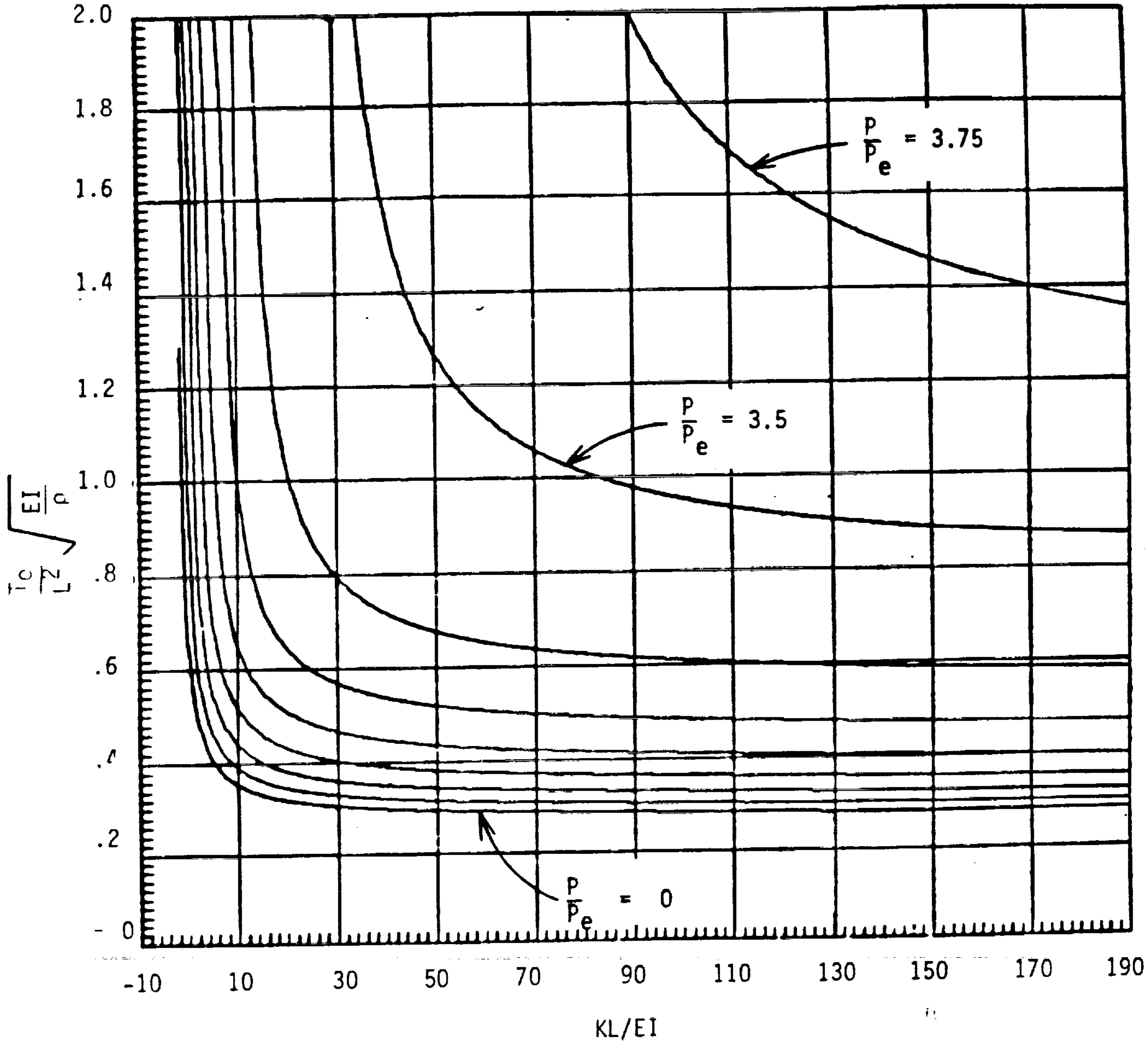


Figure 10. Graph of Equation (h) With the Same P/P_e increments as Figure 9, but With a Smaller Range for KL/EI.

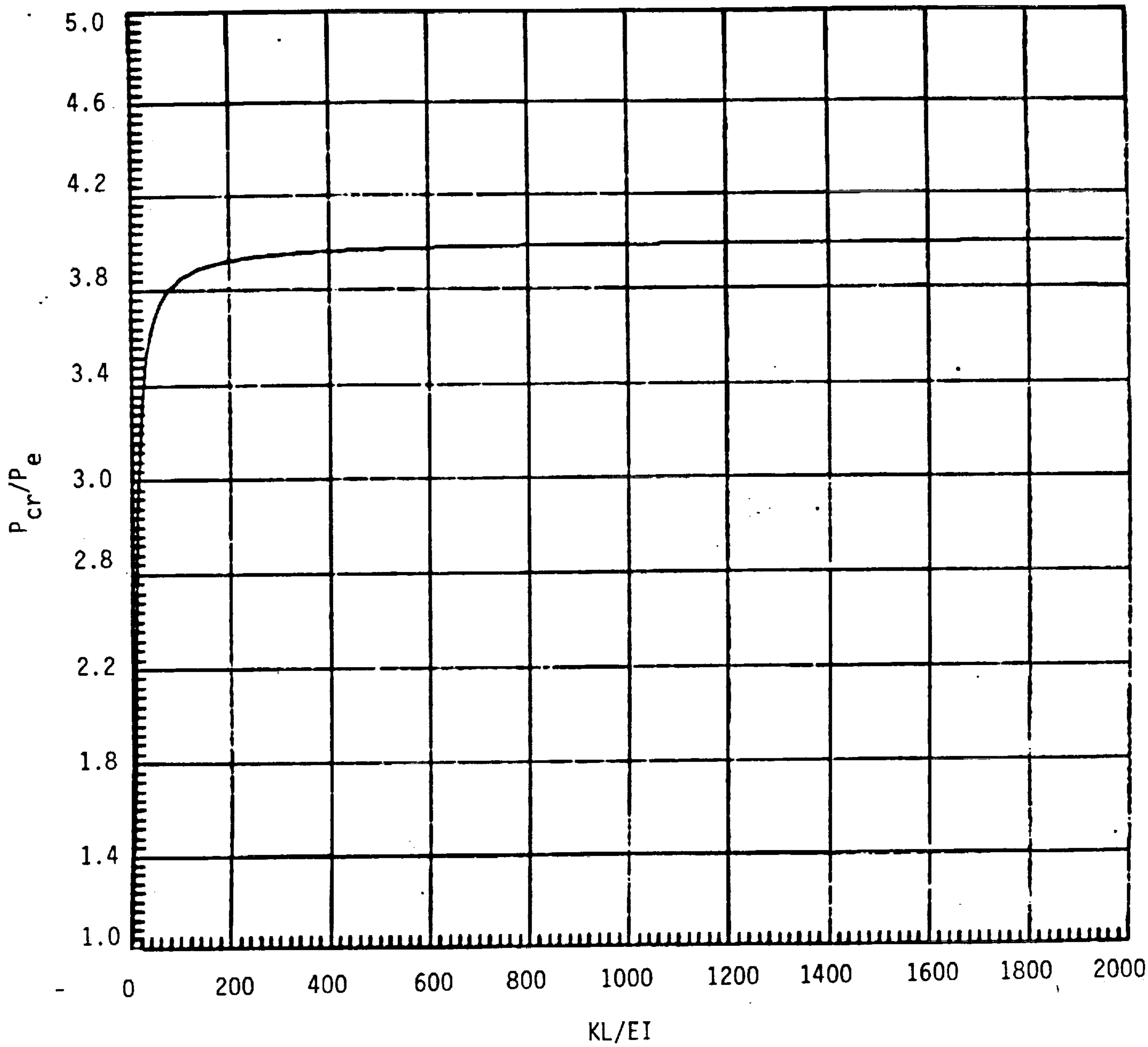


Figure 11. Plot of P_{cr}/P_e vs KL/EI for Equal End Stiffness With KL/EI Varying From 0 to 2000

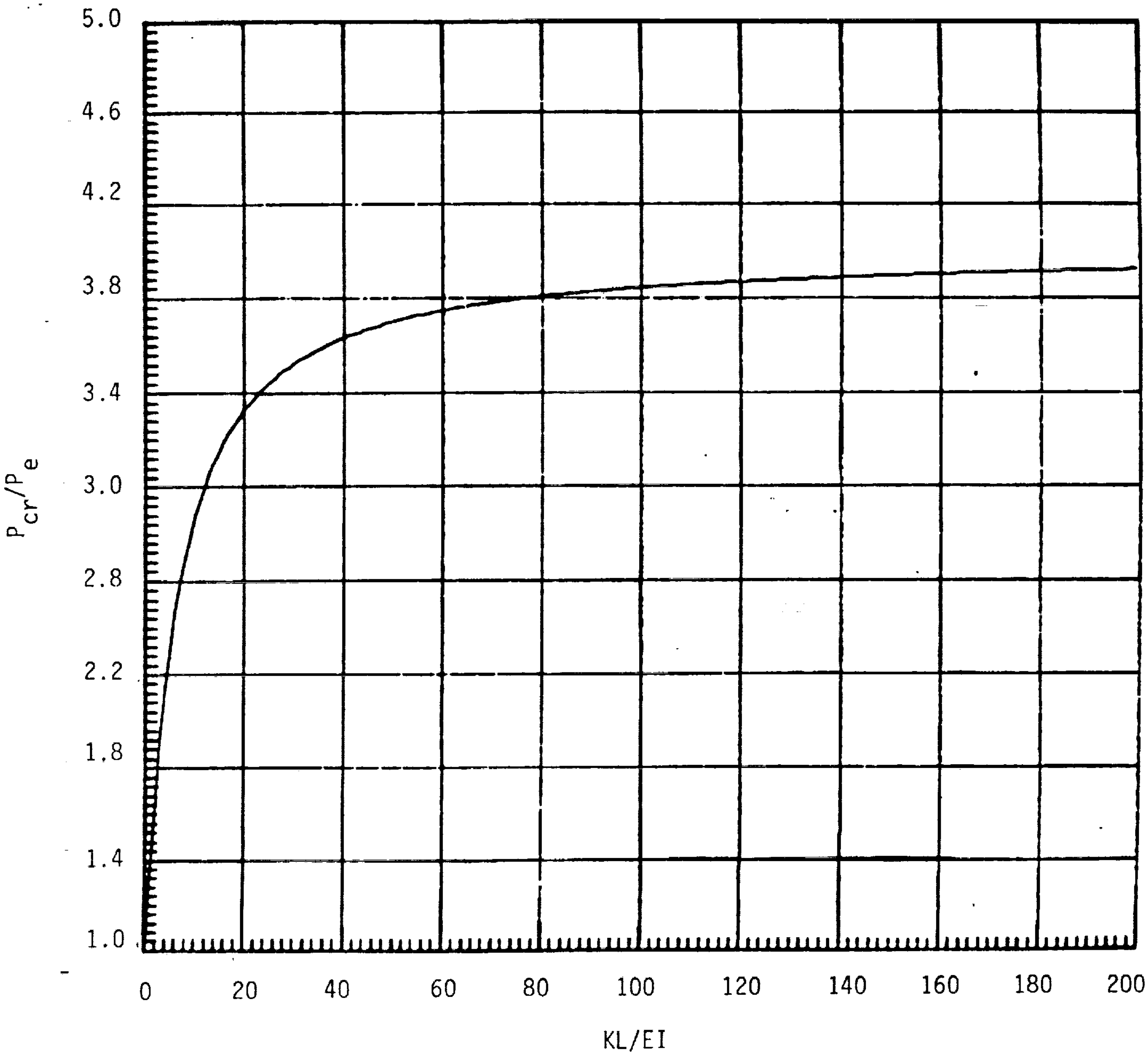


Figure 12. Plot of P_{cr}/P_e vs KL/EI for Equal End Stiffness With KL/EI Varying From 0 to 200

APPENDIX A

APPENDIX A

NATURAL FREQUENCY AND BUCKLING LOAD

A.1 Equal End Stiffnesses

To obtain the natural frequency consider equation (1) with no pulsating force, $S_o = 0$. Then the differential equation (1) becomes;

$$EI \frac{\partial^4 w}{\partial x^4} + P_o \frac{\partial^2 w}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} = 0 \qquad \text{(a)}$$

This partial differential equation can be separated into two ordinary differential equations by separation of variables. Letting $w(x,t) = W(x)T(t)$ we get;

$$\frac{d^4 W}{dx^4} + k^2 \frac{d^2 W}{dx^2} - \lambda W = 0 \qquad \text{(b)}$$

$$\frac{d^2 T}{dt^2} + \frac{C}{\rho} \frac{dT}{dt} + \omega^2 T = 0 \qquad \text{(c)}$$

where $\omega = \sqrt{\frac{EI\lambda}{\rho}}$ (natural circular frequency) (d)

$$k = \sqrt{\frac{P_o}{EI}} \tag{e}$$

The exact solution to equation (b) leads to a transendental equation which cannot be solved explicitly. Approximate solutions have been formulated and one in particular will be discussed here (ref. 2). In the solution of (b), from ref. 2, a shape function was chosen to satisfy the boundary conditions then the error was minimized by invoking the Galerkin criterion to solve for λ . The shape function used was;

$$W(x) = A\left[\sin \frac{\pi x}{L} + B(1 - \cos \frac{2\pi x}{L})\right] \tag{f}$$

where B can be determined from the boundary conditions, equation (2) and (3);

$$B = \frac{KL}{4\pi EI} \tag{g}$$

After invoking Galerkins criterions an explicit expression for λ is obtained. By using equation (d) to get the undamped circular frequency and noting that the natural frequency is;

$$f = \frac{\omega}{2\pi} \text{ (cps)}$$

we get the natural frequency of a column with equal end restraints to be;

$$f = \frac{1}{2L} \sqrt{\frac{EI}{\rho} \; \frac{12(\pi EI)^2\left(\frac{\pi^2}{L^2} - \frac{P}{EI}\right) + 32\ EIKL\left(\frac{5\pi^2}{2L^2} - \frac{P}{EI}\right) + 3(KL)^2\left(\frac{4\pi^2}{L^2} - \frac{P}{EI}\right)}{12(\pi EI)^2 + 32\ EIKL + \frac{9}{4}(KL)^2}}$$

As P goes to the buckling load P_{cr}, the natural frequency goes to zero. Therefore by setting $f = 0$ we get the buckling load to be;

$$P_{cr} = \frac{\pi^2 EI}{L^2}\left[\frac{12(\pi EI)^2 + 80\ EIKL + 12(KL)^2}{12(\pi EI)^2 + 32\ EIKL + 3(KL)^2}\right]$$

The frequency and buckling equations above can be put into a useful graph form by setting the independent variable to be KL/EI. By doing this the following equations are formed;

$$\frac{T_o}{L^2}\sqrt{\frac{EI}{\rho}} = \frac{2}{\pi}\sqrt{\frac{12\pi^2 + 32\left(\frac{KL}{EI}\right) + \frac{9}{4}\left(\frac{KL}{EI}\right)^2}{12\pi^2\left(1 - \frac{P}{P_e}\right) + 32\left(\frac{KL}{EI}\right)\left(\frac{5}{2} - \frac{P}{P_e}\right) + 3\left(\frac{KL}{EI}\right)^2\left(4 - \frac{P}{P_e}\right)}} \qquad \text{(h)}$$

$$\frac{P_{cr}}{P_e} = \frac{12\pi^2 + 80\left(\frac{KL}{EI}\right) + 12\left(\frac{KL}{EI}\right)^2}{12\pi^2 + 32\left(\frac{KL}{EI}\right) + 3\left(\frac{KL}{EI}\right)^2} \qquad \text{(i)}$$

For equation (i), when $K = 0$ we get the buckling load to be;

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

which is correct for a pinned-pinned column as $K = \infty$ we get the buckling load to be;

$$P_{cr} = 4\,\frac{\pi^2 EI}{L^2}$$

which is correct for a fixed-fixed column. Since the upper and lower bounds of the equation are correct, and since the Galerkin method minimizes the error continuously through the domain, then equation (J) is a very good approximation to the buckling load of a column with equal end restraints. Equation h is graphed on figures 9 and 10, and equation i is graphed on figure 11.

A.2 Unequal End Stiffnesses

To obtain the natural frequency and buckling load of a column with unequal end stiffnesses a polynominal shape function was used with the Galerkin criterion. The differential equation (b) was solve using the following shape function;

$$W = a_o + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4$$

To satisfy the boundary conditions the above shape function becomes;

$$W = a_1 \Big[X + \frac{(K_1 K_2 L^4 + 6 \text{ EI } K_1 L^3)}{(2 \text{ EI } K_2 L^4 + 12 (\text{EI})^2 L^3)} X^2$$

$$- \frac{(2K_1 K_2 L^3 + 6 \text{ EI } K_2 L^2 + 10 \text{ EI } K_1 L^2 + 24 (\text{EI})^2 L)}{(2 \text{ EI } K_2 l^4 + 12 (\text{EI})^2 L^3)} X^3$$

$$+ \frac{(K_1 K_2 L^2 + 4 \text{ EI } K_2 L + 4 \text{ EI } K_1 L + 12 (\text{EI})^2}{(2 \text{ EI } K_2 L^4 + 12 (\text{EI})^2 L^3)} X^4\Big]$$

C-2

After invoking Galerkins criterion, the natural frequency of a column with unequal end stiffnesses is obtained.

$$f = \frac{1}{2\pi}\sqrt{\frac{EI}{\rho} \quad \frac{(C_4L^4 \; + C_3L^3 + \ldots + C_o) - P\,(B_6L^6 + B_5L^5 + \ldots + B_2L^2)}{(D_8L^8 + D_7L^7 + \ldots + D_4L^4)}}$$

where;

$C_o = 435456\ (EI)^5$

$C_1 = 199584\ (EI)^4\ (K_1 + K_2)$

$C_2 = 18144\ (EI)^3\ [K_1^{\,2} + 13\ K_1\ K_2/3 + K_2^{\,2}]$

$C_3 = 6552\ (EI)^2\ [K_1K_2^{\,2} + K_1^{\,2}K_2]$

$C_4 = 504\ EI\ K_1^{\,2}K_2^{\,2}$

$B_2 = 44064\ (EI)^4$

$B_3 = 11232\ (EI)^3\ (K_1 + K_2)$

$B_4 = 864\ (EI)^2\ [K_1^{\,2} + 35\ K_1\ K_2/12 + K_2^{\,2}]$

$B_5 = 180\ EI\ (K_1\ K_2^{\,2} + K_1^{\,2}\ K_2)$

$B_6 = 12\ K_1^{\,2}K_2^{\,2}$

$D_4 = 4464\ (EI)^5$

$D_5 = 1140\ (EI)^4\ (K_1 + K_2)$

$D_6 = 76\ (EI)^3\ [K_1^{\,2} + 68\ K_1\ K_2/19 + K_2^{\,2}]$

$D_7 = 17\ (EI)^2\ (K_1\ K_2^{\,2} + K_1^{\,2}\ K_2)$

$D_8 = EI\ K_1^{\,2}\ K_2^{\,2}$

As P goes to the buckling load P_{cr}, the natural frequency goes to zero. Therefore by setting f = 0 we get the buckling load to be;

$$P_{cr} = \frac{(C_4 L^4 + C_3 L^3 + \ldots + C_o)}{(B_6 L^6 + B_5 L^5 + \ldots + B_2 L^2)}$$

APPENDIX B

APPENDIX B

THE COMPUTER PROGRAM

The program was written in Fortran 5 and the program listing is on page , Appendix B. There are seven main components of the program.

1. **Constants;** This is where the constants in the program are set up including the columns properties and several dumby variables that are used in the program.

2. **Initial Imperfections;** This is where the initial imperfections are set up using equation 21.

3. **First and Second Data Points;** This portion defines the first two data points. The first being at time zero and the second at the first time step using equation 24.

4. **Remaining Data Points;** This section defines the remaining data points using equation 25.

   During each time increment the deflection of all eleven nodes are calculated but only the center node (node 1) is saved for all time steps. This reduces the amount of array storage needed for each run which intern enables an increased number of time steps to be saved per run due to computer storage limitations.

5. **Check Data**; This routine checks the data to determine if the response is stable or unstable. Essentially the routine checks each peak and if two successive peaks are less than the preceding peak then the response is stable. Due to the wide range of possible responses, this routine is not 100% reliable. Therefore this routine is only an indication of the response and should not be used as a definite check on the stability of the response.

6. **Plot Results**; This routine plots the dimensionless displacement vs dimensionless time. This routine is particular to the computer being used and it therefore cannot be used on any other computer that does not have the plotting subroutines available.

7. **Stiffness Matrix Subroutine**; This subroutine evaluates the stiffness matrix as described by equation (20).

One critical considerations in this analysis is the time step to be used. The program was tested with constants from Section 3.1 and $P/P_{cr} = .5$, $S/P_{cr} = .5$, $K = 2000$, $\Omega = 20$ and the time step was varied between .001 to .0035. Figure 4 shows the maximum mid-span deflection, w_c (in.) vs time, t (sec.) and it shows that the maximum time step that can be used is .0033 sec. At $\Delta T = .0034$ sec. and greater the analysis becomes numerically unstable. The critical time step was particular to the variables used and it is recommended that the time interval be no larger than one percent of the natural period of the column. The initial imperfection was chosen to be two orders of magnitude smaller than the tolerance of a typical column.

```
*************************************************************
*                                                           *
*                    COMPUTER PROGRAM                       *
*        FINITE DIFFERENCE ROUTINE TO DETERMINE THE         *
*        DYNAMIC RESPONSE OF A COLUMN TO A DEAD LOAD        *
*      AND A PULSATING LOAD WITH UNEQUAL END RESTRAINTS     *
*                                                           *
*************************************************************
      PROGRAM COLBUCK
      DIMENSION WI(11),K(11,11),WBAR(11),XD(5000),YD(5000)
      DIMENSION ZZ1(11),ZZ2(11),ZZ3(11),ZZ5(11),WIM1(11),WIM2(11)
      REAL KK1,KK2,L,IX,K,H
      CHARACTER*20 XCHAR
      CHARACTER*20 YCHAR
*
  5   PRINT *,'INPUT OMEGA,S'
      READ *,OMEGA,S
*
************************ CONSTANTS **************************
*
      DT=.0020
      PI=3.1415927
      DELTA=.00001
      E=30E+6
      IX=.0021
      L=144.
      H=1/12.
      KK1=87500.0
      KK2=87500.0
      AA=.0859
      P=58.8
      TO=.260413
      RHO=6.3E-5
      CCO=0.0
      DTB=DT/TO
      EI=E*IX
      Z2=RHO*L**4/(EI*TO**2)
      Z3=CCO*L**4/(EI*TO)
      Q1=(KK1*L*H/(2.0*EI)-1)/(KK1*L*H/(2.0*EI)+1)
      Q2=(KK2*L*H/(2.0*EI)-1)/(KK2*L*H/(2.0*EI)+1)
      IFLAG=0
*
****************** INITIAL IMPERFECTIONS ********************
*
      BI=0.0
      WBAR(1)=-(PI**2)*L*DELTA/AA
      DO 10 I=2,10,2
      BI=BI+1
      WBAR(I)=WBAR(1)*SIN(PI*(6+BI)/12)
 10   WBAR(I+1)=WBAR(I)
*
```

```
*
***************** FIRST & SECOND DATA POINTS *****************
*
      DO 20 I=1,11
      WIM1(I)=0.0
 20   WI(I)=((P+S)*L**2*WBAR(I)*DTB**2)/(2*Z2*EI)
*
      YD(1)=WIM1(1)
      YD(2)=WI(1)
      XD(1)=0.0
      XD(2)=DT
*
******************* REMAINING DATA POINTS ******************
*
      DO 30  I=3,5000
      DO 35 J=1,11
      WIM2(J)=WIM1(J)
  35  WIM1(J)=WI(J)
      XD(I)=DT*(I-1)
      T=DT*(I-2)
      Z1=(P+S*COS(OMEGA*T))*L**2/EI
      CALL XKM(Z1,H,Q1,Q2,K)
      DO 40 II=1,11
      SUM=0
      DO 50 JJ=1,11
 50   SUM=SUM+K(II,JJ)*WIM1(JJ)
 40   ZZ3(II)=SUM
      DO 60 J=1,11
      ZZ1(J)=2*WIM1(J)-WIM2(J)
      ZZ2(J)=Z1*WBAR(J)+ZZ3(J)+(Z3/DTB)*(WIM1(J)-WIM2(J))
      ZZ5(J)=(ZZ2(J)*DTB**2)/(Z2+Z3*DTB/2)
 60   WI(J)=ZZ1(J)-ZZ5(J)
      YD(I)=WI(1)
 30   CONTINUE
*
**************** CHECK DATA *******************
*
      YDPKNEW=0.0
      ICOUNT=0
      DO 55 I=2,4999
      YB=YD(I-1)
      YN=YD(I)
      YA=YD(I+1)
      IF(YB.LT.YN.AND.YA.LT.YN)THEN
      YDPKOLD=YDPKNEW
      YDPKNEW=YN
      IF(YDPKNEW.GT.YDPKOLD)ICOUNT=0
      IF(YDPKNEW.LT.YDPKOLD)ICOUNT=ICOUNT+1
      IF(ICOUNT.EQ.2)GOTO 65
      END IF
  55  CONTINUE
      PRINT *,'*** UNSTABLE ***'
      IFLAG=2
      GOTO 75
  65  PRINT *,'*** STABLE ***'
```

```
      IFLAG=1
  75  CONTINUE
      IF(IFLAG.EQ.2)GOTO 200
 200  CONTINUE
*
      PRINT *,'   TYPE'
      PRINT *,'   1) TO PLOT'
      PRINT *,'   2) TO TRY AGAIN'
      READ *,IIFLAG
      IF(IIFLAG.EQ.2)GOTO 5
*
********************* PLOT RESULTS *************************
*
      XCHAR='T/TO                                    '
      YCHAR='WL/A                                    '
      CALL PSEUDO
      CALL INFOPLT(1,5000,XD,1,YD,1,.0,
     *0.0,.0,.0,.0,13,XCHAR,16,YCHAR,0)
      CALL CALPLT(0,0,99)
*
*
      STOP
      END
*
***************** STIFFNESS MATRIX SUBROUTINE ****************
*
      SUBROUTINE XKM(Z1,H,Q1,Q2,K)
      DIMENSION K(11,11)
      REAL K,H,H2,H4
      H2=1.0/H**2
      H4=1.0/H**4
      A=6*H4-2*Z1*H2
      B=Z1*H2-4*H4
      C=H4
      D1=H4*(6+Q1)-2*Z1*H2
      D2=H4*(6+Q2)-2*Z1*H2
      K(1,1)=A
      K(1,2)=B
      K(1,3)=B
      K(1,4)=C
      K(1,5)=C
      K(1,6)=0.0
      K(1,7)=0.0
      K(1,8)=0.0
      K(1,9)=0.0
      K(1,10)=0.0
      K(1,11)=0.0
      K(2,2)=A
      K(2,3)=C
      K(2,4)=B
      K(2,5)=0.0
      K(2,6)=C
      K(2,7)=0.0
      K(2,8)=0.0
      K(2,9)=0.0
```

```
      K(2,10)=0.0
      K(2,11)=0.0
      K(3,3)=A
      K(3,4)=0.0
      K(3,5)=B
      K(3,6)=0.0
      K(3,7)=C
      K(3,8)=0.0
      K(3,9)=0.0
      K(3,10)=0.0
      K(3,11)=0.0
      K(4,4)=A
      K(4,5)=0.0
      K(4,6)=B
      K(4,7)=0.0
      K(4,8)=C
      K(4,9)=0.0
      K(4,10)=0.0
      K(4,11)=0.0
      K(5,5)=A
      K(5,6)=0.0
      K(5,7)=B
      K(5,8)=0.0
      K(5,9)=C
      K(5,10)=0.0
      K(5,11)=0.0
      K(6,6)=A
      K(6,7)=0.0
      K(6,8)=B
      K(6,9)=0.0
      K(6,10)=C
      K(6,11)=0.0
      K(7,7)=A
      K(7,8)=0.0
      K(7,9)=B
      K(7,10)=0.0
      K(7,11)=C
      K(8,8)=A
      K(8,9)=0.0
      K(8,10)=B
      K(8,11)=0.0
      K(9,9)=A
      K(9,10)=0.0
      K(9,11)=B
      K(10,10)=D2
      K(10,11)=0.0
      K(11,11)=D1
      DO 15 I=1,11
      DO 15 J=1,11
15    K(J,I)=K(I,J)
      RETURN
      END
```

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16. Abstract

The aim of this investigation is to conduct a theoretical study of the dynamic behavior of columns with partial end restraints and loads consisting of a dead load and a pulsating load. The differential equation is solved using a lumped impulse recurrence formula relative to time coupled with a finite difference discretization along the member length. A computer program is written from which the first critical frequencies are found as a function of end stiffness. The case of a pinned ended column compares very well with the exact solution. Also, the natural frequency and buckling load formulas are derived for equal and unequal end restraints.

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