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Causality of Plasma Permittivity with Loss and Restoring Terms

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CAUSALITY OF PLASMA PERMITTIVITY WITH LOSS AND RESTORING TERMS

ABSTRACT

The causality of the complex permittivity for a plasma model, in which both loss and restoring terms are included, is shown. Discussions as to what occurs to this causality property when the loss and restoring terms vanish are presented. Results are shown to agree with earlier work.

INTRODUCTION

The expression representing the complex permittivity of a plasma material is derived through the equation of motion of electrons in the presence of an applied electric field [1-2]. In many plasma models, the material is assumed inhomogeneous in the direction normal to the plane layer and homogeneous in the unbounded lateral directions. The inhomogeneity of the material is expressed in the complex permittivity representation through the plasma and collision frequencies. An electron density which is identified with the plasma frequency and a loss term which is identified with collision frequency vary as one moves into the material. It is this spatial dependency of the plasma and collision frequencies which require numerical evaluation of Maxwell's equations in the inhomogeneous plasma material as a function of the applied electric field frequency.

Since the complex permittivity is dispersive, its frequency behavior for a fixed plasma profile is also of interest. In particular, the complex permittivity should be causal; that is, the time response for the complex permittivity must exist only for times greater than zero. The expressions given in the literature representing complex permittivity [1-3], which are derived with only losses (collision frequency) assumed in the model, are shown to be noncausal [4]. It was shown in reference 4 that the system can be made causal by the addition of a term proportional to the impulse function $\delta(\omega)$.

The purpose of this paper is to show that the complex permittivity representation, in which both loss and restoring terms are included in the plasma model, obeys the principle of causality. By a proper limiting procedure as the restoring term vanishes, the same impulsive term that appears in the solution given in reference 4 is recovered.

PROOF OF CAUSALITY

The differential equation which represents the dynamics of electron motion in a plasma or dielectric material in the presence of an applied electric is given as [5]

$$m \frac{d^2x(t)}{dt^2} + m\,\nu \frac{dx(t)}{dt} + b\ x(t) = -\ q\ \mathcal{E}(t) \tag{1}$$

where m is the mass of an electron, q is the charge of an electron, ν is the collision frequency (corresponds to loss) of the plasma, b is the coefficient of the restoring force term, x(t) is the displacement of the electrons, and $\mathcal{E}(t)$ is the applied electric field. The derivation of the complex permittivity based on equation (1) is given in reference 5. Therefore, without proof, the complex permittivity for this model is given as

$$\varepsilon(\omega) = \varepsilon_{o}\left[\ 1 - \frac{\omega_p^2\ (\omega^2 - \beta^2)}{(\omega^2-\beta^2)^2 + \omega^2\nu^2} - j\ \frac{\omega_p^2\ \omega\ \nu}{(\omega^2-\beta^2)^2 + \omega^2\nu^2}\ \right] \tag{2}$$

where ω_p is the plasma frequency, ν is the collision frequency of the plasma, $\beta = b/m$, ε_o is the free space permittivity, and ω is the frequency of the applied electric field.

Rewrite equation (2) as

$$\varepsilon(\omega) = \varepsilon_o - \frac{\omega_p^2\ \varepsilon_o}{\omega^2-\beta^2-j\omega\nu} \tag{3}$$

The inverse Fourier transform is defined as

$$\bar{\varepsilon}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon(\omega)\ e^{j\omega t} d\omega \tag{4}$$

Substituting equation (3) into equation (4),

$$\bar{\varepsilon}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\ \varepsilon_o - \frac{\omega_p^2\varepsilon_o}{\omega^2-\beta^2-j\omega\nu}\ \right] e^{j\omega t} d\omega \tag{5}$$

or

$$\bar{\varepsilon}(t) = \varepsilon_o \delta(t) - \frac{\omega_p^2 \varepsilon_o}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 - \beta^2 - j\omega\nu} e^{j\omega t} d\omega \tag{6}$$

The integral in equation (6) can be readily evaluated by contour integration, once the poles of its integrand are determined.

The poles are determined by setting the denominator of the integrand in equation (6) to zero. The discriminant obtained by using the quadratic formula permits, in general, three solution sets. The solution set corresponding to the condition $\nu^2 < 4\beta^2$ is considered invalid because as β is ultimately allowed to approach zero, ν must be real. For the condition $\nu^2 > 4\beta^2$, the poles are given as

$$\omega_1 = \frac{j}{2} \left[\nu + \sqrt{\nu^2 - 4\beta^2} \right] \tag{7}$$

and

$$\omega_2 = \frac{j}{2} \left[\nu - \sqrt{\nu^2 - 4\beta^2} \right] \tag{8}$$

which are shown in figure 1.

The complex function $f(\omega) = \dfrac{e^{j\omega t}}{\omega^2 - \beta^2 - j\omega\nu}$ is meromorphic in the upper half plane of the complex-ω plane and analytic in the lower half plane of the complex-ω plane. From Cauchy-Goursat theorem, the closed contour integral of $f(\omega)$ for $t < 0$ is zero [6]; that is,

$$\oint f(\omega)\, d\omega = 0 \tag{9}$$

where the closed contour is indicated by the dashed lines in figure 1. Rewriting equation (9) over the individual contours shown in the figure:

$$\int_{-\infty}^{\infty} \frac{e^{j\omega t}}{\omega^2 - \beta^2 - j\omega\nu} d\omega + \int_{C_{r<}} \frac{e^{j\omega t}}{\omega^2 - \beta^2 - j\omega\nu} d\omega = 0 \tag{10}$$

It is easily shown that the contribution from the $C_{r<}$ integral is zero [6]. Equation (10) now becomes

$$\int_{-\infty}^{\infty} \frac{e^{j\omega t}}{\omega^2 - \beta^2 - j\omega\nu} d\omega = 0 \quad , \; t < 0 \tag{11}$$

which says that $\bar{\varepsilon}(t)$ in equation (6) is zero for $t < 0$.

For $t > 0$ the contour may be closed in the upper half plane as indicated by the solid line shown in figure 1. The closed contour integral must again be zero; that is,

$$\oint f(\omega)\, d\omega = 0 \tag{12}$$

Expanding the total contour into the individual contours as shown in figure 1:

$$\int_{-\infty}^{\infty} f(\omega)\, d\omega + \int_{\rho_1} f(\omega)\, d\omega + \int_{\rho_2} f(\omega)\, d\omega + \int_{C_{r>}} f(\omega)\, d\omega = 0 \tag{13}$$

The contribution from the contour $C_{r>}$ is zero. The contributions from second and third integrals are evaluated by residue theory. It is easily shown that

$$\int_{\rho_1} \frac{e^{j\omega t}}{\omega^2-\beta^2-j\omega\nu}\, d\omega = -\,2\pi\, \frac{e^{-\frac{t}{2}\left[\nu + \sqrt{\nu^2-4\beta^2}\right]}}{\sqrt{\nu^2-4\beta^2}} \tag{14}$$

and

$$\int_{\rho_2} \frac{e^{j\omega t}}{\omega^2-\beta^2-j\omega\nu}\, d\omega = 2\pi\, \frac{e^{-\frac{t}{2}\left[\nu - \sqrt{\nu^2-4\beta^2}\right]}}{\sqrt{\nu^2-4\beta^2}} \tag{15}$$

With these equations, equation (6) becomes

$$\bar{\varepsilon}(t) = \varepsilon_o\delta(t) + 2\,\omega_p^2\,\varepsilon_o e^{-\frac{\nu t}{2}} \left\{ \frac{\sinh\left[\frac{t}{2}\sqrt{\nu^2-4\beta^2}\right]}{\sqrt{\nu^2-4\beta^2}} \right\} \tag{16}$$

for $4\beta^2 < \nu^2$ and $t \geq 0$. For $4\beta^2 < \nu^2$ and $t < 0$, $\bar{\varepsilon}(t)$ equals zero.

The condition $4\beta^2 = \nu^2$ and $t > 0$ can be evaluated by considering the contour integration path as shown in figure 2. In this case there exists a pole of order two at $\omega = j\nu/2$. Evaluating by residue theory,

$$\bar{\varepsilon}(t) = \varepsilon_o\delta(t) - \frac{1}{2\pi}\,\omega_p^2\,\varepsilon_o\left\{2\pi j\left[\frac{d}{d\omega}\left(e^{j\omega t}\right)\right]_{\omega = j\nu/2}\right\}, \quad t \geq 0 \tag{17}$$

and, since there are no poles in the lower half plane, $\varepsilon(t)$ is zero for $t < 0$. Hence, equation (6) becomes

$$\bar{\varepsilon}(t) = \begin{cases} \varepsilon_o \delta(t) + \omega_p^2 \, \varepsilon_o t \, e^{-\frac{\nu t}{2}} & , \; t \geq 0 \\ 0 & , \; t < 0 \end{cases} \tag{18}$$

This result can also be obtained from equation (16) through a proper limiting procedure.

Letting β approach zero, that is the term corresponding to the restoring force vanishes, equation (16) for $4\beta^2 < \nu^2$ becomes

$$\bar{\varepsilon}(t) = \left\{ \begin{array}{ll} \varepsilon_o \delta(t) + \dfrac{\omega_p^2 \, \varepsilon_o}{\nu} \left[1 - e^{-\nu t} \right] & , \; t \geq 0 \\ 0 \; , \; t < 0 & \end{array} \right\} \tag{19}$$

which, once normalized with respect to ε_o, is the same as equation (7) in reference 4. Letting β approach zero in equation (18), which implies ν goes to zero because $\nu^2 = 4\beta^2$, the expression for $\bar{\varepsilon}(t)$ becomes

$$\bar{\varepsilon}(t) = \varepsilon_o \delta(t) + \omega_p^2 \, \varepsilon_o t \qquad , \; t \geq 0 \tag{20}$$

which, once normalized with respect to ε_o, is the same as equation (19) in reference 4.

It is emphasized at this point that $\bar{\varepsilon}(t)$ obeys the principle of causality whenever both the loss and restoring terms are included (see equations (16) and (18)). Allowing β to approach zero or both β and ν to approach zero in the general solutions (equation (16) and equation (18)), causal solutions are obtained (equation (19) and equation (20), respectively). However, if $\beta=0$ in equation (1), the direct derivation of $\varepsilon(\omega)$ and its inverse Fourier transform lead to a noncausal solution (see equations (4) and (5) in reference 4). Also, if $\beta = 0$ and $\nu = 0$ in equation (1), the direct derivation of $\varepsilon(\omega)$ and its inverse Fourier transform lead to a noncausal solution. Reference 4 shows, however, that each of these noncausal solutions can be made causal by the addition of an impulsive term. The next section shows, through a limiting procedure as β approaches zero in equation (2), that the proper impulsive term given in equation (6) in reference 4 is recovered.

PERMITTIVITY AS RESTORING TERM VANISHES

In this section equation (2) is examined as β approaches zero. Its inverse Fourier transform is written explicitly as

$$\bar{\varepsilon}(t) = \varepsilon_o\delta(t)+\lim_{\beta\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{-\varepsilon_o(\omega^2+j\omega\nu)}{(\omega^2-\beta^2)^2+\omega^2\nu^2}+\left[\frac{\varepsilon_o\omega_p^2\,\beta^2}{(\omega^2-\beta^2)^2+\omega^2\nu^2}\right]\right\}e^{j\omega t}d\omega \quad (21)$$

The evaluation of this integral as β approaches zero is performed by contour integration. The poles of the integrand are

$$\left.\begin{aligned}
\omega_1 &= j\,\tfrac{1}{2}\left(\sqrt{\nu^2-4\beta^2}+\nu\right)\\
\omega_2 &= j\,\tfrac{1}{2}\left(\sqrt{\nu^2-4\beta^2}-\nu\right)\\
\omega_3 &= j\,\tfrac{1}{2}\left(-\sqrt{\nu^2-4\beta^2}+\nu\right)\\
\omega_4 &= j\,\tfrac{1}{2}\left(-\sqrt{\nu^2-4\beta^2}-\nu\right)
\end{aligned}\right\} \quad (22)$$

where the condition $4\beta^2 < \nu^2$ is the only valid solution because ν is real and greater than zero. These poles are indicated in figure 3.

Denoting the first integral by I_1 and the second integral by I_2, equation (21) is rewritten as

$$\bar{\varepsilon}(t) = \varepsilon_o\delta(t) + I_1 + I_2 \quad (23)$$

where

$$I_1 = -\lim_{\beta\to 0}\frac{\varepsilon_o\omega_p^2}{2\pi}\int_{-\infty}^{\infty}\frac{(\omega^2+j\omega\nu)}{(\omega^2-\beta^2)^2+\omega^2\nu^2}e^{j\omega t}d\omega \quad (24)$$

and

$$I_2 = \lim_{\beta\to 0}\frac{\varepsilon_o\omega_p^2}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\beta^2}{(\omega^2-\beta^2)^2+\omega^2\nu^2}\right]e^{j\omega t}d\omega \quad (25)$$

Equation (24) can be evaluated in a straightforward but lengthy process using the contour shown in figure 3. For brevity, only the results are indicated:

$$I_1 = \frac{\varepsilon_o \omega_p^2}{\nu} \left[\frac{1}{2} - e^{-\nu t} \right] , \quad t \geq 0 \tag{26}$$

At this point it might appear that as β approaches zero I_2 would vanish. This is incorrect because the range of integration includes $\omega = 0$, and thus the square bracket in equation (25) varies as β^2/ω^4 and is not zero. Contour integration, using the contour shown in figure 3, yields

$$I_2 = \frac{\varepsilon_o \omega_p^2}{2\nu} \tag{27}$$

Substituting equations (26) and (27) into equation (23), the expression $\bar{\varepsilon}$ becomes

$$\bar{\varepsilon}(t) = \varepsilon_o \delta(t) + \frac{\varepsilon_o \omega_p^2}{\nu} \left(1 - e^{-\nu t} \right) , \quad t \geq 0 \tag{28}$$

An interesting result can be obtained if equation (25) is rewritten as

$$I_2 = \frac{\varepsilon_o \omega_p^2}{2\pi} \int_{-\infty}^{\infty} \lim_{\beta \to 0} \left[\frac{\beta^2}{(\omega^2 - \beta^2)^2 + \omega^2 \nu^2} \right] e^{j\omega t} d\omega \tag{29}$$

Comparing this equation with equation (27), the following definition can be made:

$$\boxed{\delta(\omega) \equiv \lim_{\beta \to 0} \frac{1}{\pi} \left[\frac{\beta^2 \nu}{(\omega^2 - \beta^2)^2 + \beta^2 \nu^2} \right]} \tag{30}$$

This is one of the many different definitions which can be made for impulse functions. Definitions of others are given in reference 7. It is observed that as β approaches zero, therefore, equation (2) must become

$$\varepsilon(\omega) = \varepsilon_o \left[1 - \frac{\omega_p^2}{\omega^2 + \nu^2} + \pi \frac{\omega_p^2}{\nu} \delta(\omega) - j \frac{\omega_p^2 \left[\frac{\nu}{\omega} \right]}{\omega^2 + \nu^2} \right] \tag{31}$$

which, once normalized with respect to ε, is identical to equation (6) in reference 4.

CONCLUDING REMARKS

The causality of the complex permittivity for a plasma model, in which both loss and restoring terms are included, has been shown. Whenever the loss term or the restoring term or both were allowed to vanish in the *time* representation for this plasma model, the solution still obeyed the principle of causality. However, direct solutions of the differential equations without both of these terms produced noncausal results. Through an appropriate limiting procedure for the restoring term vanishing, the impulsive term given in an earlier paper was recovered— the term required for causality.

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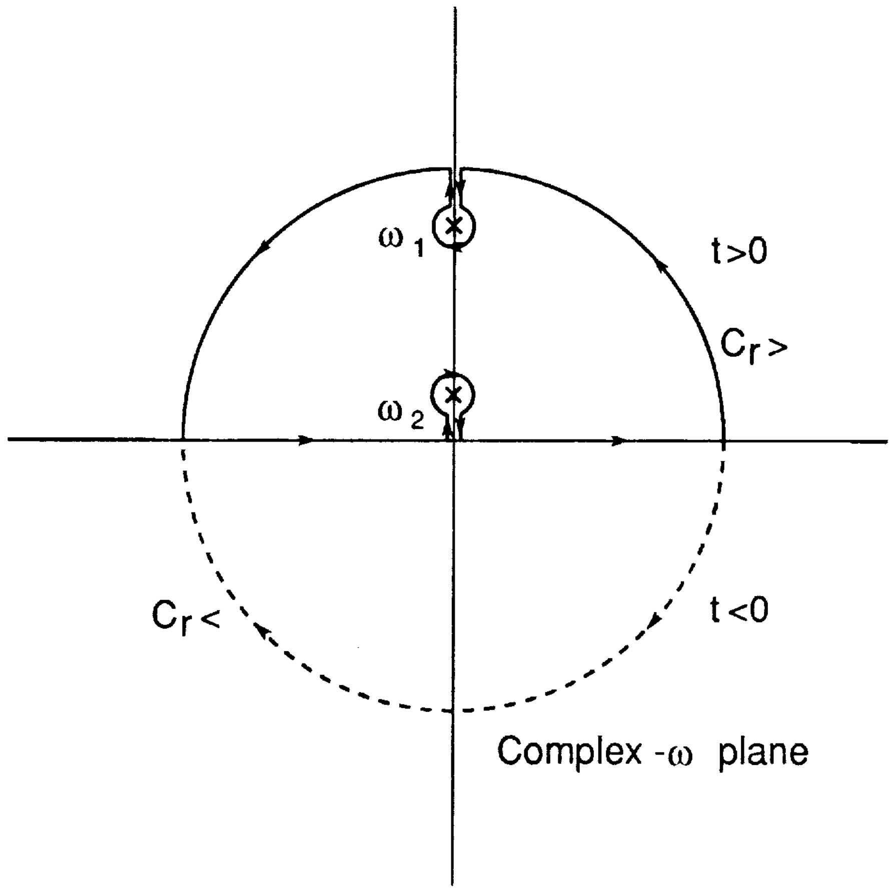

Figure 1.-Contours for equation(6) with $4\beta^2 < \upsilon^2$

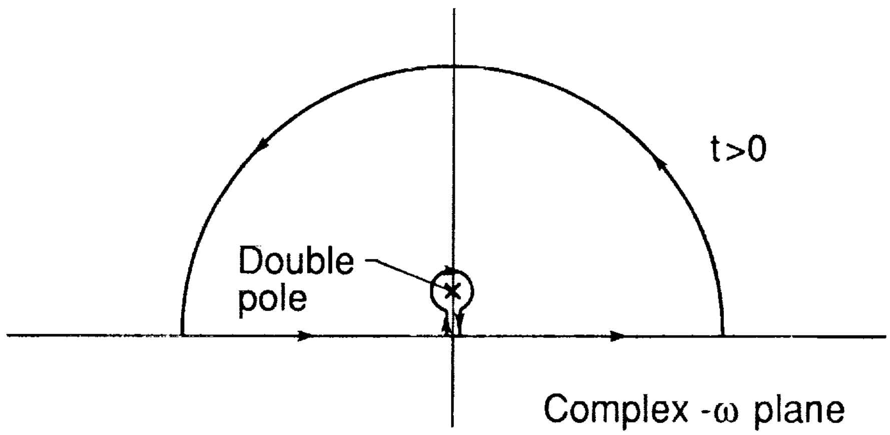

Figure 2.-Contours for equation(6) with $4\beta^2 = \upsilon^2$

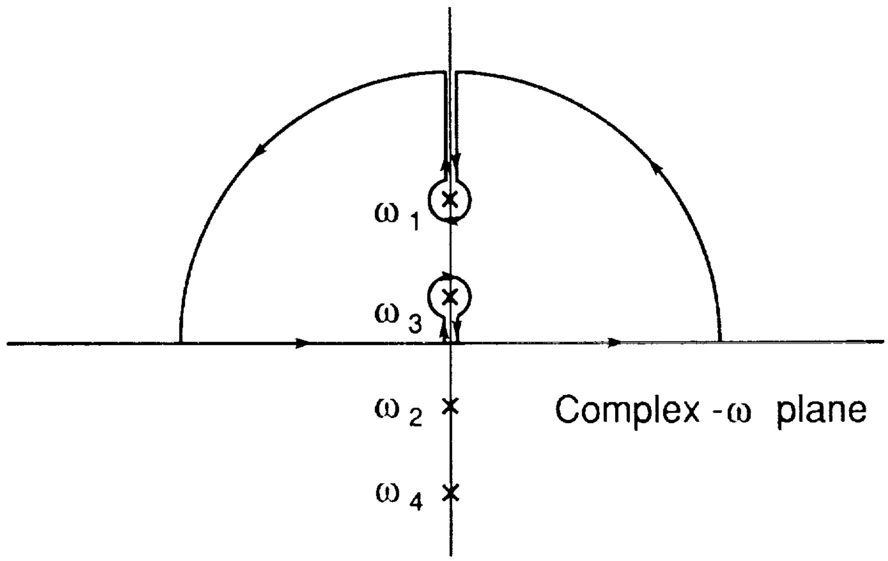

Figure 3.-Contours for equation(21)

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16. Abstract

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