

**NASA Contractor Report 182017**

**ICASE Report No. 90-26**

# **ICASE**

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**Contract No. NAS1-18605  
March 1990**

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**Operated by the Universities Space Research Association**



**National Aeronautics and  
Space Administration**

**Langley Research Center  
Hampton, Virginia 23665-5225**

**(NASA-CR-182017) ANALYTICAL METHODS FOR THE  
DEVELOPMENT OF REYNOLDS STRESS CLOSURES IN  
TURBULENCE Final Report (ICASE) 60 p**

**CSCL 20D**

**N90-19538**

**Unclas**

**G3/34 0271787**



# ANALYTICAL METHODS FOR THE DEVELOPMENT OF REYNOLDS STRESS CLOSURES IN TURBULENCE

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## ABSTRACT

Analytical methods for the development of Reynolds stress models in turbulence are reviewed in detail. Zero, one and two equation models are discussed along with second-order closures. A strong case is made for the superior predictive capabilities of second-order closure models in comparison to the simpler models. The central points of the paper are illustrated by examples from both homogeneous and inhomogeneous turbulence. A discussion of the author's views concerning the progress made in Reynolds stress modeling is also provided along with a brief history of the subject.

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\*This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.



## INTRODUCTION

Despite over a century of research, turbulence remains the major unsolved problem of classical physics. While most researchers agree that the essential physics of turbulent flows can be described by the Navier-Stokes equations, limitations in computer capacity make it impossible – for now and the foreseeable future – to directly solve these equations in the complex turbulent flows of technological interest. Hence, virtually all scientific and engineering calculations of non-trivial turbulent flows, at high Reynolds numbers, are based on some type of modeling. This modeling can take a variety of forms:

- (a) Reynolds stress models which allow for the calculation of one-point first and second moments such as the mean velocity, mean pressure and turbulent kinetic energy.
- (b) Subgrid-scale models for large-eddy simulations wherein the large, energy containing eddies are computed directly and the effect of the small scales – which are more universal in character – are modeled.
- (c) Two-point closures or spectral models which provide more detailed information about the turbulence structure since they are based on the two-point velocity correlation tensor.
- (d) Pdf models based on the joint probability density function.

Large-eddy simulations (LES) have found a variety of important geophysical applications where they have been used in weather forecasting as well as in other atmospheric studies (cf. Deardorff 1973, Clark and Farley 1984, and Smolarkiewicz and Clark 1985). Likewise, LES has shed new light on the physics of certain basic turbulent flows – which include homogeneous shear flow and channel flow – at higher Reynolds numbers that are not accessible to direct simulations (cf. Moin and Kim 1982, Bardina, Ferziger and Reynolds 1983, Rogallo and Moin 1984, and Piomelli, Ferziger and Moin 1987). Two-point closures such as the EDQNM model of Orszag (1970) have been quite useful in the analysis of homogeneous turbulent flows where they have provided new information on the structure of isotropic turbulence (cf. Lesieur 1987) and on the effect of shear and rotation (cf. Bertoglio 1982). However, there are a variety of theoretical and operational problems with two-point closures and large-eddy simulations that make their application to strongly inhomogeneous turbulent flows difficult, if not impossible – especially in irregular geometries with solid boundaries. There have been no applications of two-point closures to wall-bounded turbulent flows and virtually all such applications of LES have been in simple geometries where Van Driest damping could be used

– an empirical approach that generally does not work well when there is flow separation. Comparable problems in dealing with wall-bounded flows have, for the most part, limited pdf methods to free turbulent flows where they have been quite useful in the description of chemically reacting turbulence (see Pope 1985). Since most practical engineering flows involve complex geometries with solid boundaries – at Reynolds numbers that are far higher than those that are accessible to direct simulations – the preferred approach has been to base such calculations on Reynolds stress modeling.<sup>†</sup> This forms the motivation for the present review paper whose purpose is to put into perspective some of the more recent theoretical developments in Reynolds stress modeling.

The concept of Reynolds averaging was introduced by Sir Osborne Reynolds in his landmark turbulence research of the latter part of the nineteenth century (see Reynolds 1895). During a comparable time frame, Boussinesq (1877) introduced the concept of the turbulent or eddy viscosity as the basis for a simple time-averaged turbulence closure. However, it was not until after 1920 that the first successful calculation of a practical turbulent flow was achieved based on the Reynolds averaged Navier-Stokes equations with an eddy viscosity model. This was largely due to the pioneering work of Prandtl (1925) who introduced the concept of the mixing length as a basis for the determination of the eddy viscosity. This mixing length model led to closed form solutions for turbulent pipe and channel flows that were remarkably successful in collapsing the existing experimental data. A variety of turbulence researchers – most notably including Von Kármán (1930, 1948) – made further contributions to the mixing length approach which continued to be a highly active area of research until the post World War II period. By this time it was clear that the basic assumptions behind the mixing length approach – which makes a direct analogy between turbulent transport processes and molecular transport processes – were unrealistic; turbulent flows do not have a clear cut separation of scales. With the desire to develop more general models, Prandtl and Wieghardt (1945) tied the eddy viscosity to the turbulent kinetic energy which was obtained from a separate modeled transport equation. This was a precursor to the one equation models of turbulence – or so called  $K - \ell$  models – wherein the turbulent length scale  $\ell$  is specified empirically and the turbulent kinetic energy  $K$  is obtained from a modeled transport equation. However, these models still suffered from the deficiencies intrinsic to all eddy viscosity models: the inability to properly account for streamline curvature and history effects on the individual Reynolds stress components.

In a landmark paper by Rotta (1951), the foundation was laid for a full Reynolds stress turbulence closure which was to ultimately change the course of Reynolds stress modeling.

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<sup>†</sup>In fact, the only alternative of comparable simplicity is the vorticity transport theory of Taylor (1915); a three-dimensional vorticity covariance closure along these lines has been recently pursued by Bernard and co-workers (cf. Bernard and Berger 1982).

This new approach of Rotta – which is now referred to as second order or second moment closure – was based on the Reynolds stress transport equation. By making use of some of the statistical ideas of Kolmogorov from the 1940's – and by introducing some entirely new ideas – Rotta succeeded in closing the Reynolds stress transport equation. This new Reynolds stress closure, unlike eddy viscosity models, accounted for both history and nonlocal effects on the evolution of the Reynolds stress tensor – features whose importance had long been known. However, since this approach required the solution of an additional six transport equations for the individual components of the Reynolds stress tensor, it was not to be computationally feasible for the next few decades to solve complex engineering flows based on a full second-order closure. By the 1970's, with the wide availability of high speed computers, a new thrust in the development and implementation of second-order closure models began with the work of Daly and Harlow (1970) and Donaldson (1972). In an important paper, Launder, Reece and Rodi (1975) developed a new second-order closure model that improved significantly on the earlier work of Rotta (1951). This paper developed more systematic models for the pressure-strain correlation and turbulent transport terms; a modeled transport equation for the turbulent dissipation rate was solved in conjunction with this Reynolds stress closure. However, more importantly, Launder, Reece and Rodi (1975) showed how second-order closure models could be calibrated and applied to the solution of practical engineering flows. When the Launder, Reece and Rodi (1975) model is contracted and supplemented with an eddy viscosity representation for the Reynolds stress, a two-equation model (referred to as the  $K - \epsilon$  model) is obtained which is identical to that derived by Hanjalic and Launder (1972) a few years earlier. Because of the substantially lower computational effort required, the  $K - \epsilon$  model is still one of the most commonly used turbulence models for the solution of practical engineering problems.

Subsequent to the publication of the paper by Launder, Reece and Rodi (1975), a variety of turbulence modelers have continued research on second-order closures. Lumley (1978) introduced the important constraint of realizability and made significant contributions to the modeling of the pressure-strain correlation. Launder and co-workers continued to expand on the refinement and application of second-order closure models to problems of significant engineering interest (see Launder 1989). Speziale (1985, 1987a) exploited invariance arguments – along with consistency conditions for solutions of the Navier-Stokes equations in a rapidly rotating frame – to develop new models for the rapid pressure-strain correlation. Haworth and Pope (1986) developed a second-order closure model starting from the pdf based Langevin equation. Reynolds (1988) has attempted to develop models for the rapid pressure-strain correlation by using Rapid Distortion Theory.

In this paper, analytical methods for the derivation of Reynolds stress models will be

reviewed. Zero, one and two-equation models will be considered along with second-order closures. Two approaches to the development of models will be discussed:

- (1) *The continuum mechanics approach* which is typically based on a Taylor expansion. Invariance constraints – as well other consistency conditions such as Rapid Distortion Theory (RDT) and realizability – are then used to simplify the model. The remaining constants are evaluated by reference to benchmark physical experiments.
- (2) *The statistical mechanics approach* which is based on the construction of an asymptotic expansion. Unlike in the continuum mechanics approach, here the constants of the model are calculated explicitly. The two primary examples of this approach are the two-scale DIA models of Yoshizawa (1984) and the Renormalization Group (RNG) models of Yakhot and Orszag (1986).

The basic methodology of these two techniques will be examined, however, more emphasis will be placed on the continuum mechanics approach since there is a larger body of literature on this method and since it has been the author's preferred approach. The strengths and weaknesses of a variety of Reynolds stress models will be discussed in detail and illustrated by examples. A strong case will be made for the superior predictive capabilities of second-order closures in comparison to the older zero, one and two-equation models. However, some significant deficiencies in the structure of second-order closures that still remain will be pointed out. These issues, as well as the author's views concerning possible future directions of research, will be discussed in the sections to follow.

## BASIC EQUATIONS OF REYNOLDS STRESS MODELING

We will consider the turbulent flow of a viscous, incompressible fluid with constant properties (limitations of space will not allow us to discuss compressible turbulence modeling in any detail). The governing field equations are the Navier-Stokes and continuity equations which are given by

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \quad (1)$$

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (2)$$

where  $u_i$  is the velocity vector,  $p$  is the modified pressure (which can include a gravitational potential), and  $\nu$  is the kinematic viscosity of the fluid. In (1) - (2), the Einstein summation convention applies to repeated indices.



The velocity and pressure are decomposed into mean and fluctuating parts as follows:

$$u_i = \bar{u}_i + u'_i, \quad p = \bar{p} + p'. \quad (3)$$

It is assumed that any flow variables  $\phi$  and  $\psi$  obey the Reynolds averaging rules (cf. Tennekes and Lumley 1972):

$$\overline{\phi'} = \overline{\psi'} = 0 \quad (4)$$

$$\overline{\phi\psi} = \bar{\phi} \bar{\psi} + \overline{\phi'\psi'} \quad (5)$$

$$\overline{\phi'\bar{\psi}} = \overline{\psi'\bar{\phi}} = 0. \quad (6)$$

In a statistically steady turbulence, the mean of a flow variable  $\phi$  can be taken to be the simple time average

$$\bar{\phi} = \bar{\phi}^{(T)} \equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \phi(\mathbf{x}, t) dt, \quad (7)$$

whereas for a spatially homogeneous flow, a volume average can be used

$$\bar{\phi} = \bar{\phi}^{(V)} \equiv \lim_{V \rightarrow \infty} \frac{1}{V} \int_V \phi(\mathbf{x}, t) d^3x. \quad (8)$$

For more general turbulent flows that are neither statistically steady nor homogeneous, the mean of any flow variable  $\phi$  is taken to be the ensemble mean

$$\bar{\phi} = \bar{\phi}^{(E)} \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \phi^{(k)}(\mathbf{x}, t) \quad (9)$$

where an average is taken over  $N$  repeated experiments. The ergodic hypothesis is assumed to apply – namely, in a statistically steady turbulent flow it is assumed that

$$\bar{\phi}^{(T)} = \bar{\phi}^{(E)} \quad (10)$$

and in a homogeneous turbulent flow it is assumed that

$$\bar{\phi}^{(V)} = \bar{\phi}^{(E)}. \quad (11)$$

The Reynolds equation – which physically corresponds to a balance of mean linear momentum – takes the form

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \nu \nabla^2 \bar{u}_i - \frac{\partial \tau_{ij}}{\partial x_j} \quad (12)$$

where

$$\tau_{ij} = \overline{u'_i u'_j} \quad (13)$$

is the Reynolds stress tensor. Equation (12) is obtained by substituting the decompositions (3) into the Navier-Stokes equation (1) and then taking an ensemble mean. The mean continuity equation is given by

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (14)$$

and is obtained by simply taking the ensemble mean of (2). Equations (12) - (14) do not represent a closed system for the determination of the mean velocity  $\bar{u}_i$  and mean pressure  $\bar{p}$  due to the additional six unknowns contained within the Reynolds stress tensor. The problem of Reynolds stress closure is to tie the Reynolds stress tensor to the mean velocity field in some physically consistent fashion.

In order to gain greater insight into the problem of Reynolds stress closure, we will now consider the governing field equations for the fluctuation dynamics. The fluctuating momentum equation - from which  $u'_i$  is determined - takes the form

$$\frac{\partial u'_i}{\partial t} + \bar{u}_j \frac{\partial u'_i}{\partial x_j} = -u'_j \frac{\partial u'_i}{\partial x_j} - u'_j \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial p'}{\partial x_i} + \nu \nabla^2 u'_i + \frac{\partial \tau_{ij}}{\partial x_j} \quad (15)$$

and is obtained by subtracting (12) from (1) after the decompositions (3) are introduced. The fluctuating continuity equation, which is obtained by subtracting (14) from (2), is given by

$$\frac{\partial u'_i}{\partial x_i} = 0. \quad (16)$$

Equations (15) - (16) have solutions for the fluctuating velocity  $u'_i$  that are of the general mathematical form

$$u'_i(\mathbf{x}, t) = \mathcal{F}_i[\bar{\mathbf{u}}(\mathbf{y}, s), \mathbf{u}'(\mathbf{y}, 0), \mathbf{u}'(\mathbf{y}, s)|_{\partial \mathcal{V}}; \mathbf{x}, t] \quad \mathbf{y} \in \mathcal{V}, \quad s \in (-\infty, t) \quad (17)$$

where  $\mathcal{F}_i[\cdot]$  denotes a functional,  $\mathcal{V}$  is the volume of the fluid, and  $\partial \mathcal{V}$  is its bounding surface. In alternative terms, the fluctuating velocity is a *functional of the global history of the mean velocity field* with an implicit dependence on its own initial and boundary conditions. Here we use the term functional in its broadest mathematical sense, namely, any quantity determined by a function. From (17), we can explicitly calculate the Reynolds stress tensor  $\tau_{ij} \equiv \overline{u'_i u'_j}$  which will also be a functional of the global history of the mean velocity. However, there is a serious problem in regard to the dependence of  $\tau_{ij}$  on the initial and boundary conditions for the fluctuating velocity as discussed by Lumley (1970). There is no hope for a workable Reynolds stress closure if there is a detailed dependence on such initial and boundary conditions. For turbulent flows that are sufficiently far from solid boundaries - and sufficiently far evolved in time past their initiation - it is not unreasonable to assume that the initial and boundary conditions on the fluctuating velocity (beyond those for  $\tau_{ij}$ )

merely set the length and time scales of the turbulence. Hence, with this crucial assumption, we obtain the expression

$$\tau_{ij}(\mathbf{x}, t) = \mathcal{F}_{ij}[\bar{\mathbf{u}}(\mathbf{y}, s), \ell_0(\mathbf{y}, s), \tau_0(\mathbf{y}, s); \mathbf{x}, t] \quad \mathbf{y} \in \mathcal{V}, \quad s \in (-\infty, t) \quad (18)$$

where  $\ell_0$  is the turbulent length scale,  $\tau_0$  is the turbulent time scale, and the functional  $\mathcal{F}_{ij}$  depends implicitly on the initial and boundary conditions for  $\tau_{ij}$  (see Lumley 1970 for a more detailed discussion of these points). Equation (18) serves as the cornerstone of Reynolds stress modeling. Eddy viscosity models, which are of the form

$$\tau_{ij} = -\nu_T \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \quad (19)$$

(where the turbulent or eddy viscosity  $\nu_T \propto \ell_0^2/\tau_0$ ) represent one of the simplest examples of (18).

Since we will be discussing second-order closure models later, it would be useful at this point to introduce the Reynolds stress transport equation as well as the turbulent dissipation rate transport equation. The latter equation plays an important role in many commonly used Reynolds stress models where the turbulent dissipation rate is used to build up the turbulent length and time scales. If we denote the fluctuating momentum equation (15) in operator form as

$$\mathcal{L}u'_i = 0, \quad (20)$$

then the Reynolds stress transport equation is obtained from the second moment

$$\overline{u'_i \mathcal{L}u'_j + u'_j \mathcal{L}u'_i} = 0 \quad (21)$$

whereas the turbulent dissipation rate is obtained from the moment

$$2\nu \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial}{\partial x_j} (\mathcal{L}u'_i)} = 0. \quad (22)$$

More explicitly, the Reynolds stress transport equation (21) is given by (cf. Hinze 1975)

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik} \frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk} \frac{\partial \bar{u}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij} - \frac{\partial C_{ijk}}{\partial x_k} + \nu \nabla^2 \tau_{ij} \quad (23)$$

where

$$\Pi_{ij} \equiv \overline{p' \left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right)} \quad (24)$$

$$\varepsilon_{ij} = 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} \quad (25)$$

$$C_{ijk} \equiv \overline{u'_i u'_j u'_k} + \overline{p' u'_i \delta_{jk}} + \overline{p' u'_j \delta_{ik}} \quad (26)$$

are the pressure-strain correlation, dissipation rate correlation and third-order diffusion correlation, respectively. On the other hand, the turbulent dissipation rate transport equation (22) is given by

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} = & \nu \nabla^2 \varepsilon - 2\nu \frac{\overline{\partial u'_i}{\partial x_j} \frac{\partial u'_k}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_k}} \\
& - 2\nu \frac{\overline{\partial u'_j}{\partial x_i} \frac{\partial u'_j}{\partial x_k} \frac{\partial \bar{u}_i}{\partial x_k}} - 2\nu \overline{u'_k \frac{\partial u'_i}{\partial x_j} \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_j}} \\
& - 2\nu \frac{\overline{\partial u'_i}{\partial x_k} \frac{\partial u'_i}{\partial x_m} \frac{\partial u'_k}{\partial x_m}} - \nu \frac{\partial}{\partial x_k} \left( \overline{u'_k \frac{\partial u'_i}{\partial x_m} \frac{\partial u'_i}{\partial x_m}} \right) \\
& - 2\nu \frac{\partial}{\partial x_k} \left( \overline{\frac{\partial p'}{\partial x_m} \frac{\partial u'_k}{\partial x_m}} \right) - 2\nu^2 \frac{\overline{\partial^2 u'_i}{\partial x_k \partial x_m} \frac{\partial^2 u'_i}{\partial x_k \partial x_m}}
\end{aligned} \tag{27}$$

where  $\varepsilon \equiv \frac{1}{2} \varepsilon_{ii}$  is the scalar dissipation rate. The seven higher-order correlations on the right-hand-side of (27) correspond to three physical effects: the first four terms give rise to the production of dissipation, the next two terms represent the turbulent diffusion of dissipation, and the last term represents the turbulent dissipation of dissipation.

Finally, before closing this section, it would be useful to briefly discuss two constraints that have played a central role in the formulation of modern Reynolds stress models: realizability and frame invariance. The constraint of realizability was rigorously introduced by Lumley (see Lumley 1978, 1983 for a more detailed discussion). It requires that a Reynolds stress model yield positive component energies, i.e., that

$$\tau_{\alpha\alpha} \geq 0, \quad \alpha = 1, 2, 3 \tag{28}$$

for any given turbulent flow. The inequality (28) (where Greek indices are used to indicate that there is no summation) is a direct consequence of the definition of the Reynolds stress tensor given by (13). It was first shown by Lumley that realizability could be satisfied identically in homogeneous turbulent flows by Reynolds stress transport models; this is accomplished by requiring that whenever a component energy  $\tau_{\alpha\alpha}$  vanishes, its time rate  $\dot{\tau}_{\alpha\alpha}$  also vanishes.

Donaldson (1968) was probably the first to advocate the unequivocal use of coordinate invariance in turbulence modeling. This approach, which Donaldson termed "invariant modeling," was based on the Reynolds stress transport equation and required that all modeled terms be cast in tensor form. Prior to the 1970's it was not uncommon for turbulence models to be proposed that were incapable of being uniquely put in tensor form (hence, these older models could not be properly extended to more complex flows, particularly to ones involving curvilinear coordinates). The more complicated question of frame invariance – where

*time-dependent* rotations and translations of the reference frame are accounted for – was first considered by Lumley (1970) in an interesting paper. A more comprehensive analysis of the effect of a change of reference frame was conducted by the author in a series of papers published during the 1980's (see Speziale 1989a for a detailed review of these results). In an arbitrary non-inertial reference frame, which can undergo *arbitrary time-dependent* rotations and translations relative to an inertial framing, the fluctuating momentum equation takes the form

$$\frac{\partial u'_i}{\partial t} + \bar{u}_j \frac{\partial u'_i}{\partial x_j} = -u'_j \frac{\partial u'_i}{\partial x_j} - u'_j \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial p'}{\partial x_i} + \nu \nabla^2 u'_i + \frac{\partial \tau_{ij}}{\partial x_j} - 2e_{ijk} \Omega_j u'_k \quad (29)$$

where  $e_{ijk}$  is the permutation tensor and  $\Omega_i$  is the rotation rate of the reference frame relative to an inertial framing (see Speziale 1989a). From (29), it is clear that the evolution of the fluctuating velocity only depends directly on the motion of the reference frame through the Coriolis acceleration; translational accelerations – as well as centrifugal and angular accelerations – only have an indirect effect through the changes that they induce in the mean velocity field. Consequently, closure models for the Reynolds stress tensor must be form invariant under the *extended* Galilean group of transformations

$$\mathbf{x}^* = \mathbf{x} + \mathbf{c}(t) \quad (30)$$

which allows for an arbitrary *translational acceleration* –  $\ddot{\mathbf{c}}$  of the reference frame relative to an inertial framing  $\mathbf{x}$ .

In the limit of two-dimensional turbulence (or a turbulence where the ratio of the fluctuating to mean time scales  $\tau_0/T_0 \ll 1$ ), the Coriolis acceleration is derivable from a scalar potential that can be absorbed into the fluctuating pressure (or neglected) yielding complete frame-indifference (see Speziale 1981, 1983). This invariance under arbitrary time-dependent rotations and translations of the reference frame specified by

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t) \quad (31)$$

(where  $\mathbf{Q}(t)$  is any *time-dependent* proper-orthogonal rotation tensor) is referred to as Material Frame Indifference (MFI) – the term that has been traditionally used for the analogous manifest invariance of constitutive equations in continuum mechanics. For general three-dimensional turbulent flows where  $\tau_0/T_0 = \mathcal{O}(1)$ , MFI does not apply as first pointed out by Lumley (1970). However, the Coriolis acceleration in (29) can be combined with the mean velocity in such a way that frame-dependence enters *exclusively* through the appearance of the intrinsic or absolute mean vorticity defined by (see Speziale 1989a)

$$\bar{W}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right) + e_{mji} \Omega_m. \quad (32)$$

This result, along with the constraint of MFI in the two-dimensional limit, restricts the allowable form of models considerably.

## ZERO-EQUATION AND ONE-EQUATION MODELS BASED ON AN EDDY VISCOSITY

In the simplest continuum mechanics approach – whose earliest formulations have often been referred to as phenomenological models – the starting point is equation (18). Invariance under the extended Galilean group of transformations (30) – which any physically sound Reynolds stress model must obey – can be satisfied identically by models of the form

$$\tau_{ij}(\mathbf{x}, t) = \mathcal{F}_{ij}[\bar{\mathbf{u}}(\mathbf{y}, s) - \bar{\mathbf{u}}(\mathbf{x}, s), \ell_0(\mathbf{y}, s), \tau_0(\mathbf{y}, s); \mathbf{x}, t] \quad \mathbf{y} \in \mathcal{V}, \quad s \in (-\infty, t). \quad (33)$$

The variables  $\bar{\mathbf{u}}(\mathbf{y}, s) - \bar{\mathbf{u}}(\mathbf{x}, s)$ ,  $\ell_0(\mathbf{y}, s)$  and  $\tau_0(\mathbf{y}, s)$  can be expanded in a Taylor series as follows:

$$\begin{aligned} \bar{\mathbf{u}}(\mathbf{y}, s) - \bar{\mathbf{u}}(\mathbf{x}, s) &= (y_i - x_i) \frac{\partial \bar{\mathbf{u}}}{\partial x_i} + \frac{(y_i - x_i)(y_j - x_j)}{2!} \frac{\partial^2 \bar{\mathbf{u}}}{\partial x_i \partial x_j} + \\ &\quad (s - t)(y_i - x_i) \frac{\partial^2 \bar{\mathbf{u}}}{\partial t \partial x_i} + \dots \end{aligned} \quad (34)$$

$$\begin{aligned} \ell_0(\mathbf{y}, s) &= \ell_0 + (y_i - x_i) \frac{\partial \ell_0}{\partial x_i} + (s - t) \frac{\partial \ell_0}{\partial t} + \frac{(s - t)^2}{2!} \frac{\partial^2 \ell_0}{\partial t^2} + \\ &\quad \frac{(y_i - x_i)(y_j - x_j)}{2!} \frac{\partial^2 \ell_0}{\partial x_i \partial x_j} + (s - t)(y_i - x_i) \frac{\partial^2 \ell_0}{\partial t \partial x_i} + \dots \end{aligned} \quad (35)$$

$$\begin{aligned} \tau_0(\mathbf{y}, s) &= \tau_0 + (y_i - x_i) \frac{\partial \tau_0}{\partial x_i} + (s - t) \frac{\partial \tau_0}{\partial t} + \frac{(s - t)^2}{2!} \frac{\partial^2 \tau_0}{\partial t^2} + \\ &\quad \frac{(y_i - x_i)(y_j - x_j)}{2!} \frac{\partial^2 \tau_0}{\partial x_i \partial x_j} + (s - t)(y_i - x_i) \frac{\partial^2 \tau_0}{\partial t \partial x_i} + \dots \end{aligned} \quad (36)$$

where terms up to the second order are shown and it is understood that  $\bar{\mathbf{u}}$ ,  $\ell_0$  and  $\tau_0$  on the r.h.s. of (34) - (36) are evaluated at  $\mathbf{x}$  and  $t$ . After splitting  $\tau_{ij}$  into isotropic and deviatoric parts – and applying elementary dimensional analysis – the following expression is obtained:

$$\tau_{ij} = \frac{2}{3} K \delta_{ij} - \frac{\ell_0^2}{\tau_0^2} \hat{\mathcal{F}}_{ij}[\bar{\mathbf{v}}(\mathbf{y}, s) - \bar{\mathbf{v}}(\mathbf{x}, s); \mathbf{x}, t] \quad \mathbf{y} \in \mathcal{V}, \quad s \in (-\infty, t) \quad (37)$$

where

$$\bar{\mathbf{v}} = \frac{\tau_0 \bar{\mathbf{u}}}{\ell_0}, \quad K = \frac{1}{2} \tau_{ii} \quad (38)$$

are, respectively, the dimensionless mean velocity and the turbulent kinetic energy.  $\hat{\mathcal{F}}_{ij}$  is a traceless and dimensionless functional of its arguments. By making use of the Taylor expansions (34) - (36), it is a simple matter to show that

$$\bar{v}_i(\mathbf{y}, s) - \bar{v}_i(\mathbf{x}, s) = \frac{\tau_0}{T_0}(y_j^* - x_j^*) \left( \frac{\partial \bar{u}_i}{\partial x_j} \right)^* + \mathcal{O} \left( \frac{\tau_0^2}{T_0^2} \right) \quad (39)$$

where

$$y_i^* - x_i^* \equiv \frac{y_i - x_i}{\ell_0}, \quad \left( \frac{\partial \bar{u}_i}{\partial x_j} \right)^* = T_0 \frac{\partial \bar{u}_i}{\partial x_j} \quad (40)$$

are dimensionless variables of order one given that  $T_0$  is the time scale of the mean flow. If, analogous to the molecular fluctuations of most continuum flows, we assume that there is a complete separation of scales such that

$$\frac{\tau_0}{T_0} \ll 1, \quad \frac{\ell_0}{L_0} \ll 1, \quad (41)$$

equation (37) can then be localized in space and time. Of course, it is well-known that this constitutes an over-simplification; the molecular fluctuations of most continuum flows are such that  $\tau_0/T_0 \leq 10^{-6}$  whereas with turbulent fluctuations,  $\tau_0/T_0$  can be of  $\mathcal{O}(1)$ .

By making use of (39) - (41), equation (37) can be localized to the approximate form

$$\tau_{ij} = \frac{2}{3} K \delta_{ij} - \frac{\ell_0^2}{\tau_0^2} G_{ij}(\bar{v}_{k,l}) \quad (42)$$

where

$$\bar{v}_{k,l} \equiv \frac{\tau_0}{T_0} \left( \frac{\partial \bar{u}_k}{\partial x_l} \right)^* \quad (43)$$

is the dimensionless mean velocity gradient. Since the tensor function  $G_{ij}$  is symmetric and traceless (and since  $\bar{v}_{i,j}$  is traceless) it follows that - to the first order in  $\tau_0/T_0$  - form invariance under a change of coordinates simplifies (42) to (cf. Smith 1971):

$$\tau_{ij} = \frac{2}{3} K \delta_{ij} - \nu_T \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \quad (44)$$

where

$$\nu_T \equiv \ell_0^2 / \tau_0 \quad (45)$$

is the eddy viscosity. While the standard eddy viscosity model (44) comes out of this derivation when only first order terms in  $\tau_0/T_0$  are maintained, anisotropic eddy viscosity (or viscoelastic) models are obtained when second-order terms are maintained. These more complicated models will be discussed in the next section.

Eddy viscosity models are not closed until prescriptions are made for the turbulent length and time scales in (45). In zero equation models, both  $\ell_0$  and  $\tau_0$  are prescribed algebraically.

The earliest example of a successful zero-equation model is Prandtl's mixing length theory (see Prandtl 1925). By making analogies between the turbulent length scale and the mean free path in the kinetic theory of gases, Prandtl argued that  $\nu_T$  should be of the form

$$\nu_T = \ell_m^2 \left| \frac{d\bar{u}}{dy} \right| \quad (46)$$

for a plane shear flow where the mean velocity is of the form  $\mathbf{U} = \bar{u}(y)\mathbf{i}$ . In (46),  $\ell_m$  is the "mixing length" which represents the distance traversed by a small lump of fluid before losing its momentum. Near a plane solid boundary, it was furthermore assumed that

$$\ell_m = \kappa y \quad (47)$$

where  $\kappa$  is the Von Kármán constant (this result can be obtained from a first-order Taylor series expansion since  $\ell_m$  must vanish at a wall). When (46) - (47) are used in conjunction with the added assumption that the shear stress is approximately constant in the near wall region, the celebrated "law of the wall" is obtained:

$$u^+ = \frac{1}{\kappa} \ln y^+ + C \quad (48)$$

where  $y^+$  is measured normal from the wall and

$$u^+ \equiv \frac{\bar{u}}{u_\tau}, \quad y^+ = \frac{y u_\tau}{\nu} \quad (49)$$

given that  $u_\tau$  is the friction velocity and  $C$  is a dimensionless constant. Equation (48) (with  $\kappa \doteq 0.4$  and  $C \doteq 4.9$ ) was remarkably successful in collapsing the experimental data for turbulent pipe and channel flows for a significant range of  $y^+$  varying from 30 to 1,000 (see Schlichting 1968 for an interesting review of these results). The law of the wall is still heavily used to this day as a boundary condition in the more sophisticated turbulence models for which it is either difficult or too expensive to integrate directly to a solid boundary.

During the 1960's and 1970's, with the dramatic emergence of computational fluid dynamics, some efforts were made to generalize mixing length models to three-dimensional turbulent flows. With such models, Reynolds averaged computations could be conducted with any existing Navier-Stokes computer code that allowed for a variable viscosity. Prandtl's mixing length theory (46) has two straightforward extensions to three dimensional flows: the strain rate form

$$\nu_T = \ell_m^2 (2\overline{S}_{ij}\overline{S}_{ij})^{\frac{1}{2}} \quad (50)$$

where  $\overline{S}_{ij} \equiv \frac{1}{2}(\partial\bar{u}_i/\partial x_j + \partial\bar{u}_j/\partial x_i)$  is the mean rate of strain tensor, or the vorticity form

$$\nu_T = \ell_m^2 (\overline{\omega}_i\overline{\omega}_i)^{\frac{1}{2}} \quad (51)$$



where  $\bar{\omega}_i = \epsilon_{ijk} \partial \bar{u}_k / \partial x_j$  is the mean vorticity vector. The former model (50) is due to Smagorinsky (1963) and has been primarily used as a subgrid scale model for large-eddy simulations; the latter model (51) is due to Baldwin and Lomax (1978) and has been widely used for Reynolds averaged aerodynamic computations. Both models – which collapse to Prandtl's mixing length theory (46) in a plane shear flow – have the primary advantage of their computational ease of application. They suffer from the disadvantage of the need for an ad hoc prescription of the turbulent length scale in each problem solved as well as from the complete neglect of history effects. Furthermore, they do not provide for the computation of the turbulent kinetic energy which is a crucial measure of the intensity of the turbulence (such zero-equation models only allow for the calculation of the mean velocity and mean pressure).

One-equation models were developed in order to eliminate some of the deficiencies cited above, namely, to provide for the computation of the turbulent kinetic energy and to account for some limited nonlocal and history effects in the determination of the eddy viscosity. In these one-equation models of turbulence, the eddy viscosity is assumed to be of the form (see Kolmogorov 1942 and Prandtl and Wiegardt 1945)

$$\nu_T = K^{\frac{1}{2}} \ell \quad (52)$$

where the turbulent kinetic energy  $K$  is obtained from a modeled version of its exact transport equation

$$\frac{\partial K}{\partial t} + \bar{u}_i \frac{\partial K}{\partial x_i} = -\tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \epsilon - \frac{\partial}{\partial x_i} \left( \frac{1}{2} \overline{u'_k u'_k u'_i} + \overline{p' u'_i} \right) + \nu \nabla^2 K. \quad (53)$$

Equation (53), which is obtained by a simple contraction of (23), can be closed once models for the turbulent transport and dissipation terms (i.e., the second and third terms on the r.h.s. of (53)) are provided. Consistent with the assumption that there is a clear-cut separation of scales (i.e., that the turbulent transport processes parallel the molecular ones), the turbulent transport term is modeled by a gradient transport hypothesis, i.e.,

$$\frac{1}{2} \overline{u'_k u'_k u'_i} + \overline{p' u'_i} = -\frac{\nu_T}{\sigma_K} \frac{\partial K}{\partial x_i} \quad (54)$$

where  $\sigma_K$  is a dimensionless constant. By simple scaling arguments – analogous to those made by Kolmogorov (1942) – the turbulent dissipation rate  $\epsilon$  is usually modeled as follows

$$\epsilon = C^* \frac{K^{\frac{3}{2}}}{\ell} \quad (55)$$

where  $C^*$  is a dimensionless constant. A closed system of equations for the determination of  $\bar{u}_i$ ,  $\bar{p}$  and  $K$  is obtained once the turbulent length scale  $\ell$  is specified empirically. It should

be mentioned that the modeled transport equation for the turbulent kinetic energy specified by equations (53) - (55) cannot be integrated to a solid boundary. Either wall functions must be used or low-Reynolds-number versions of (53) - (55) must be substituted (cf. Norris and Reynolds 1975 and Reynolds 1976). It is interesting to note that Bradshaw, Ferriss and Atwell (1967) considered an alternative one-equation model, based on a modeled transport equation for the Reynolds shear stress  $\overline{u'v'}$ , which seemed to be better suited for turbulent boundary layers.

Since zero and one equation models have not been in the forefront of turbulence modeling research for the past twenty years, we will not present the results of any illustrative calculations (the reader is referred to Cebeci and Smith 1974 and Rodi 1980 for some interesting examples). The primary deficiencies of these models are twofold: (a) the use of an eddy viscosity, and (b) the need to provide an ad hoc specification of the turbulence length scale. This latter deficiency in regard to the length scale makes zero and one equation models incomplete; the two-equation models that will be discussed in the next section were the first complete turbulence models (i.e., models that only require the specification of initial and boundary conditions for the solution of problems). Nonetheless, despite these deficiencies, zero and one equation models have made some important contributions to the computation of practical engineering flows. Their simplicity of structure – and reduced computing times – continue to make them the most commonly adopted models for complex aerodynamic calculations (see Cebeci and Smith 1968 and Johnson and King 1984 for two of the most popular such models).

## TWO-EQUATION MODELS

A variety of two-equation models – which are among the most popular Reynolds stress models for scientific and engineering calculations – will be discussed in this section. Models of the  $K - \varepsilon$ ,  $K - \ell$  and  $K - \omega$  type will be considered based on an isotropic and anisotropic eddy viscosity. Both the continuum mechanics and statistical mechanics approach for deriving such two-equation models will be discussed.

The feature that distinguishes two-equation models from zero- or one-equation models is that *two separate modeled transport equations* are solved for the turbulent length and time scales (or for any two linearly independent combinations thereof). In the standard  $K - \varepsilon$  model – which is probably the most popular such model – the length and time scales are built up from the turbulent kinetic energy and dissipation rate as follows (see Hanjalic and Launder 1972 and Launder and Spalding 1974):

$$\ell_0 \propto \frac{K^{\frac{3}{2}}}{\varepsilon}, \quad \tau_0 \propto \frac{K}{\varepsilon}.$$

Separate modeled transport equations are solved for the turbulent kinetic energy  $K$  and turbulent dissipation rate  $\varepsilon$ . In order to close the exact transport equation for  $K$ , only a model for the turbulent transport term on the r.h.s. of (53) is needed; consistent with the overriding assumption that there is a clear-cut separation of scales, the gradient transport model (54) is used. The exact transport equation for  $\varepsilon$ , given by (27), can be rewritten in the form

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} = \nu \nabla^2 \varepsilon + \mathcal{P}_\varepsilon + \mathcal{D}_\varepsilon - \Phi_\varepsilon \quad (56)$$

where  $\mathcal{P}_\varepsilon$  represents the production of dissipation (given by the first four correlations on the r.h.s. of (27)),  $\mathcal{D}_\varepsilon$  represents the turbulent diffusion of dissipation (given by the next two correlations on the r.h.s. of (27)), and  $\Phi_\varepsilon$  represents the turbulent dissipation of dissipation (given by the last term on the r.h.s. of (27)). Again, consistent with the underlying assumption (41), a gradient transport hypothesis is used to model  $\mathcal{D}_\varepsilon$ :

$$\mathcal{D}_\varepsilon = \frac{\partial}{\partial x_i} \left( \frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right) \quad (57)$$

where  $\sigma_\varepsilon$  is a dimensionless constant. The production of dissipation and dissipation of dissipation are modeled as follows:

$$\mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon(b_{ij}, \frac{\partial \bar{u}_i}{\partial x_j}, K, \varepsilon) \quad (58)$$

$$\Phi_\varepsilon = \Phi_\varepsilon(K, \varepsilon). \quad (59)$$

Eqs. (58) - (59) are based on the physical reasoning that the production of dissipation is governed by the level of anisotropy  $b_{ij} \equiv (\tau_{ij} - \frac{2}{3}K\delta_{ij})/2K$  in the Reynolds stress tensor and the mean velocity gradients (scaled by  $K$  and  $\varepsilon$  which determine the length and time scales) whereas the dissipation of dissipation is determined by the length and time scales alone (an assumption motivated by isotropic turbulence). By a simple dimensional analysis it follows that

$$\Phi_\varepsilon = C_{\varepsilon 2} \frac{\varepsilon^2}{K} \quad (60)$$

where  $C_{\varepsilon 2}$  is a dimensionless constant. Coordinate invariance coupled with a simple dimensional analysis yields

$$\begin{aligned} \mathcal{P}_\varepsilon &= -2C_{\varepsilon 1} \varepsilon b_{ij} \frac{\partial \bar{u}_i}{\partial x_j} \\ &\equiv -C_{\varepsilon 1} \frac{\varepsilon}{K} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} \end{aligned} \quad (61)$$

as the leading term in a Taylor expansion of (58) assuming that  $\|\mathbf{b}\|$  and  $\tau_0/T_0$  are small ( $C_{\varepsilon 1}$  is a dimensionless constant). Equation (61) was originally postulated based on the

simple physical reasoning that the production of dissipation should be proportional to the production of turbulent kinetic energy (cf. Hanjalic and Launder 1972). A composition of these various modeled terms yields the standard  $K - \varepsilon$  model (cf. Launder and Spalding 1974):

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - \nu_T \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \quad (62a)$$

$$\nu_T = C_\mu \frac{K^2}{\varepsilon} \quad (62b)$$

$$\frac{\partial K}{\partial t} + \bar{u}_i \frac{\partial K}{\partial x_i} = -\tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_i} \left( \frac{\nu_T}{\sigma_K} \frac{\partial K}{\partial x_i} \right) + \nu \nabla^2 K \quad (62c)$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} = -C_{\varepsilon 1} \frac{\varepsilon}{K} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - C_{\varepsilon 2} \frac{\varepsilon^2}{K} + \frac{\partial}{\partial x_i} \left( \frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right) + \nu \nabla^2 \varepsilon. \quad (62d)$$

Here, the constants assume the approximate values of  $C_\mu = 0.09$ ,  $\sigma_K = 1.0$ ,  $\sigma_\varepsilon = 1.3$ ,  $C_{\varepsilon 1} = 1.44$  and  $C_{\varepsilon 2} = 1.92$  which are obtained, for the most part, by comparisons of the model predictions with the results of physical experiments on equilibrium turbulent boundary layers and the decay of isotropic turbulence. It should be noted that the standard  $K - \varepsilon$  model (62) cannot be integrated to a solid boundary; either wall functions or some form of damping must be implemented (see Patel, Rodi and Scheuerer 1985 for an extensive review of these methods).

At this point, it would be useful to provide some examples of the performance of the  $K - \varepsilon$  model in some benchmark, homogeneous turbulent flows as well as in a non-trivial, inhomogeneous turbulent flow. It is a simple matter to show that in isotropic turbulence where

$$\tau_{ij} = \frac{2}{3}K(t)\delta_{ij}, \quad \varepsilon_{ij} = \frac{2}{3}\varepsilon(t)\delta_{ij}$$

the  $K - \varepsilon$  model predicts the following rate of decay of the turbulent kinetic energy (cf. Reynolds 1987):

$$K(t) = K_0 [1 + (C_{\varepsilon 2} - 1)\varepsilon_0 t / K_0]^{-1/(C_{\varepsilon 2} - 1)}. \quad (63)$$

Equation (63) indicates a power law decay where  $K \sim t^{-1.1}$  – a result that is not far removed from what is observed in physical experiments (cf. Comte-Bellot and Corrsin 1971).

Homogeneous shear flow constitutes another classical turbulent flow that has been widely used to evaluate models. In this flow, an initially isotropic turbulence is subjected to a constant shear rate  $S$  with mean velocity gradients

$$\frac{\partial \bar{u}_i}{\partial x_j} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (64)$$

The time evolution of the turbulent kinetic energy obtained from the standard  $K - \varepsilon$  model is compared in Figure 1 with the large-eddy simulation of Bardina, Ferziger and Reynolds (1983) (here,  $K^* \equiv K/K_0$  is the dimensionless kinetic energy and  $t^* \equiv St$  is the dimensionless time). In so far as the equilibrium states are concerned, the standard  $K - \varepsilon$  model predicts that (see Speziale and Mac Giolla Mhuiris 1989a):  $(b_{12})_\infty = -0.217$  and  $(SK/\varepsilon)_\infty = 4.82$  in comparison to the experimental values of  $(b_{12})_\infty = -0.15$  and  $(SK/\varepsilon)_\infty = 6.08$ , respectively<sup>†</sup> (see Tavoularis and Corrsin 1981). Consistent with a wide range of physical and numerical experiments, the standard  $K - \varepsilon$  model predicts that the equilibrium structure of homogeneous shear flow is *universal* (i.e., attracts all initial conditions) in the phase space of  $b_{ij}$  and  $SK/\varepsilon$ . Hence, from Figure 1 and the equilibrium results given above, it is clear that the  $K - \varepsilon$  model yields a qualitatively good description of shear flow; the specific quantitative predictions, however, could be improved upon.

As an example of the performance of the standard  $K - \varepsilon$  model in a more complicated inhomogeneous turbulence, the case of turbulent flow past a backward facing step at a Reynolds number  $Re \sim 100,000$  will be presented (the same test case considered at the 1980/81 AFOSR-HTTM Stanford Conference on Turbulence; it corresponds to the experimental test case of Kim, Kline, and Johnston 1980). In Figures 2(a) - (b) the mean flow streamlines and turbulence intensity profiles predicted by the  $K - \varepsilon$  model are compared with the experimental data of Kim, Kline, and Johnston (1980). The standard  $K - \varepsilon$  model - integrated using a single log wall layer starting at  $y^+ = 30$  - predicts a reattachment point of  $x/\Delta H \doteq 5.7$  in comparison to the experimental mean value of  $x/\Delta H \doteq 7.0$ . This error, which is of the order of 20%, is comparable to that which occurs in the predicted turbulence intensities (see Figure 2(b) and Speziale and Ngo 1988 for more detailed comparisons). However, Avva, Kline and Ferziger (1988) reported an improved prediction of  $x/\Delta H \doteq 6.3$  for the reattachment point by using a fine mesh and a three-layer log wall region.

Recently, Yakhot and Orszag (1986) derived a  $K - \varepsilon$  model based on RNG methods. In this approach, an expansion is made about an equilibrium state with known Gaussian statistics by making use of the correspondence principle. Bands of high wavenumbers (i.e., small scales) are systematically removed and space is rescaled. The dynamical equations for the renormalized (large-scale) velocity field account for the effect of the small scales that have been removed through the presence of an eddy viscosity. The removal of only the smallest scales gives rise to subgrid scale models for large-eddy simulations; the removal of successively larger and larger scales ultimately gives rise to Reynolds stress models. In the high Reynolds number limit, the RNG based  $K - \varepsilon$  model of Yakhot and Orszag (1986) is identical in form to the standard  $K - \varepsilon$  model (62). However, the constants of the model are *calculated*

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<sup>†</sup>Here,  $(\cdot)_\infty$  denotes the equilibrium value obtained in the limit as  $t \rightarrow \infty$ .

explicitly by the theory to be:  $C_\mu = 0.0837$ ,  $C_{\epsilon 1} = 1.063$ ,  $C_{\epsilon 2} = 1.7215$ ,  $\sigma_K = 0.7179$  and  $\sigma_\epsilon = 0.7179$ . Beyond having the attractive feature of no undetermined constants, the RNG  $K - \epsilon$  model of Yakhot and Orszag (1986) automatically bridges the eddy viscosity to the molecular viscosity as a solid boundary is approached, eliminating the need for the use of empirical wall functions or Van Driest damping. It must be mentioned, however, that some problems with the specific numerical values of the constants in the RNG  $K - \epsilon$  model have recently surfaced. In particular, the value of  $C_{\epsilon 1} = 1.063$  is dangerously close to  $C_{\epsilon 1} = 1$  which constitutes a singular point of the  $\epsilon$ -transport equation. For example, the growth rate  $\lambda$  of the turbulent kinetic energy (where  $K \sim e^{\lambda t^*}$  for  $\lambda t^* \gg 1$ ) predicted by the  $K - \epsilon$  model in homogeneous shear flow is given by (see Speziale and Mac Giolla Mhuiris 1989a):

$$\lambda = \left[ \frac{C_\mu (C_{\epsilon 2} - C_{\epsilon 1})^2}{(C_{\epsilon 1} - 1)(C_{\epsilon 2} - 1)} \right]^{\frac{1}{2}} \quad (65)$$

which becomes singular when  $C_{\epsilon 1} = 1$ . Consequently, the value of  $C_{\epsilon 1} = 1.063$  derived by Yakhot and Orszag (1986) yields excessively large growth rates for the turbulent kinetic energy in homogeneous shear flow in comparison to both physical and numerical experiments (see Speziale, Gatski and Mac Giolla Mhuiris 1989).

One of the major deficiencies of the standard  $K - \epsilon$  model lies in its use of an eddy viscosity model for the Reynolds stress tensor. Eddy viscosity models have two major deficiencies associated with them: (a) they are purely dissipative and, hence, cannot account for Reynolds stress relaxation effects, and (b) they are oblivious to the presence of rotational strains (e.g., they fail to distinguish between the physically distinct cases of plane shear, plane strain, and rotating plane shear). In an effort to overcome these deficiencies, a considerable research effort has been directed toward the development of nonlinear or anisotropic generalizations of eddy viscosity models. By keeping second-order terms in the Taylor expansions (34) - (36), subject to invariance under the extended Galilean group (30), a more general representation for the Reynolds stress tensor is obtained:

$$\begin{aligned} \tau_{ij} = & \frac{2}{3} K \delta_{ij} - 2 \frac{\ell_0^2}{\tau_0} \bar{S}_{ij} + \alpha_1 \ell_0^2 \left( \bar{S}_{ik} \bar{S}_{kj} - \frac{1}{3} \bar{S}_{mn} \bar{S}_{mn} \delta_{ij} \right) \\ & + \alpha_2 \ell_0^2 \left( \bar{\omega}_{ik} \bar{\omega}_{kj} - \frac{1}{3} \bar{\omega}_{mn} \bar{\omega}_{mn} \delta_{ij} \right) + \alpha_3 \ell_0^2 (\bar{S}_{ik} \bar{\omega}_{jk} + \bar{S}_{jk} \bar{\omega}_{ik}) \\ & + \alpha_4 \ell_0^2 \left( \frac{\partial \bar{S}_{ij}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{S}_{ij} \right) \end{aligned} \quad (66)$$

where

$$\bar{S}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right), \quad \bar{\omega}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right) \quad (67)$$

are the mean rate of strain and mean vorticity tensors ( $\alpha_1 \dots \alpha_4$  are dimensionless constants; in the linear limit as  $\alpha_i \rightarrow 0$ , the eddy viscosity model (44) is recovered). When  $\alpha_4 = 0$ , the deviatoric part of (66) is of the general form  $D\tau_{ij} = A_{ijkl}\partial\bar{u}_k/\partial x_l$  (where  $A_{ijkl}$  depends algebraically on the mean velocity gradients) and, hence, the term “anisotropic eddy viscosity model” has been used in the literature. These models are probably more accurately characterized as “nonlinear” or “viscoelastic” corrections to the eddy viscosity models. Lumley (1970) was probably the first to systematically develop such models (with  $\alpha_4 = 0$ ) wherein he built up the length and time scales from the turbulent kinetic energy, turbulent dissipation rate, and the invariants of  $\bar{S}_{ij}$  and  $\bar{\omega}_{ij}$ . Saffman (1977) proposed similar anisotropic models which were solved in conjunction with modeled transport equations for  $K$  and  $\omega^2$  (where  $\omega \equiv \epsilon/K$ ). Pope (1975) and Rodi (1976) developed alternative anisotropic eddy viscosity models from the Reynolds stress transport equation by making an equilibrium hypothesis. Yoshizawa (1984, 1987) derived a more complete two-equation model – with a nonlinear correction to the eddy viscosity of the full form of (66) – by means of a two-scale Direct Interaction Approximation (DIA) method. In this approach, Kraichnan’s DIA formalism (cf. Kraichnan 1964) is combined with a scale expansion technique where the slow variations of the mean fields are distinguished from the fast variations of the fluctuating fields by means of a scale parameter. The length and time scales of the turbulence are built up from the turbulent kinetic energy and dissipation rate for which modeled transport equations are derived. These modeled transport equations are identical in form to (62c) and (62d) except for the addition of higher-order cross diffusion terms. The numerical values of the constants are derived directly from the theory (as with the RNG  $K - \epsilon$  model). However, in applications it has been found that these values need to be adjusted (see Nisizima and Yoshizawa 1987).

Speziale (1987b) developed a nonlinear  $K - \epsilon$  model based on a simplified version of (66) obtained by invoking the constraint of MFI in the limit of two-dimensional turbulence. In this model – where the length and time scales are built up from the turbulent kinetic energy and dissipation rate – the Reynolds stress tensor is modeled as<sup>§</sup>

$$\begin{aligned} \tau_{ij} = & \frac{2}{3}K\delta_{ij} - 2C_\mu\frac{K^2}{\epsilon}\bar{S}_{ij} - 4C_D C_\mu^2\frac{K^3}{\epsilon^2}(\bar{S}_{ik}\bar{S}_{kj} \\ & - \frac{1}{3}\bar{S}_{mn}\bar{S}_{mn}\delta_{ij}) - 4C_E C_\mu^2\frac{K^3}{\epsilon^2}(\overset{\circ}{\bar{S}}_{ij} - \frac{1}{3}\overset{\circ}{\bar{S}}_{mm}\delta_{ij}) \end{aligned} \quad (68)$$

where

$$\overset{\circ}{\bar{S}}_{ij} = \frac{\partial\bar{S}_{ij}}{\partial t} + \bar{u}_k\frac{\partial\bar{S}_{ij}}{\partial x_k} - \frac{\partial\bar{u}_i}{\partial x_k}\bar{S}_{kj} - \frac{\partial\bar{u}_j}{\partial x_k}\bar{S}_{ki} \quad (69)$$

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<sup>§</sup>It is interesting to note that Rubinstein and Barton (1989) recently derived an alternative version of this model – which neglects the convective derivative in (69) – by using the RNG method of Yakhot and Orszag.

is the frame-indifferent Oldroyd derivative of  $\overline{S}_{ij}$  and  $C_D = C_E = 1.68$ . Equation (68) can also be thought of as an approximation for turbulent flows where  $\tau_0/T_0 \ll 1$  since MFI (which (68) satisfies identically) becomes exact in the limit as  $\tau_0/T_0 \rightarrow 0$ . This model was shown by Speziale (1987b) and Speziale and Mac Giolla Mhuiris (1989a) to yield much more accurate predictions for the normal Reynolds stress anisotropies in turbulent channel flow and homogeneous shear flow (the standard  $K - \varepsilon$  model erroneously predicts that  $\tau_{xx} = \tau_{yy} = \tau_{zz} = \frac{2}{3}K$ ). As a result of this feature, the nonlinear  $K - \varepsilon$  model is capable of predicting turbulent secondary flows in non-circular ducts unlike the standard  $K - \varepsilon$  model which erroneously predicts a unidirectional mean turbulent flow (see Figures 3(a) - (c)). Comparably good predictions of turbulent secondary flows in a rectangular duct were obtained much earlier by Launder and Ying (1972), Gessner and Po (1976) and Demuren and Rodi (1984) who used the nonlinear algebraic Reynolds stress model of Rodi (1976). Due to the more accurate prediction of normal Reynolds stress anisotropies – and the incorporation of weak relaxation effects – the nonlinear  $K - \varepsilon$  model of Speziale (1987b) was also able to yield improved results for turbulent flow past a backward facing step (compare Figures 4(a) - (b) with Figures 2(a) - (b)). Most notably, the nonlinear  $K - \varepsilon$  model predicts reattachment at  $x/\Delta H \doteq 6.4$  – a value which is more in line with the experimental value of  $x/\Delta H \doteq 7.0$  (as shown earlier, the standard  $K - \varepsilon$  model yields a value of  $x/\Delta H \doteq 5.7$  when a single log wall layer is used).

Alternative two-equation models based on the solution of a modeled transport equation for an integral length scale (the  $K - \ell$  model) or the reciprocal time scale (the  $K - \omega$  model) have also been considered during the past fifteen years. In the  $K - \ell$  model (see Mellor and Herring 1973) a modeled transport equation is solved for the integral length scale  $\ell$  defined by

$$\ell(\mathbf{x}, t) = \frac{1}{2K} \int_{-\infty}^{\infty} \frac{R_{ii}(\mathbf{x}, \mathbf{r}, t)}{4\pi|\mathbf{r}|^2} d^3\mathbf{r} \quad (70)$$

where  $R_{ij} \equiv \overline{u'_i(\mathbf{x}, t)u'_j(\mathbf{x} + \mathbf{r}, t)}$  is the two-point velocity correlation tensor. The typical form of the modeled transport equation for  $\ell$  is as follows:

$$\frac{\partial(K\ell)}{\partial t} + \overline{u}_i \frac{\partial(K\ell)}{\partial x_i} = \frac{\partial}{\partial x_i} \left[ (\nu + \beta_1 K^{\frac{1}{2}} \ell) \frac{\partial}{\partial x_i} (K\ell) + \beta_2 K^{\frac{3}{2}} \ell \frac{\partial \ell}{\partial x_i} \right] - \beta_3 \ell \tau_{ij} \frac{\partial \overline{u}_i}{\partial x_j} - \beta_4 K^{\frac{3}{2}} \quad (71)$$

where  $\beta_1, \dots, \beta_4$  are empirical constants. Equation (71) is derived by integrating the contracted form of a modeled transport equation for the two-point velocity correlation tensor  $R_{ij}$  (see Wolfshtein 1970). Mellor and co-workers have utilized this  $K - \ell$  model – with an eddy viscosity of the form (52) – in the solution of a variety of engineering and geophysical fluid dynamics problems (see Mellor and Herring 1973 and Mellor and Yamada 1974 for a more thorough review). It has been argued – and correctly so – that it is more sound to



base the turbulent macroscale on the integral length scale (70) rather than on the turbulent dissipation rate which only formally determines the turbulent microscale. However, for homogeneous turbulent flows, it is a simple matter to show that this  $K - \ell$  model is *equivalent* to a  $K - \varepsilon$  model where the constants  $C_\mu$ ,  $C_{\varepsilon 1}$  and  $C_{\varepsilon 2}$  assume slightly different values (cf. Speziale 1989b). Furthermore, the modeled transport equation (71) for  $\ell$  has comparable problems to the modeled  $\varepsilon$ -transport equation in so far as integrations to a solid boundary are concerned (either wall functions or wall damping must be used). Consequently, at their current stage of development, it does not appear that this type of  $K - \ell$  model offers any significant advantages over the  $K - \varepsilon$  model.

Wilcox and co-workers have developed two-equation models of the  $K - \omega$  type (see Wilcox and Traci 1976 and Wilcox 1988). In this approach, modeled transport equations are solved for the turbulent kinetic energy  $K$  and reciprocal turbulent time scale  $\omega \equiv \varepsilon/K$ . The modeled transport equation for  $\omega$  is of the form

$$\frac{\partial \omega}{\partial t} + \bar{u}_i \frac{\partial \omega}{\partial x_i} = -\gamma_1 \frac{\omega}{K} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \gamma_2 \omega^2 + \frac{\partial}{\partial x_i} \left( \frac{\nu_T}{\sigma_\omega} \frac{\partial \omega}{\partial x_i} \right) + \nu \nabla^2 \omega \quad (72)$$

where  $\nu_T = \gamma^* K / \omega$  and  $\gamma_1, \gamma_2, \gamma^*$  and  $\sigma_\omega$  are constants. Equation (72) is obtained by making the same type of assumptions in the modeling of the exact transport equation for  $\omega$  that were made in developing the modeled  $\varepsilon$ -transport equation (62d). For homogeneous turbulent flows, there is little difference between the  $K - \varepsilon$  and  $K - \omega$  models. The primary difference between the two models is in their treatment of the transport terms: the  $K - \varepsilon$  model is based on a gradient transport hypothesis for  $\varepsilon$  whereas the  $K - \omega$  model uses the same hypothesis for  $\omega$  instead. Despite the fact that  $\omega$  is singular at a solid boundary, there is some evidence to suggest that the  $K - \omega$  model is more computationally robust for the integration of turbulence models to a wall (i.e., there is the need for less empirical damping; see Wilcox 1988).

## SECOND-ORDER CLOSURE MODELS

### *Theoretical Background*

Although two-equation models represent the first simple and complete Reynolds stress models to be developed, they still have significant deficiencies that make their application to complex turbulent flows precarious. As mentioned earlier, the two-equation models of the eddy viscosity type have the following major deficiencies: (a) the inability to properly account for streamline curvature, rotational strains and other body force effects and (b) the neglect of nonlocal and history effects on the Reynolds stress anisotropies. Most of these deficiencies are intimately tied to the assumption that there is a clear-cut separation of scales

at the second moment level (i.e., the level of the Reynolds stress tensor). This can be best illustrated by the example of homogeneous shear flow presented in the previous section. For this problem, the ratio of fluctuating to mean time scales is given by

$$\frac{\tau_0}{T_0} \equiv \frac{SK}{\epsilon} \doteq 4.8$$

for the  $K - \epsilon$  model. This is in flagrant conflict with the assumption that  $\tau_0/T_0 \ll 1$  which was crucial for the derivation of the  $K - \epsilon$  model! While some of the deficiencies cited above can be partially overcome by the use of two-equation models with a nonlinear algebraic correction to the eddy viscosity, major improvements can only be achieved by higher-order closures – the simplest of which are second-order closure models.

Second-order closure models are based on the Reynolds stress transport equation (23). Since this transport equation automatically accounts for the convection and diffusion of Reynolds stresses, second-order closure models (unlike eddy viscosity models) are able to account for strong nonlocal and history effects. Furthermore, since the Reynolds stress transport equation contains convection and production terms that adjust themselves automatically in turbulent flows with streamline curvature or a system rotation (through the addition of scale factors or Coriolis terms), complex turbulent flows involving these effects are usually better described.

In order to develop a second-order closure, models must be provided for the higher-order correlations  $C_{ijk}$ ,  $\Pi_{ij}$ , and  $\epsilon_{ij}$  on the right-hand-side of the Reynolds stress transport equation (23). These terms, sufficiently far from solid boundaries, are typically modeled as follows:

- (1) The third-order transport term  $C_{ijk}$  is modeled by a gradient transport hypothesis which is based on the usual assumption that there is a clear-cut separation of scales between mean and fluctuating fields.
- (2) The pressure-strain correlation  $\Pi_{ij}$  and the dissipation rate correlation  $\epsilon_{ij}$  are modeled based on ideas from homogeneous turbulence wherein the departures from isotropy are assumed to be small enough to allow for a Taylor series expansion about a state of isotropic turbulence.

Near solid boundaries, either wall functions or wall damping are used in a comparable manner to that discussed in the last section. One important point to note is that the crucial assumption of separation of scales is made only at the *third moment level*. This leads us to the *raison d'être* of second-order closure modeling: *Since crude approximations for the second moments in eddy viscosity models can often yield adequate approximations for first*

order moments (i.e.,  $\Pi$  and  $\bar{p}$ ) it may follow that crude approximations for the third-order moments can yield adequate approximations for the second-order moments in Reynolds stress transport models.

The pressure-strain correlation  $\Pi_{ij}$  plays a crucial role in determining the structure of most turbulent flows. Virtually all of the models for  $\Pi_{ij}$  that have been used in conjunction with second-order closure models are based on the assumption of local homogeneity. For homogeneous turbulent flows, the pressure-strain correlation takes the form

$$\Pi_{ij} = A_{ij} + M_{ijkl} \frac{\partial \bar{u}_k}{\partial x_l} \quad (73)$$

where

$$A_{ij} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \overline{\left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right) \frac{\partial^2 u'_k u'_l}{\partial y_k \partial y_l} \frac{dV(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}}$$

$$M_{ijkl} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right) \frac{\partial u'_l}{\partial y_k} \frac{dV(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}}$$

Here, the first term on the r.h.s. of (73) is referred to as the slow pressure-strain whereas the second term is called the rapid pressure-strain. It has been shown that  $A_{ij}$  and  $M_{ijkl}$  are functionals – in time and wavenumber space – of the energy spectrum tensor (cf. Weinstock 1981 and Reynolds 1987). In a one-point closure, this suggests models for  $A_{ij}$  and  $M_{ijkl}$  that are functionals of the Reynolds stress tensor and turbulent dissipation rate. Neglecting history effects, the simplest such models are of the form

$$A_{ij} = \varepsilon \mathcal{A}_{ij}(\mathbf{b}) \quad (74)$$

$$M_{ijkl} = K \mathcal{M}_{ijkl}(\mathbf{b}). \quad (75)$$

These algebraic models – based on the assumptions stated above – are obtained by using simple dimensional arguments combined with the fact that  $\Pi_{ij}$  vanishes in the limit of isotropic turbulence (a constraint identically satisfied if  $\mathcal{A}_{ij}(0) = 0$  and  $\mathcal{M}_{ijkl}(0) = 0$ ). Virtually all of the models for the pressure-strain correlation that have been used in conjunction with second-order closure models are of the form (73) - (75).

Lumley (1978) was probably the first to systematically develop a general representation for the pressure-strain correlation based on (73) - (75). It can be shown that invariance under a change of coordinates – coupled with the assumption of analyticity about the isotropic state  $b_{ij} = 0$  – restricts (73) to be of the form<sup>¶</sup>

<sup>¶</sup>This representation, obtained by using the results of Smith (1971) on isotropic tensor functions, is actually somewhat more compact than that obtained by Lumley and co-workers.

$$\begin{aligned}
\Pi_{ij} = & a_0 \varepsilon b_{ij} + a_1 \varepsilon (b_{ik} b_{kj} - \frac{1}{3} II \delta_{ij}) + a_2 K \overline{S}_{ij} \\
& + (a_3 tr \mathbf{b} \cdot \overline{\mathbf{S}} + a_4 tr \mathbf{b}^2 \cdot \overline{\mathbf{S}}) K b_{ij} + (a_5 tr \mathbf{b} \cdot \overline{\mathbf{S}} \\
& + a_6 tr \mathbf{b}^2 \cdot \overline{\mathbf{S}}) K (b_{ik} b_{kj} - \frac{1}{3} II \delta_{ij}) + a_7 K (b_{ik} \overline{S}_{jk} \\
& + b_{jk} \overline{S}_{ik} - \frac{2}{3} tr \mathbf{b} \cdot \overline{\mathbf{S}} \delta_{ij}) + a_8 K (b_{ik} b_{kl} \overline{S}_{jl} \\
& + b_{jk} b_{kl} \overline{S}_{il} - \frac{2}{3} tr \mathbf{b}^2 \cdot \overline{\mathbf{S}} \delta_{ij}) + a_9 K (b_{ik} \overline{\omega}_{jk} \\
& + b_{jk} \overline{\omega}_{ik}) + a_{10} K (b_{ik} b_{kl} \overline{\omega}_{jl} + b_{jk} b_{kl} \overline{\omega}_{il})
\end{aligned} \tag{76}$$

where

$$a_i = a_i(II, III), \quad i = 0, 1, \dots, 10$$

$$II = b_{ij} b_{ij}, \quad III = b_{ik} b_{kl} b_{li}.$$

and  $tr(\cdot)$  denotes the trace. The eigenvalues  $b^{(\alpha)}$  of  $b_{ij}$  are bounded as follows (see Lumley 1978)

$$-\frac{1}{3} \leq b^{(\alpha)} \leq \frac{2}{3}, \quad \alpha = 1, 2, 3 \tag{77}$$

and for many engineering flow,  $\|\mathbf{b}\|_2 \equiv |b^{(\alpha)}|_{\max} < 0.25$ . Hence, it would seem that a low order Taylor series truncation of (76) could possibly provide an adequate approximation. To the first order in  $b_{ij}$ :

$$\Pi_{ij} = -C_1 \varepsilon b_{ij} + C_2 K \overline{S}_{ij} + C_3 K \left( b_{ik} \overline{S}_{jk} + b_{jk} \overline{S}_{ik} - \frac{2}{3} b_{mn} \overline{S}_{mn} \delta_{ij} \right) + C_4 K (b_{ik} \overline{\omega}_{jk} + b_{jk} \overline{\omega}_{ik}) \tag{78}$$

which is the form used in the Launder, Reece and Rodi (1975) model. In the Launder, Reece and Rodi (LRR) model, the constants  $C_1$ ,  $C_3$ , and  $C_4$  were calibrated based on the results of return to isotropy and homogeneous shear flow experiments. The constant  $C_2$  was chosen to be consistent with the value obtained by Crow (1968) from RDT for an irrotationally strained turbulence starting from an initially isotropic state. This yielded the following values for the constants in the simplified version of the LRR model:  $C_1 = 3.6$ ,  $C_2 = 0.8$ ,  $C_3 = 0.6$  and  $C_4 = 0.6$ . It should be noted that the representation for the slow pressure-strain correlation in the LRR model is the Rotta (1951) return to isotropy model with the Rotta constant  $C_1$  adjusted from 2.8 to 3.6 (a value which is in the range of what can be extrapolated from

physical experiments). This model – consistent with experiments – predicts that an initially anisotropic, homogeneous turbulence relaxes gradually to an isotropic state after the mean velocity gradients are removed.

An even simpler version of (78) was proposed by Rotta (1972) wherein  $C_3 = C_4 = 0$ . This model has been used by Mellor and co-workers for the calculation of a variety of engineering and geophysical flows (see Mellor and Herring 1973 and Mellor and Yamada 1974). Research during the past decade has focused attention on the development of nonlinear models for  $\Pi_{ij}$ . Lumley (1978) and Shih and Lumley (1985) developed a nonlinear model by using the constraint of realizability discussed earlier. Haworth and Pope (1986) developed a nonlinear model for the pressure-strain correlation based on the Langevin equation used in the pdf description of turbulence. This model – which was cubic in the anisotropy tensor – was calibrated based on homogeneous turbulence experiments and was shown to perform quite well for a range of such flows. Speziale (1987a) developed a hierarchy of second-order closure models that were consistent with the MFI constraint in the limit of two-dimensional turbulence\* (this constraint was also made use of by Haworth and Pope 1986 in the development of their second-order closure). Launder and his co-workers (cf. Fu, Launder and Tselepidakis 1987 and Craft, et al. 1989) have developed new nonlinear models for the pressure-strain correlation based on the use of realizability combined with a calibration using newer homogeneous turbulence experiments. Reynolds (1988) has attempted to develop models that are consistent with RDT and the author has been analyzing models based on a dynamical systems approach (see Speziale and Mac Giolla Mhuiris 1989a, 1989b and Speziale, Sarkar and Gatski 1990).

The modeling of the dissipation rate tensor, at high turbulence Reynolds numbers, is usually based on the Kolmogorov hypothesis of isotropy where

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} \quad (79)$$

given that  $\varepsilon \equiv \nu \overline{\partial u'_i / \partial x_j \partial u'_i / \partial x_j}$  is the scalar dissipation rate. Here, the turbulent dissipation rate  $\varepsilon$  is typically taken to be a solution of the modeled transport equation

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} = -C_{\varepsilon 1} \frac{\varepsilon}{K} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - C_{\varepsilon 2} \frac{\varepsilon^2}{K} + C_\varepsilon \frac{\partial}{\partial x_i} \left( \frac{K}{\varepsilon} \tau_{ij} \frac{\partial \varepsilon}{\partial x_j} \right) \quad (80)$$

where  $C_{\varepsilon 1} = 1.44$ ,  $C_{\varepsilon 2} = 1.92$  and  $C_\varepsilon = 0.15$ . Equation (80) is identical to the  $\varepsilon$ -transport equation used in the  $K - \varepsilon$  model with one exception: the turbulent diffusion term is *anisotropic*. Hence, the logic used in deriving (80) is virtually the same as that used in

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\* MFI in the limit of two-dimensional turbulence can be satisfied identically by (76) if  $a_{10} = -3a_9 + 12$ ; see Speziale (1987a).

deriving the modeled  $\varepsilon$ -transport equation for the  $K - \varepsilon$  model. Near solid boundaries, anisotropic corrections to (79) have been proposed that are typically of the algebraic form (see Hanjalic and Launder 1976)

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} + 2\varepsilon f_s b_{ij} \quad (81)$$

where  $f_s$  is a function of the turbulence Reynolds number  $Re_t \equiv K^2/\nu\varepsilon$ . Equation (81) – which can be thought of as a first-order Taylor expansion about a state of isotropic turbulence – is solved in conjunction with (80) where the model coefficients are taken to be functions of  $Re_t$  as a solid boundary is approached (cf. Hanjalic and Launder 1976). As an alternative to (81), the isotropic form (79) can be used in a wall bounded flow if suitable wall functions are used to bridge the outer and inner flows.

One major weakness of the models (80) - (81) is their neglect of rotational strains. For example, in a rotating isotropic turbulence, the modeled  $\varepsilon$ -transport equation (80) yields the *same decay rate* independent of the rotation rate of the reference frame. In stark contrast to this result, physical and numerical experiments indicate that the decay rate of the turbulent kinetic energy can be considerably reduced by a system rotation – the inertial waves generated by the rotation disturb the phase coherence needed to cascade energy from the large scales to the small scales (see Wigeland and Nagib 1978 and Speziale, Mansour and Rogallo 1987). A variety of modifications to (80) have been proposed during the last decade to account for rotational strains (see Pope 1978, Hanjalic and Launder 1980, and Bardina, Ferziger and Rogallo 1985). However, these modifications have tended to be “one problem” corrections which gave rise to difficulties when other flows were considered. It was recently shown by the author that all of these modified  $\varepsilon$ -transport equations are more ill-behaved than the standard model (80) for general homogeneous turbulent flows in a rotating frame (e.g., they fail to properly account for the stabilizing effect of a strong system rotation on a homogeneously strained turbulent flow; see Speziale 1989b).

At this point it should be mentioned that in the second-order closure models of Mellor and co-workers, the dissipation rate is modeled as in equation (55) and a modeled transport equation for the integral length scale (70) is solved which is identical in form to (71). When this model has been applied to wall bounded turbulent flows it has typically been used in conjunction with wall functions. In addition, it should also be mentioned that second-order closure models along the lines of the  $K - \omega$  model of Wilcox and co-workers have been considered (here a modeled transport equation for the reciprocal time scale  $\omega \equiv \varepsilon/K$  is solved; cf. Wilcox 1988).

In order to complete these second-order closures, a model for the third-order diffusion correlation  $C_{ijk}$  is needed. Since this is a third-order moment, the simplifying assumption

of gradient transport (which is generally valid only when there is a clear-cut separation of scales) is typically made. Hence, all of the commonly used second-order closures are based on models for  $C_{ijk}$  of the form:

$$C_{ijk} = -C_{ijklmn} \frac{\partial \tau_{lm}}{\partial x_n}$$

where the diffusion tensor  $C_{ijklmn}$  can depend anisotropically on  $\tau_{ij}$ . For many incompressible turbulent flows, the pressure diffusion terms in  $C_{ijk}$  can be neglected in comparison to the triple velocity correlation  $\overline{u_i u_j u_k}$ . Then, the symmetry of  $C_{ijk}$  under an interchange of any of its three indices immediately yields the form

$$C_{ijk} = -C_s \frac{K}{\varepsilon} \left( \tau_{im} \frac{\partial \tau_{jk}}{\partial x_m} + \tau_{jm} \frac{\partial \tau_{ik}}{\partial x_m} + \tau_{km} \frac{\partial \tau_{ij}}{\partial x_m} \right) \quad (82)$$

which was first obtained by Launder, Reece and Rodi (1975) via an alternative analysis based on the transport equation for  $\overline{u_i u_j u_k}$ . Equation (82) is sometimes used in its isotropized form

$$C_{ijk} = -\frac{2}{3} C_s \frac{K^2}{\varepsilon} \left( \frac{\partial \tau_{jk}}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_k} \right) \quad (83)$$

(cf. Mellor and Herring 1973). The constant  $C_s$  was chosen to be 0.11 by Launder, Reece and Rodi (1975) based on comparisons with experiments on thin shear flows.

### Examples

Now, by the use of some illustrative examples, a case will be made for the superior predictive capabilities of second-order closures in comparison to zero, one and two equation models. First, to demonstrate the ability of second-order closure models to describe Reynolds stress relaxation effects, we will consider the return to isotropy problem. In this problem, an initially anisotropic, homogeneous turbulence – generated by the application of constant mean velocity gradients – gradually relaxes to a state of isotropy after the mean velocity gradients are removed. By introducing the transformed dimensionless time  $\tau$  (where  $d\tau = \varepsilon dt / 2K$ ), the modeled Reynolds stress transport equation can be written in the equivalent form

$$\frac{db_{ij}}{d\tau} = 2b_{ij} + \mathcal{A}_{ij} \quad (84)$$

where  $\mathcal{A}_{ij}$  is the dimensionless slow pressure strain correlation. Since the rapid pressure-strain and transport terms vanish in this problem – and since the dissipation rate can be absorbed into the dimensionless time  $\tau$  – only a model for the slow pressure strain correlation is needed as indicated in (84). In Figure 5, the predictions of the LRR model (where  $\mathcal{A}_{ij} = -C_1 b_{ij}$  and the Rotta constant  $C_1 = 3.0$ ) for the time evolution of the second invariant

of the anisotropy tensor are compared with the experimental data of Choi and Lumley (1984) for the relaxation from plane strain case. It is clear from this figure that this simple second-order closure model does a reasonably good job in reproducing the experimental trends which show a gradual return to isotropy where  $II \rightarrow 0$  as  $\tau \rightarrow \infty$ . This is in considerable contrast to eddy viscosity (or nonlinear algebraic stress) models which erroneously predict that  $II = 0$  for  $\tau > 0$ ! Further improvements can be obtained with second-order closures based on nonlinear models for the slow pressure strain correlation. A simple quadratic model was recently proposed by Sarkar and Speziale (1990) where

$$A_{ij} = -C_1 b_{ij} + C_2 (b_{ik} b_{kj} - \frac{1}{3} II \delta_{ij}) \quad (85)$$

with  $C_1 = 3.4$  and  $C_2 = 4.2$ . This model does a better job in reproducing the trends of the Choi and Lumley (1984) experiment (see Figure 6). Most notably, the quadratic model (85) yields curved trajectories in the  $\xi - \eta$  phase space (where  $\xi = III^{\frac{1}{3}}$ ,  $\eta = II^{\frac{1}{2}}$ ) that are well within the range of experimental data; any linear or quasilinear model where  $C_2 = 0$  erroneously yields straight line trajectories in the  $\xi - \eta$  phase space as clearly shown in Figure 6.

As alluded to earlier, second-order closure models perform far better than eddy viscosity models in rotating turbulent flows. To illustrate this point, a comparison of the predictions of the standard  $K - \varepsilon$  model and the Launder, Reece and Rodi model will be made for the problem of homogeneous turbulent shear flow in a rotating frame. This problem represents a non-trivial test of turbulence models since a system rotation can have either a stabilizing or destabilizing effect on turbulent shear flow. The most basic type of plane shear flow in a rotating frame will be discussed where

$$\frac{\partial \bar{u}_i}{\partial x_j} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega_i = (0, 0, \Omega) \quad (86)$$

(see Figure 7). For the case of pure shear flow ( $\Omega = 0$ ), the Launder, Reece and Rodi model yields substantially improved predictions over the  $K - \varepsilon$  model for the equilibrium values of  $b_{ij}$  and  $SK/\varepsilon$  as shown in Table 1. Since the standard  $K - \varepsilon$  model is frame-indifferent, it erroneously yields solutions for rotating shear flow that are independent of  $\Omega$ . Second-order closure models, on the other hand, yield rotationally-dependent solutions due to the effect of the Coriolis acceleration. For any homogeneous turbulent flow in a rotating frame, second-order closure models take the form (cf. Speziale 1989a)

$$\dot{\tau}_{ij} = -\tau_{ik} \frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk} \frac{\partial \bar{u}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij} - 2(\tau_{ik} e_{mkj} \Omega_m + \tau_{jk} e_{mki} \Omega_m) \quad (87)$$



where the mean vorticity tensor  $\overline{\omega}_{ij}$  in the model for  $\Pi_{ij}$  (see equation (78)) is replaced with the intrinsic mean vorticity tensor  $\overline{W}_{ij}$  defined in (32). The equations of motion for the LRR model are obtained by substituting (86) into (87) and the modeled  $\varepsilon$ -transport equation

$$\dot{\varepsilon} = -C_{\varepsilon 1} \frac{\varepsilon}{K} \tau_{ij} \frac{\partial \overline{u}_i}{\partial x_j} - C_{\varepsilon 2} \frac{\varepsilon^2}{K} \quad (88)$$

which is not directly affected by rotations. A complete dynamical systems analysis of these nonlinear ordinary differential equations – which are typically solved for initial conditions that correspond to a state of isotropic turbulence – was conducted recently by Speziale and Mac Giolla Mhuiris 1989a. It was found that  $\varepsilon/SK$  and  $b_{ij}$  have finite equilibrium values that are independent of the initial conditions and only depend on  $\Omega$  and  $S$  through the dimensionless ratio  $\Omega/S$ . There are two equilibrium solutions for  $(\varepsilon/SK)_{\infty}$ : one where  $(\varepsilon/SK)_{\infty} = 0$  which exists for *all*  $\Omega/S$  and one where  $(\varepsilon/SK)_{\infty} > 0$  which only exists for an intermediate band of  $\Omega/S$  (see Figure 8(a)). The trivial equilibrium solution is predominantly associated with solutions for  $K$  and  $\varepsilon$  that undergo a power law decay with time; the non-zero equilibrium solution (ellipse ACB on the bifurcation diagram shown in Figure 8(a)) is associated with unstable flow wherein  $K$  and  $\varepsilon$  undergo an exponential time growth at the same rate. The two solutions exchange stabilities in the interval AB (i.e., this is a degenerate type of transcritical bifurcation). In stark contrast to these results, the standard  $K - \varepsilon$  model erroneously predicts the *same* equilibrium value for  $(\varepsilon/SK)_{\infty}$  independent of  $\Omega/S$  (see Figure 8(b)). In Figures 9(a) - (c), the time evolution of the turbulent kinetic energy predicted by the standard  $K - \varepsilon$  model and the LRR model are compared with the large-eddy simulations of Bardina, Ferziger and Reynolds (1983). It is clear that the second-order closure model is able to properly account for the stabilizing or destabilizing effect of rotations on shear flow whereas the  $K - \varepsilon$  model erroneously predicts results that are independent of the rotation rate  $\Omega$ . The LRR model predicts that there is unstable flow (where  $K$  and  $\varepsilon$  grow exponentially) only for rotation rates lying in the intermediate range  $-0.1 \leq \Omega/S \leq 0.39$  whereas linear stability analyses indicate unstable flow for  $0 \leq \Omega/S \leq 0.5$ . Similar improved results using second-order closures have been recently obtained by Gatski and Savill (1989) for curved homogeneous shear flow.

Finally, an example of an inhomogeneous wall-bounded turbulent flow will be given. The problem of rotating channel flow recently considered by Launder, Tselepidakis and Younis (1987) represents a challenging example. In this problem a turbulent channel flow is subjected to a steady spanwise rotation (see Figure 10). Physical and numerical experiments (see Johnston, Halleen and Lezius 1972 and Kim 1983) indicate that Coriolis forces arising from a system rotation cause the mean velocity profile  $\overline{u}(y)$  to become asymmetric about the channel centerline. In Figure 11, the mean velocity profile computed by Launder, Tselepi-

dakis and Younis (1987) using the Gibson and Launder (1978) second-order closure model is compared with the results of the  $K - \epsilon$  model and the experimental data of Johnston, Halleen and Lezius (1972) for a Reynolds number  $Re = 11,500$  and a rotation number  $Ro = 0.21$ . From this figure, it is clear that the second-order closure model yields a highly asymmetric mean velocity profile that is well within the range of the experimental data. The standard  $K - \epsilon$  model erroneously predicts the *same* symmetric mean velocity profile as in an inertial framing (where  $\Omega = 0$ ) as shown in Figure 11. Comparable improvements in the prediction of curved turbulent shear flows have been obtained by Gibson and Rodi (1981) and Gibson and Younis (1986) using second-order closure models. Likewise, turbulent flows involving buoyancy effects have been shown to be better described by second-order closure models (cf. Zeman and Lumley 1976, 1979). In these problems, the Coriolis terms on the r.h.s. of the Reynolds stress transport equation (87) are replaced with the body force term

$$\beta(g_i \overline{T' u_j'} + g_j \overline{T' u_i'}) \quad (89)$$

where  $\beta$  is the coefficient of thermal expansion and  $g_i$  is the acceleration due to gravity. The temperature-velocity correlation  $\overline{T' u_i'}$  (also called the Reynolds heat flux) is modeled by a gradient transport hypothesis or is obtained from a modeled version of its transport equation.

While second-order closure models constitute, by far, the most promising approach in Reynolds stress modeling, it must be said that they have not progressed to the point where reliable quantitative predictions can be made for a variety of turbulent flows. To illustrate this point we again cite the case of rotating shear flow. As shown earlier, the phase space portrait of second-order closures is far superior to that of any two-equation model of the eddy viscosity type (i.e., the second-order closures properly predict that there is unstable flow only for an intermediate band of rotation rates; see Figure 8). However, the specific quantitative predictions of a wide variety of existing second-order closures were recently shown by Speziale, Gatski and Mac Giolla Mhuiris (1989) to be highly contradictory in rotating shear flow for a significant range of  $\Omega/S$  (see Figure 12). Comparable problems with the reliability of predictions when second-order closure models are integrated directly to a solid boundary persist so that a variety of modifications – which usually involve the introduction of empirical wall damping that depends on the turbulence Reynolds number as well as the unit normal to the wall – continue to be proposed along alternative lines (cf. Launder and Shima 1989, Lai and So 1989 and Shih and Mansour 1990).

In the opinion of the author, there are two major areas of development that are direly needed in order to improve the predictive capabilities of second-order closures:

- (1) The introduction of improved transport models for the turbulence length scale which

incorporate at least some limited *two-point and directional information* (e.g., through some appropriate integral of the two-point velocity correlation tensor  $R_{ij}$ ). In conjunction with this research, the use of gradient transport models should be re-examined. Although Donaldson and Sandri (1981) developed a tensor length scale along these lines, it was recently shown by Speziale (1989b) that the specific form of the model that they chose can be collapsed to the standard  $\varepsilon$ -transport model in homogeneous flows.

- (2) The need for asymptotically consistent low turbulence Reynolds number extensions of existing models that can be robustly integrated to a solid boundary. Existing models use ad hoc damping functions based on  $Re_\tau$  and have an implicit dependence on the unit normal to the wall which does not allow for the proper treatment of geometrical discontinuities such as those that occur in the square duct or back-step problems. Furthermore, the *nonlinear* effect of both rotational and irrotational strains need to be accounted for in the modeling of near wall anisotropies in the dissipation.

## CONCLUDING REMARKS

There has been a tendency to be overly pessimistic about the progress made in Reynolds stress modeling during the past few decades. It must be remembered that the first complete Reynolds stress models – cast in tensor form and supplemented only with initial and boundary conditions – were developed less than twenty years ago. Progress was at first stymied by the lack of adequate computational power to properly explore full Reynolds stress closures in non-trivial turbulent flows – a deficiency that was not overcome until the late 1960's. Then, by 1980 – with an enormous increase in computer capacity – efforts were shifted toward direct and large eddy simulations of the Navier-Stokes equations. Furthermore, the interest in coherent structures (cf. Hussain 1983) and alternative theoretical approaches based on nonlinear dynamics (e.g., period doubling bifurcations as a route to chaos; cf. Swinney and Gollub 1981) that crystalized during the late 1970's have also shifted attention away from Reynolds stress modeling, as well as the general statistical approach for that matter. While progress has been slow, this is due in large measure to how intrinsically difficult the problem is. The fact that real progress has been made, however, cannot be denied. Many of the turbulent flows considered in the last section – which were solved without the introduction of any further ad hoc empiricisms – could not be properly analyzed by the Reynolds stress models that were available before 1970.

Some discussion is warranted concerning the goals and limitations of Reynolds stress modeling. Under the best of circumstances, Reynolds stress models can only provide accurate in-

formation about first and second one-point moments (e.g., the mean velocity, mean pressure, and turbulence intensity) which, quite often, is all that is needed for design purposes. Since Reynolds stress modeling constitutes a low-order one-point closure, it intrinsically cannot provide detailed information about flow structures. Furthermore, since spectral information needs to be indirectly built into Reynolds stress models, a given model cannot be expected to perform well in a variety of turbulent flows where the spectrum of the energy containing eddies is changing dramatically. However, to criticize Reynolds stress models purely on the grounds that they are not based rigorously on solutions of the full Navier-Stokes equations would be as childish as criticizing exact solutions of the Navier-Stokes equations for not being rigorously derivable from the Boltzmann equation or, for that matter, from quantum mechanics. The more appropriate question is whether or not a Reynolds stress model can be developed that will provide adequate engineering answers for the mean velocity, mean pressure and turbulence intensities in a significant range of turbulent flows that are of technological interest. To obtain accurate predictions for these quantities in *all* possible turbulent flows will probably require nothing short of solving the full Navier-Stokes equations. Such a task will not be achievable in the foreseeable future, if ever at all (cf. Hussaini, Speziale and Zang 1989). To gain an appreciation for the magnitude of such an endeavor, consider the fact that economically feasible direct simulations of turbulent pipe flow at a Reynolds number of 500,000 – a turbulent flow which, although non-trivial, is far from the most difficult encountered – would require a computer that is 10 million times faster than the Cray YMP!

While second-order closures represent the most promising approach in Reynolds stress modeling, much work remains to be done. The two problem areas mentioned in the previous section – namely, the development of transport models for an anisotropic integral length scale and the development of more asymptotically consistent methods for the integration of second-order closures to a solid boundary – are of utmost importance. In fact, the latter issue of near wall modeling is so crucial that deficiencies in it – along with associated numerical stiffness problems – are primarily responsible for the somewhat misleading critical evaluations of second-order closures that arose out of the 1980-81 AFOSR-HTTM Stanford Conference on Turbulence (see Kline, Cantwell and Lilley 1981). Another area that urgently needs attention is the second-order closure modeling of compressible turbulent flows. Until recently, most compressible second-order closure modeling has consisted of Favre-averaged, variable-density extensions of the incompressible models (cf. Cebeci and Smith 1974). However, with the current thrust in compressible second-order modeling at NASA Langley and NASA Ames, some new compressible modeling ideas – such as dilatational dissipation – have come to the forefront (see Sarkar et al. 1989 and Zeman 1990). Much more work in this area is needed, however.

Reynolds stress modeling should continue to steadily progress – complementing numerical simulations of the Navier-Stokes equations and alternative theoretical approaches. In fact, with anticipated improvements in computer capacity, direct numerical simulations should begin to play a pivotal role in the screening and calibration of turbulence models. Furthermore, from the theoretical side, statistical mechanics approaches such as RNG could be of considerable future use in the formulation of new models (unfortunately, at their current stage of development, it does not appear that they can reliably calibrate turbulence models for use in complex flows). Although Reynolds stress models provide information only about a limited facet of turbulence, this information can have such important scientific and engineering applications that they are likely to remain a part of turbulence research for many years to come.

## ACKNOWLEDGEMENTS

The author would like to thank Drs. T. B. Gatski, N. Mac Giolla Mhuiris and S. Sarkar for their assistance with some of the computations presented in this paper. Helpful comments by Dr. M. Y. Hussaini are also gratefully acknowledged. This research was supported under NASA Contract NAS1-18605 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

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<i>Equilibrium Values</i>	<i>Standard K-<math>\epsilon</math> Model</i>	<i>Launder, Reece &amp; Rodi Model</i>	<i>Experiments</i>
$(b_{11})_{\infty}$	0	0.193	0.201
$(b_{22})_{\infty}$	0	-0.096	-0.147
$(b_{12})_{\infty}$	-0.217	-0.185	-0.150
$(SK/\epsilon)_{\infty}$	4.82	5.65	6.08

Table 1. Comparison of the predictions of the standard K- $\epsilon$  model and the Launder, Reece, and Rodi model with the experiments of Tavoularis and Corrsin (1981) on homogeneous shear flow.

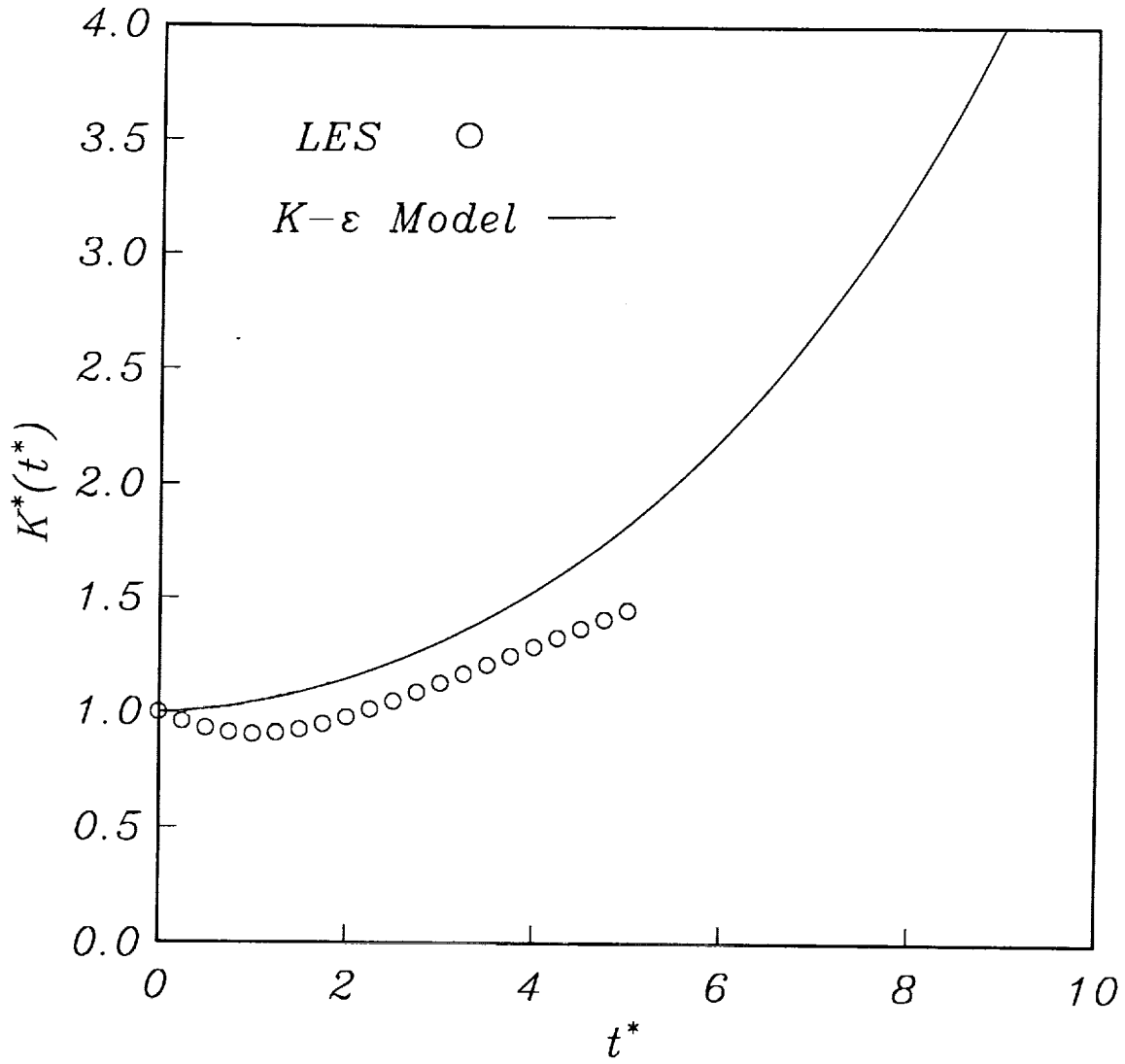


Figure 1. Time evolution of the turbulent kinetic energy in homogeneous shear flow. Comparison of the predictions of the standard  $K - \epsilon$  model with the large-eddy simulation of Bardina, Ferziger, and Reynolds (1983) for  $\epsilon_0/SK_0 = 0.296$ .



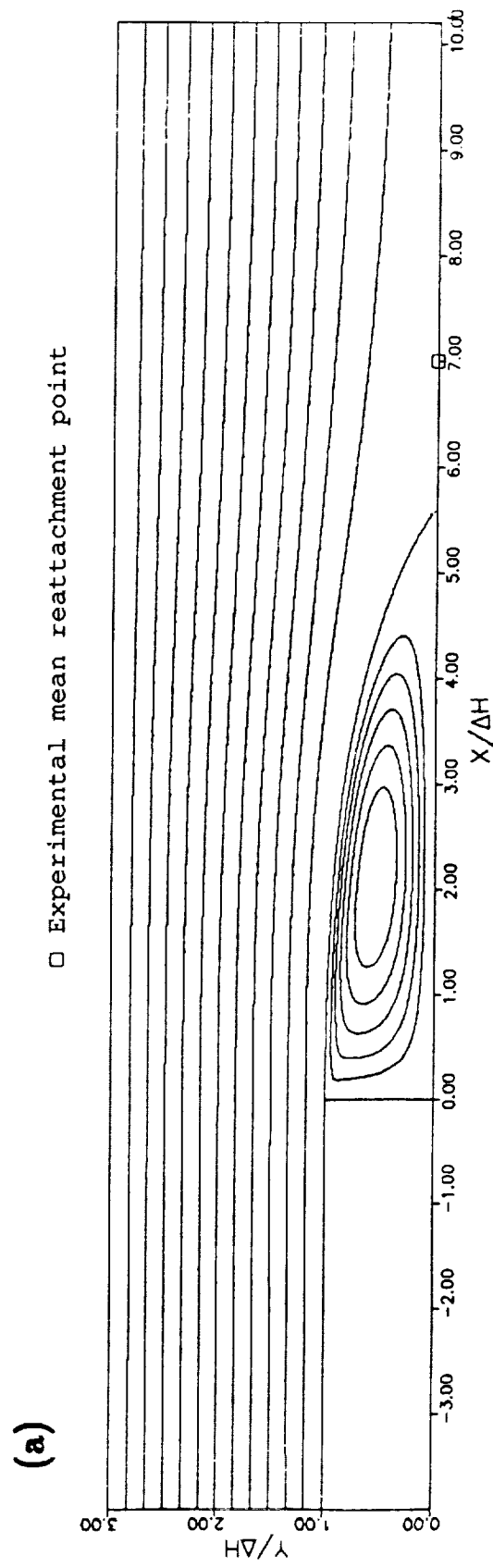


Figure 2. Comparison of the predictions of the standard  $K - \epsilon$  model with the experiments of Kim, Kline, and Johnston (1980) for turbulent flow past a backward facing step: (a) mean flow streamlines, (b) turbulence intensity profiles.

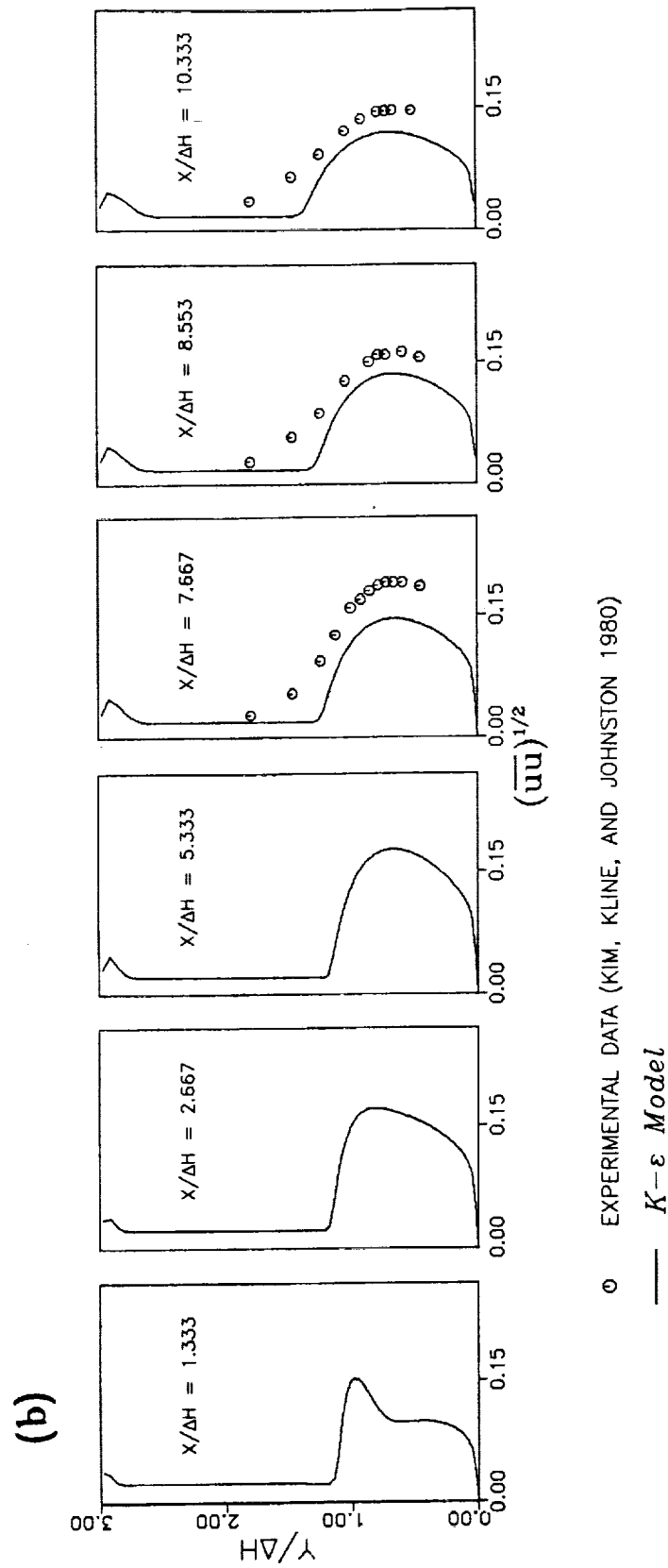


Figure 2. Comparison of the predictions of the standard  $K - \epsilon$  model with the experiments of Kim, Kline, and Johnston (1980) for turbulent flow past a backward facing step: (a) mean flow streamlines, (b) turbulence intensity profiles.

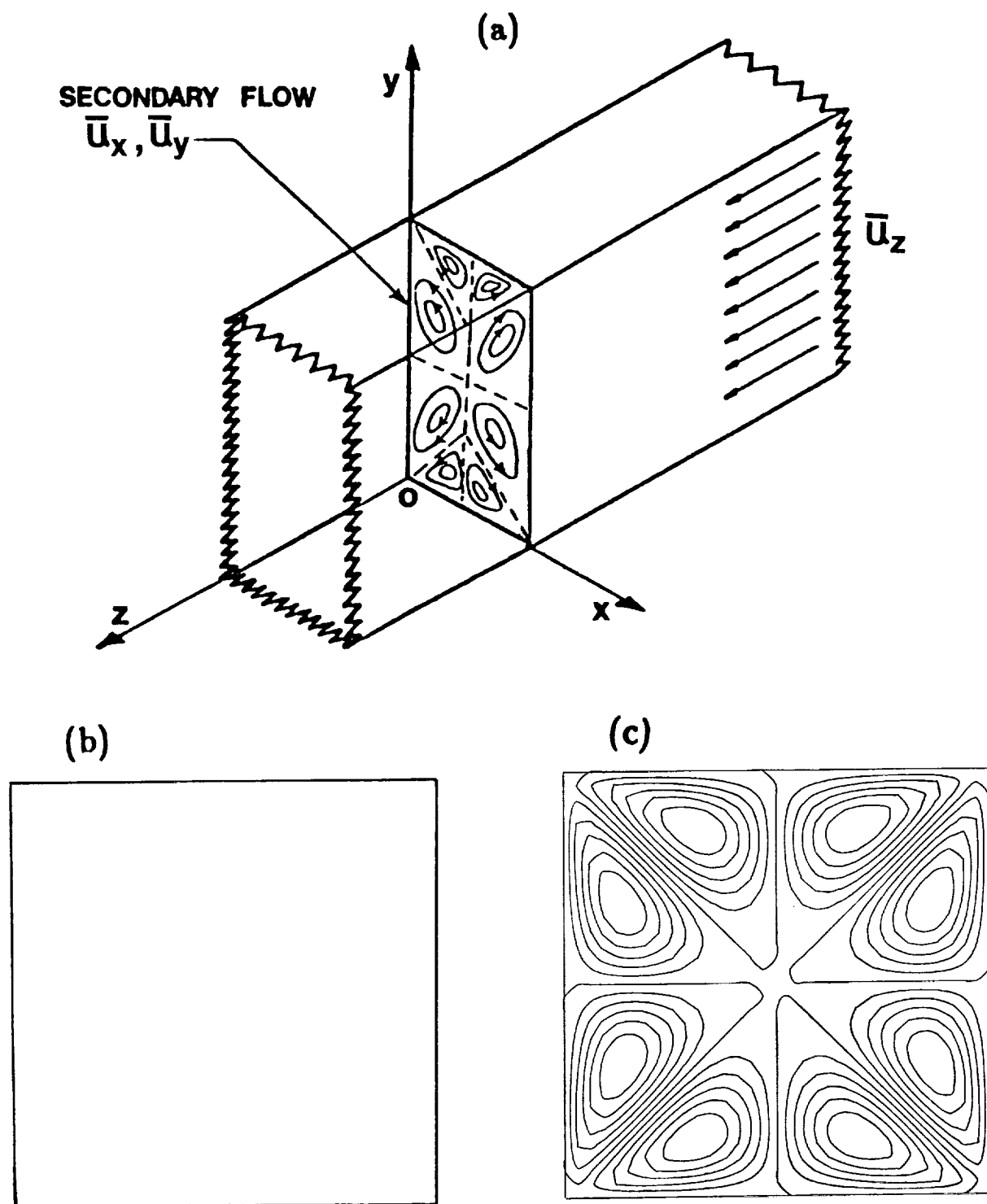


Figure 3. Turbulent secondary flow in a rectangular duct: (a) experiments, (b) standard  $K - \epsilon$  model, and (c) the nonlinear  $K - \epsilon$  model of Speziale (1987b).

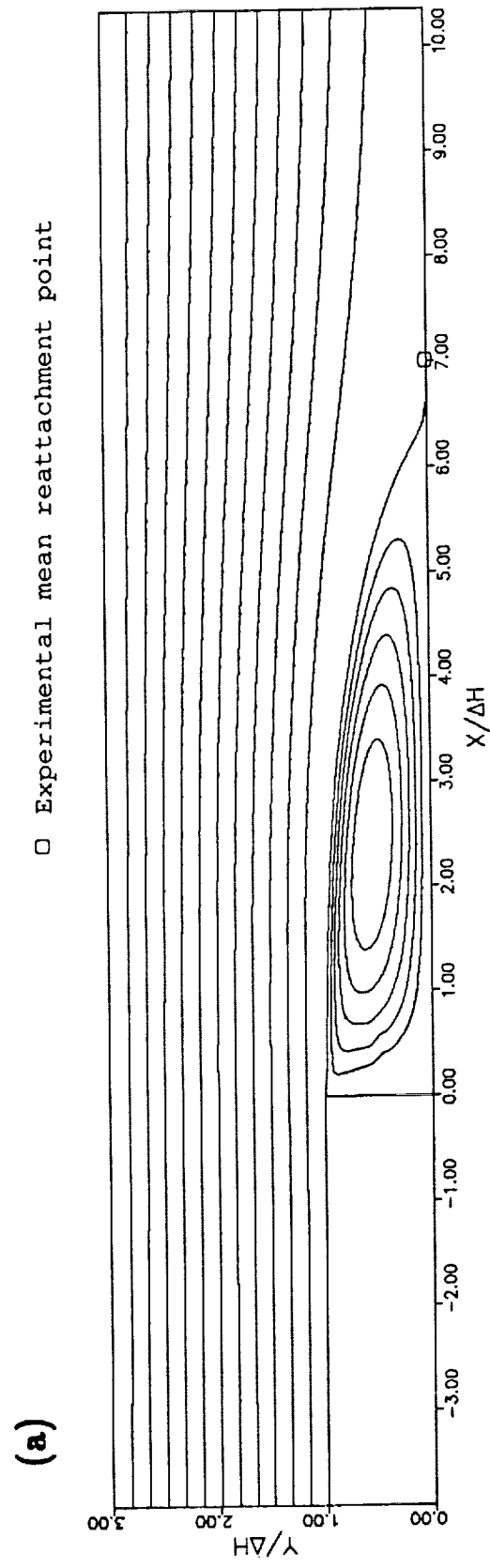


Figure 4. Comparison of the predictions of the nonlinear  $K - \epsilon$  model of Speziale (1987b) with the experiments of Kim, Kline, and Johnston (1980) for turbulent flow past a backward facing step: (a) mean flow streamlines, (b) turbulence intensity profiles.

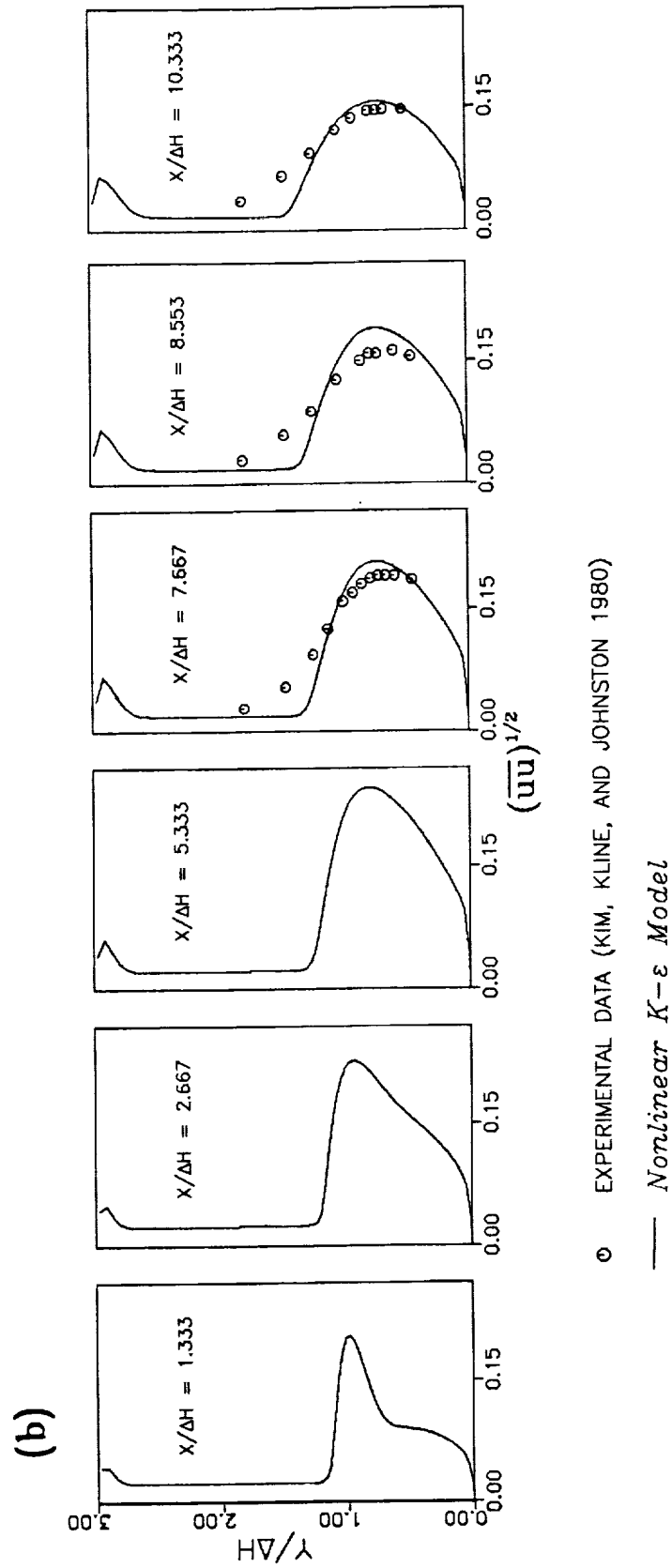


Figure 4. Comparison of the predictions of the nonlinear  $K - \varepsilon$  model of Speziale (1987b) with the experiments of Kim, Kline, and Johnston (1980) for turbulent flow past a backward facing step: (a) mean flow streamlines, (b) turbulence intensity profiles.

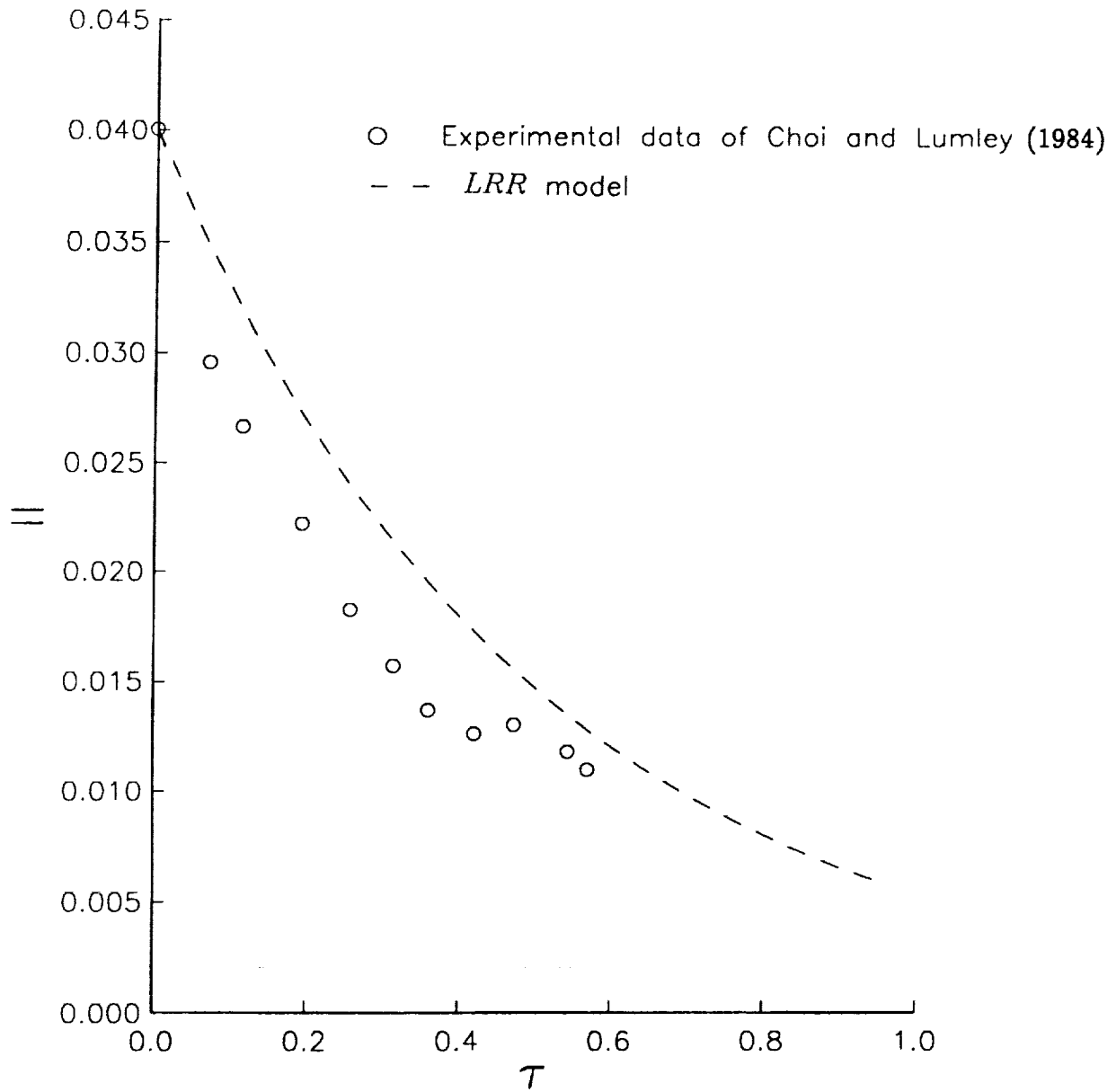


Figure 5. Time evolution of the second invariant of the anisotropy tensor for the return to isotropy problem: Comparison of the predictions of the Launder, Reece, and Rodi model with the experimental data of Choi and Lumley (1984) for the relaxation from plane strain.

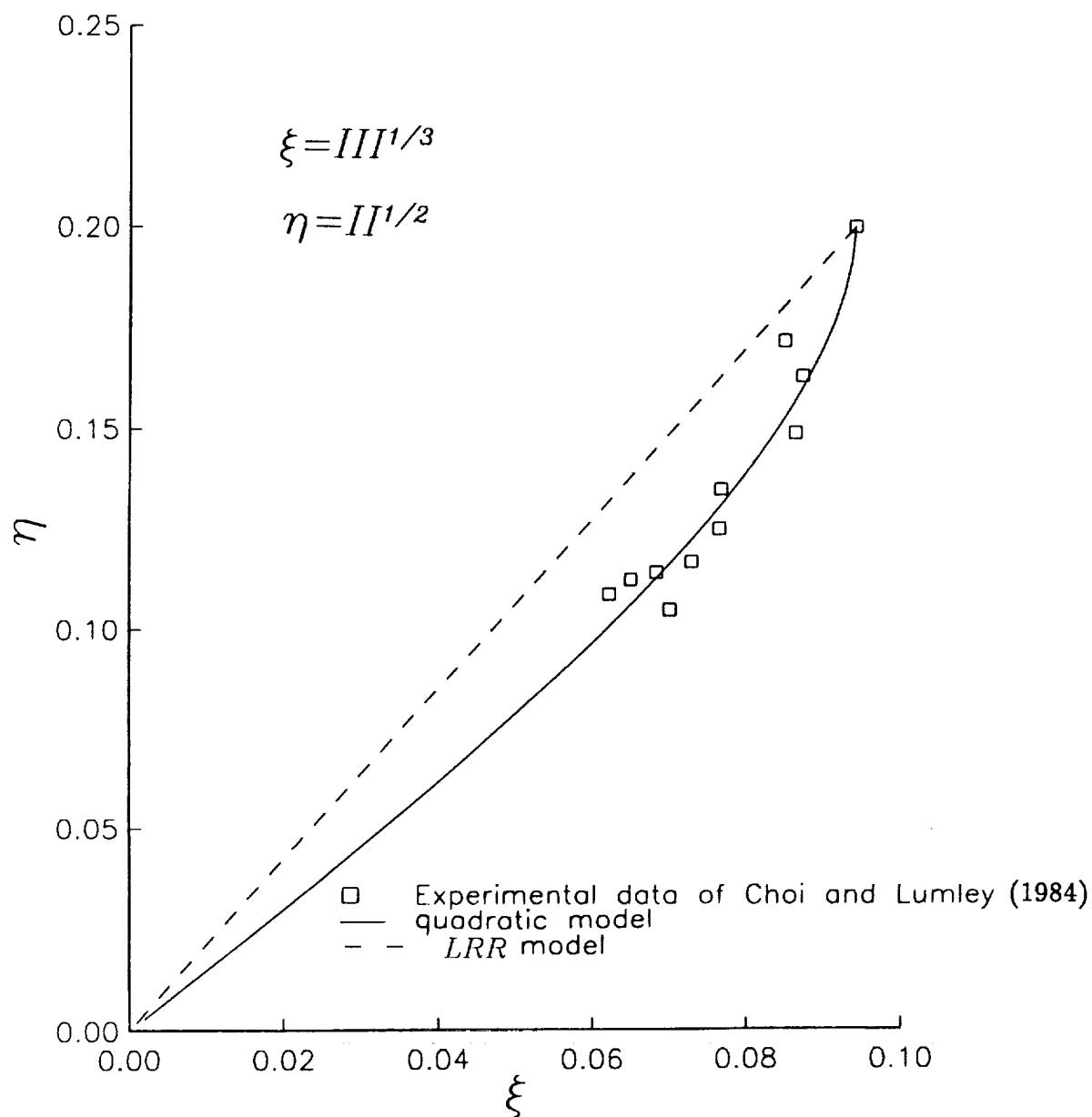


Figure 6. Phase space portrait of the return to isotropy problem: Comparison of the predictions of the Launder, Reece, and Rodi model and the quadratic model of Sarkar and Speziale (1990) with the experimental data of Choi and Lumley (1984) for the relaxation from plane strain.

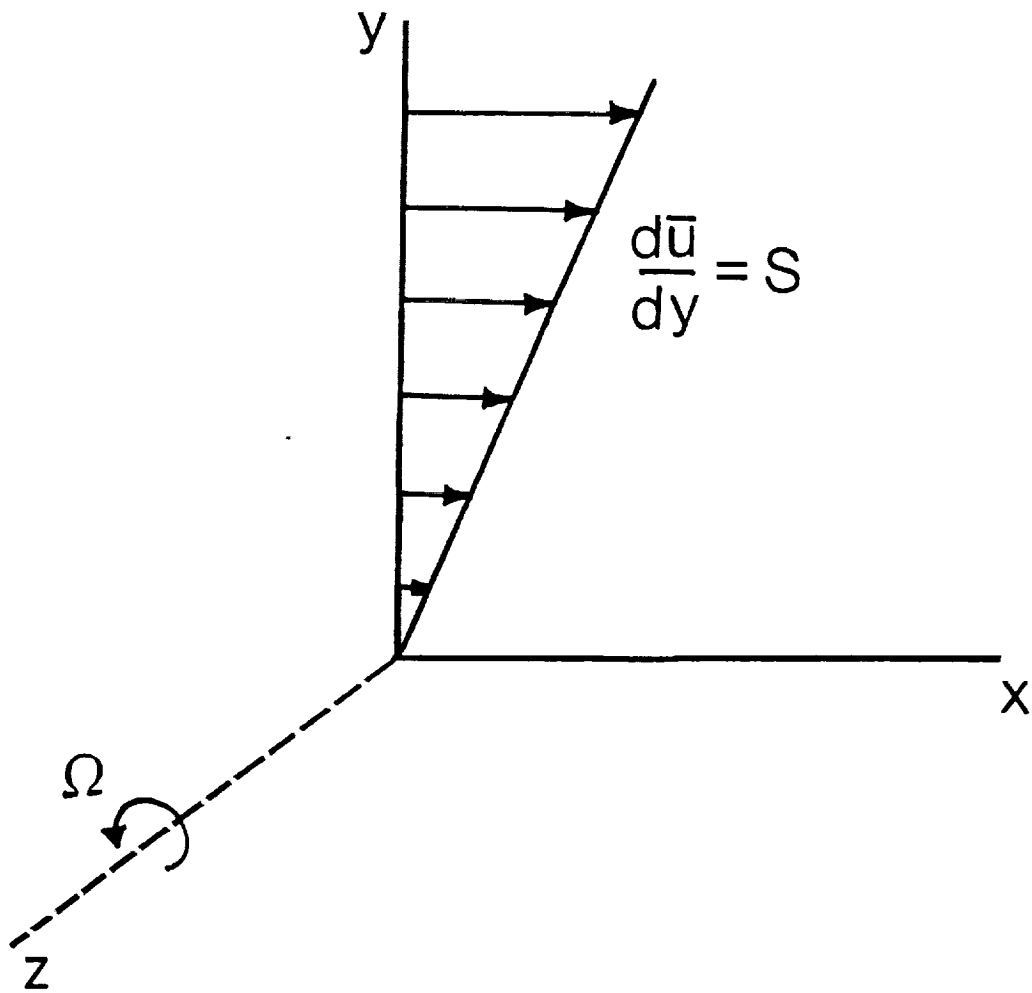


Figure 7. Homogeneous shear flow in a rotating frame.



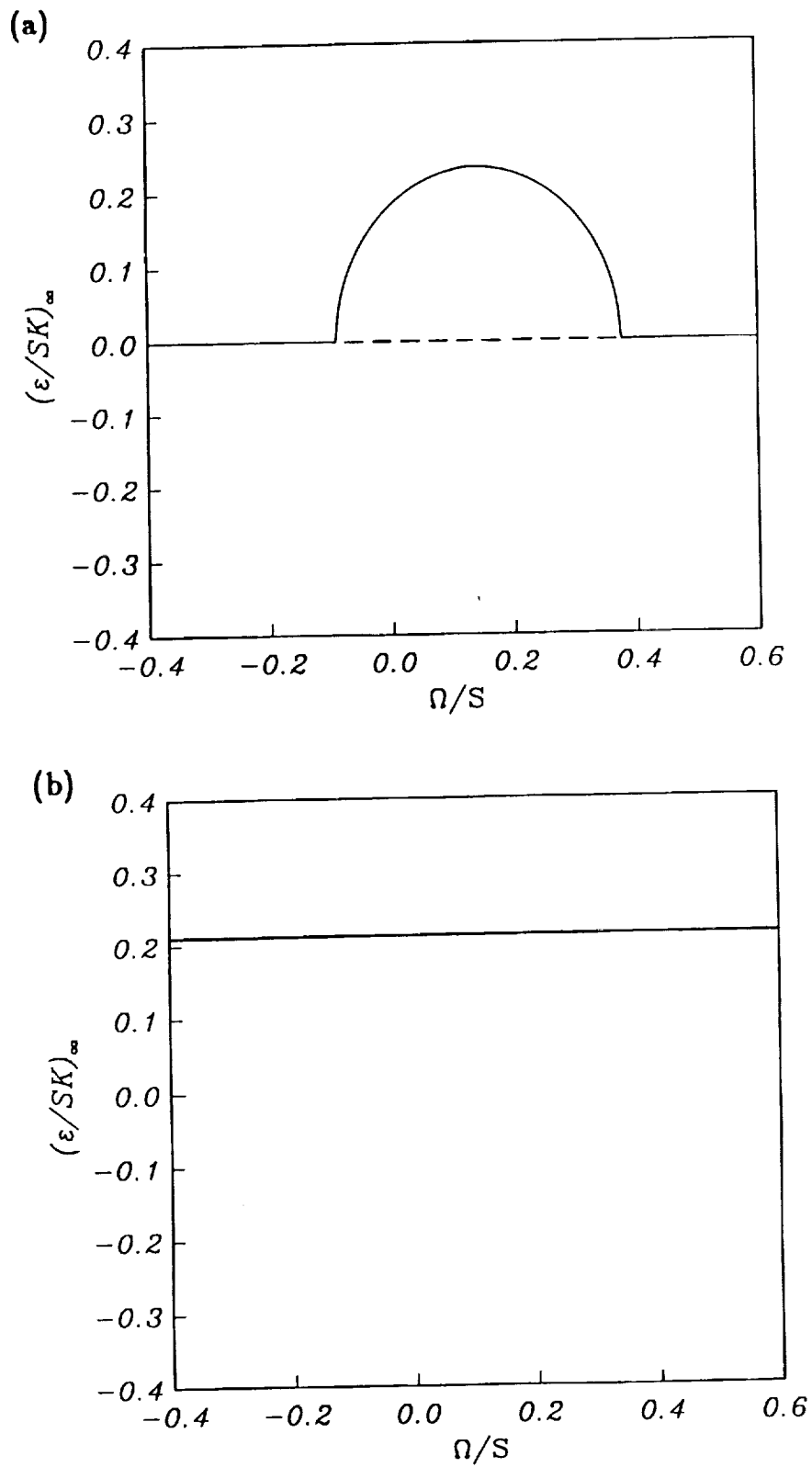


Figure 8. Bifurcation diagram for rotating shear flow: (a) Launder, Reece, and Rodi model, (b) standard  $K - \varepsilon$  model.

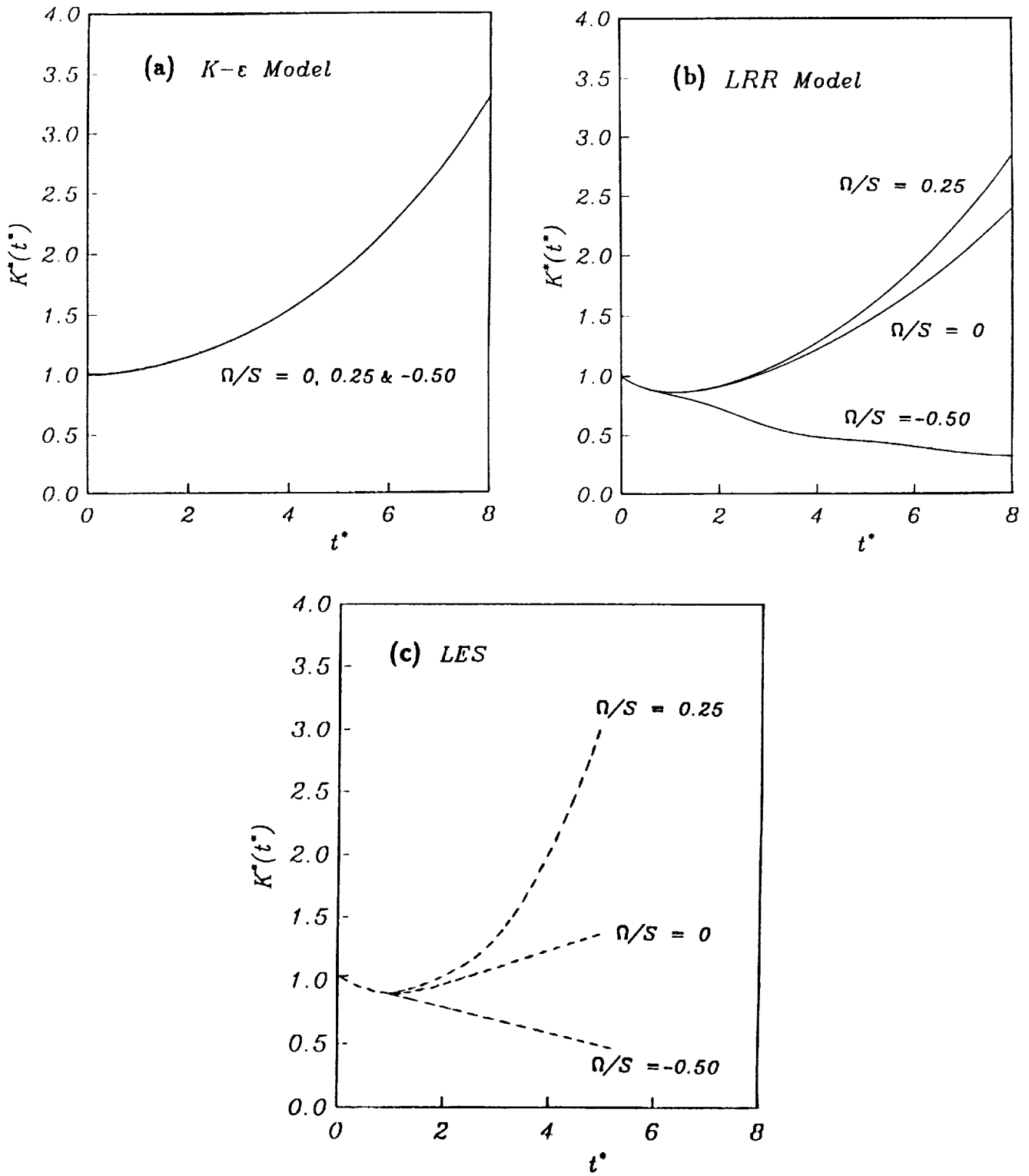


Figure 9. Time evolution of the turbulent kinetic energy in rotating shear flow for  $\epsilon_0/SK_0 = 0.296$ : (a) standard  $K - \epsilon$  model, (b) Launder, Reece, and Rodi model, and (c) large-eddy simulations of Bardina, Ferziger, and Reynolds (1983).

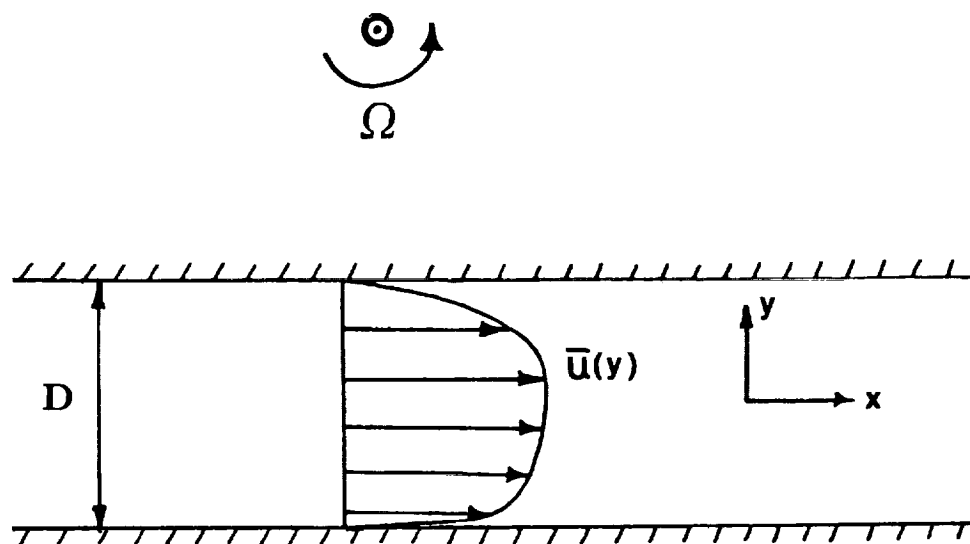


Figure 10. Fully-developed turbulent channel flow in a rotating frame.

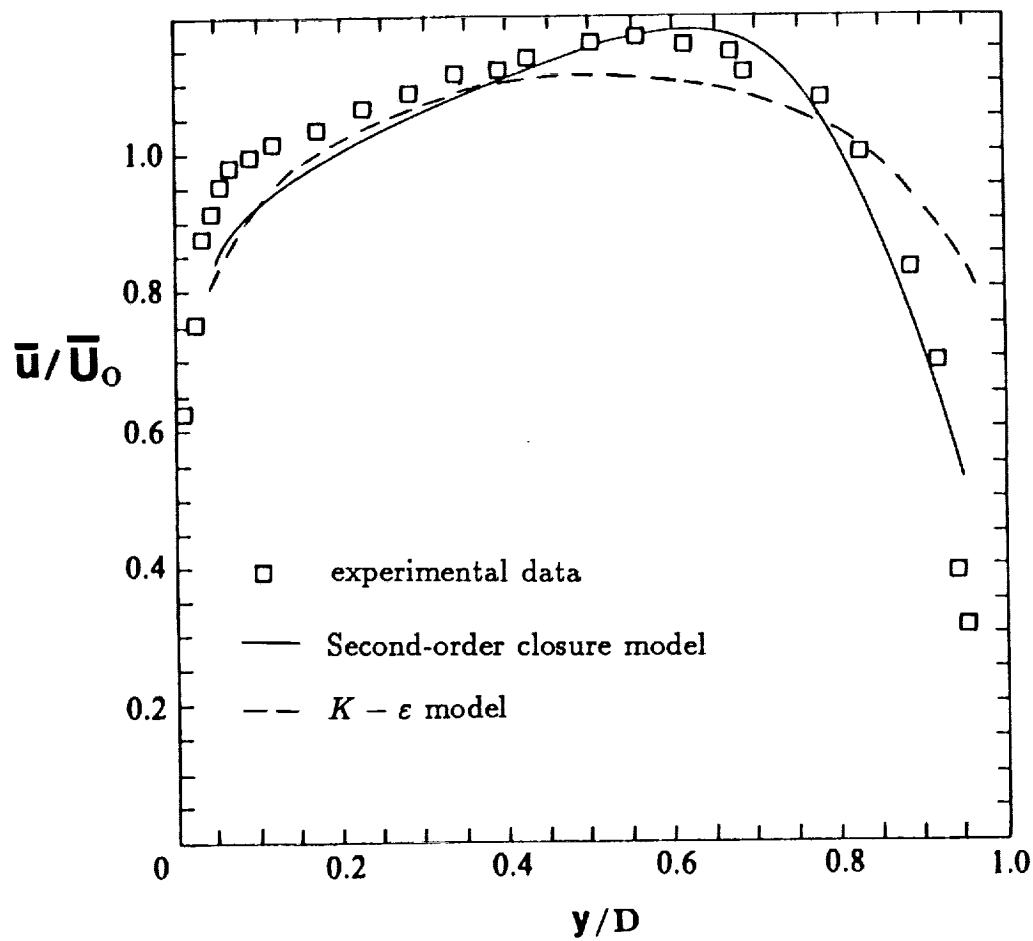


Figure 11.

Comparison of the mean velocity predictions of the second-order closure model of Gibson and Launder (1978) and the standard  $K - \epsilon$  model with the experimental data of Johnston, Halleen, and Lezius (1972) on rotating channel flow.

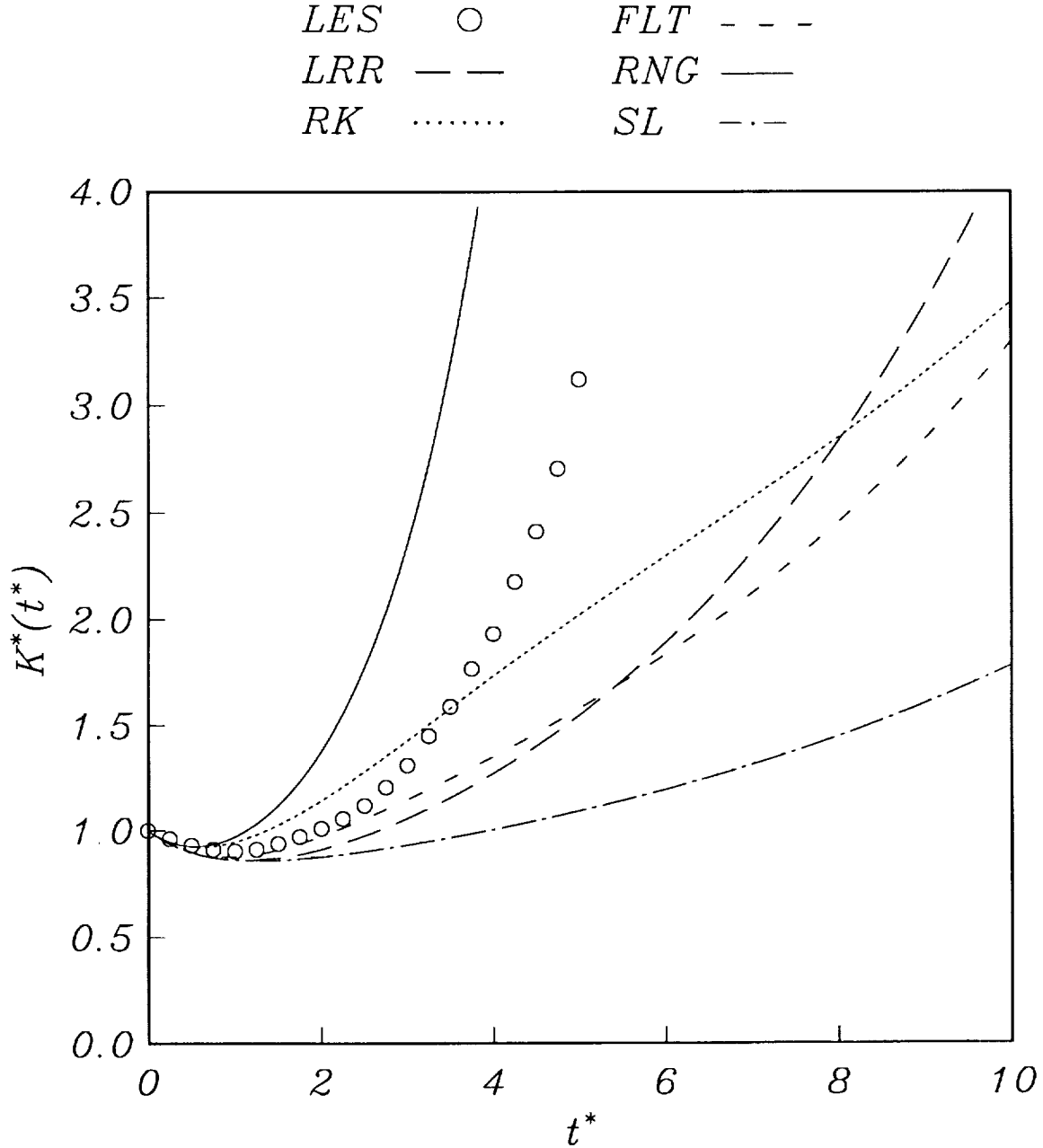


Figure 12. Comparison of the predictions of a variety of second-order closure models for the time evolution of the turbulent kinetic energy in rotating shear flow;  $\Omega/S = 0.25$ ,  $\varepsilon_0/SK_0 = 0.296$ .  $LES \equiv$  large-eddy simulations of Bardina, Ferziger, and Reynolds (1983);  $LRR \equiv$  Launder, Reece, and Rodi model;  $RK \equiv$  Rotta Kolmogorov model of Mellor and Herring (1973);  $FLT \equiv$  Fu, Launder, and Tselepidakis (1987) model;  $RNG \equiv$  renormalization group model of Yakhot and Orszag;  $SL \equiv$  Shih and Lumley (1985) model.



## Report Documentation Page

1. Report No. NASA CR-182017 ICASE Report No. 90-26	2. Government Accession No.	3. Recipient's Catalog No.	
4. Title and Subtitle  ANALYTICAL METHODS FOR THE DEVELOPMENT OF REYNOLDS STRESS CLOSURES IN TURBULENCE		5. Report Date March 1990	
		6. Performing Organization Code	
7. Author(s)  Charles G. Speziale		8. Performing Organization Report No. 90-26	
		10. Work Unit No. 505-90-21-01	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		11. Contract or Grant No. NAS1-18605	
		13. Type of Report and Period Covered Contractor Report	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225		14. Sponsoring Agency Code	
15. Supplementary Notes Langley Technical Monitor: Richard W. Barnwell  To appear in the Annual Review of Fluid Mechanics  <u>Final Report</u>			
16. Abstract  Analytical methods for the development of Reynolds stress models in turbulence are reviewed in detail. Zero, one and two equation models are discussed along with second-order closures. A strong case is made for the superior predictive capabilities of second-order closure models in comparison to the simpler models. The central points of the paper are illustrated by examples from both homogeneous and inhomogeneous turbulence. A discussion of the author's views concerning the progress made in Reynolds stress modeling is also provided along with a brief history of the subject.			
17. Key Words (Suggested by Author(s))  Turbulence Modeling; K- $\epsilon$ model; two-equation models; second-order closures		18. Distribution Statement  34 - Fluid Mechanics and Heat Transfer  Unclassified - Unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of pages 59	22. Price A04