THIN-PLATE SPLINE QUADRATURE OF GEODETIC INTEGRALS

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ABSTRACT

Thin-plate spline functions – known for their flexibilty. and fidelity
in representing experimental data – are especially well-suited for the
numerical integration of geodetic integrals in the area where the
integration is most sensitive to the data, i.e. in the immediate
vicinity of the evaluation point. Spline quadrature rules are derived
for the contribution of a circular innermost zone to Stokes's formula,
to the formulae of Vening Meinesz, and to the recursively evaluated
operator L_n in the analytical continuation solution of Molodensky's
problem. These rules are exact for interpolating thin-plate splines.

In cases where the integration data are distributed irregularly, a
system of linear equations needs to be solved for the quadrature
coefficients. Formulae are given for the terms appearing in these
equations. In case the data are regularly distributed, the coefficients
may be determined once-and-for-all. Examples are given of some fixed-
point rules. With such rules successive evaluation, within a circular
disk, of the terms in Molodensky's series becomes relatively easy.

The spline quadrature technique presented here complements other
techniques such as ring integration for intermediate integration
zones.

Quadrature rules are sought approximating the contribution of a

circular innermost zone to the evaluation of Stokes's formula, the

formulae of Vening Meinesz, and the L_1 gradient operator in the series

analytical continuation solution of Molodensky's problem. The rules

are to be of the form

$$\zeta_I(x) = \frac{r_0}{\gamma_0} \sum_{i=1}^{N} a_i \Delta g(\alpha_i);$$

$$\begin{pmatrix} \xi_I(x) \\ \eta_I(x) \end{pmatrix} = \frac{1}{\gamma_0} \sum_{i=1}^{N} \begin{pmatrix} b_{1i} \\ b_{21} \end{pmatrix} \Delta g(\alpha_i);$$

$$L_{1,I} f(x) = \frac{1}{\gamma_0} \sum_{i=1}^{N} c_i \{f(\alpha_i) - f(x)\};$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ are plane co-ordinates (with the 1-axis north and 2-axis east), r_0 is the radius of the innermost zone, and the subscript I indicates innermost zone contribution. $\{r_0\alpha_i\}$, $i=1,..,N$, are distinct points in the innermost zone (not all on a single straight line) where the data are given. The coefficients in each instance are chosen to make the integration exact for thin-plate splines with nodes at the α_i, and exact for constant and linear functions in the nullspace of these splines. The thin-plate spline kernel function associated with function evaluation at α_i is $|y - \alpha_i|^2 \log_e |y - \alpha_i|$.

The quadrature weights for Stokes's formula are obtained from the solution of the linear equations

$$\sum_{i=1}^{N} a_i |\alpha_j - \alpha_i|^2 \log_e |\alpha_j - \alpha_i| + b_0 + b_1 \alpha_{1j} + b_2 \alpha_{2j} = d_j(x), \quad j = 1, \ldots, N;$$

$$\sum_{i=1}^{N} a_i = e_0(x); \qquad \sum_{i=1}^{N} a_i \alpha_{1i} = e_1(x); \qquad \sum_{i=1}^{N} a_i \alpha_{2i} = e_2(x);$$

where

$$d_j(x) = \frac{1}{2\pi} \iint_{|x-y| \leq 1} \frac{|y - \alpha_j|^2 \log_e |y - \alpha_j|}{|x - y|} \, dm(y) = -\frac{1}{9} + \frac{4}{9} |x - \alpha_j|^3;$$

$$e_0(x) = \frac{1}{2\pi} \iint_{|x-y| \leq 1} \frac{1}{|x - y|} \, dm(y) = 1;$$

$$e_1(x) = \frac{1}{2\pi} \iint_{|x-y| \leq 1} \frac{y_1 - x_1}{|x - y|} \, dm(y) = 0;$$

$$e_2(x) = \frac{1}{2\pi} \iint_{|x-y| \leq 1} \frac{y_2 - x_2}{|x - y|} \, dm(y) = 0.$$

Similar looking systems of equations give the quadrature weights for the spline approximation of Vening Meinesz'z formula and the L_1 operator. In the last case non-zero constant functions are not admissable, and the thin-plate spline kernel function needs some modification.

Numerical examples show that thin-plate spline quadrature can be very effective in evaluating the three integrals of Stokes, Vening Meinesz and the L_1 gradient operator.

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