NASA Contractor Report 182028
ICASE Report No. 90–27

ICASE

CELL AVERAGING CHEBYSHEV METHODS FOR HYPERBOLIC PROBLEMS

Wei Cai
David Gottlieb
Ami Harten

Contract No. NAS1–18605
March 1990

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, Virginia 23665–5225

Operated by the Universities Space Research Association

(NASA-CR-182028) CELL AVERAGING CHEBYSHEV METHODS FOR HYPERBOLIC PROBLEMS Final Report (ICASE) 23 p CSCL 12A

N90-20768

Unclas
G3/64 0275320

National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

CELL AVERAGING CHEBYSHEV METHODS FOR HYPERBOLIC PROBLEMS[1]

Wei Cai
Department of Mathematics
University of North Carolina at Charlotte
Charlotte, NC 28223

David Gottlieb[2]
Division of Applied Mathematics
Brown University
Providence, RI 02912

Ami Harten [2]
Tel-Aviv University
Tel-Aviv

ABSTRACT

This paper describes a cell averaging method for the Chebyshev approximations of first order hyperbolic equations in conservation form. We present formulas for transforming between pointwise data at the collocation points and cell averaged quantities, and vice-versa. This step, trivial for the finite difference and Fourier methods, is nontrivial for the global polynomials used in spectral methods. We then prove that the cell averaging methods presented are stable for linear scalar hyperbolic equations and present numerical simulations of shock-density wave interaction using the new cell averaging Chebyshev methods.

[1]This work has been supported in part by AFOSR grant 85-0303 and by NSF grant DMS-8810150. The second author was partially supported by DARPA-URI contract N00014-86-k0754.

[2]Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the authors were in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

1 Introduction

In this paper we introduce a new spectral technique for the numerical solution of nonlinear hyperbolic equations. This technique, as almost every modern finite difference scheme for shock computations, is based on the cell averaged form of the equations. This is essential for finite difference shock capturing techniques and it is our experience that it plays an essential role in a successful spectral simulation of problems that involve shock waves [1].

Consider the nonlinear hyperbolic equation

$$\begin{cases} U_t + F_x(U) = 0, & x \in [-1,1] \\ U(x,0) = U_0(x), & \end{cases} \tag{1.1}$$

with appropriate boundary conditions.

The cell averaged form of (1.1) is obtained by integrating (1.1) between any two points $-1 \leq a < b \leq 1$ to get

$$\frac{\partial}{\partial t}\bar{U} \equiv \frac{\partial}{\partial t}\frac{1}{b-a}\int_a^b U(x)\,dx = -\frac{1}{b-a}(F(U(b)) - F(U(a))). \tag{1.2}$$

Let $\bar{u}(x,t)$ be an approximation to $\bar{U}(x,t)$ at time t. Following Harten [4] we express the approximation to $\bar{U}(x,t+\tau)$ by

$$\bar{u}(x,t+\tau) = \mathcal{A}E(t)\mathcal{R}(\cdot\,;\bar{u}), \tag{1.3}$$

where $\mathcal{A}$ is the cell averaging operator and $E(t)$ is the exact time evolution operator corresponding to (1.1). Throughout the paper we will not distinguish between $\mathcal{R}(\cdot;\bar{u})$ and $\mathcal{R}\bar{u}$. The operator $\mathcal{R}(\cdot;\bar{u})$ is of extreme importance, it represents the way we reconstruct u from its given cell average values $\bar{u}_{j-\frac{1}{2}} = \frac{1}{x_{j-1}-x_j}\int_{x_j}^{x_{j-1}} u(x)\,dx$, where $\{x_j\}_{j=1}^{N}$ are the grid points. For finite difference schemes u is a piecewise polynomial of low degree, so that the reconstruction itself is simple. It becomes complicated only if one imposes also the requirement that the reconstruction should be essentially nonoscillatory. In [1] we have presented an essentially non-oscillatory Fourier method based on the cell averaging formulation (1.2). In that case the transformations between the cell averages and the point values are simple and can be carried out efficiently by the Fast Fourier Transformations (FFT). This can be attributed to the fact that the boundary conditions are periodic and that the cell average of a trigonometric function is proportional to the function itself. However, for Chebyshev methods, the cell averaging operation (denoted by the operator $\mathcal{A}$) is not simple nor is $\mathcal{R}(\cdot\,;\bar{u})$. As a matter of fact not only the formulation but also the implementation of $\mathcal{A}$ and $\mathcal{R}$ is not straightforward.

In this paper we formulate the cell averaging technique for the Chebyshev method. We will discuss its stability for linear problems and show an example of its applicability to nonlinear systems of equations by simulating the problem of shock-density wave interaction. The cell averaging formulation is an essential part of the numerical code.

The outline of the paper is as follows: In Section 2 we show how to reconstruct efficiently point values of a polynomial from its cell averages and vice versa.In Section 3 we introduce the new numerical technique and show its stability for linear problems. Section 4 is devoted to numerical results obtained by using the new method.

2 Cell Averages and Point values

In this section we will discuss the cell averaging operator $\mathcal{A}$ and the reconstruction operator $\mathcal{R}$ in the context of the Chebyshev methods. In these methods the approximations are taken from the space of polynomials of degree N. It is therefore clear that $\mathcal{A}$ and $\mathcal{R}$, when restricted to the polynomial space, can be expressed as matrices $\mathbf{A_N}$ and $\mathbf{R_N}$. We will give the explicit formulas for these matrices. We start by discussing the operator $\mathcal{A}$.

Assume that $f(x)$ is in $C^r[-1,1]$, $r \geq 0$. Let $x_j = \cos\left(\frac{j\pi}{N}\right)$, $0 \leq j \leq N$ be the Chebyshev-Lobatto points in [-1,1]. For later use, define $x_{j-\frac{1}{2}} = \cos\left(\frac{(j-\frac{1}{2})\pi}{N}\right)$, $1 \leq j \leq N$. The cell averaged function $\bar{f}(x)$ of $f(x)$ is defined as

$$\bar{f}(x) = \mathcal{A}f =: \frac{1}{h_2(x) - h_1(x)} \int_{h_1(x)}^{h_2(x)} f(x)\,dx \qquad \text{for } -1 < x < 1, \tag{2.1}$$

where

$$
\begin{aligned}
h_1(x) &= \cos(\cos^{-1} x - \frac{\Delta\theta}{2}), \\
h_2(x) &= \cos(\cos^{-1} x + \frac{\Delta\theta}{2}), \\
\Delta\theta &= \frac{\pi}{N}.
\end{aligned}
\tag{2.2}
$$

The reason for the definition in (2.1) is that

$$
\bar{f}(x_{j-\frac{1}{2}}) = \frac{1}{x_{j-1} - x_j} \int_{x_j}^{x_{j-1}} f(x)\, dx, \qquad \text{for } 1 \leq j \leq N. \tag{2.3}
$$

As stated before, we are interested in $\mathcal{A}$ operating in a polynomial space. In Lemma 1, we show that the result of $\mathcal{A}$ on a polynomial is still a polynomial of the same degree.

Lemma 1 *Let $U_k(x) = \frac{1}{k+1} T'_{k+1}(x)$, $k \geq 0$, be the second kind of the Chebyshev polynomials. Then*

$$
\bar{U}_k(x) = \mathcal{A} U_k = \sigma_k U_k(x), \tag{2.4}
$$

where

$$
\sigma_k = \frac{\sin(k+1)\frac{\Delta\theta}{2}}{(k+1)\sin\frac{\Delta\theta}{2}}. \tag{2.5}
$$

Proof: Substituting $U_k(x)$ in the right hand side of (2.1) and making the transformation $x = \cos\theta$, $0 \leq \theta \leq \pi$, we have

$$
\bar{U}_k(x) = \frac{1}{\cos(\cos^{-1} x + \frac{\Delta\theta}{2}) - \cos(\cos^{-1} x - \frac{\Delta\theta}{2})} \int_{\cos(\cos^{-1} x - \frac{\Delta\theta}{2})}^{\cos(\cos^{-1} x + \frac{\Delta\theta}{2})} U_k(x)\, dx.
$$

Since $T_k(x) = \cos(k\theta)$, then $U_k(x) = \frac{\sin(k+1)\theta}{(k+1)\sin\theta}$ and therefore

$$
\begin{aligned}
\bar{U}_k(x) &= \frac{1}{2(k+1)\sin\frac{\Delta\theta}{2}\sin\theta} \int_{\theta-\frac{\Delta\theta}{2}}^{\theta+\frac{\Delta\theta}{2}} \sin(k+1)\theta\, d\theta \\
&= \frac{1}{2(k+1)\sin\frac{\Delta\theta}{2}\sin\theta} (-1) \frac{\cos(k+1)\theta}{k+1} \Big|_{\theta-\frac{\Delta\theta}{2}}^{\theta+\frac{\Delta\theta}{2}} \\
&= \frac{\sin(k+1)\frac{\Delta\theta}{2}}{(k+1)\sin\frac{\Delta\theta}{2}} \left(\frac{1}{k+1} \frac{\sin(k+1)\theta}{\sin\theta} \right),
\end{aligned}
$$

i.e.

$$\bar{U}_k(x) = \sigma_k U_k(x).$$

Q.E.D.

¿From Lemma 1 one gets

Corollary 1 *The cell averaged function of any polynomial of order N is a polynomial of the same order.*

Proof: any polynomial of degree N has the expression

$$f(x) = \sum_{k=0}^{N} a_k U_k(x), \tag{2.6}$$

where a_k are constants.

Therefore, by Lemma 1

$$\bar{f}(x) = \sum_{k=0}^{N} \sigma_k a_k U_k(x). \tag{2.7}$$

Q.E.D.

Thus, Lemma 1 gives explicitly the eigenvalues σ_k of the matrix $\mathbf{A_N}$ (the restriction of $\mathcal{A}$ to polynomials of degree N). These eigenvalues are uniformly bounded from above and below, in fact

$$\frac{2}{\pi} \leq \sigma_k \leq \frac{\pi}{2}, \qquad 0 \leq k \leq N. \tag{2.8}$$

If $f(x)$ is a polynomial of degree N, then it is uniquely determined by its values $f(x_j)$, $j = 0, \cdots, N$. So theoretically $\bar{f}(x_{j-\frac{1}{2}})$, $j = 1, \cdots, N$ can be determined. Therefore the transformation from $f(x_j)$, $j = 0, \cdots, N$ to $\bar{f}(x_{j-\frac{1}{2}})$, $j = 1, \cdots, N$ is well defined and we only need to address the issue of its efficient implementation.

In general it is known that

$$f(x) = \sum_{k=0}^{N} a_k T_k(x), \tag{2.9}$$

where

$$\begin{aligned} a_k &= \frac{2}{N\bar{c}_k}\sum_{j=0}^{N} f(x_j)T_k(x_j), \\ \bar{c}_0 &= \bar{c}_N = 2, \qquad \bar{c}_k = 1 \text{ if } k \neq 0 \text{ or } N. \end{aligned} \tag{2.10}$$

Alternatively

$$f(x) = \sum_{k=0}^{N} b_k U_k(x), \tag{2.11}$$

$$\begin{aligned} b_k &= \frac{a_k}{2}, \qquad k = N-1 \text{ or } N, \\ b_k &= \frac{1}{2}(\bar{c}_k a_k - a_{k+2}), \qquad 0 \leq k \leq N-2 \end{aligned} \tag{2.12}$$

and therefore by corollary 1

$$\bar{f}(x) = \sum_{k=0}^{N} \sigma_k b_k U_k(x). \tag{2.13}$$

Now $\{\bar{f}(x_{j-\frac{1}{2}})\}_{j=1}^{N}$ are obtained by substituting $x_{j-\frac{1}{2}}$ in (2.13) (this can be carried out using the FFT).

We note that equations (2.9) - (2.13) describe how to get the vector $\bar{f}(x_{j-\frac{1}{2}}),\ j = 1, \cdots, N$ from the vector $f(x_i),\ i = 0, \cdots, N$. Denote by $\mathbf{A_N}$ the $N \times (N+1)$ matrix defined by this transformation. We note that $\mathbf{A_N}$ can be written explicitly. In fact the polynomial $f(x)$ has an unique representation as

$$f(x) = \sum_{j=0}^{N} f(x_j) g_j(x), \tag{2.14}$$

where

$$g_j(x) = \frac{(1-x^2)T_N'(x)(-1)^{j+1}}{\bar{c}_j N^2 (x - x_j)} = \frac{2}{\bar{c}_j N}\sum_{k=0}^{N} \frac{T_k(x_j)T_k(x)}{\bar{c}_k}$$

with $\bar{c}_k$ defined in (2.10) .

It follows upon substituting (2.14) in (2.1) and using the fact that $\bar{T}_0(x) = U_0(x)$, $\bar{T}_1(x) = 2\sigma_1 U_1(x)$ and $\bar{T}_k(x) = \frac{1}{2}(\sigma_k U_k(x) - \sigma_{k-2}U_{k-2}(x))$ for $k \geq 2$,

$$\bar{f}(x) = \sum_{\ell=0}^{N} f(x_\ell)\bar{g}_\ell(x), \tag{2.15}$$

where

$$\bar{g}_\ell(x) = \frac{1}{2} + 2\sigma_1 T_1(x_\ell)U_1(x) + \frac{2}{\bar{c}_\ell N}\sum_{k=2}^{N} \frac{T_k(x_\ell)(\sigma_k U_k(x) - \sigma_{k-2}U_{k-2}(x))}{2\bar{c}_k} \tag{2.16}$$

with σ_k defined in (2.5).

Setting $x = x_{j-\frac{1}{2}}$ in (2.16)

$$(\mathbf{A_N})_{j\ell} = \bar{g}_\ell(x_{j-\frac{1}{2}}), \qquad 0 \le \ell \le N,\ 1 \le j \le N. \tag{2.17}$$

Thus we have outlined two procedures to get $\bar{f}(x_{j-\frac{1}{2}})$ from $f(x_j)$, one uses the FFT and another uses matrix vector multiplications.

We are now ready to discuss the reconstruction operator $\mathcal{R}(\cdot\,;\bar{f})$. Note that in the beginning of the solution process (1.3), we only have the values $\bar{f}_{j-\frac{1}{2}}$, $j = 1, \cdots, N$, thus we need another piece of information in order to define uniquely the N-th degree polynomial $f(x)$. This piece of information is provided by the boundary condition. For simplicity, we assume that the boundary condition is of the form

$$f(1) = f(x_0) = f_0. \tag{2.18}$$

The reconstruction is done in two steps. Define first a $(N-1)$-th polynomial $\bar{f}_{N-1}(x)$ which collocates $\bar{f}(x)$ at $\{x_{j-\frac{1}{2}}\}_{j=1}^{N}$, i.e. $\bar{f}_{N-1}(x_{j-\frac{1}{2}}) = \bar{f}(x_{j-\frac{1}{2}})$, $1 \le j \le N$, it is readily verified that

$$\bar{f}_{N-1}(x) = \sum_{k=0}^{N-1} c_k T_k(x), \tag{2.19}$$

where

$$c_k = \frac{2}{\bar{c}_k N}\sum_{j=1}^{N} \bar{f}(x_{j-\frac{1}{2}})T_k(x_{j-\frac{1}{2}}), \qquad \bar{c}_k \text{ is same as in (2.10).} \tag{2.20}$$

Alternatively we have

$$\bar{f}_{N-1}(x) = \sum_{k=0}^{N-1} b_k U_k(x), \tag{2.21}$$

where

$$\begin{aligned} b_k &= \frac{c_k}{2}, \qquad k = N-2,\ N-1, \\ b_k &= \frac{1}{2}(\bar{c}_k c_k - c_{k+2}), \qquad 0 \le k \le N-3. \end{aligned} \tag{2.22}$$

Now, by Corollary 1

$$f_{N-1}(x) = \sum_{k=0}^{N-1} \frac{b_k}{\sigma_k} U_k(x) \tag{2.23}$$

is a polynomial of degree $N-1$ such that $\mathcal{A} f_{N-1} = \bar{f}_{N-1}$.

Generally, $f_{N-1}(x)$ does not satisfy the boundary condition (2.18). There are two ways to modify $f_{N-1}(x)$ so that the boundary condition (2.18) is satisfied. In the first way we can add to $f_{N-1}(x)$ an N-th degree polynomial $Q(x)$ such that

$$\bar{Q}(x_{j-\frac{1}{2}}) = 0, \qquad 1 \le j \le N, \tag{2.24}$$

and

$$f_{N-1}(x) + Q(1) = f_0. \tag{2.25}$$

It can be verified that in order to satisfy (2.24)

$$Q(x) = c\left((1-x^2)T_N'(x)\right)'. \tag{2.26}$$

Let $f(x)$ be the sum of $f_{N-1}(x)$ and $Q(x)$,

$$\begin{aligned} f(x) &= f_{N-1}(x) + c\left((1-x^2)T_N'(x)\right)' \\ &= f_{N-1}(x) - c[xT_N'(x) + N^2 T_N(x)]. \end{aligned} \tag{2.27}$$

The last equality follows from the Chebyshev equation, and the constant c is now determined by the condition $f(x_0) = f_0$, i.e.

$$c = -\frac{1}{2N^2}\left(f_0 - f_{N-1}(1)\right). \tag{2.28}$$

Finally given $\bar{f}(x_{j-\frac{1}{2}})$, $j = 1, \cdots, N$ and f_0 we can get

$$f(x_i) = f_{N-1}(x_i) - c\left[x_i T_N'(x_i) + N^2 T_N(x_i)\right], \qquad i = 0, \cdots, N \tag{2.29}$$

where $f_{N-1}(x_i)$ can be evaluated from (2.23) by using the FFT.

Note that in this procedure we change the values of $f_{N-1}(x_i)$ at all the grid points.

Denote by $\mathbf{R_N}$ the $(N+1)\times(N+1)$ matrix transforming f_0 and $\bar{f}(x_{j-\frac{1}{2}})$, $1\le j\le N$ to $f(x_j)$, $0\le j\le N$, i.e.

$$(f(x_0),\cdots,f(x_N))^\top = \mathbf{R_N}\left(f(x_0),\bar{f}(x_{\frac{1}{2}}),\cdots,\bar{f}(x_{N-\frac{1}{2}})\right)^\top . \tag{2.30}$$

As before we can write $\mathbf{R_N}$ explicitly. Equation (2.19) can be rewritten as

$$\bar{f}_{N-1}(x)=\sum_{j=1}^{N}\bar{f}(x_{j-\frac{1}{2}})\bar{h}_j(x), \tag{2.31}$$

where $\bar{h}_j(x_{k-\frac{1}{2}})=\delta_{jk}$ and $h_j(x)$ are polynomials of degree $N-1$; explicitly

$$\bar{h}_j(x)=(-1)^{j+1}N\sin\left((j-\frac{1}{2})\frac{\pi}{N}\right)\frac{T_N(x)}{x-x_{j-\frac{1}{2}}}. \tag{2.32}$$

It can be shown that $\bar{h}_j(x)$ is the cell averaged polynomial of

$$h_j(x)=\sum_{k=0}^{N-1}\lambda_k U_k(x) \tag{2.33}$$

with λ_k defined as

$$\begin{aligned}
\lambda_k &= \frac{1}{N\sigma_k}T_k(x_{j-\frac{1}{2}}), && \text{if } k=N-2,N-1\ , \\
\lambda_k &= \frac{1}{N\sigma_k}(T_k(x_{j-\frac{1}{2}})-T_{k+2}(x_{j-\frac{1}{2}})), && \text{if } 0\le k\le N-3\ .
\end{aligned} \tag{2.34}$$

As a result of (2.31) and (2.33) polynomial $f_{N-1}(x)$ takes the form

$$f_{N-1}(x)=\sum_{j=1}^{N-1}\bar{f}(x_{j-\frac{1}{2}})h_j(x), \tag{2.35}$$

and by (2.27)

$$f(x)=f_{N-1}(x)-\frac{1}{2N^2}\left(f_0-f_{N-1}(1)\right)\left((1-x^2)T_N'(x)\right)'.$$

Using (2.31) we get

$$\begin{aligned}
f(x) &= \sum_{j=1}^{N}\bar{f}(x_{j-\frac{1}{2}})h_j(x)-\frac{1}{2N^2}\left(f_0-\sum_{j=1}^{N}\bar{f}(x_{j-\frac{1}{2}})h_j(1)\right)\left((1-x^2)T_N'(x)\right)' \qquad (2.36)\\
&= \sum_{j=1}^{N}\bar{f}(x_{j-\frac{1}{2}})\left[h_j(x)+\frac{h_j(1)}{2N^2}\left((1-x^2)T_N'(x)\right)'\right]-\frac{f_0}{2N^2}\left((1-x^2)T_N'(x)\right)'
\end{aligned}$$

Substituting $x = x_i$ into (2.36) gives

$$(\mathbf{R_N})_{ij} = \begin{cases} \delta_{1j}, & \text{if } i = 0, \\ -\frac{1}{2N^2}\left((1-x_i^2)T_N'(x_i)\right)', & \text{if } j = 0, \\ h_j(x_i) + \frac{h_j(1)}{2N^2}\left((1-x_i^2)T_N'(x_i)\right)', & 1 \le i, j \le N. \end{cases} \tag{2.37}$$

To summarize we state

Lemma 2 *Let $f(x) = \mathcal{R}(\cdot\,; \bar{f})$ be the N-th polynomial defined in (2.27) then*

$$(\mathcal{A}f)(x_{j-\frac{1}{2}}) = \bar{f}(x_{j-\frac{1}{2}}) \qquad \text{for } 1 \le j \le N. \tag{2.38}$$

A different way to modify $f_{N-1}(x)$ in (2.23), in order to satisfy the boundary condition (2.18), is to add to it a polynomial $Q_1(x)$ of degree N such that the point values $f_{N-1}(x_i)$ remain unchanged and the new polynomial satisfies the boundary condition. Thus instead of (2.27) we define

$$f(x) = f_{N-1}(x) + (f_0 - f_{N-1}(1))\frac{(1+x)T_N'(x)}{2N^2}. \tag{2.39}$$

Computationally (2.39) is simpler than (2.23) (2.27). However, note that in this case

$$(\mathcal{A}f)(x_{j-\frac{1}{2}}) \neq \bar{f}(x_{j-\frac{1}{2}}), \tag{2.40}$$

which is in contrast to (2.38). The matrix corresponding to (2.40) can be formed similarly as in (2.37).

3 Cell Averaging Chebyshev (CAC) Method and Linear Stability

In this section we will establish the stability of the Cell Averaging Methods, presented in Section 2, when applied to a first order scalar hyperbolic equation. It is tempting to try to obtain stability in the L^1 norm because of the way the method (1.3) is presented. However we will only give the stability estimate based on a weighted Chebyshev norm.

Consider the initial boundary value problem of the scalar hyperbolic equation

$$\begin{cases} U_t = U_x, & x \in [-1,1], \\ U(x,0) = U_0(x), \\ U(1,t) = 0. \end{cases} \tag{3.1}$$

The cell averaged form of (3.1) is

$$\frac{\partial}{\partial t}\bar{U}(x,t) + \frac{1}{h_2(x) - h_1(x)}\left[U(h_2(x),t) - U(h_1(x),t)\right] = 0, \tag{3.2}$$

where $h_1(x)$ and $h_2(x)$ are defined in (2.2).

We follow the notation (1.3). Let the N-th degree polynomial $\bar{u}(x,t)$ be the approximation to (3.2). Our aim is to find the error equation for $\mathcal{R}(\cdot;\bar{u})$ defined either by (2.27) or (2.39). ¿From (1.3)

$$\bar{u}(x,t+\tau) = \mathcal{A}E\mathcal{R}\bar{u}(x,t). \tag{3.3}$$

Applying the reconstruction operator $\mathcal{R}$ we get

$$\mathcal{R}\bar{u}(x,t+\tau) = \mathcal{R}\mathcal{A}E\mathcal{R}\bar{u}(x,t) \tag{3.4}$$

¿From equation (3.4) it is clear that $\mathcal{R}\bar{u}$ satisfies exactly the equation

$$\frac{\partial \mathcal{R}\bar{u}}{\partial t} = \frac{\partial \mathcal{R}\bar{u}}{\partial x} + \tau\left[(1-x^2)T_N'\right]' \tag{3.5}$$

for the reconstruction procedure (2.27) and

$$\frac{\partial \mathcal{R}\bar{u}}{\partial t} = \frac{\partial \mathcal{R}\bar{u}}{\partial x} + \tau_1(1+x)T_N' \tag{3.6}$$

for (2.39). τ and τ_1 are quantities depending on time t.

Note that (3.6) is the error equation for the Chebyshev collocation method. For a scalar linear equation the CAC method corresponding to (3.6) is equivalent to the known collocation method [2]. It remains to investigate the stability of (3.5). For simplicity, in the remainder of this section we denote $\mathcal{R}\bar{u}$ by u. From the construction of $\mathcal{R}$ in section 2 we know that $u(x,t)$ satisfies the boundary condition in (3.1), i.e. $u(1,t) = 0$.

It is interesting to note from (3.6) that the CAC method with reconstruction operator (2.27) is equivalent - for constant coefficient case - to the collocation method where the grid points are the zeros of the polynomial

$$Q(x) = \left[(1-x^2)T_N'(x)\right]'. \tag{3.7}$$

Note that by using standard identities

$$Q(x) = -\left[xT_N'(x) + N^2 T_N(x)\right]. \tag{3.8}$$

The term τ in the right hand side of (3.5) is determined by the boundary condition $u(1,t) = \mathcal{R}\bar{u}(1,t)$. In fact substituting $x = 1$ in the equation (3.5) and noting that $\frac{\partial u}{\partial t}(1,t) = 0$ we get for τ

$$\tau = \frac{u_x(1,t)}{2N^2} \tag{3.9}$$

An alternative expression can be obtained by equating the highest coefficient on both sides of (3.5), thus if u is expanded as

$$u(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x) \tag{3.10}$$

then

$$\tau = -\frac{1}{N(N+1)} \frac{\partial \hat{u}_N}{\partial t} \tag{3.11}$$

Before stating the main stability result of this paper, we state the following lemma

Lemma 3 *If $f(x) = \sum_{k=0}^{4N-1} a_k T_k(x)$, then*

$$\frac{\pi}{N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} f(x_j) = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx + \pi a_{2N} \tag{3.12}$$

where $\bar{c}_j$ is defined in (2.10); x_j $0 \le j \le N$ are the Chebyshev-Lobatto points.

Proof: see [3].

The main stability result follows from the following lemma

Lemma 4 *Let $U(x,t)$ be the solution of (3.2) and $u(x,t) = \mathcal{R}\bar{u}$ be the approximation to $U(x,t)$ by the CAC method (1.3) with the reconstruction operator (2.27) or (2.30), then*

$$\frac{\partial}{\partial t}\left[\frac{\pi}{2N}\sum_{j=1}^{N}\frac{1}{\bar{c}_j}\frac{1+x_j}{1-x_j}u^2(x_j,t) + \frac{\pi N}{2(N+1)}\hat{u}_N^2(t)\right] \le 0. \tag{3.13}$$

Proof: We multiply (3.5) by $\frac{1}{\bar{c}_j}\frac{1+x_j}{1-x_j}u(x_j,t)$ and sum from the points $x_N = -1$ to x_1 to get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\pi}{2N}\sum_{j=1}^{N}\frac{1}{\bar{c}_j}\frac{1+x_j}{1-x_j}u^2(x_j,t) &= \frac{\pi}{N}\sum_{j=1}^{N}\frac{1}{\bar{c}_j}\frac{1+x_j}{1-x_j}u(x_j,t)u_x(x_j,t)\\ &- \tau N^2\frac{\pi}{N}\sum_{j=1}^{N}\frac{1}{\bar{c}_j}\frac{1+x_j}{1-x_j}u(x_j,t)T_N(x_j)\end{aligned} \tag{3.14}$$

We treat the two terms in the right hand side of (3.14) separately.

$$\begin{aligned} I &= \frac{\pi}{N}\sum_{j=1}^{N}\frac{1}{\bar{c}_j}\frac{1+x_j}{1-x_j}u(x_j,t)u_x(x_j,t)\\ &= \frac{\pi}{N}\sum_{j=0}^{N}\frac{1}{\bar{c}_j}\frac{1+x_j}{1-x_j}u(x_j,t)u_x(x_j,t) + \frac{\pi}{N}u_x^2(1,t).\end{aligned} \tag{3.15}$$

By noting the exactness of the Gauss Lobatto formula one gets

$$I = \int_{-1}^{1}\frac{1+x}{1-x}u(x,t)\frac{\partial}{\partial x}u(x,t)\frac{dx}{\sqrt{1-x^2}} + \frac{\pi}{N}u_x^2(1,t) \tag{3.16}$$

and integration by part yields

$$I = -\int_{-1}^{1}\frac{1+\frac{x}{2}}{\sqrt{1-x^2}}\frac{u^2(x,t)}{(1-x)^2}\,dx + \frac{\pi}{N}u_x^2(1,t). \tag{3.17}$$

We use again the Gauss Lobatto formula to get

$$I = -\frac{\pi}{N}\sum_{j=0}^{N}\frac{1}{\bar{c}_j}(1+\frac{x_j}{2})\frac{u^2(x_j,t)}{(1-x_j)^2} + \frac{\pi}{N}u_x^2(1,t) \tag{3.18}$$

and therefore

$$I \le -\frac{3}{4}\frac{\pi}{N}u_x^2(1,t) + \frac{\pi}{N}u_x^2(1,t) = \frac{\pi}{4N}u_x^2(1,t). \tag{3.19}$$

We turn now to the second term in (3.14)

$$II = -\tau N^2\frac{\pi}{N}\sum_{j=1}^{N}\frac{1}{\bar{c}_j}\frac{1+x_j}{1-x_j}u(x_j,t)T_N(x_j) \tag{3.20}$$

or

$$II = -\tau N^2 \frac{\pi}{N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} \frac{1+x_j}{1-x_j} u(x_j,t) T_N(x_j) - \tau N^2 \frac{\pi}{N} u_x(1,t). \tag{3.21}$$

Using (3.9) and (3.11) for τ one gets

$$II = \frac{N}{N+1} \frac{\partial \hat{u}_N}{\partial t} \frac{\pi}{N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} \frac{1+x_j}{1-x_j} u(x_j,t) T_N(x_j) - \frac{\pi}{2N} u_x^2(1,t). \tag{3.22}$$

As $\frac{1+x}{1-x} u(x,t) T_N(x)$ is a polynomial of degree less than $4N-1$, one gets using lemma 3

$$\frac{\pi}{N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} \frac{1+x_j}{1-x_j} u(x_j,t) T_N(x_j) = \int_{-1}^{1} \frac{1+x}{1-x} \frac{u T_N}{\sqrt{1-x^2}}\, dx + \pi a_{2N}, \tag{3.23}$$

where a_{2N} is the $2N$ coefficient in the expansion of $\frac{1+x}{1-x} u(x,t) T_N(x)$

It can be easily verified that

$$\int_{-1}^{1} \frac{1+x}{1-x} \frac{u T_N}{\sqrt{1-x^2}}\, dx = -\frac{\pi}{2} \hat{u}_N, \tag{3.24}$$

$$a_{2N} = -\tfrac{1}{2} \hat{u}_N, \tag{3.25}$$

so

$$II = -\pi \frac{N}{N+1} \hat{u}_N \frac{\partial \hat{u}_N}{\partial t} - \frac{\pi}{2N} u_x^2(1,t). \tag{3.26}$$

Using (3.19) and (3.26) we get

$$I + II \le -\pi \frac{N}{N+1} \hat{u}_N \frac{\partial \hat{u}_N}{\partial t} - \frac{\pi}{4N} u_x^2(1,t). \tag{3.27}$$

Substituting (3.27) into (3.14) yields (3.13). This completes the proof of the lemma. Q.E.D.

We note that from Lemma 3

$$\frac{\pi}{2N} \sum_{j=1}^{N} \frac{1}{\bar{c}_j} \frac{1+x_j}{1-x_j} u^2(x_j,t) = \frac{1}{2} \int_{-1}^{1} \frac{1+x}{1-x} \frac{u^2(x,t)}{\sqrt{1-x^2}}\, dx - \frac{\pi}{4} \hat{u}_N^2(t), \tag{3.28}$$

so that we can finally state that

Theorem 1 *Let $u(x,t) = \mathcal{R}\bar{u}$ be the solution of the CAC method with the reconstruction operator (2.27) or (2.30), then*

$$\frac{1}{2} \int_{-1}^{1} \frac{1+x}{1-x} \frac{u^2(x,t)}{\sqrt{1-x^2}}\, dx + \frac{2N-1}{4(N+1)} \pi \hat{u}_N^2(t) \le \frac{1}{2} \int_{-1}^{1} \frac{1+x}{1-x} \frac{u^2(x,0)}{\sqrt{1-x^2}}\, dx + \frac{2N-1}{4(N+1)} \pi \hat{u}_N^2(0). \tag{3.29}$$

4 Numerical Results

In this section we apply the CAC spectral method (1.3) to the one dimensional gas dynamics equations. The time evolution is done by the Runge-Kutta type method. Each step of the Runge-Kutta scheme is done as follows:

Fully Discretized CAC method

Step 1: Reconstruction:

given $\bar{u}^n_{j-\frac{1}{2}}$, $j = 1, \cdots, N$, we use the boundary condition and the matrix $\mathbf{R_N}$ to find the point values u_j, $j = 1, \cdots N$ as suggested in (2.27) or (2.30);

Step 2: Solution in time:

we update the values $\bar{u}^{n+1}_{j-\frac{1}{2}}$, $j = 1, \cdots, N$ using the forward scheme,

$$\bar{u}^{n+1}_{j-\frac{1}{2}} = \bar{u}^n_{j-\frac{1}{2}} - \frac{\Delta t}{x_{j-1} - x_j}\left[F(u^n_{j-1}) - F(u^n_j)\right],\ j = 1, \cdots, N, \tag{4.1}$$

where Δt is the time step.

The reconstruction operator $\mathbf{R_N}$ yields spectrally accurate point value approximations to the exact solutions if the exact solutions are smooth, thus $\bar{u}_{j-\frac{1}{2}}$, $j = 1, \cdots, N$ can be expected to approximate their cell averages accurately. However, if the exact solutions are discontinuous, the point values u_j, $j = 0, \cdots, N$ obtained by $\mathbf{R_N}$ will be oscillatory as the result of the Gibbs phenomenon. In [1] we proposed a practical way to obtain essentially nonoscillatory spectral reconstruction to a discontinuous function from its oscillatory Fourier approximations. The key idea there is to augment the Fourier space by adding simple discontinuous functions whose locations and magnitudes of discontinuities are approximations to those of the shock waves in the numerical solutions. In our computations of CAC method, we extend this idea along the same line to obtain essentially nonoscillatory reconstructions. The estimates on these reconstructions will be appearing in a separate work. We refer the reader to ([1]) for further details.

Now consider the Riemann problem for the Euler equations for a polytropic gas

$$\mathbf{U}_t(x,t) + \mathbf{f}_x(\mathbf{U}) = 0, \tag{4.2}$$

where $\mathbf{U}(\mathbf{x},\mathbf{t})$ and $\mathbf{f}(\mathbf{U})$ are defined as

$$\begin{aligned} \mathbf{U} &= (\rho, M, E)^{\top}, \qquad -1 \le x \le 1, \\ \mathbf{f}(\mathbf{U}) &= q\mathbf{U} + (0, P, qP)^{\top}, \end{aligned} \tag{4.3}$$

where

$$P = (\gamma - 1)(E - \frac{1}{2}\rho q^2), \qquad M = \rho q, \tag{4.4}$$

with $\gamma = 1.4$ and the initial conditions are as follows

$$\begin{aligned} (\rho_L, q_L, P_L) &= (3.857143, 2.629369, 10.33333) \qquad \text{when } x < -0.8, \\ (\rho_R, q_R, P_R) &= (1 + \epsilon \sin 5\pi x, 0, 1) \qquad \text{when } x > -0.8, \end{aligned} \tag{4.5}$$

where $\epsilon = 0.2$.

The solutions to (4.2) - (4.5) model the interaction between a moving shock wave and disturbances. Note that in the right state of the density a sinuous perturbation of magnitude $\epsilon = 0.2$ is superimposed upon a constant state. ¿From linear analysis it can be shown that the disturbances will interact with the shock wave. A density wave of different frequency and magnitude will emerge behind the shock wave. Also the disturbance in the density field will perturb the velocity and pressure fields behind the shock wave. The numerical solution of this Riemann problem mandates a high order scheme in order to capture the fine structures in the solutions for a correct interpretation of the physical process. We test this problem with second order MUSCL scheme [5] and third order point value version ENO finite difference scheme [6] and the CAC spectral method proposed in this paper. Our numerical results have shown clearly the advantage of a higher order numerical scheme for problems with complicated structures.

Figure 1 - 3 show the density profiles obtained by the three methods mentioned above. Figure 1 is the result using CAC spectral method. The solid lines are the solutions taken

from [6] using the third order ENO finite difference method with $N = 800$ which we take as the exact solutions . Figure 2 and Figure 3 are the results with the second order MUSCL scheme [5] and the third order ENO finite difference scheme respectively. In all three cases we use the same amount of mesh points $N = 220$. All the results are plotted at the same time $t = 0.3$.

ACKNOWLEDGMENT

We would like to thank Prof. Chi-wang Shu for many useful discussions.

References

[1] W. Cai, D. Gottlieb, C. W. Shu, "Essentially Non-Oscillatory Spectral Fourier method for Shock Wave Calculations" Math. Comput., Vol. 52, No. 186, April 1989, pp. 389-410.

[2] D. Gottlieb and E. Turkel, "Topics in Spectral Methods", Lecture Notes in Mathematics, No. 1127, edited by A. Dold and B. Eckman, 1983 , pp. 115 - 155

[3] D. Funaro and D. Gottlieb " A New Method of Imposing Boundary Conditions in Pseudospectral Approximation of Hyperbolic Equations" Math. Compt. December 1988

[4] A. Harten, "On the nonlinearity of modern shock capturing schemes" ICASE Report No. 86-69 (October 1986)

[5] S. Osher, "Convergence of Generalized MUSCL Schemes," SIAM J. Numer. Anal., v. 22, 1985, pp. 947-961.

[6] C.-W. Shu and S. Osher, "Efficient Implementation of Essentially Nonoscillatory Shock-Capturing Schemes, II," J. Compt. Phys., V. 83, No. 1, July 1989.

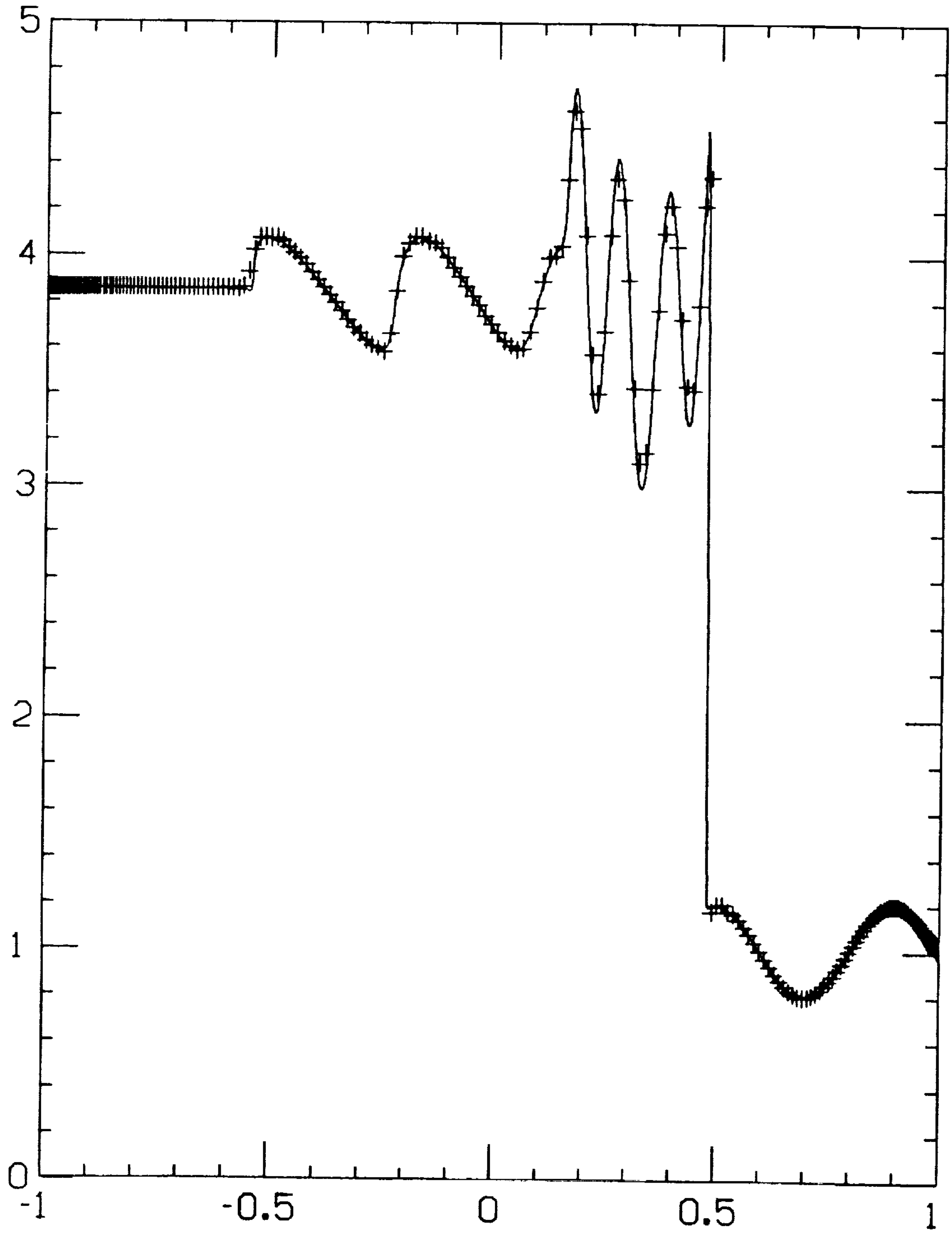

Figure 1: The CAC spectral method: density, N = 220, time t = 0.3. (+) - numerical solutions, solid line - exact solutions.

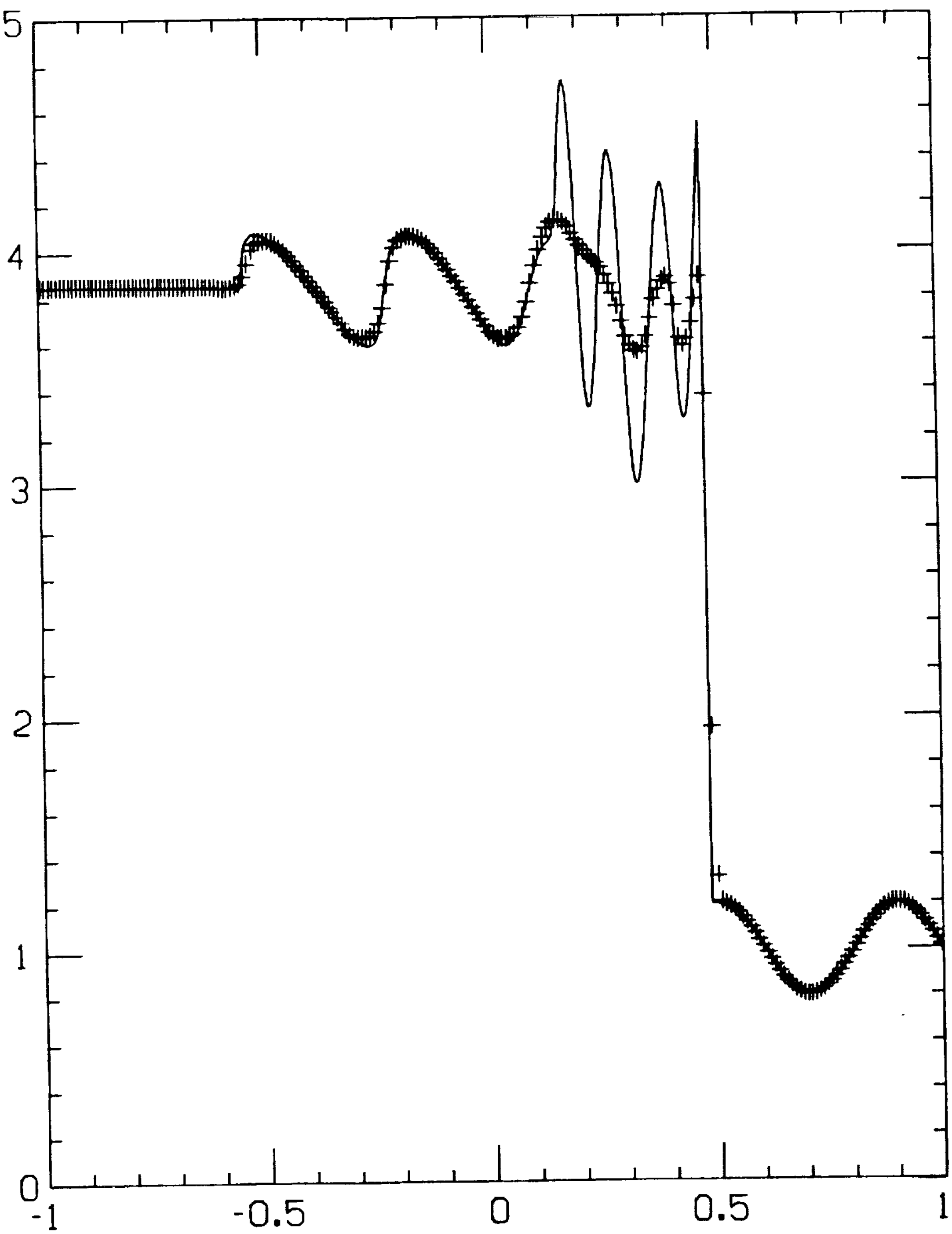

Figure 2: The second order MUSCL scheme: density, N = 220, time t = 0.3. (+) - numerical solutions, solid line - exact solutions.

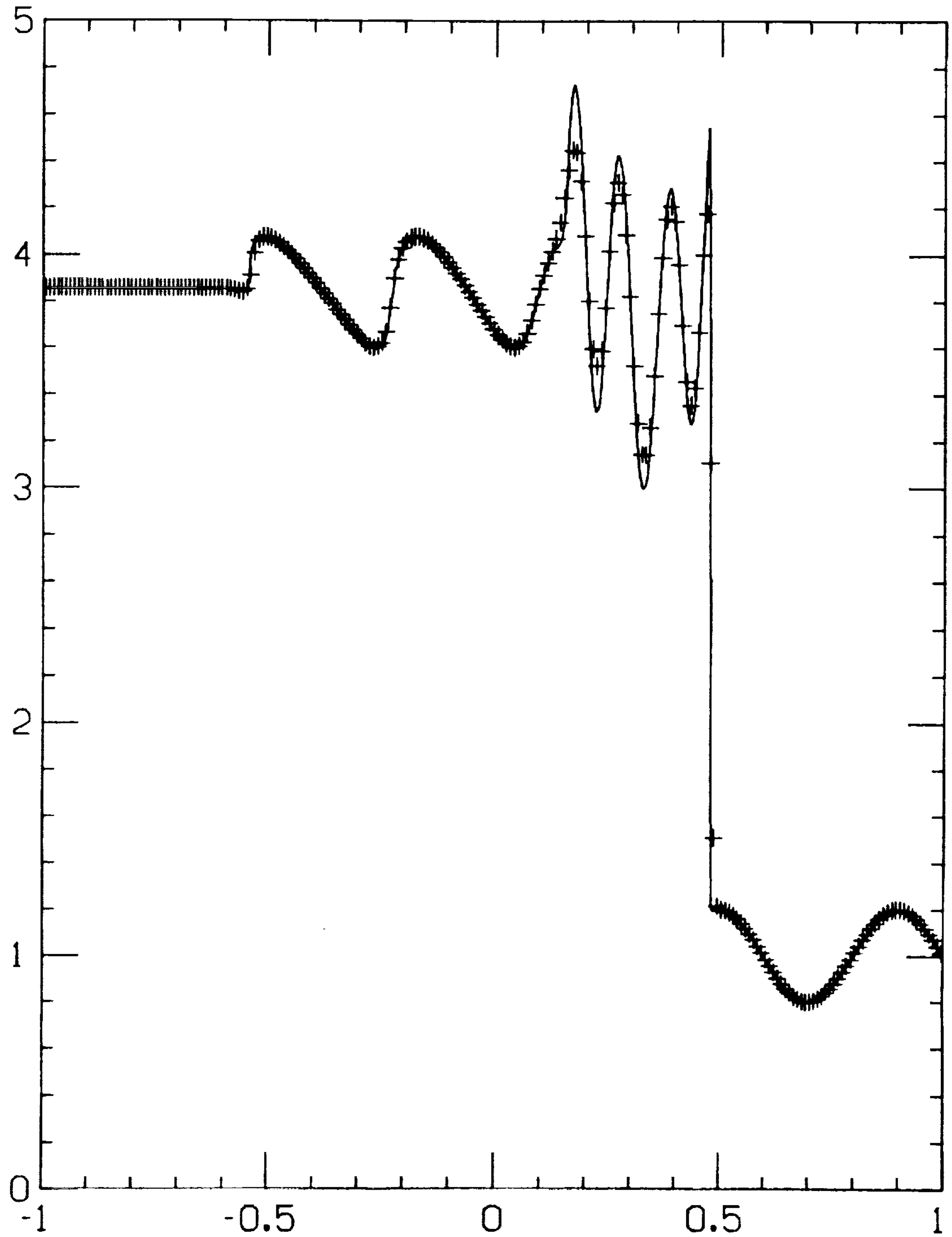

Figure 3: The third order ENO scheme: density, N = 220, time t = 0.3. (+) - numerical solutions, solid line - exact solutions.

NASA National Aeronautics and Space Administration

Report Documentation Page

1. Report No. NASA CR-182028 ICASE Report No. 90-27	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle CELL AVERAGING CHEBYSHEV METHODS FOR HYPERBOLIC PROBLEMS		5. Report Date March 1990
		6. Performing Organization Code
7. Author(s) Wei Cai, David Gottlieb, Ami Harten		8. Performing Organization Report No. 90-27
		10. Work Unit No. 505-90-21-01
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering, Mail Stop 132C, NASA Langley Research Center, Hampton, VA 23665-5225		11. Contract or Grant No. NAS1-18605
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Langley Research Center, Hampton, VA 23665-5225		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor:
Richard W. Barnwell

To appear in Computer and Math with Application; special issue: Advance in P.D.E.

Final Report

16. Abstract

This paper describes a cell averaging method for the Chebyshev approximations of first order hyperbolic equations in conservation form.

We present formulas for transforming between pointwise data at the collocation points and cell averaged quantities, and vice-versa. This step, trivial for the finite difference and Fourier methods, is nontrivial for the global polynomials used in Spectral methods.

We then prove that the cell averaging methods presented are stable for linear scalar hyperbolic equations and present numerical simulations of shock-density wave interaction using the new cell averaging Chebyshev methods.

17. Key Words (Suggested by Author(s)) pseudospectral methods, cell averaging techniques, conservation laws	18. Distribution Statement 64 - Numerical Analysis; Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	22	A03

NASA FORM 1626 OCT 86