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MODELING OF VORTEX-INDUCED OSCILLATIONS BASED ON INDICIAL RESPONSE APPROACH

Khiem Van Truong

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Ames Research Center
Moffett Field, California 94035

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16. Abstract

Abstract: The aim of this research is the application of the indicial response approach, a modeling approach originally used for studying nonlinear problems in flight dynamics, to the study of vortex-induced oscillations phenomena. The indicial response of the velocity field is derived for the problem studied with emphasis on physical postulates involved. A full account of fluid dynamics effects is taken by considering the incompressible Navier-Stokes equations. The theory is applied to the particular case of flow past a cylinder in periodically forced motion to derive some salient features. The indicial response approach is shown to be equivalent to a currently popular approach based on the use of the amplitude equation. Jump and hysteresis phenomena that experiments indicate occur within the lock-in regime (where the velocity field oscillates at the forcing frequency) are captured by our approach.

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1. Introduction

The aim of this research is the application of the indicial response approach, a modeling approach originally used for studying nonlinear problems in flight dynamics and exposed in a recent review article [1], to the study of vortex-induced oscillations phenomena. The problem chosen, vortex shedding from a bluff body, is itself a subject with many practical applications and has attracted the attention of researchers for over several decades. In spite of the long intense interest, researchers in this field agree that an adequate rationally based mathematical model of the phenomena still does not exist [2]. The phenomenon of vortex shedding enters into a class of problems involving a periodic equilibrium state which is the result of a bifurcation from a previous steady equilibrium state as a parameter exceeds some critical value. The occurrence is called a Hopf bifurcation. In [1] it was shown on the basis of physical reasoning that the amplitude and frequency of the periodic equilibrium state are determined by time invariant conditions and it was suggested that specification of its phase demands a proper consideration of fluid dynamics effects.

To that end, this study has two purposes. The first purpose is to develop the indicial response approach appropriate for the problem considered. The indicial response approach, which takes its origin from linear theory, enables one to write the response of a (linear) system as a superposition integral of step responses. This approach transposed to nonlinear systems in flight dynamics gives an analogous result in terms of a generalized superposition integral of elementary indicial responses [1]. We give in this report a derivation of the above result in a suitable form for the problem considered, e.g. in terms of the appropriate physical quantity which is the velocity field. Such a derivation provides us the opportunity of analyzing the mathematical postulates involved, which until now haven't been properly discussed. We demonstrate that this main theoretical result can be applied first to the simple situation of a time-invariant equilibrium state. The indicial response of the velocity field is derived directly from the equations governing the fluid motion. These are taken to be the incompressible Navier-Stokes equations. Results for the aerodynamic response are shown to confirm the form, obtained previously by a variety of approaches [1]. The analysis is then directed to the new situation where the equilibrium state is periodic in time. The second purpose is to apply the results of the analysis to the particular case of flow past a cylinder in periodically forced motion to derive some salient features. The indicial response approach is shown to be equivalent to the currently popular amplitude equation approach.

2. Generalized superposition integral based on indicial response approach.

We derive the indicial response approach with emphasis on the physical postulates involved.

For generality, consider an aircraft that has started from rest in the distant past with fixed axial velocity U_0 and zero vertical velocity (cf. Fig. 1). Its motion is referred to an X, Y coordinate system that is fixed in space. It passes through the origin at the arbitrarily chosen initial instant $\xi = 0$, maintaining the constant axial velocity U_0 and simultaneously translating vertically, with vertical velocity v_C at the center of gravity being an arbitrary function of time ξ. The angle-of-attack α is defined as the angle between the resultant velocity vector and the aircraft's longitudinal axis:

$$\alpha = tan^{-1}\frac{v_C(\xi)}{U_0} \tag{1}$$

Let us note that we specify a constant axial velocity U_0 to be in accord with normal operating conditions in wind- or water-tunnel experiments, wherein the uniform oncoming flow, normally held at constant velocity, would supply the corresponding value of U_0. To form the indicial response, we need to consider two motions. In the first one, the aircraft undergoes a variation of angle of attack $\alpha(\xi)$ from time zero to a time $\xi = \tau$ (cf. Fig. 2). Subsequent to time τ, the angle of attack is held constant at $\alpha(\tau)$. The first motion history therefore is designated as $\alpha_\tau(\xi)$:

$$\alpha_\tau(\xi) \begin{cases} = \alpha(\xi) & : \quad if\ 0 < \xi < \tau \\ = \alpha(\tau) & : \quad if\ \xi \geq \tau \end{cases} \tag{2}$$

Notice that the subscript τ is used to distinguish the above motion history from the natural motion history $\alpha(\xi)$. In the second motion, the aircraft undergoes the same angle of attack history $\alpha(\xi)$ up to time τ. Subsequent to τ, the angle of attack again is held constant, but is given an incremental step change $\Delta\alpha$ over its previous value of $\alpha(\tau)$. The second motion, designated $\alpha_\tau^*(\xi)$, is represented as

$$\alpha^*_\tau(\xi) = \alpha_\tau(\xi) + \epsilon.\eta_\tau(\xi) \tag{3}$$

with:

$$\epsilon = \Delta\alpha \tag{4}$$

$$\eta_\tau(\xi) = \begin{cases} 0 & : \quad if\ 0 < \xi < \tau \\ 1 & : \quad if\ \xi \geq \tau \end{cases} \tag{5}$$

Let us define by $\vec{u}(\vec{x}, t, \tau)$($\vec{x}$: spatial coordinates, t: time of observation subsequent to τ) the velocity field in the neighborhood of the aircraft.

The indicial response approach is established by making use of the following postulates.

. <u>Postulate</u> §1: Corresponding to a motion history belonging to the family of motion histories $\alpha_\tau(\xi)$ with $\tau \in [0, t]$, there exists a velocity field $\vec{u}(\vec{x}, t, \tau)$ well defined and supposed known by some means.

In mathematical terms, one could say that there exists a mapping of the space of scalar quantities $\alpha_\tau(\xi)$ with $\tau \in [0, t]$ to the space of vectors $\vec{u}(\vec{x}, t, \tau)$. However, in physical terms used in this report, we simply say that there exists a velocity response $\vec{u}$ to a motion history α_τ. An analogous postulate is held about the existence of velocity fields associated with the family of motion histories $\alpha^*_\tau(\xi)$ with $\tau \in [0,t]$:

.<u>Postulate</u> §2: To each member of the family of motion histories $\alpha^*_\tau(\xi)$ with $\tau \in [0,t]$, there exists a velocity response $\vec{u}[\vec{x}, \alpha^*_\tau(\xi); t, \tau]$. Such velocity response reproduces the velocity response corresponding to the motion history $\alpha_{\tau+\Delta\tau}(\xi)$ within a negligible error of $\vartheta(\Delta\alpha^2)$:

$$\vec{u}[\alpha_\tau(\xi) + \Delta\alpha.\eta_\tau(\xi)] = \vec{u}[\alpha_{\tau+\Delta\tau}(\xi)] + \vartheta(\Delta\alpha^2) \qquad i = 1, \ldots, n \tag{6}$$

The velocity of the angle-of-attack $\dot{\alpha}(\tau) = \frac{\Delta\alpha}{\Delta\tau}$ is supposed to be defined for each value of $\tau \in [0,t]$. A further postulate is made about the existence of the indicial response of the velocity field:

. <u>Postulate</u> § 3: For every value of $\tau \in [0,t]$, there exists an indicial response $\vec{u}_\alpha[\vec{x}, \alpha(\xi); t, \tau]$ defined as:

$$\vec{u}_\alpha[\vec{x}, \alpha(\xi); t, \tau] = lim_{\Delta\alpha\to 0}\frac{1}{\Delta\alpha}\left\{\vec{u}[\vec{x}, \alpha_\tau(\xi) + \Delta\alpha.\eta_\tau(\xi); t, \tau] - \vec{u}[\vec{x}, \alpha_\tau(\xi); t, \tau]\right\} \tag{7}$$

Given a motion history $\alpha(\xi : \xi \in [-\infty, t])$, it is possible to choose an initial time t_0 and make a time partition $[t_0, t_1, \ldots, t_n]$ of the time interval $[t_0, t]$ such that:

$$t_n = t \ , \ n\Delta t = t \ . \ t_i - t_{i-1} = \Delta t \qquad (i = 1, \ldots, n) \tag{8}$$

One has the following relations:

$$\left.\begin{array}{rcl}\vec{u}_1[\alpha_{t_0}(\xi)+\Delta\alpha.\eta_{t_0}(\xi)]-\vec{u}_0[\alpha_{t_0}(\xi)] & = & \vec{u}_\alpha[\alpha_{t_0}(\xi)]\Delta\alpha.+\vartheta(\Delta\alpha^2)\\ \vec{u}_2[\alpha_{t_1}(\xi)+\Delta\alpha.\eta_{t_1}(\xi)]-\vec{u}_1[\alpha_{t_1}(\xi)] & = & \vec{u}_\alpha[\alpha_{t_1}(\xi)]\Delta\alpha+\vartheta(\Delta\alpha^2)\\ & \vdots & \\ \vec{u}_n[\alpha_{t_{n-1}}(\xi)]+\Delta\alpha.\eta_{t_{n-1}}(\xi)]-\vec{u}_{n-1}[\alpha_{t_{n-1}}(\xi)] & = & \vec{u}_\alpha[\alpha_{t_{n-1}}(\xi)]\Delta\alpha+\vartheta(\Delta\alpha^2)\end{array}\right\} \tag{9}$$

Notice that the additional dependence of $\vec{u}_i$ on $\vec{x},t,t_i(i=1,\ldots,n)$ is omitted for reason of abreviation.

By summing up the relations (9) and using postulate §2, one gets

$$\vec{u}_n[\alpha_{t_n}(\xi)]-\vec{u}_0[\alpha_{t_0}(\xi)]=\sum_{i=1}^{n}\vec{u}_\alpha[\alpha_{t_i}(\xi)]\Delta\alpha+\vartheta(\Delta\alpha^2) \tag{10}$$

i.e.:

$$\vec{u}[\vec{x},\alpha_t(\xi);t,t]=\vec{u}_0[\vec{x},\alpha_{t_0}(\xi);t,t_0]+\int_{t_0}^{t}d\tau\;\vec{u}_\alpha[\vec{x},\alpha_\tau(\xi);t,\tau]\frac{d\alpha}{d\tau}+\vartheta(\Delta\alpha^2) \tag{11}$$

The relation (11) constitutes our main result that will be used in the subsequent sections. Notice that it relies on three physical postulates which have to be satisfied.

3. Time-invariant equilibrium state

We wish to derive the aerodynamic response on the basis of relation (11). In the following subsections, we shall demonstrate, on the basis of the Navier-Stokes equations, that it is possible to find analytical expressions of the velocity responses over the time interval $t - \tau > 0$ to the motion histories $\alpha_\tau(\xi)$ and $\alpha^*_\tau(\xi)$.

3.1 <u>Construction of the velocity response for the motion history $\alpha_\tau(\xi)$</u>

For simplicity, we neglect compressibility effects and assume that the fluid motion is governed by the Navier-Stokes equations for an incompressible fluid. In a coordinate system attached to the body, the Navier-Stokes equations have an additional term $\dot{\vec{v}}_C$ (where $\vec{v}_C$ is the vertical velocity at the mass center) to account for the acceleration of the coordinate system relative to inertial space:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\vec{\nabla})\vec{u} + \vec{\nabla} p - \nu \nabla^2 \vec{u} = -\dot{\vec{v}}_C \tag{12}$$

Here, $\vec{u}$ is the velocity field, p the pressure and ν the kinematic viscosity. We wish to derive a form for the velocity response to the motion $\alpha_\tau(\xi)$ over the time interval $t - \tau > 0$. Since $\alpha_\tau(\xi) = \alpha(\tau) =$ constant for $t - \tau > 0$, boundary conditions determining the velocity response over the interval $t - \tau > 0$ are perfectly steady. Consequently, we expect the velocity response to attain an equilibrium state as the interval becomes large, i.e., as $t - \tau \to \infty$. The response of the velocity field can be decomposed into an equilibrium state $\vec{u}_{equil}$ and a transient component $\vec{u}_{trans}$ that decays as time increases. The principal condition that we impose in this section is that the equilibrium state $\vec{u}_{equil}$ be time-invariant. Thus, as time is referred to τ (the instant specifying when the angle-of-attack is to be kept constant thereafter), we have:

$$\left.\begin{aligned} \vec{u}(\vec{x}, t-\tau > 0) &= \vec{u}_{equil}(\vec{x}) + \vec{u}_{trans}(\vec{x}, t-\tau > 0) \\ p(\vec{x}, t-\tau > 0) &= p_{equil}(\vec{x}) + p_{trans}(\vec{x}, t-\tau > 0) \end{aligned}\right\} \tag{13}$$

To simplify the notation, we define:

$$t_+ = t - \tau > 0 \tag{14}$$

Substituting (13) into (12), we get for the equations governing $\vec{u}_{trans}$

$$\left.\begin{aligned} \frac{\partial \vec{u}_{trans}}{\partial t_+} + (\vec{u}_{equil}.\vec{\nabla})\vec{u}_{trans} + (\vec{u}_{trans}.\vec{\nabla})\vec{u}_{equil} + \vec{\nabla} p_{trans} - \nu \nabla^2 \vec{u}_{trans} & \\ + (\vec{u}_{trans}.\vec{\nabla})\vec{u}_{trans} + \dot{\vec{v}}_C(\tau).(1 - H(t_+)) &= 0 \\ \vec{\nabla}.\vec{u}_{trans} &= 0 \end{aligned}\right\} \tag{15}$$

where $H(t_-)$ is the Heaviside function:

$$H(t_+) = \begin{cases} 0 & : \quad t_+ < 0 \\ 1 & : \quad t_+ \geq 0 \end{cases} \tag{16}$$

with $\vec{u}_{trans} = 0$ on appropriate boundaries $\partial\Omega$.

To find a form for $\vec{u}_{trans}$, we shall first consider a small disturbance $\vec{u}'$ which satisfies the linearized version of (15):

$$\left.\begin{aligned} \frac{\partial}{\partial t_+}\vec{u}' + (\vec{u}_{equil}.\vec{\nabla})\vec{u}' + (\vec{u}'.\vec{\nabla})\vec{u}_{equil} + \vec{\nabla}p' - \nu\nabla^2\vec{u}' &= -\dot{\vec{v}}_C(\tau).(1 - H(t_+)) \\ \vec{\nabla}.\vec{u}' &= 0 \end{aligned}\right\} \quad (17)$$

with $\vec{u}' = 0$ on $\partial\Omega$. Equations (17) constitute a linear eigenvalue problem. Since $\vec{u}_{equil}$ is independent of time, the eigensolutions $\vec{\gamma}_n$ are also independent of time. It is possible to choose them such that [3]

$$\vec{u}' = \vec{u}_n' = \exp(\lambda_n t_+)\vec{\gamma}_n(\vec{x}) \quad , \quad \vec{\nabla}.\vec{\gamma}_n = 0 \quad , \quad \vec{\gamma}_n|_{\partial\Omega} = 0 \qquad (18)$$

From equation (18), the corresponding expression for p' is:

$$p' = \exp(\lambda_n t_+)p_n'(\vec{x}) \qquad (19)$$

The eigensolutions $\vec{\gamma}_n(\vec{x})(n = 1, \ldots, N)$ are associated with the eigenvalues λ_n, the latter all having negative real parts in the physical situation under consideration. There is an equation adjoint to (17) having a set of eigensolutions $\vec{\gamma}_n^*$ such that [3]

$$\left.\begin{aligned} \langle\vec{\gamma}_i, \vec{\gamma}_j^*\rangle &= \delta_{ij} & \forall\, i, j \\ \langle\vec{\gamma}_i, \overline{\vec{\gamma}_j^*}\rangle &= 0 & \forall\, i, j \end{aligned}\right\} \qquad (20)$$

where the brackets $\langle \quad \rangle$ denote the scalar product over space. The eigensolutions $\vec{\gamma}_n(\vec{x})$ span a complete functional space and one can use this fact to construct a suitable solution form for $\vec{u}_{trans}(\vec{x}, t_+)$. Returning now to the full nonlinear equation (15) governing $\vec{u}_{trans}$, let us assign $\vec{u}_{trans}$ the form resulting from its projection onto the functional space of the $\vec{\gamma}_n$:

$$\vec{u}_{trans}(\vec{x}, t_+) = \sum_{n=1}^{N} \left(d_n(t_+)\vec{\gamma}_n(\vec{x}) + \bar{d}_n(t_+)\bar{\vec{\gamma}}_n(\vec{x}) \right) \tag{21}$$

where the barred terms denote complex conjugates (c.c.). It will be convenient to break the pressure term p_{trans} in (13) into two parts: one following the form of equation (21), and the other (denoted $\tilde{p}$) corresponding to the nonlinear part of the disturbance equations:

$$p_{trans}(\vec{x}, t_+) = \sum_{n=1}^{N} \left(d_n(t_+)p_n^{'}(\vec{x}) + \bar{d}_n(t_+)\bar{p}_n^{'}(\vec{x}) \right) + \tilde{p}(\vec{x}, t_+) \tag{22}$$

After inserting (21) and (22) into (15) and eliminating the set of terms that identically satisfies the linear form (17), we find that the coefficients $d_n(t_+)$ satisfy the following equation:

$$\sum_n \left(\dot{d}_n(t_+) - \lambda_n d_n \right) \vec{\gamma}_n + \sum_n (\dot{\bar{d}}_n - \bar{\lambda}_n \bar{d}_n)\bar{\vec{\gamma}}_n + \vec{\nabla}\tilde{p} = -\sum_n \sum_m \Big(d_n d_m (\vec{\gamma}_n . \vec{\nabla})\vec{\gamma}_m +$$
$$d_n \bar{d}_m (\vec{\gamma}_n . \vec{\nabla})\bar{\vec{\gamma}}_m + \bar{d}_n d_m (\bar{\vec{\gamma}}_n . \vec{\nabla})\vec{\gamma}_m + \bar{d}_n \bar{d}_m (\bar{\vec{\gamma}}_n . \vec{\nabla})\bar{\vec{\gamma}}_m - \dot{\vec{v}}_C(\tau).(1 - H(t_+)) \Big) \tag{23}$$

Multiplying this equation by the vector $\bar{\vec{\gamma}}_j^{*}(\vec{x})$, integrating over space and using the properties of adjoint vectors defined by (20), we obtain

$$\dot{d}_j(t_+) - \lambda_j d_j = -\sum_n \sum_m \Big(d_n d_m \langle (\vec{\gamma}_n . \vec{\nabla})\vec{\gamma}_m, \vec{\gamma}_j^{*} \rangle + d_n \bar{d}_m \langle (\vec{\gamma}_n . \vec{\nabla})\bar{\vec{\gamma}}_m, \vec{\gamma}_j^{*} \rangle +$$

$$\bar{d}_n d_m \langle(\bar{\vec{\gamma}}_n.\vec{\nabla})\vec{\gamma}_m, \vec{\gamma}_j^\star\rangle + \bar{d}_n \bar{d}_m \langle(\bar{\vec{\gamma}}_n.\vec{\nabla})\bar{\vec{\gamma}}_m, \vec{\gamma}_j^\star\rangle\Big)$$

$$-c_j(\alpha(\tau)).(1 - H(t_+)) \qquad j = 1,\ldots,N \tag{24}$$

with:

$$c_j(\alpha(\tau)) = \int_V d^3x\, \vec{\gamma}_j^\star . \frac{\dot{\vec{v}}_C}{\| \dot{\vec{v}}_C \|} \tag{25}$$

Notice that the pressure gradient term $\vec{\nabla}\tilde{p}$ in (23) makes no appearance in (24). The properties of $\vec{\gamma}_n$ and $\vec{\gamma}_n^\star$ (solenoidal and vanishing on $\partial\Omega$) ensure that scalar products of gradient terms (such as $\vec{\nabla}\tilde{p}$) and $\vec{\gamma}_n^\star$ will be identically zero.

By defining appropriate coefficients $A_{jnm}, B_{jnm}, C_{jnm}$ and D_{jnm}, the system of equations (24) can be rewritten as:

$$\dot{d}_j(t_+) - \lambda_j(\alpha(\tau))d_j(t_+) = \sum_n \sum_m \Big(A_{jnm}(\alpha(\tau))d_n(t_+)d_m(t_+) +$$

$$B_{jnm}(\alpha(\tau))d_n(t_+)\bar{d}_m(t_+) + C_{jnm}(\alpha(\tau))\bar{d}_n(t_+)d_m(t_+) + D_{jnm}(\alpha(\tau))\bar{d}_n(t_+)\bar{d}_m(t_+)\Big)$$

$$-c_j(\alpha(\tau)).(1 - H(t_+)), \qquad j = 1,\ldots,N \tag{26}$$

Thus, it appears from the system of equations (26) that it is required to fix the values of $d_j(t_+ = 0)(j = 1,\ldots,N)$ at some origin of time.

The values of $d_j(t_+ = 0)$ must be determined from a match of the velocity fields on either side of $\xi = \tau$:

$$\vec{u}(\vec{x}, \xi = \tau_-) = \vec{u}(\vec{x}, \xi = \tau_+) \tag{27}$$

On the negative side of $\xi = \tau$, in general the velocity field is the resultant of the entire past history of the motion $\alpha(\xi)$ up to the ''present'' time $\xi = \tau$. Through specification of the d_j at $t_+ = 0$ in terms of this velocity field, the velocity response for $t_+ \geq 0$ acknowledges its dependence on the past motion. The representation of this past motion in a suitable way is all that remains to be done to realize a complete characterization of the velocity response for $t_+ \geq 0$ and thus to satisfy postulate §1. We shall defer a discussion of this important step until we have derived the velocity response to the motion $\alpha_\tau^*(\xi)$ and formed the indicial response. For the present, we simply designate the dependence of $d_j(t_+ = 0)$ on the past motion as a functional:

$$d_j(t_+ = 0) = d_j\left[\alpha\left(\xi : \xi \in [0, \tau]\right), t_+ = 0\right] \tag{28}$$

3.2 Construction of the velocity response for the motion history $\alpha_\tau^*(\xi)$

The velocity response fo the motion history $\alpha_\tau^*(\xi)$ can be constructed in an analogous way to that in the previous section. It is possible to demonstrate that the velocity response to the motion history $\alpha_\tau^*(\xi)$ reproduces the velocity response to the motion history $\alpha_{\tau+\Delta\tau}(\xi)$ within some negligible error of $\vartheta(\Delta\alpha^2)$ for time $t \geq \tau + \Delta\tau$. Postulate §2 is therefore satisfied. We refer for complete details to reference [4]. It is shown that the velocity response to the motion history $\alpha_\tau^*(\xi)$ is:

$$\begin{aligned}\vec{u}[\vec{x},\alpha_\tau^*(\xi)] &= \vec{u}_{equil}(\vec{x},\alpha(\tau)-\Delta\alpha)+\sum_n d_{2n}[\alpha_\tau^*(\xi),t_+]\vec{\gamma}_n(\vec{x},\alpha(\tau)+\Delta\alpha)\\ &+ c.c. \end{aligned} \tag{29}$$

3.3 Indicial and total responses of the velocity field

The indicial response of the velocity field can be obtained as

$$\vec{u}_\alpha = \lim_{\Delta\alpha\to 0}\frac{\vec{u}[\vec{x},\alpha_\tau^*(\xi)]-\vec{u}[\vec{x},\alpha_\tau(\xi)]}{\Delta\alpha} \tag{30}$$

Indeed, it is permissible to make the operation defined by the right-hand-side member of equation (30) on the analytical expressions obtained for $\vec{u}[\alpha_\tau(\xi)]$ and $\vec{u}[\alpha_\tau^*(\xi)]$ in the previous subsections. Therefore, we have satisfied postulate §3. The operation in equation (30) leads to the following expression for $\vec{u}_\alpha$:

$$\begin{aligned}\vec{u}_\alpha = \lim_{\Delta\alpha\to 0}\frac{\vec{u}_{equil}(\vec{x},\alpha(\tau)-\Delta\alpha)-\vec{u}_{equil}(\vec{x},\alpha(\tau))}{\Delta\alpha}+\lim_{\Delta\alpha\to 0}\frac{1}{\Delta\alpha}\cdot\\ \sum_n\left(d_{2n}[\alpha_2(\xi)]\vec{\gamma}_n(\vec{x},\alpha(\tau)+\Delta\alpha)-d_n[\alpha_1(\xi)]\vec{\gamma}_n(\vec{x},\alpha(\tau))\right)+c.c.\end{aligned} \tag{31}$$

One gets:

$$\begin{aligned}\vec{u}_\alpha &= \frac{\partial}{\partial\alpha}\vec{u}_{equil}(\vec{x},\alpha(\tau))+\sum_n d_n[\alpha(\xi)]\frac{\partial}{\partial\alpha}\vec{\gamma}_n+\lim_{\Delta\alpha\to 0}\frac{1}{\Delta\alpha}\\ &\sum_n\left(d_{2n}[\alpha_2(\xi)]-d_n[\alpha_1(\xi)]\right)\vec{\gamma}_n+c.c.\end{aligned} \tag{32}$$

The total response of the velocity field is obtained by summing the indicial response along the motion history $\alpha_t(\xi)$ according to equation (11):

$$\vec{u}(\vec{x},t) = \vec{u}_{equil}(\vec{x},\alpha(t)) + \int_0^t d\tau \frac{d\alpha}{d\tau} \left\{ \sum_n \left(d_n[\alpha_\tau(\xi)] \frac{\partial}{\partial\alpha} \vec{\gamma}_n(\vec{x},\alpha(\tau)) \right) \right. - \\ + \left. \sum_n \lim_{\Delta\alpha\to 0} \left(\frac{d_{2n}[\alpha_\tau^*(\xi)] - d_n[\alpha_\tau(\xi)]}{\Delta\alpha} \right) \vec{\gamma}_n \right\} + c.c. \qquad (33)$$

It has the following analytical form

$$\vec{u}(\vec{x},t) = \vec{u}_{equil}(\vec{x},\alpha(t)) + \int_0^t d\tau \frac{d\alpha}{d\tau} \mathcal{F}\left[d_j\left[\alpha_\tau\left(\xi\right), t_+ = 0\right]; t_+\right] \qquad (34)$$

with the functional dependence of $\mathcal{F}$ determined by the initial condition $d_j(t_+ = 0)(j = 1,\ldots,N)$.

3.4 Determination of the initial condition $d_j(t_+ = 0)(j = 1,\ldots,N)$

As it appears in equation (33), the value of the velocity field contains terms appearing under an integration over time such as

$$\exp\{\lambda_j(\alpha(\tau))(t - \tau)\} \qquad j = 1,\ldots,N$$

due to the behavior of $d_j[\alpha_\tau(\xi)](j = 1,\ldots,n)$ (cf. [4]). Hence, only a recent past motion history is taken into account in determining the value of the velocity field. Under such circumstances, one can consider a fit of the recent past motion history to a known motion history of the system. A motion history of the system is provided, for instance, by making experiments under external periodic forcing . It can be shown, either theoretically or experimentally, that the response of the periodically forced system is periodic in time with the same period ν as for the external forcing:

$$\begin{aligned}\vec{u}(\vec{x},\tau_-) &= \sum_n \vec{a}_n e^{in\tau_-} \\ &\simeq \vec{a}_1(\vec{x})e^{i\nu\tau_-} = \vec{u}(\vec{x},\alpha(\tau),\dot{\alpha}(\tau)) \end{aligned} \tag{35}$$

The choice of $d_j(t_+ = 0)$ is governed by the following equation, derived from equation (27)

$$\vec{u}(\vec{x},\tau_-) = \vec{u}_{equil}(\vec{x},\tau_+) + \sum_n d_n(\xi = \tau_+)\vec{\gamma}_n(\vec{x},\alpha(\tau_+)) \tag{36}$$

i.e. by using the properties of adjoint vectors:

$$d_j(\xi = \tau_+) = \langle\vec{u}(\vec{x}.\tau_-),\vec{\gamma}_j^*(\vec{x},\alpha(\tau_+))\rangle - \langle\vec{u}_{equil}(\vec{x},\tau_+),\vec{\gamma}_j^*(\vec{x},\alpha(\tau_+))\rangle \tag{37}$$

The analytical dependence of $d_j(t_+ = 0)$ is therefore:

$$d_j(t_+ = 0) \simeq d_j(\alpha(\tau),\dot{\alpha}(\tau)) \tag{38}$$

For slow motion where $\dot{\alpha}(\tau) \simeq \alpha(\tau)$, one can expand d_j:

$$d_j(t_+ = 0) = d_j(\alpha(\tau),0) + \dot{\alpha}(\tau)\frac{\partial d_j}{\partial\alpha(\tau))} + \vartheta(\dot{\alpha}^2(\tau)) \tag{39}$$

3.5 Determination of the lift force - Discussions

Using relation (39) valid for slow motion, one can rewrite relation (34) as:

$$\vec{u}(\vec{x},t) \simeq \vec{u}_{equil}(\vec{x},\alpha(t)) + \int_0^t d\tau \frac{d\alpha}{d\tau} \mathcal{F}(\alpha(\tau),\dot{\alpha}(\tau),t_+) \tag{40}$$

It is possible to derive the pressure from the velocity field by using the Navier-Stokes equations. The value of the lift force L is obtained by taking the value of the pressure on the body surface (∂B) and by integrating over a solid angle:

$$L = \int_{-\pi}^{+\pi} p(\vec{x},t)|_{\partial B} \sin(\theta) d\theta \tag{41}$$

Hence, the lift coefficient is of the following form

$$C_L(t) = C_L^{equil}[\alpha(t)] - \int_0^{t_+} d\tau \frac{d\alpha}{d\tau} \mathcal{F}\left[d_j\left[\alpha_\tau(\xi), t_+ = 0\right], t_+, \tau\right] \tag{42}$$

where the value of the function $\mathcal{F}$ approaches zero as $t-\tau \to \infty$, due to the properties of $d_n(t_+)$. The relation (42) was derived in previous work on the basis of functional expansion approach [1]: the function $\mathcal{F}$ was called the deficiency function. It was used for various applications in aircraft dynamics (cf. for instance [5]).The value of the lift force can be approximated as:

$$\begin{aligned} C_L(t) &\simeq C_L^{equil}[\alpha(t)] - \int_0^t d\tau \frac{d\alpha}{d\tau} \mathcal{F}(\alpha(\tau),\dot{\alpha}(\tau),t_+) \\ &\simeq C_L^{equil}[\alpha(t)] - \int_0^t d\tau \frac{d\alpha}{d\tau} \mathcal{F}(\alpha(\tau),t_+) + \vartheta(\dot{\alpha}^2(t)) \end{aligned} \tag{43}$$

Such relation is used in most mathematical models in flight dynamics.

4. Periodic equilibrium state

As one of the values of the real part of the eigenvalues $\lambda_j (j = 1, \dots, N)$ becomes positive for increasing Reynolds number, the time-invariant equilibrium state loses its stability. A Hopf bifurcation occurs: the time-invariant equilibrium state is replaced by a time-varying equilibrium state. The analytical development in the present case follows the same procedure as in the previous section. However, it is more laborious, due to the presence of an additional parameter which is the phase $\phi(\tau)$ of the periodic equilibrium state. We shall refer to reference [4] for complete details. We give the result obtained for the lift force

$$
\begin{aligned}
C_L(t) &= \frac{\partial}{\partial t}\int_0^t d\tau C_L^{equil} - \int_0^t d\tau \left(\frac{\partial \phi}{\partial \tau} + \ddot{\alpha}\frac{\partial \phi}{\partial \dot{\alpha}}\right)\frac{C_L^{equil}}{\partial \phi} \\
&+ \int_0^t d\tau \dot{\alpha}(\tau) C_{L_\alpha}^{trans}[\alpha(\tau), \dot{\alpha}(\tau); t - \tau]
\end{aligned}
\tag{44}
$$

where C_L^{equil} denotes the equilibrium component of the lift force and C_L^{trans} denotes its transient part. In the case of vortex-induced vibrations, the value of C_L^{equil} can be deduced from experimental data on a stationary cylinder:

$$C_L^{equil} = A_\infty[\alpha(\tau)]\sin\{k_S[\alpha(\tau)](t-\tau) + \phi[\alpha(\tau), \dot{\alpha}(\tau), \tau]\} \qquad (45)$$

with A_∞ and k_S known from experimental data.

The contribution into $C_L(t)$ contains terms associated with C_L^{equil} (the first two terms of the right-hand-side member of equation (44)). Their contribution may exceed the value of $C_L^{equil}(\alpha(t), \phi(t); t, \tau = t)$, as shown by our own numerical simulation, based on the following expression for $\phi(\tau)$:

$$\phi(\tau) = A_P\alpha(\tau) + B_P\dot{\alpha}(\tau) \qquad (46)$$

Such relation for $\phi(\tau)$ is suggested by physical reasoning [4]. It appears clearly that relation (43), giving the lift force in the first physical situation with time-invariant equilibrium state, is not adequate for the periodic equlibrium state. Modeling of systems in flight dyanmics involving a new regime (for instance, stall phenomenon) requires an appropriate change of the analytical expression of the lift force.

5. Predictions for a periodically forced cylinder near the Hopf bifurcation: relationship with the amplitude equation approach.

It appears from relation (44) the the lift force depends generally on the motion history of angle of attack, which is itself dependent on the value of the lift force (through the governing equation of motion of the elastically mounted cylinder). So, generally the equations governing the angle of attack and the lift force are coupled. In the special case of a periodically forced cylinder, one controls externally the motion and therefore the angle-of-attack. Hence, the only independent variable of the system is constituted by the lift force, i.e. the velocity field. One can simplify furthermore the analysis by restricting it just near the Hopf bifurcation. Such a restriction corresponds to the hypothesis of the amplitude equation approach.

The equations describing the present physical situation correspond to equations (26) slightly modified: the terms $c_j.(1-H(t_+))$ are to be replaced by appropriate terms associated with external forcing (cf. [4]). Due to the restriction of the analysis to the neighborhood of the critical situation, only the equation corresponding to the projection onto the critical eigenvalue $\bar{\gamma}_0$ (associated with the eigenvalue ω_0) is interesting to study. We make use of the theoretical results of Elphick et al. [6] to carry out the simplification of this equation to get its ''normal form'' (cf. reference [4] for complete details). The final equation obtained is:

$$\frac{dA}{dt} = (a + i\omega_0)A - b|A|^2 A + cf_0 e^{i\omega t} \tag{47}$$

where a, b are complex constants, c is a real constant, f_0 is the forcing amplitude and ω is the external frequency.

It can be shown that equations (47) are derived from the following modified Van der Pol equation:

$$\ddot{A} - (\alpha - \gamma \dot{A}^2)\dot{A} + \omega_0^2 A - \eta A^3 = f_0 \sin(\omega t) \tag{48}$$

Such an equivalence was already noticed by Provansal [7]. However, the author hadn't realized its relationship with oscillator models of vortex-induced oscillations. Indeed, one can show that the lift force generated by vortex-shedding behaves as the amplitude of the velocity field.

We have therefore justified the choice of Van der Pol equation as the model of the lift force generated by vortex-shedding from fluid dynamics considerations (Navier-Stokes equations). Furthermore, our analysis supports the analytical form of the fluid oscillator proposed by Skop and Griffin [8]. We have also analysed the amplitude equation (47) to show that there is jump and hysteresis phenomena inside the lock-in regime (where the body oscillates at the forcing frequency).

6. Conclusion

In the present study, we have taken some effort to improve our mathematical modeling approach in order to incorporate a Hopf bifurcation. We have developed the indicial response of the velocity field, based on the incompressible Navier-Stokes equations for the regimes of time-invariant and periodic equilibrium state.

The theory has been applied to the particular case of flow past a cylinder in periodically forced motion. It has been shown that the indicial response approach is equivalent to the currently popular amplitude equation approach, near the Hopf bifurcation. By recognizing that the lift force behaves like the amplitude of the velocity field, we have been able to justify the choice of a Van-der-Pol equation which was used empirically to model the lift force generated by vortex-shedding. We have been also able to capture jump and hysteresis phenomena that experiments indicate occur within the lock-in regime (where the velocity field oscillates at the forcing frequency).

It is hoped that our modeling approach will have some beneficial implications for mathematical modeling of other configurations of self-induced vibrations that bear common features.

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LEGEND.

Fig.1.- Maneuver referred to space-fixed (X,Y) and moving (x,y) coordinates, the latter attached to the airplane.

Fig.2.- Formation of indicial response.

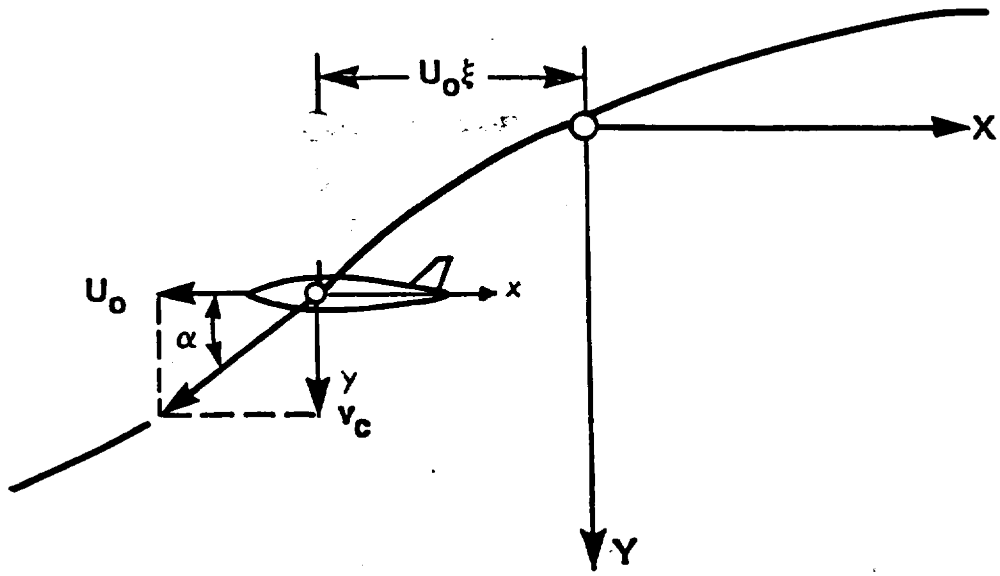
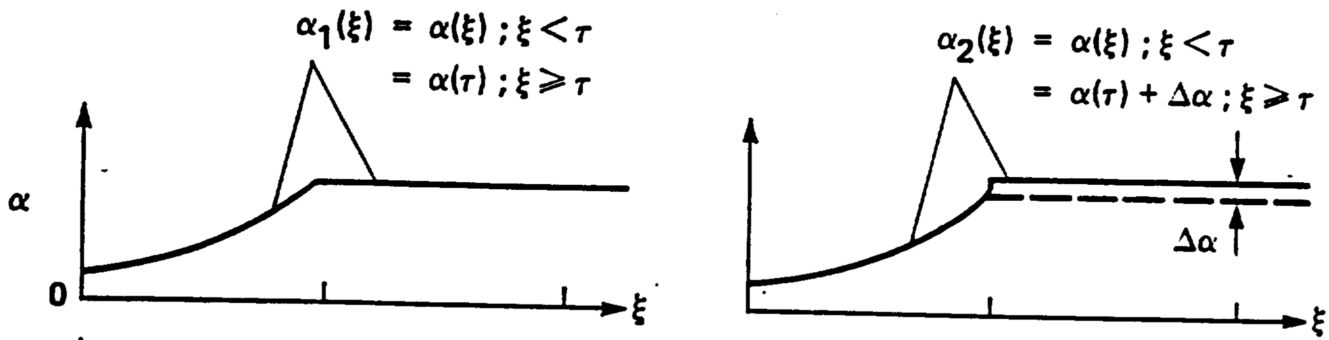


FIG. 1. —



$\alpha_1(\xi) = \alpha(\xi)\,;\ \xi < \tau$
$= \alpha(\tau)\,;\ \xi \geqslant \tau$
α
0
ξ
$\alpha_2(\xi) = \alpha(\xi)\,;\ \xi < \tau$
$= \alpha(\tau) + \Delta\alpha\,;\ \xi \geqslant \tau$
$\Delta\alpha$
ξ

FIG. 2.-