NONLINEAR STRAIN-DISPLACEMENT RELATIONS AND FLEXIBLE MULTIBODY DYNAMICS

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ABSTRACT

This paper considers dynamics of chains of flexible bodies undergoing large rigid body motions, but small elastic deflections. The role of nonlinear strain-displacement relations in the development of the motion equations correct to first order in elastic deflections is investigated. The general form of these equations linearized only in the small elastic deflections is presented, and the relative significance of various nonlinear terms is studied both analytically and through the use of numerical simulations. Numerical simulations are performed for a two link chain constrained to move in the plane, subject to hinge torques. Each link is modeled as a thin beam. Slew maneuver simulation results are compared for models with and without properly modeled kinematics of deformation. The goal of this case study is to quantify the importance of the terms in the equations of motion which arise from the inclusion of nonlinear strain-displacement relations. It is concluded that unless the consistently linearized equations in elastic deflections and speeds are available and necessary, the inconsistently (prematurely) linearized equations should be replaced in all cases by "ruthlessly" linearized equations: equations in which all nonlinear terms involving the elastic deflections and speeds are ignored.

1 INTRODUCTION

In recent years a fundamental limitation of the finite element formulation of flexible deformations in flexible multibody simulation programs such as TREETOPS[1], DISCOS[2], etc. has been pointed out [3,4,5]. This limitation could be characterized as a premature linearization of velocity expressions that is implicit in a linear finite element or modal formulation of the motion equations for flexible bodies [4]. Kane et al. [6] demonstrated this flaw of such an approach numerically by simulating a simple system consisting of a flexible beam attached to a rigid base spinning in the plane. This simulation yielded the surprising and intuitively wrong result of the beam diverging during a spin up maneuver [3]. Probably because of this simple example the "prematurely linearized" equations of motion are said to lack the "spin-stiffening effect." Further study of this simple system showed that this limitation of the traditional approach does not significantly affect simulation results for some maneuvers typical of space dynamical systems, such as repositioning slewing maneuvers and station keeping [4].

This paper presents a general discussion of the equations of motion of a flexible multibody system undergoing motion with large rigid body rates (not just angular velocities) and rigid body configuration changes, but with infinitesimal elastic deformations. The generic equations of motion for such systems are presented; linearized only in elastic deformation amplitudes. The implicit assumption here is that this is a useful set of motion equations: nonlinear in rigid body rates and coordinates but linear in flexible coordinates and rates. Some of the terms in these motion equations cannot be obtained with linear kinematics of elastic deformation (i.e., the traditional linear finite element or modal formulation). This paper illuminates the form of these practically unobtainable (in the general case) terms, evaluates their relative importance and examines the possible consistent simplifications of the motion equations.

The practical impact of these simplifications is investigated with the use of two case studies. The consistent equations of motion, derived with nonlinear strain-displacement relations, are explicitly given for two systems: 1) a single Bernoulli-Euler beam cantilevered to a rigid base undergoing large rigid body, but small flexible motions in the plane; and, 2) a two-link, revolute, planar, flexible manipulator consisting of three rigid bodies (shoulder, elbow and end effector) connected by two Bernoulli-Euler beams. These two examples will serve to emphasize the general discussion and to quantify the magnitudes of the neglected and retained terms in the equations of motion. This will be done both analytically and through comparison of simulation results.

Before proceeding to a more general discussion of the "correct" linearized equations of motion, a brief explanation of how these equations are obtained through proper linearization, and of the role of nonlinear strain-displacement relations is in order.

2 NONLINEAR STRAIN-DISPLACEMENT AND PROPER LINEARIZATION

The problem that concerns us is that of obtaining the correctly linearized equations of motion for an important class of systems which exhibit large rigid body motions but small elastic deflections. By far the most common practice to date is to handle the flexibility through discretization of the desired continuous system. This is achieved by representing the solution as a finite series of time-dependent generalized elastic coordinates multiplied by space-dependent functions, as in an assumed modes approach, or in a finite element formulation [7]. Whichever formulation one uses, in light of the class of systems under study, the next step is to assume that these elastic coordinates, together with the generalized elastic speeds, are infinitesimally small. In other words, we assume these coordinates and speeds to be small enough so that only terms linear in them are kept in the equations of motion, as terms of second order or higher are negligible.

Now that our goal is clearly stated, it should be an easy matter to obtain the linearized equations of motion as long as we consistently drop all terms nonlinear in the elastic coordinates and the corresponding generalized elastic speeds. Of course, the word "consistently" is the catch. When do we linearize? That is to say: Does it matter at what step in our derivation of the equations of motion we start to linearize? To answer this question we have to consider the process by which we derive these equations.

Let us consider two of the more widely known methods to derive motion equations for complex systems: Lagrange's equations of motion [8] and Kane's dynamical equations [9]. Lagrange's equations for a holonomic system with n generalized coordinates q_k:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k, \qquad (k = 1, \ldots, n)$$

$$L = T - V \tag{1}$$

where the Lagrangian L is a function of the system kinetic and potential energies (T and V respectively). Q_k are n generalized non-potential forces. The important thing to note is that using this method to derive motion equations requires differentiating both the potential and kinetic energies of the system with respect to the generalized coordinates and speeds. If these q_i and u_i $(=dq_i/dt)$ were to represent our generalized elastic coordinates, we see that the above differentiations imply that some terms linear in q_i and u_i in the energy expressions become terms of zeroth order in q_i and u_i. More importantly, we see that terms of second order in the generalized coordinates and speeds in the energy expressions become terms of first order in the resulting equations of motion. Clearly then in order to obtain equations of motion correct to first order in q_i and u_i we need to have energy expressions for our system that are correct to second order in these same elastic generalized coordinates and speeds. More specifically the requirement demands in general that the expressions for displacements and velocities used in determining potential and kinetic energies be correct to second order in the elastic coordinates and speeds. Only by doing this can we ensure consistent linearization.

Kane's dynamical equations for a holonomic system of n particles with n generalized speeds u_i:

$$F_r + F_r^* = 0 \qquad (r = 1,\dots,n)$$
$$F_r^* = \sum_{i=1}^{n} \mathbf{v}_r^{P_i} \bullet \mathbf{R}_i^* \quad , \quad \mathbf{R}_i^* = -\, m_i \vec{a}_i$$
$$\mathbf{v}^{P_i} = \sum_{i=1}^{n} \mathbf{v}_r^{P_i} u_i + \mathbf{v}_t \quad , \quad u_i = \dot{q}_i \tag{2}$$

where F_r is the generalized active force, F_r^* is the generalized inertia force, $\mathbf{R}_i^*$ is the inertia force for particle P_i in an inertial reference frame, and $\mathbf{v}^{P_i}$ is the velocity of this particle in the same frame. $\mathbf{v}_r^{P_i}$ is the r-th partial velocity of particle P_i in the inertial frame. Using a similar argument as above, it is easy to see that since the partial velocity has to be correct to first order in the generalized coordinates and speeds, and again this term is obtained through differentiation of the velocity with respect to the generalized speeds, the velocity has to be correct to second order in q_i and u_i until we form the partial velocities. This is necessary if we want our equations of motion to be consistently linear in the generalized coordinates and speeds.

3 FORM OF THE EQUATIONS OF MOTION FOR A CHAIN OF ELASTIC BODIES

The equations of motion of an open chain of elastic bodies can be expressed quite generally as [10]:

$$\begin{bmatrix} M_{RR}(x,q) & M_{RE}(x,q) \\ M_{ER}(x,q) & M_{EE}(x,q) \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} T_{C,R} \\ T_{C,E} \end{bmatrix} + \begin{bmatrix} T_{ext,R} \\ T_{ext,E} \end{bmatrix} + \begin{bmatrix} F_R(x,q,\dot{x},u) \\ F_E(x,q,\dot{x},u) \end{bmatrix}, \qquad u = \dot{q} \tag{1}$$

where x is a vector of rigid body generalized coordinates; q is a vector of the elastic generalized coordinates; M_{RR}, M_{RE}, M_{ER}, M_{EE} form the configuration-dependent mass matrix; T_C is a vector of control forces (as in joint-torque actuators in a manipulator); T_{ext} is a vector of other generalized external forces; K_{EE} is a constant stiffness matrix (see equation (2) below) and F is a vector of nonlinear inertial (coriolis and centripetal) forces.

We are often interested in the important class of systems for which the elastic deformations remain small so we can ignore terms of second order in q and u. Strictly speaking this requires that $\|q\|$ and $\|u\|$ be infinitesimally small. However we know that if, for example, our flexible body is modelled as a beam, it is sufficient that the elastic deformations do not exceed one tenth of the length of the beam in order to use linear Bernoulli-Euler beam theory. At any rate, given this assumption of small elastic deflections, we could expand the previous equation in order to show more explicitly the form of the nonlinear terms:

$$\begin{bmatrix} M_{RR}(x,q) & M_{RE}(x,q) \\ M_{ER}(x,q) & M_{EE}(x,q) \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} T_{C,R} \\ T_{C,E} \end{bmatrix} + \begin{bmatrix} T_{ext,R} \\ T_{ext,E} \end{bmatrix}$$
$$- \begin{bmatrix} 0 & 0 \\ 0 & K_{EE} \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \sum_{i=1}^{n} f_{1ii}(x)\dot{x}_i^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} f_{1ij}(x)\dot{x}_i\dot{x}_j$$

$$+\frac{1}{2}\sum_{i=1}^{n} f_{2ii}(x)q\dot{x}_i^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} f_{2ij}(x)q\dot{x}_i\dot{x}_j$$

$$+\sum_{i=1}^{n} f_{3i}(x)\dot{q}\dot{x}_i-\begin{bmatrix} M^*_{RR}(x,q) \\ M^*_{ER}(x,q)\end{bmatrix}\ddot{x},$$

$$\left(M^*_{RR}(x,q)\right)_{ij}=m^T_{1ij}(x)q,\quad \left(M^*_{ER}(x,q)\right)_{ij}=m^T_{2ij}(x)q \tag{2}$$

where n above is the number of rigid body coordinates; f_{1ij} is a column matrix, but f_{2ij} and f_{3i} are n+m by m matrices, where m is the number of elastic coordinates.

In light of the discussion in the previous section, we could further write:

$$\begin{aligned} f_{2ij}(x) &= f^1_{2ij}(x)+f^2_{2ij}(x)\\ m_{1ij}(x) &= m^1_{1ij}(x)+m^2_{1ij}(x)\\ m_{2ij}(x) &= m^1_{2ij}(x)+m^2_{2ij}(x)\end{aligned} \tag{3}$$

where the superscript ^{2} terms can only be obtained through the use of displacement and velocity expressions, in the development of the equations of motion, that are accurate to second order in the elastic generalized coordinates and speeds (q and u). This requires the use of nonlinear strain-displacement relations [11,12], nonlinear kinematic constraints [6,4], or the use of a nonlinear "geometric stiffening" term appended to the incorrectly linearized equations of motion [5,13].

The complexity of equations (2) and the difficulty involved in obtaining the nonlinear terms have prompted attempts at simplification. It is common [10] for example to assume small velocities and drop all terms nonlinear in rates. This results in rate-linear motion equations that greatly simplify the dynamicist's task. It has been pointed out [14], however, that in the case of n-link rigid manipulators in any configuration, the velocity and acceleration terms of the dynamic equations have the same relative significance at any speed of movement. The fact that the omission of these terms does not significantly affect simulation results is attributed to the fact that gravity and joint friction usually overpower inertial terms. These results have not been extended to chains of flexible bodies. One might argue that in some limit (i.e., vanishingly small q) the equations of motion of the flexible multibody system should reduce to those of the rigid multibody system. Then it seems that a good case could be made for the inclusion of at least nonlinear terms in the rigid body rates in our rate-linear equations (i.e., $f_{1ij}(x)$), particularly considering the fact that future, fast, space manipulators with low joint frictions are part of the class of systems under consideration.

Faced with this, we can proceed in two ways with respect to the equations of motion: we can be consistent, or we can be selective. To be consistent requires keeping all terms of order $\|q\|$ and $\|u\|$ in equation (2), i.e., no simplification. We also encounter the problem that the superscript ^{2} terms in equations (3) are not readily available in the general case since they depend on nonlinear elastic theory or nonlinear kinematics of deformation for their derivation. We could just make do with the superscript ^{1} terms in equations (3) (standard approach) but this would not be consistent nor justifiable since there is no *a priori* reason to guarantee $\|f_{2ij}{}^1\| >> \|f_{2ij}{}^2\|$, for example.

We are forced then to be selective, at least in the general case. Now we have to rely mostly on experience and simulation to determine which terms are important and which negligible under given conditions. Using the simple example of a beam radially cantilevered to a spinning hub, an empirical speed limit has been proposed beyond which the standard linear finite element or modal formulations of the model give erroneous results [5]. This limit is specified as follows: the magnitude of the spinning rate of the system has to be one order of magnitude less than the fundamental bending frequency of the beam. We use this simulation result to claim the following:

unless the equations of motion exact to first order in elastic generalized coordinates and speeds are available, we are limited to rigid body angular rates less than an order of magnitude lower than the lowest fundamental bending frequency of our system. In view of this, we might be able to model our system accurately enough by just keeping the rate-linear equations together with terms f_{1ij}. In other words, we might as well drop all nonlinear terms involving elastic coordinates and speeds. We further claim that speed or acceleration limits also exist in translational rates or accelerations, arising from those mass matrix terms that depend on q and cannot be obtained through the standard approaches. In the next section we investigate these claims analytically and through simulation.

In what follows we shall refer to three types of models for a flexible multibody system under study. The "consistent" model will be that which retains all terms to first order in q and u, that is, the consistently linearized model. The "inconsistent" model shall be that obtained through linear kinematics of deformation, that is, one whose equations omit the superscript ^{2} terms mentioned above. Finally, a "ruthlessly linearized," or simply "ruthless" model, shall be one in which the nonlinear terms which include elastic coordinates and speeds are ignored, including those terms in the mass matrix which depend on elastic coordinates. In other words, in our "ruthless" model equations we ignore terms f_{2ij} and f_{3i}, and we assume the mass matrix depends on rigid body configuration only.

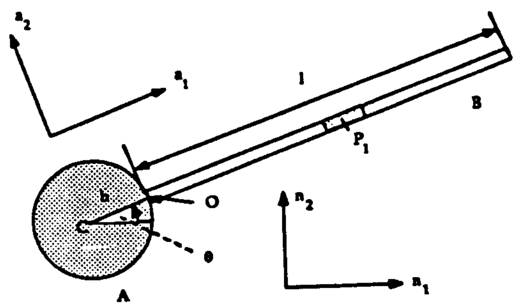

Fig. 1: Single Beam Attached to a Moving Base

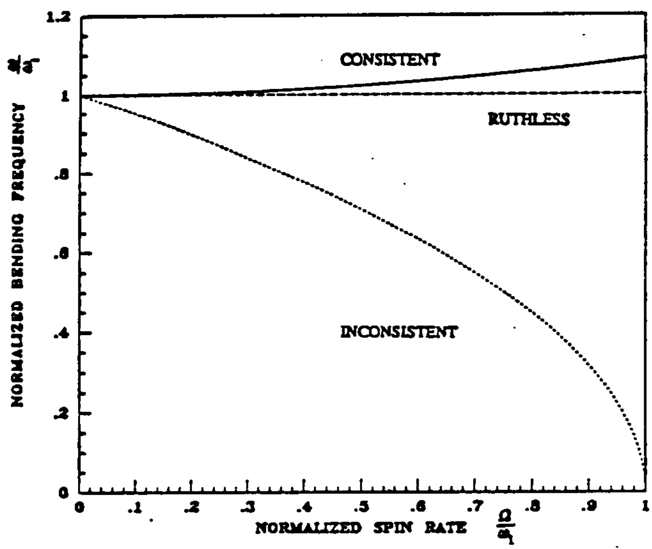

Fig. 2: Fundamental Bending Frequency of Spinning Beam vs Spin Rate

4 Two Case Studies

4.1 One flexible body example

Consider a simple, slender, uniform beam cantilevered to a rigid body free to move in the plane (see Fig. 1). The frame N, defined by the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, is inertial, and we introduce the rotating frame A defined by the unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, attached to body A and whose $\mathbf{a}_1$ axis lies along the undeformed neutral axis of the beam B initially. The consistently linearized equations of motion for this system have been derived using Kane's dynamical equations together with nonlinear strain-displacement relations. Shear and rotary inertia effects have been ignored (i.e., slender beam assumption). The equations of motion, exact to first-order in generalized elastic coordinates and speeds are [15]:

$$
\begin{bmatrix}
m_A + m_B & 0 & -\sum_{i=1}^{n} E_i q_i & \cdots & 0 & \cdots \\
0 & m_A + m_B & bm_B + e & \cdots & E_i & \cdots \\
-\sum_{i=1}^{n} E_i q_i & bm_B + e & b^2 m_B + 2eb + I_B + {}'I_A & \cdots & bE_i + F_i & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \iddots \\
-\sum_{i=1}^{n} \mu_{ij} q_i & E_j & bE_j + F_j & \cdots & G_{ij} & \cdots \\
\vdots & \vdots & \vdots & \iddots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
\dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \vdots \\ \ddot{q}_i \\ \vdots
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
(m_A + m_B) v_2 v_3 + v_3^2 (bm_B + e) + 2v_3 \sum_{i=1}^{n} E_i \dot{q}_i \\
-(m_A + m_B) v_3 v_1 + v_3^2 \sum_{i=1}^{n} E_i q_i \\
-v_2 v_3 \sum_{i=1}^{n} E_i q_i - v_3 v_1 (bm_B + e) \\
\vdots \\
- v_3 v_1 E_j + v_3^2 \sum_{i=1}^{n} G_{ij} q_i - v_3^2 \sum_{i=1}^{n} (b\mu_{ij} + \eta_{ij}) q_i - v_2 v_3 \sum_{i=1}^{n} \mu_{ij} q_i \\
\vdots
\end{bmatrix}
+
$$

$$
\begin{bmatrix}
\cdots & 0 & \cdots \\
\cdots & 0 & \cdots \\
\cdots & 0 & \cdots \\
\ddots & \vdots & \iddots \\
\cdots & -H_{ij} & \cdots \\
\iddots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
\vdots \\ q_i \\ \vdots
\end{bmatrix}
\qquad (1)
$$

where following Kane et al. [16] we have defined:

$$
m_B \equiv \int_0^L \rho dx, \quad e \equiv \int_0^L x\rho dx, \quad I_B \equiv \int_0^L x^2 \rho dx
$$

$$
H_{ij} \equiv \int_0^L EI\phi''_{2i}(x)\phi''_{2j}(x)dx, \quad E_i \equiv \int_0^L \phi_{2i}(x)\rho dx
$$

$$
F_i \equiv \int_0^L x\phi_{2i}(x)\rho dx, \quad G_{ij} \equiv \int_0^L \phi_{2i}(x)\phi_{2j}(x)\rho dx
$$

$$
(i, j = 1, \ldots, n)
$$

and we have further defined:

$$\mu_{ij} \equiv \int_0^L \rho \int_0^x \phi'_{2i}(\sigma)\phi'_{2j}(\sigma) d\sigma dx,$$

$$\eta_{ij} \equiv \int_0^L x\rho \int_0^x \phi'_{2i}(\sigma)\phi'_{2j}(\sigma) d\sigma dx$$

In deriving the above equations, we have assumed no external forces act on the beam for simplicity. m_A is the mass of the rigid body A; v_1 and v_2 are translational speeds of body A in the directions of $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively; v_3 is the rotational speed of body A; q_i are the n generalized elastic coordinates, where we have discretized the transverse elastic deformation of the beam using assumed modes.

Specializing equation (1) to the prescribed motion of uniform rotation of the base,

$$v_1 = v_2 = 0, \quad v_3 = \Omega = \text{constant}$$

we get:

$$\sum_{i=1}^{n} G_{ij}\ddot{q}_i + \sum_{i=1}^{n}\left[H_{ij} + \Omega^2(b\mu_{ij} + \eta_{ij} - G_{ij})\right]q_i = 0 \qquad (j = 1, \ldots, n) \tag{2}$$

Note that the terms μ_{ij} and η_{ij} cannot be obtained using linear kinematics of deformation but are obtained through the use of nonlinear strain-displacement relations. The term

$$\Omega^2(b\mu_{ij} + \eta_{ij})$$

is known as the geometric stiffness matrix for this specialized rigid body motion (rotation). It is easy to observe from just this analytical study that in the absence of these terms our equations lose stiffness with increasing angular rates Ω, since:

$$(K)_{ij} = H_{ij} - \Omega^2 G_{ij} = \int_0^L EI\phi''_{2i}(x)\phi''_{2j}(x)dx - \Omega^2\int_0^L \rho\phi_{2i}(x)\phi_{2j}(x)dx \tag{3}$$

We note that for a variety of mode shapes [17]:

$$\int_0^L \phi_{2i}(x)\phi_{2j}(x)dx = \int_0^L \phi''_{2i}(x)\phi''_{2j}(x)\left(\frac{L}{\lambda_i}\right)^2\left(\frac{L}{\lambda_j}\right)^2 dx = \begin{cases} L, & i = j \\ 0, & i \neq j \end{cases}$$

$$\lambda_i^2 = \omega_i L^2 \sqrt{\frac{\rho}{EI}} \tag{4}$$

so

$$(K)_{ij} = \begin{cases} \omega_i^2 \rho L - \Omega^2 \rho L, & i = j \\ 0, & i \neq j \end{cases}$$

and it is clear that as long as $\Omega << \omega_i$ the incorrect de-stiffening effect will not be apparent. This clearly evidences the angular speed limit mentioned at the end of the previous section. As shall be seen, this rotational rigid body rate limit is the most restrictive on the validity of other than the correctly linearized equations for the maneuvers considered. Figure 2 shows the first bending eigenfrequency as a function of Ω, as predicted by the three modelling approaches, each using the same assumed modes.

If we now specialize equation (1) to the prescribed motion of constant translational acceleration:

$$du_1/dt = g = \text{constant}, \quad v_2 = v_3 = 0$$

we obtain:

$$\sum_{i=1}^{n} G_{ij}\ddot{q}_i + \sum_{i=1}^{n} \left(H_{ij} - g\mu_{ij}\right) q_i = 0 \qquad (j = 1, \ldots, n) \tag{5}$$

μ_{ij} is in this case the operative geometric stiffness matrix. Now we can see that for g large enough, we again obtain de-stiffening. Another way of looking at it is to realize that for g large enough the stiffness matrix becomes non-positive definite which implies that the beam buckles due to its own weight. The predicted buckling is correct, and would have been lost with the inconsistently or the ruthlessly linearized approaches. This suggests that a translational rate or acceleration limit also exists beyond which our model is again grossly incorrect if we do not use equations exact to first order. Inspection of equation (1) suggests that another rate limit exists, this one on the product of rigid body rotational and translational rates, $v_2 v_3$. Banerjee's recent work [13] suggests that as many as 12 such independent limits on rigid body motion exist and must be considered for general three-dimensional motion. It is doubtful that for typical stop-to stop slew and repositioning maneuvers these limits become operative, since the magnitude of the applied forces is limited by the requirement that the flexible body not deform excessively. This anticipation motivates the case study of the next section.

4.2 Two-link flexible arm example

Consider a two-link flexible, revolute arm composed of three rigid bodies connected by two slender uniform beams (see Fig. 3). The equations of motion for this system were derived using Kane's dynamical equations together with nonlinear strain-displacement relations. The equations are thus exact to first order in the beams' elastic generalized coordinates and speeds. We ignore independent axial extensions of the beams (i.e., the axial strain at the neutral axis is assumed zero for each beam). The elbow joint is actually modelled as two bodies: one attached to link 1 in a cantilevered way and the other, free to rotate with respect to the first, with link 2 attached to it in a cantilevered way also. Both elbow bodies are rigid and share the hinge point but their mass centers are allowed to be offset from the hinge point. As in the previous examples, the continuous transverse displacements of the beams are discretized using an assumed modes approach. One percent modal damping is added to the model to represent structural damping. Actuation is assumed in the form of shoulder and elbow torques. The equations of motion for this system, which are quite extensive, are available in [12].

An examination of the equations of motion for this system makes it clear that the complexity of the mass matrix and nonlinear inertial terms could be diminished immensely by dropping most terms involving the generalized elastic coordinates and speeds (i.e., the "ruthless" case). It would be advantageous to know, then, if these terms have a significant effect on the dynamics of a chain of elastic bodies if we are limited to the low rotational speeds and translational accelerations encountered during slew maneuvers with joint torques limited by the requirement that flexible deflections remain small. In order to investigate this, we propose to examine the three models mentioned at the end of section three, to note, consistent, inconsistent, and ruthless, as applied to the two link arm. The complexity of the motion equations precludes an analytical study akin to the one presented in the previous section. We will therefore rely entirely on numerical simulation and compare the performance of the three "models" of the arm when it is subjected to a smooth slew maneuver.

5 NUMERICAL SIMULATION

The motion equations for the two-link, flexible, planar manipulator, consistently linearized in small elastic deflections and speeds, were programmed in FORTRAN and implemented in a VAXstation 2000. The code allows for a maximum of four cantilevered modes per beam. The mass matrix is inverted using LU decomposition and a Runge-Kutta fourth-order scheme with adaptive time step is utilized for the time integration [18]. Energy and angular momentum checks are built into the simulation. Table 1 shows the physical properties assumed for the arm (see also Fig. 3). These numbers were chosen to mimic an actual experimental testbed built at Martin Marietta by Dr. Eric Schmitz [19].

Physical Properties of Planar Manipulator with Two Flexible Links			
Mass of shoulder body (kg)	20.0	Length of link 1 (m)	0.9144
Mass density of link 1 (kg/m)	1.33937	Length of link 2 (m)	0.9144
Mass of elbow body (kg)	14.0		
Mass density of link 2 (kg/m)	0.669685	Other lengths (see Fig. 3):	
Mass of tip body (kg)	2.0	b_1 (m)	0.0762
		b_{21} (m)	0.0762
Moments of Inertia (about axis perpendicular to plane):		b_{22} (m)	0.0127
Shoulder body (kgm^2)	0.01	b_t (m)	0.0508
Elbow body (kgm^2)	0.03		
Tip body (kgm^2)	0.01		

Table 1: Physical Properties of the Two-link Manipulator

All the trajectories presented below were run in open loop after the torques had been computed from the inverse dynamics problem (given the desired angular trajectories) assuming rigid links for the manipulator. For all simulation runs, only two assumed modes per link were used, since this gave adequate results and was computationally much cheaper than running the full four modes per link. With two modes per link, the first two system vibration frequencies were obtained to within three percent of the value obtained using four modes per link.

In the following, reference is made to time-scaled trajectories. Time-scaling of nominal trajectories [20] is achieved by replacing time as the independent variable by the new variable

$$r = \alpha t, \qquad \alpha > 0$$

where α is a constant. When α is greater than one, the trajectory is sped up, while if α is less than one it is slowed down. From this it is apparent that rates scale like α, while accelerations, and thus torques, scale like the square of α. These relations are used in the following sections to select a scaling factor that yields a desired maximum value of angular rate, or a given maximum value of torque to obtain desired maximum link tip deflections.

5.1 A Smooth Slew Maneuver

In section 4.1 it was predicted that the more severe limit on the validity of other than consistently linearized equations would be the limit on rigid body angular rates. In the case of chains of flexible bodies, as exemplified by the two-link manipulator in this simulation, the nonlinearity arising from dependence on configuration makes it difficult to select a "characteristic" angular rate in the general case. This suggests a case by case approach. Several different slews were compared in reference [12]. We report here only one, perhaps the most interesting.

The smooth trajectory is obtained by assuming a form of the joint angular time histories of the equivalent rigid manipulator that is quintic in time. This allows the specification of angle, angular rate and angular acceleration at the initial and final times of the trajectories [21]. This trajectory is a deployment maneuver. Both links undergo significant rotational motion, and the outboard link translates. Fig. 4 shows the computed torques for the nominal trajectory.

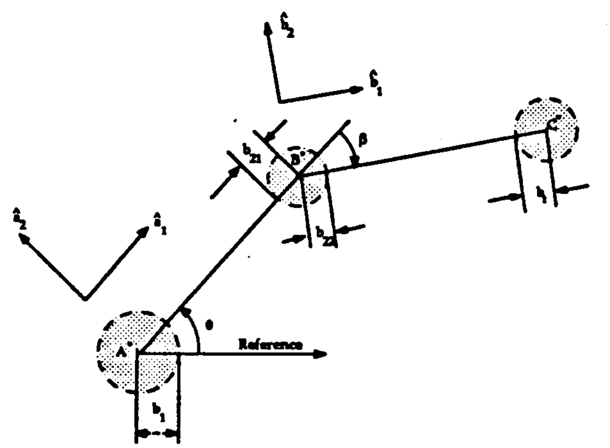

Fig. 3: Schematic of two-link Planar Manipulator

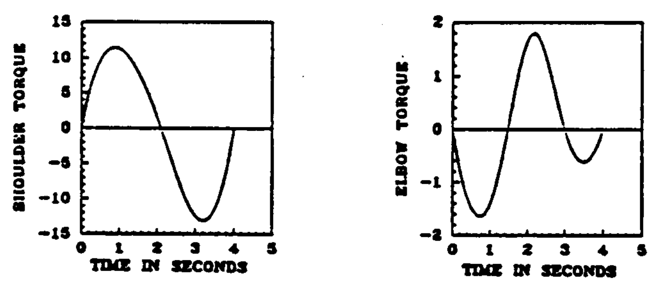

Fig. 4: Computed Torques for Second Smooth Trajectory (Nm)

Figure 5 shows the nominal trajectory. In this figure, plots for both the ruthless and the consistent models have been overlaid. The inconsistent model fails for this case. The failure is a numerical divergence during time integration. This is perhaps related to the fact that the angular rates for the nominal maneuver are as large as thirty percent of the "fundamental vibration frequency." The system frequencies for the manipulator with locked joints are seen to fluctuate from 3 rad/sec to 4 rad/sec for corresponding elbow relative angles of zero to 135 degrees [12]. For this reason, in the case at hand reference is made to "one" fundamental frequency, and it is assumed that it lies in the range specified above and is about 3.5 rad/sec. For the two models shown, the agreement is again excellent for the shoulder angle and tip deflections, with the elbow angular position being off by only a maximum of ten percent relative error.

Convergence of all three models is achieved if the nominal maneuver is slowed down (α=0.1685). Figure 6 shows that for this case, all three models yield identical results. This confirms the predictions in section 4.1, and further suggests that within the limit of validity of the inconsistent model, the ruthless is as good as the inconsistent, and actually better since it is much easier to obtain. Note that the angular rates are well within ten percent of the fundamental.

The last case considered consists of the nominal trajectory scaled upwards in time (α=1.2). As in the previous section, it was desired to reach the "hard" simulation limit of link tip deflections of about ten percent of the link lengths. As Figure 7 shows, only the ruthless model did not fail under the given speed-up of the trajectory, even though tip deflections should only be about four percent of link lengths. This divergence is traceable to a near singularity of the mass matrix, related to the fact that the links are modeled with distributed mass, while they are in fact nearly "massless springs." The manipulator mass distribution is dominated by the elbow and the tip mass. Reference [12] reports that the ratio of the largest to smallest mass matrix singular values is on the order of 10^6, and that this range is configuration dependent. It is not known why this numerical ill-conditioning of the

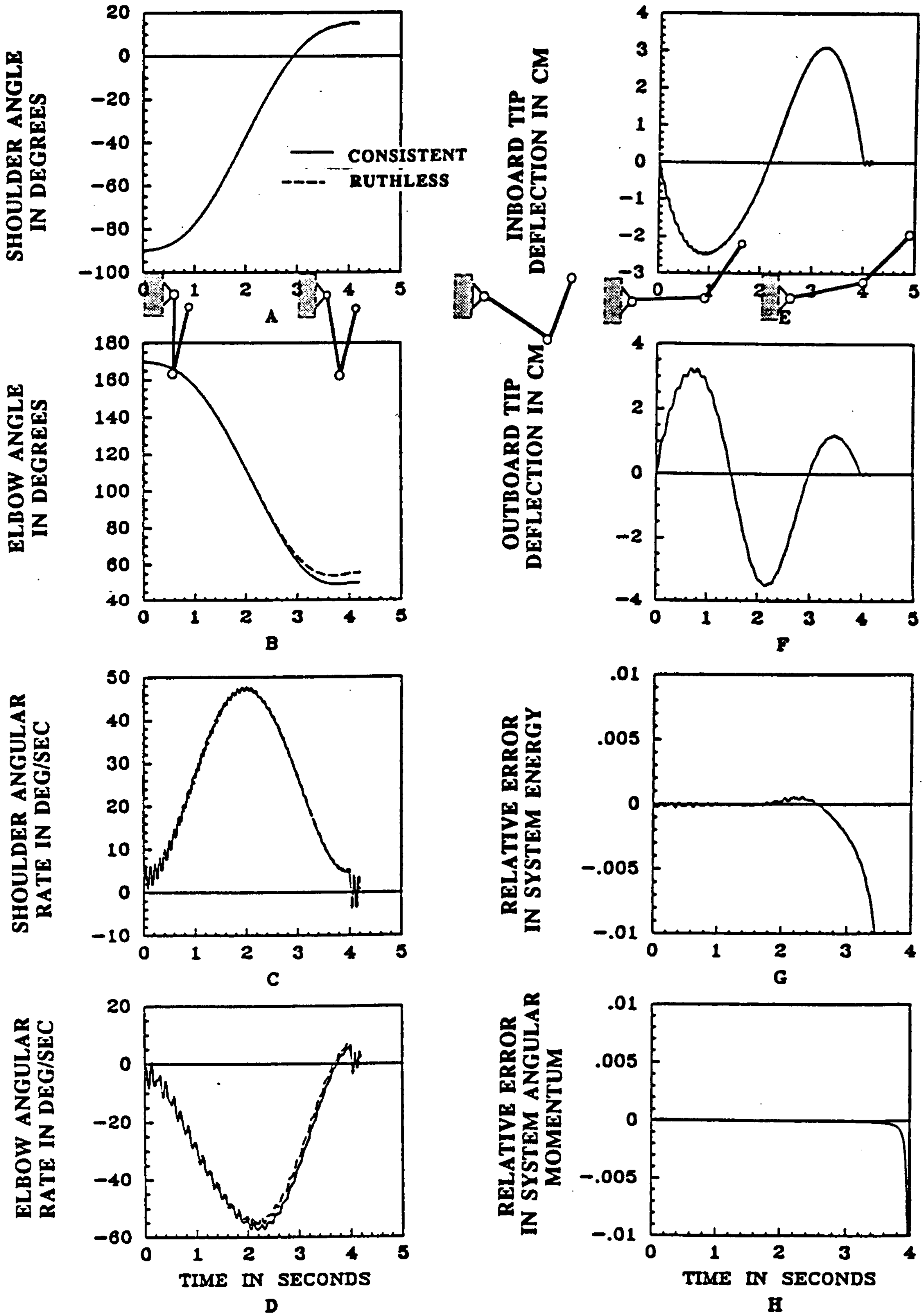

Fig. 5: Smooth Slew Maneuver with $\alpha=1.0$

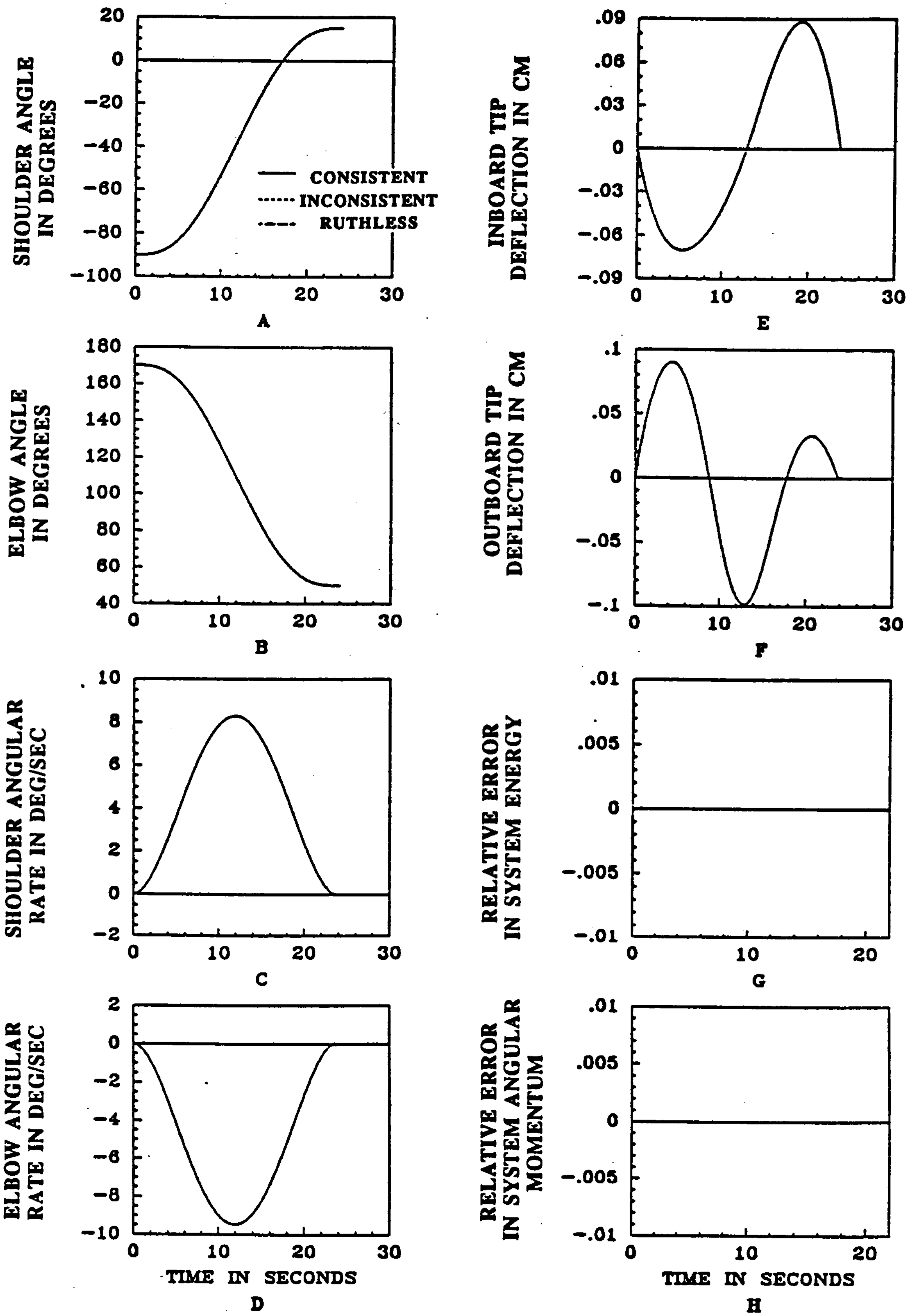

Fig. 6: Smooth Slew Maneuver with $\alpha=0.1685$

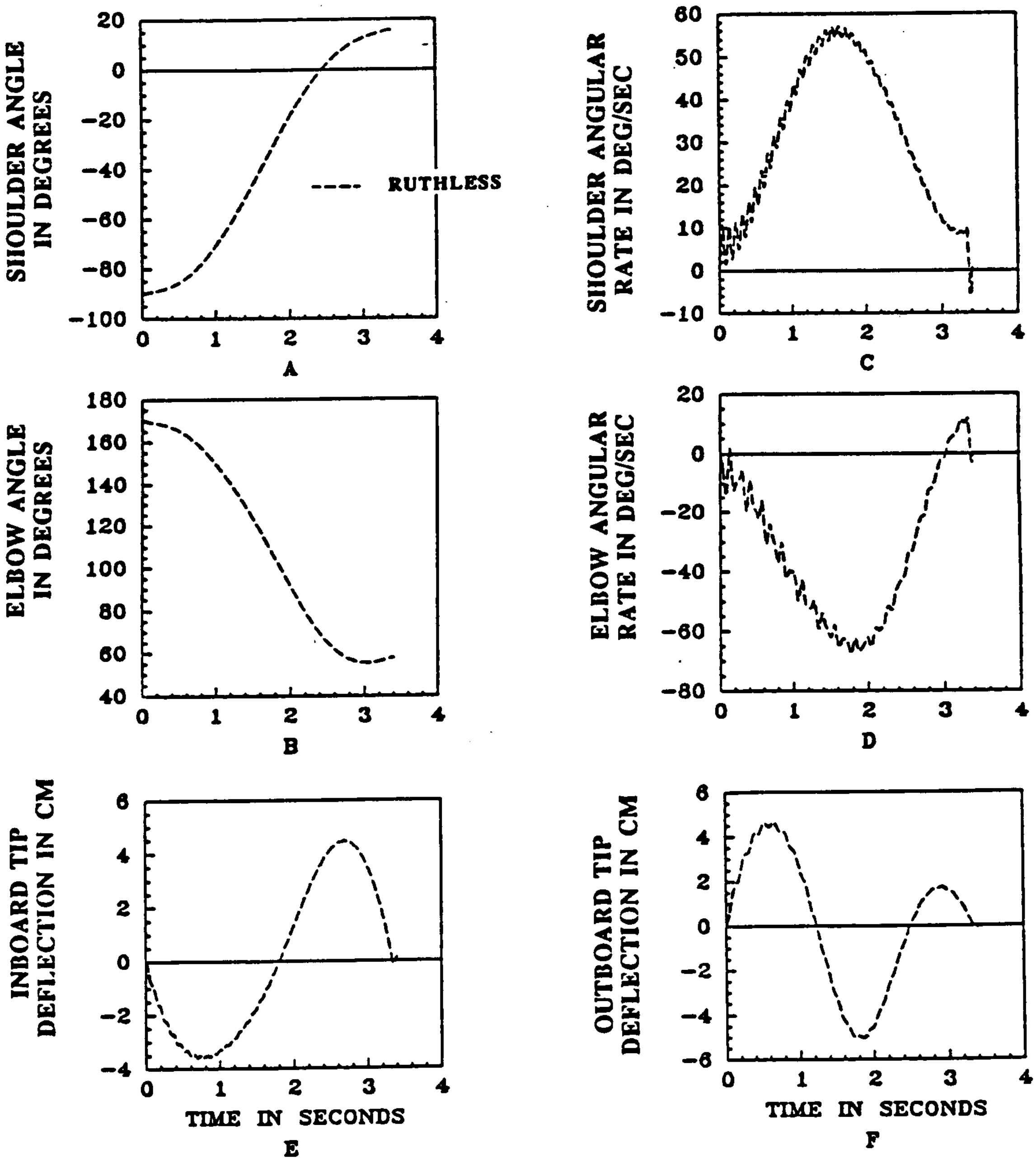

Fig. 7: Smooth Slew Maneuver with $\alpha=1.2$

mass matrix appears to be exacerbated by including dependence upon elastic deformation, as in the consistent model.

In summary, the above results again show a strong correlation between the limit of validity of the inconsistent model and the maximum values of angular rates. In this case, however, no "characteristic" rigid body rate is apparent. The limit at which the inconsistent model fails seems to be even before any of the two angular rates (shoulder or elbow) reach ten percent of the fundamental vibration frequency. A strong point can still be made, nevertheless, in that the ruthless model is as good as the inconsistent whenever the inconsistent is valid, and more conservative since the ruthless model does not fail. Even at high angular rates the ruthless model yields results that are quantitatively very close to the consistent model results.

Finally, it is worth pointing out the excellent agreement in the tip deflections for both links, in all cases. This is probably due to the fact that the equations of motion (see [12]) are elastically decoupled, and, while inertially coupled, the elastic degrees of freedom mass matrix (M_{EE} of section 3) does not depend on elastic nonlinear terms due to linearization. Also, the agreement in shoulder angle and angular rates is remarkable. This indicates that, depending on what state variable is of interest in a given trajectory, the ruthless model will be as good as the more cumbersome consistent model. In all cases, the ruthless is more conservative and better conditioned than the inconsistent model.

5.2 Other Trajectories

The above qualitative results for the smooth slew maneuver have also been confirmed for two other maneuvers: a smooth maneuver in which the elbow motion is kept to a minimum and the two-link manipulator is slewed in extended configuration like a "beam"; and a time-optimal, bang-bang control slew maneuver where joint torques were assumed limited. Details of these results, together with an analysis of numerical considerations, can be found in [12].

6 SUMMARY AND CONCLUSIONS

Having looked into the general form of the linearized dynamics equations for chains of flexible bodies undergoing large rigid body motions, but small elastic deflections, we concluded that some terms cannot be obtained through the use of linear strain-displacement relations. These terms were seen to be critical in the simple rotating beam example as they provide the geometric stiffness terms necessary to obtain physical results. The absence of these terms in inconsistently linearized equations limits their validity to relatively gentle rigid body motions. The fact that these terms are unobtainable for the general case of an arbitrary flexible body led us to consider possible simplifications of the general motion equations, consistent with such restrictions on their applicability.

The two alternative models studied, the ruthlessly linearized model and the inconsistent model, are subject to several limits in applicability. While the consistent model requires we keep elastic coordinates and speeds small, the two alternative models will only be accurate if we further maintain low rigid body angular rates. There also exists some translational acceleration or speed limit that needs to be considered, although for the cases studied this limit was of no consequence. Within the domain of validity of both simplified models, it appears the ruthless model yields results as accurate as the correct consistently linearized model. In addition, preliminary results promise that the ruthless model will result in large reductions in computational time in the simulation of large flexible multibody systems. This coupled to the simplification of the dynamicist's task inherent in the adoption of ruthlessly linearized models makes this option an attractive alternative.

From the above it is clear that the inconsistent model should never be used. Further, the more cumbersome and hard to obtain consistent model should only be used when necessary (i.e., when the domain of validity of the simplified models is exceeded). Finally, it is our strong belief that the ruthless model deserves widespread use.

Simulation misbehavior at certain trajectories for relatively high rigid body angular rates was tracked down to numerical ill-conditioning of the configuration dependent mass matrix. This problem was attributed to modelling error inherent in choosing cantilever (clamped-free) modes to model the flexible deflections of the manipulator links. In ref. [12] it is suggested that this results in effectively modelling one (or more) massless degrees of freedom. Thus it became apparent that physical modelling of bodies, and adequate selection of assumed modes and numerical procedures, can be as important as sensible simplifications of the motion equations.

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