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A Finite Element Based Method for Solution of Optimal Control Problems

Robert R. Bless [1], Dewey H. Hodges [2], and Anthony J. Calise [3]

School of Aerospace Engineering
Georgia Institute of Technology, Atlanta, GA 30332

Abstract

A temporal finite element based on a mixed form of the Hamiltonian weak principle is presented for optimal control problems. The mixed form of this principle contains both states and costates as primary variables that are expanded in terms of elemental values and simple shape functions. Unlike other variational approaches to optimal control problems, however, time derivatives of the states and costates do not appear in the governing variational equation. Instead, the only quantities whose time derivatives appear therein are virtual states and virtual costates. Also noteworthy among characteristics of the finite element formulation is the fact that in the algebraic equations which contain costates, they appear linearly. Thus, the remaining equations can be solved iteratively without initial guesses for the costates; this reduces the size of the problem by about a factor of two. Numerical results are presented herein for an elementary trajectory optimization problem which show very good agreement with the exact solution along with excellent computational efficiency and self-starting capability. The goal of this work is to evaluate the feasibility of this approach for real-time guidance applications. To this end, a simplified two-stage, four-state model for an advanced launch vehicle application is presented which is suitable for finite element solution.

Introduction

Future space transportation and deployment needs are critically dependent on the development of reliable and economical launch vehicles that will provide flexible, routine access to orbit. A particular requirement now receiving attention is that for an advanced technology heavy-lift vehicle. Future space transportation systems will need to place large payloads – 100,000 to 150,000 pounds – into low Earth orbit at an order of magnitude lower cost per pound. Such systems will also require on-board algorithms that maximize system performance as measured by autonomy, mission flexibility, in-flight adaptability, reliability, accuracy and payload capability. They must be computationally efficient, robust, self-starting, and capable of functioning independently of ground control. Also, the algorithms must be designed with the anticipation that the launch vehicle will undergo evolutionary growth [1].

One approach to optimal guidance consists of repeatedly solving a two-point boundary-value problem that results from the traditional necessary conditions for optimality in an optimal control problem formulation. The vehicle state at discrete instants of time along a trajectory can be viewed as a new starting condition, and the remainder of the trajectory is reoptimized for that condition. The open loop optimal control is applied for a short interval of time, and feedback is introduced by reoptimization at the next time instant. This process presupposes that the two-point boundary-value problem can be reliably solved in a time interval that is small compared to the control update interval. Knowledge of the previous solution helps in providing a good starting point for the optimization process; however, no method has been demonstrated that can operate reliably in a real time environment.

This paper examines a finite element approach to addressing this problem. Hamilton's principle has traditionally been used in *analytical* mechanics as a method of obtaining the governing equations of motion for

[1] Graduate Research Assistant. Student Member, AIAA.

[2] Professor. Associate Fellow, AIAA.

[3] Professor. Member, AIAA.

continuous systems. In the 1970's a form of Hamilton's principle called Hamilton's law of varying action was first used by Bailey (see, for example, [2]) to obtain direct solutions to dynamics problems in the time domain, thus introducing Hamilton's principle into *computational* mechanics. During the last decade, it was shown by Borri *et al.* [3] and by Peters and Izadpanah [4] that these direct methods, when expressed in a weak form, could be competitive with numerical solution of the corresponding ordinary differential equations. Later work by Borri *et al.* [5] has shown that a mixed form of the weak principle has further computational advantages, namely that shape functions can be chosen from a far less restrictive class of functions.

In [6], Hodges and Bless have shown that optimal control problems can be solved in a virtually identical way to that of the mixed form of Hamilton's weak principle. Hence, the method as used in [6] and the present paper has been called the weak Hamiltonian method for optimal control problems. Finite element methods have some advantages over other solution procedures; one advantage is that finite element methods provide the possibility for development of algorithms which converge reliably. The present method, at least for the problems investigated to date, is essentially self-starting. This meets a key operational requirement for on-board algorithms. However, application of the finite element method to optimal control problems is rather new. For example, Patten [7] used a Ritz-Galerkin technique with Lagragean interpolation polynomials. One advantage of the present formulation is the allowance for a simpler choice of shape functions. The computational savings which may stem from this are now under investigation.

In this paper, a weak form governing optimal control problems is derived, and a finite element procedure is outlined for the solution of such problems. Numerical results for the solution of a simple trajectory optimization problem are presented and compared with the exact solution to demonstrate the accuracy and efficiency of the weak Hamiltonian finite element formulation. In anticipation of applying the present method to optimal guidance of a rocket booster, a simplified two-stage model suitable for this problem is presented. In [6] a one-stage model was analyzed and finite element results were compared to a numerical solution obtained using a multiple shooting method [8]. Of particular interest are the self-starting operation and the performance in terms of execution time and accuracy versus the number of elements used to represent the time span of the trajectory.

Weak Principle for Optimal Control

A definite analogy exists between the mixed formulation of Hamilton's weak principle in dynamics and the first variation of the performance index in optimal control theory. Specifically, there is an analogy between the generalized coordinates and generalized momenta in dynamics and the states and co-states in optimal control theory. Only a brief development of the weak Hamiltonian method for optimal control problems is presented herein. More details on the development and the analogy with dynamics problems may be found in [6].

General Development

We start with a performance index taken from Eq. (2.8.4) of Bryson and Ho [9]. Its first variation will be taken in a standard manner, except that states, costates, and controls will have arbitrary variations. Rather than setting its first variation equal to zero, however, it will be set equal to an expression which contains the terms that are necessary to transform all boundary conditions to the natural or "weak" type. The final weak form is then obtained by integration of this equation by parts in such a way that no derivatives of states or costates appear.

It should be noted that the fundamental relationships are not being changed. To make certain of this, we will ensure that the resulting formulation produces the Euler-Lagrange equations and boundary conditions which have already been established in optimal control theory (see, for example, [9], Eqs. 2.8.15 – 2.8.21).

In order to clearly understand what is meant by a "weak" formulation and the derivation of the weak formulation that is to follow, we first study a more simple problem. Let us start with a functional of the form

$$J = \int_A^B F(y, y', x)dx \tag{1}$$

where A and B are fixed numbers. The necessary conditions for an extremal are defined by

$$\delta J = \int_A^B \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right) dx = 0 \tag{2}$$

Introducing $\partial F/\partial y' = f$ for notational convenience and integrating by parts we obtain

$$\int_A^B \left(\frac{\partial F}{\partial y} - f' \right) \delta y \, dx + f\delta y \Big|_A^B = 0 \tag{3}$$

The integrand in the above equation is the familiar Euler-Lagrange equation. The trailing term leads to the boundary conditions. If y is specified at $x = A$ or $x = B$, then $\delta y = 0$ at $x = A$ or $x = B$ respectively. This is referred to as a *strong* boundary condition. If y is not specified at one of the endpoints, then $f = 0$ at that endpoint. This is referred to as a natural or *weak* boundary condition. The key points to remember are that the trailing term itself is zero at each endpoint and that specifying y requires that $\delta y = 0$ at a point.

In our weak formulation, we want all the boundary conditions to be of the weak type. Thus, even if $y(A) = 0$ then $\delta y(A) \neq 0$. To allow for this mathematically, we introduce a new variable $\hat{f}$ which represents the discrete value of f at an endpoint. The variation of J is now set equal to $\hat{f}\delta y|_A^B$ yielding

$$\delta J = \int_A^B \left(\frac{\partial F}{\partial y}\delta y + f\delta y' \right) dx = \hat{f}\delta y \Big|_A^B \tag{4}$$

This is referred to as a weak form. If we integrate by parts, we obtain

$$\int_A^B \left(\frac{\partial F}{\partial y} - f' \right) \delta y \, dx = (\hat{f} - f)\, \delta y \Big|_A^B \tag{5}$$

Note that the Euler-Lagrange equation is the same as before and that the two boundary conditions are that $f = \hat{f}$ at the two endpoints; thus, the trailing terms are still constrained to be zero, but in a weak sense and δy need not ever vanish.

The advantages of the weak formulation are not apparent from the above discussion. However, when we apply this type of formulation to problems in dynamics (see, for example [6]), or optimal control theory [6], and use finite elements, then we have a powerful problem-solving tool.

Consider a system defined by a set of n states x and a set of m controls u. Furthermore, let the system be governed by a set of state equations of the form $\dot{x} = f(x, u, t)$. We may denote elements of the performance index, J, with an integrand $L(x, u, t)$ and a discrete function of the final states and time $\phi[x(t_f), t_f]$. In addition, any terminal constraints placed on the states may be placed in the set of q functions $\psi[x(t_f), t_f]$ and adjoined to the performance index by a set of q discrete Lagrange multipliers ν. Finally, we will adjoin the state equations to the performance index with a set of Lagrange multiplier functions $\lambda(t)$ which are referred to as costates. This yields a performance index of the form

$$J = \int_{t_0}^{t_f} (\lambda^T \dot{x} - L - \lambda^T f)\, dt - \phi\Big|_{t_f} - \nu^T \psi\Big|_{t_f} \tag{6}$$

Taking the first variation of J and setting it equal to an expression chosen so that all boundary conditions are of the weak type, one obtains

$$
\begin{aligned}
\delta J = \int_{t_0}^{t_f} & \left[\delta\lambda^T(\dot{x} - f) + \delta\dot{x}^T\lambda - \delta L - \delta f^T\lambda\right] dt \\
& + \delta t_f \left(\lambda^T \dot{x} - L - \lambda^T f - \frac{\partial\phi}{\partial t} - \nu^T \frac{\partial\psi}{\partial t}\right)\Big|_{t_f} \\
& - \delta x_f^T \hat{\lambda}_f - \delta\nu^T \psi\Big|_{t_f} \\
& = \delta t_f \left(\lambda^T \dot{x}\right)\Big|_{t_f} - \delta x_0^T \hat{\lambda}_0 - \delta\lambda^T \left(\hat{x} - x\right)\Big|_{t_0}^{t_f}
\end{aligned}
\tag{7}
$$

where

$$
\hat{\lambda}_f \equiv \left[\left(\frac{\partial\phi}{\partial x}\right)^T + \left(\frac{\partial\psi}{\partial x}\right)^T \nu\right]\Bigg|_{t_f}
\tag{8}
$$

The right hand side of Eq. (7) contains terms necessary to form all of the proper boundary conditions as natural ones. The quantities $\hat{x}$ and $\hat{\lambda}$ are discrete values of the states and co-states at the initial (with subscript 0) and final times (with subscript f). Depending on the problem, these values will either be specified or left as unknowns.

From Eq. (7), we can directly write down a weak formulation. Before this is done, however, let us examine this expression to ensure that it produces the correct Euler-Lagrange equations and boundary conditions. Integrating the $\delta\dot{x}^T$ term in Eq. (7) by parts and expanding the variation of L, one obtains

$$
\begin{aligned}
\int_{t_0}^{t_f} & \Bigg\{\delta\lambda^T(\dot{x} - f) - \delta u^T \left[\left(\frac{\partial L}{\partial u}\right)^T + \left(\frac{\partial f}{\partial u}\right)^T \lambda\right] \\
& - \delta x^T \left[\left(\frac{\partial L}{\partial x}\right)^T + \left(\frac{\partial f}{\partial x}\right)^T \lambda + \dot{\lambda}\right]\Bigg\} dt \\
& - \delta\nu^T \psi\Big|_{t_f} - \delta t_f \left(L + \lambda^T f + \frac{\partial\phi}{\partial t} + \nu^T \frac{\partial\psi}{\partial t}\right)\Big|_{t_f} \\
& + \delta x_f^T \left(\lambda_f - \hat{\lambda}_f\right) - \delta x_0^T \left(\lambda_0 - \hat{\lambda}_0\right) \\
& + \delta\lambda_f^T \left(\hat{x}_f - x_f\right) - \delta\lambda_0^T \left(\hat{x}_0 - x_0\right) = 0
\end{aligned}
\tag{9}
$$

where x_0, λ_0, x_f, and λ_f represent the values of those functions at the initial and final times, respectively. The coefficients of $\delta\lambda^T$, δx^T, and δu^T in the integrand are the three correct Euler-Lagrange equations, Eqs. 2.8.15 – 2.8.17 from [9]. There are also six trailing terms in Eq. (9) from which the boundary conditions can be determined. The equations corresponding to the first four and the sixth of these terms correspond to the correct boundary conditions in [9]. Namely, the requirement for the coefficient of $\delta\nu^T$ to vanish yields Eq. (2.8.21). The requirement for the coefficient of δt_f to vanish is equivalent to Eq. (2.8.20). The requirement for the coefficient of δx_f^T to vanish shows that the final value of λ equals $\hat{\lambda}_f$ as given in Eq.(8), which corresponds to Eq. (2.8.19). If $\hat{\lambda}_0$ is chosen as zero, the requirement for the coefficient of δx_0^T to vanish requires the initial value of λ to equal zero; on the other hand, the requirement for the coefficient of $\delta\lambda_0^T$ to vanish requires the initial value of x to equal $\hat{x}_0$, in accordance with Eq. (2.8.18). Finally, the requirement for the coefficient of $\delta\lambda_f^T$ to vanish demands that the final value of x equal the discrete value $\hat{x}_f$; this has no counterpart in [9] since the elements of $\hat{x}_f$ are usually unknown.

Having satisfied our requirement that none of the fundamental equations are altered, we may now derive our weak formulation from Eq. (7). In order to allow for the simplest possible shape functions when we introduce a finite element discretization, we do not want time derivatives of x and λ to appear in the weak formulation. Therefore, we integrate the $\dot{x}$ term by parts in Eq. (7) yielding

$$
\begin{aligned}
&\int_{t_0}^{t_f} \left\{ \delta \dot{x}^T \lambda - \delta x^T \left[\left(\frac{\partial L}{\partial x} \right)^T + \left(\frac{\partial f}{\partial x} \right)^T \lambda \right] \right. \\
&\left. - \delta \dot{\lambda}^T x - \delta \lambda^T f - \delta u^T \left[\left(\frac{\partial L}{\partial u} \right)^T + \left(\frac{\partial f}{\partial u} \right)^T \lambda \right] \right\} dt \\
&- \delta t_f \left(L + \lambda^T f + \frac{\partial \phi}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} \right) \Big|_{t_f} - \delta \nu^T \psi \big|_{t_f} \\
&- \delta x_f^T \hat{\lambda}_f + \delta x_0^T \hat{\lambda}_0 + \delta \lambda_f^T \hat{x}_f - \delta \lambda_0^T \hat{x}_0 = 0
\end{aligned}
\tag{10}
$$

This is the governing equation for the weak Hamiltonian method for optimal control problems. It will serve as the basis for the finite element discretization described below. It should be noted that normally one will encounter various types of inequality constraints in problems that deal with optimal control. Inequality constraints will be the subject of future research.

Finite Element Solution

Note in Eq. (10) that time derivatives of δx and $\delta \lambda$ are present. However, no time derivatives of x, λ, u or δu exist. Therefore, it is possible to implement linear shape functions for δx and $\delta \lambda$ within elements and constant shape functions for x, λ, u, and δu within elements.

For simplicity, let us break up the time interval into N segments of equal length $\Delta t = \frac{t_f - t_0}{N}$. Let the values of time be given by t_i for $i = 1, 2, \ldots, N+1$ at the points where the time interval is broken, the so-called nodes. Here $t_0 = t_1$ and $t_f = t_{N+1}$. Then, introduce a nondimensional elemental time τ such that

$$
\tau = \frac{t - t_i}{t_{i+1} - t_i} = \frac{t - t_i}{\Delta t} \qquad (0 \le \tau \le 1) \tag{11}
$$

Now, in accordance with the above guidelines, and letting $i = 1, 2, \ldots, N$, we can choose simple linear shape functions

$$
\begin{aligned}
\delta x &= \delta x_{(i)}(1 - \tau) + \delta x_{(i+1)} \tau \\
\delta \lambda &= \delta \lambda_{(i)}(1 - \tau) + \delta \lambda_{(i+1)} \tau
\end{aligned}
\tag{12}
$$

where the arbitrary, discrete virtual states and virtual costates are defined at every node point. (The nodal indices are enclosed in parentheses to avoid confusion with the state column matrix index.) For the states and costates, we can choose even simpler shape functions

$$
x = \begin{cases} \hat{x}_{(i)} & \text{if } \tau = 0; \\ \bar{x}_{(i)} & \text{if } 0 < \tau < 1; \\ \hat{x}_{(i+1)} & \text{if } \tau = 1 \end{cases} \tag{13}
$$

and

$$
\lambda = \begin{cases} \hat{\lambda}_{(i)} & \text{if } \tau = 0; \\ \bar{\lambda}_{(i)} & \text{if } 0 < \tau < 1; \\ \hat{\lambda}_{(i+1)} & \text{if } \tau = 1 \end{cases} \tag{14}
$$

where in both cases, $i = 1, 2, \ldots, N$. For u and δu, since their time derivatives do not appear in the formulation, we let

$$
\begin{aligned}
u &= \bar{u}_{(i)} \\
\delta u &= \delta \bar{u}_{(i)}
\end{aligned} \tag{15}
$$

where $i = 1, 2, \ldots, N$.

Substitution of Eqs. (12) – (15) into Eq. (10) along with Eq. (11) and

$$
\begin{aligned}
t &= t_i + \tau \Delta t \\
dt &= \Delta t d\tau
\end{aligned} \tag{16}
$$

yields a set of nonlinear algebraic equations which can be assembled in accordance with standard finite element practice. For the sake of brevity, these rather lengthy equations for the general case are not written out explicitly here. They are, however, written out in [6] wherein it is noted that there are $2n(N+1)$ equations for the states and costates, mN equations for the controls, q equations for the end-point constraints, and 1 equation for the unknown time equation. In the assembly process, $\hat{x}_{(i)}$ and $\hat{\lambda}_{(i)}$ drop out of the equations for $i = 2, \ldots, N$ (*i.e.*, for all but the ends of the time interval $t_0 = t_1$ and $t_f = t_{N+1}$). Thus, the total number of unknowns is $2n(N+2)$ for the states and costates, mN for the controls, q for the ν's, and 1 for the unknown time t_{N+1}. Thus, there are $2n$ more unknowns than there are equations, which allows for one to choose, say, $\hat{x}_0$ in accordance with physical constraints and $\hat{\lambda}_f$ in accordance with Eq. (8) and solve for the rest. The resulting equations may be explicitly perturbed to obtain the Jacobian and solved iteratively by a Newton-Raphson method or by any method that is suitable for nonlinear algebraic equations with very sparse Jacobians.

It is also pointed out in [6] that $n(N+1)$ of these algebraic equations contain the $n(N+1)$ unknown costates and that these equations are linear in the costates. Thus, the costates can be solved for symbolically in terms of the other unknowns, and the remaining equations can be solved *circumventing the need for initial estimates of the costates.* This decreases the size of the problem by approximately a factor of two.

Example: A Free-Final-Time Problem

In this section a relatively simple optimal control problem is solved by means of the Hamiltonian weak formulation, and the results are compared with the exact solution. Of particular interest is the computational effort for varying numbers of elements, the ability of the method to converge for various values of the system parameters without needing new initial estimates of the unknowns, and, most important, the accuracy of the method versus the number of elements.

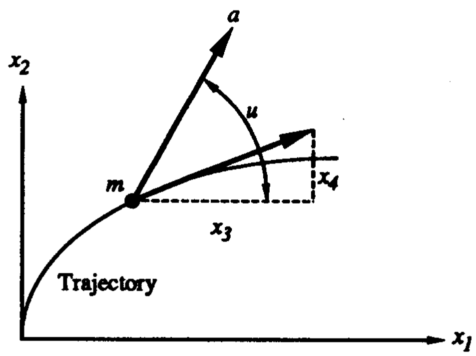

Fig. 1: Nomenclature for planar motion with thrust acceleration = a

Problem Statement

The problem is taken from [9], article 2.7, problem 9. A particle of mass m is acted upon by a thrust force of magnitude ma. Assuming planar motion and making use of an inertial coordinate system x_1 and x_2 as shown in Fig. 1, we may write the dynamical equations as

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{Bmatrix} 0 \\ 0 \\ a\cos u \\ a\sin u \end{Bmatrix} \tag{17}$$

where x_1 is the horizontal component of position, x_2 is the vertical component of position, x_3 is the horizontal component of velocity, and x_4 is the vertical component of velocity. The control is the thrust angle u, and $x(0) = \hat{x}_0 = \lfloor 0 \quad 0 \quad 0 \quad 0 \rfloor^T$. We want to obtain given values h of the final vertical component of position and U of the horizontal component of velocity. The final value of the vertical component of the velocity, must vanish, but we do not care what the final value of the horizontal component of position is. For the optimal control problem, the initial time is taken to be zero, and the final time T is to be minimized so that $\phi = 0$ and $L = 1$ which yields $J = T$. The elements of the end-point constraint function ψ must vanish where

$$\psi = \begin{Bmatrix} x_2 - h \\ x_3 - U \\ x_4 \end{Bmatrix} \tag{18}$$

In accordance with Eq. (8)

$$\hat{\lambda}_f = \begin{Bmatrix} 0 \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{Bmatrix} \tag{19}$$

Substitution of these equations into the weak form, Eq. (10), yields a system of algebraic equations whose size depends on N.

Results and Discussion

These equations are solved by a Newton-Raphson algorithm with trivial initial guesses (that are never changed regardless of input parameters) for $N = 2$. These results are then used to obtain the initial guesses for arbitrary N by simple interpolation. In all results obtained to date for this problem, no additional steps are necessary to obtain results as accurate as desired.

Representative numerical results for x_1/h versus x_2/h are presented in Fig. 2 for a case with $\frac{ah}{U^2} = 0.75$. Also, the control angle u versus dimensionless time t/T is presented in Fig. 3. Based on other results, not shown due to space limitations, accuracy for the costates is comparable to that for the states and for u. It can easily be seen that $N = 8$ gives acceptable results for all variables.

It should be noted that the computer time on a Cyber 990 is only about 2 seconds for $N = 2$, $N = 4$, and $N = 8$ and 3 seconds for $N = 16$. Thus, it is relatively insensitive to N.

Overall, the method provides very accurate results for this problem with only a few elements and for minimal computational effort. Furthermore, in results that are not presented herein, the Hamiltonian is seen to converge nicely to zero all along the trajectory as the number of elements increases.

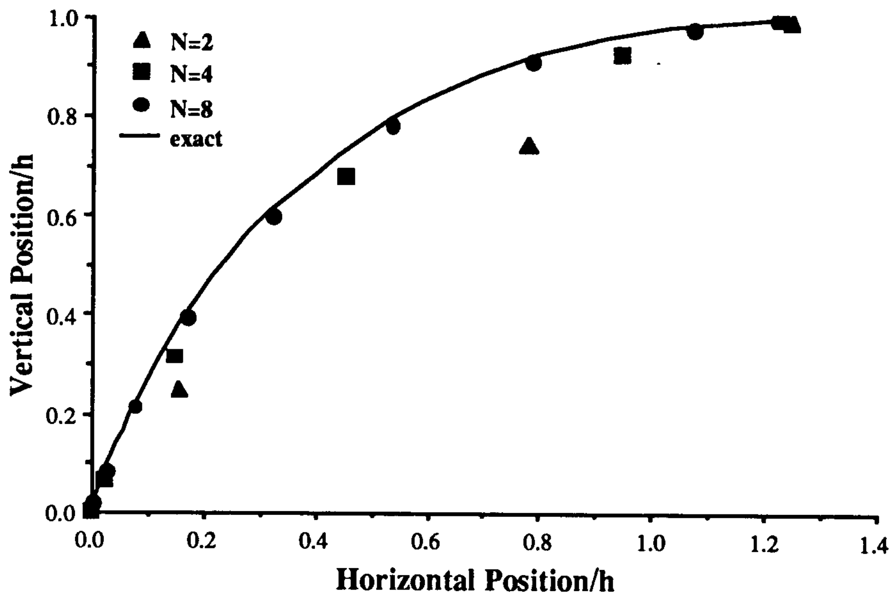

Fig. 2: Dimensionless vertical position x_2/h versus horizontal position x_1/h

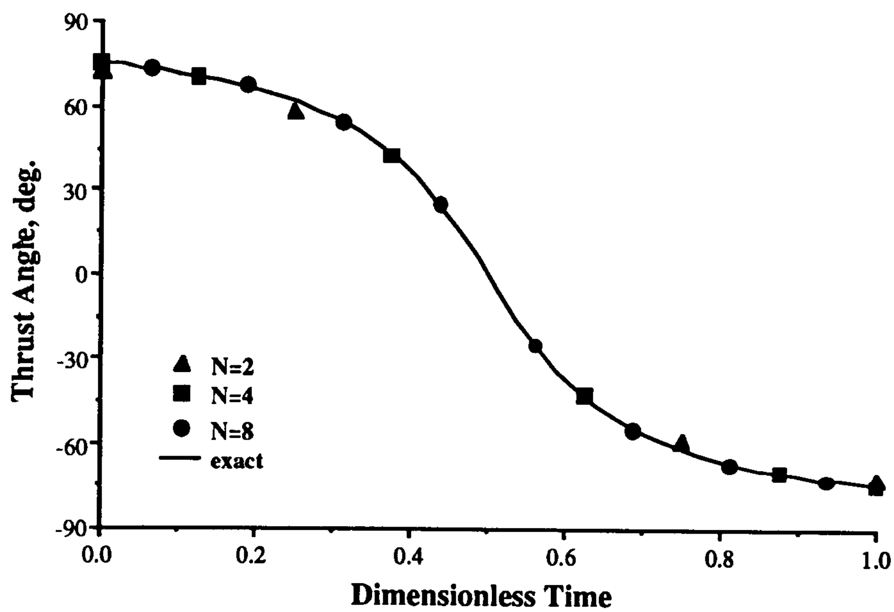

Fig. 3: Control angle u versus t/T

An Advanced Launch System Model

In this section, a model is presented which is suitable for evaluating the potential usefulness of the weak Hamiltonian finite element approach in real time guidance of an advanced launch system. A two-stage vehicle is considered that is simplified by not allowing for any inequality constraints.

We confine our attention to vertical plane dynamics of a vehicle flying over a spherical, non-rotating earth as depicted in Fig. 4. This results in the following state model for the states m (mass), h (height), E (energy per unit mass), and γ (flight-path angle):

$$\dot{m} = -\frac{T}{9.81 I_{sp}}; \qquad \dot{h} = V \sin\gamma; \qquad \dot{E} = \left(\frac{T-D}{m}\right) V$$
$$\dot{\gamma} = \left(\frac{T + qSC_{L\alpha}}{mV}\right)\alpha + \left(\frac{V}{r} - \frac{\mu}{r^2 V}\right)\cos\gamma \tag{20}$$

where T is the thrust, D is the drag, and V is the velocity. Here α, the angle of attack, has been adopted as a control variable.

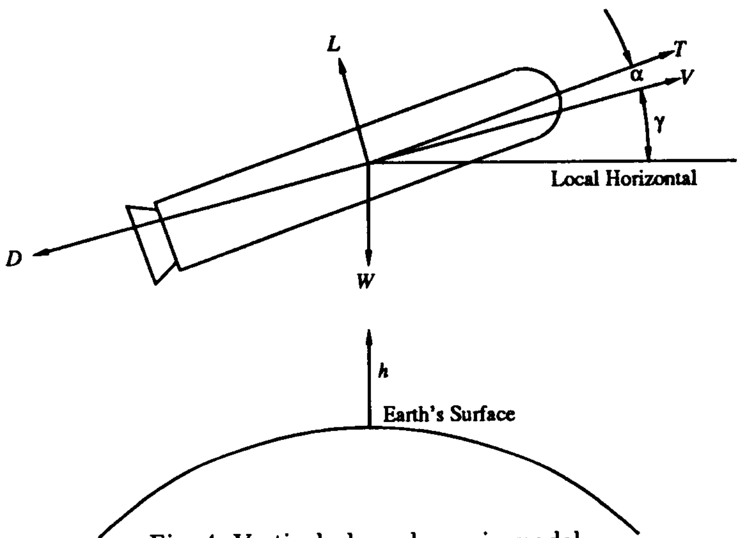

Fig. 4: Vertical plane dynamic model

The aerodynamic, propulsion, and atmospheric models are given by the following equations:

$$\begin{aligned}
T &= T_{vac} - A_e p \\
p &= p_0 (1 - 0.00002255h)^{5.256} \quad \text{for } h \leq 11000\text{m} \\
p &= p_{11} \exp\left(-\frac{h - 11000}{6350}\right) \quad \text{for } h \geq 11000\text{m} \\
r &= R_e + h; \quad V = \sqrt{2\left(E + \frac{\mu}{r}\right)}; \quad q = \frac{\rho V^2}{2} \\
\rho &= \rho_0 \exp\left(-\frac{h}{6700}\right); \quad M = \frac{V}{a} \\
D &= qS\left[C_{D0}(M) + \alpha^2 C_{N\alpha}(M)\right] \\
C_{L\alpha}(M) &= C_{N\alpha}(M) - C_{D0}(M) \\
a &= a_0\sqrt{1 - 0.00002255h} \quad \text{for } h \leq 11000\text{m} \\
a &= 295.03\text{ms}^{-2} \quad \text{for } h > 11000\text{m}
\end{aligned} \tag{21}$$

The vehicle parameters chosen for this model are based on a Saturn IB launch vehicle SA-217 [10] and are

$$\begin{aligned}
I_{sp_1} &= 263.4\text{s}; \qquad I_{sp_2} = 430.4\text{s} \\
T_{vac_1} &= 8155800\text{N}; \quad T_{vac_2} = 1186200\text{N} \\
A_{e_1} &= 8.47\text{m}^2; \qquad A_{e_2} = 5.29\text{m}^2; \quad S = 33.468\text{m}^2
\end{aligned} \tag{22}$$

where subscripts "1" and "2" refer to the first and second stages respectively. The aerodynamic coefficient data C_{D0} and $C_{N\alpha}$ are presented as functions of the Mach number M in Tables 1 and 2. The physical constants used in the above model are the earth's gravitational constant $\mu = 3.9906 \times 10^{14}\text{m}^3\text{s}^{-2}$, the earth's mean radius $R_e = 6.378 \times 10^6\text{m}$, the sea-level atmospheric pressure $p_0 = 101320\text{Nm}^{-2}$, the atmospheric pressure at 11km $p_{11} = 22637\text{Nm}^{-2}$, the sea-level density of air $\rho_0 = 1.225\text{kg m}^{-3}$, and the sea-level speed of sound in air $a_0 = 340.3\text{ms}^{-1}$.

M	C_{D0}
0.20	1.00
0.75	0.45
0.98*	0.80
1.00	0.80
1.02*	0.80
3.50	0.20
6.00	0.02

Table 1: Aerodynamic coefficient C_{D0} versus Mach number
(* denotes a common end point of two quadratic polynomial curves)

M	$C_{N\alpha}$
0.00	6.20
0.50	6.35
0.98*	7.70
1.00	7.70
1.02*	7.70
2.50	5.20
4.40*	4.70
5.00	5.50
6.00	6.00

Table 2: Aerodynamic coefficient $C_{N\alpha}$ versus Mach number
(* denotes a common end point of two quadratic polynomial curves)

The performance index is

$$J = \phi|_{t_f} = m|_{t_f} \tag{23}$$

and the final time t_f is open. The initial conditions specified are $m(0) = 5.2 \times 10^5\text{kg}$, $h(0) = 1800\text{m}$, $E(0) = -6.25 \times 10^7\text{m}^2\text{s}^{-2}$, and $\gamma(0) = 75°$. The final energy is specified as $E(t_f) = -1.25 \times 10^7\text{m}^2\text{s}^{-2}$. The burnout mass of the first stage is 192000 kg and the drop-mass of the booster is 51000 kg.

For this two-stage model, we must modify our formulation somewhat. We must accomodate for the unknown staging time t_s, the constraint on the mass at t_s (as opposed to a constraint on the states at the final time), the jump in the mass at t_s, the jump in the mass costate at t_s, the condition for a continuous Hamiltonian at t_s (continuous since ϕ and ψ are not explicit functions of time), and finally the change in state equations at t_s (due to the change in thrust). We further point out that the control u is discontinuous at the staging time. However, since only discrete mid-point values of u are solved for, the jumps are allowed to occur automatically at the nodes.

The new performance index (with $L = 0$) is

$$J = \int_0^{t_s} [\lambda^T(\dot{x} - f_1)\, dt + \int_{t_s}^{t_f} [\lambda^T(\dot{x} - f_2)\, dt - \nu_1\psi_1\Big|_{t_s} - \phi\Big|_{t_f} - \nu_2\psi_2\Big|_{t_f} \tag{24}$$

where f_1 and f_2 represent the state equations before and after t_s respectively, $\psi_1 = \hat{m}(t_s^-) - 192000$ and $\psi_2 = \hat{E}(t_f) + 12500000$.

The weak formulation is derived exactly as before and the same shape functions can be employed. This time, however, we will discretize the time from 0 to t_s with N_1 elements and the time from t_s to t_f with N_2 elements. The algebraic equations shown in [6] for the weak formulation are readily modified to account for the various jumps and state equation discontinuities. The resulting equations are

$$
\begin{aligned}
&\sum_{i=1}^{N_1}\left\{\delta x_i^T\left[-\bar{\lambda}_i - \frac{\Delta t_1}{2}\left(\frac{\partial \bar{f}_1}{\partial \bar{x}}\right)_i^T \bar{\lambda}_i\right] + \delta\lambda_i^T\left[\bar{x}_i - \frac{\Delta t_1}{2}\left(\bar{f}_1\right)_i\right]\right.\\
&\quad +\delta x_{i+1}^T\left[\bar{\lambda}_i - \frac{\Delta t_1}{2}\left(\frac{\partial \bar{f}_1}{\partial \bar{x}}\right)_i^T \bar{\lambda}_i\right] - \delta\lambda_{i+1}^T\left[\bar{x}_i + \frac{\Delta t_1}{2}\left(\bar{f}_1\right)_i\right]\\
&\quad \left.-\delta\bar{u}_i^T\left[\Delta t_1\left(\frac{\partial \bar{f}_1}{\partial \bar{u}}\right)_i^T \bar{\lambda}_i\right]\right\} - \delta t_s\left[\hat{H}(t_s^+) - \hat{H}(t_s^-)\right] - \delta\nu_1\left(\hat{m}_{N_1+1}^- - 192000\right)\\
&\quad +\delta x_1^T\hat{\lambda}_0 - \delta\lambda_1^T\hat{x}_0 - \delta x_{N_1+1}^T\left(\hat{\lambda}_{N_1+1}^- - \hat{\lambda}_{N_1+1}^+\right) + \delta\lambda_{N_1+1}^T\left(\hat{x}_{N_1+1}^- - \hat{x}_{N_1+1}^+\right)\\
&\quad +\sum_{i=N_1+1}^{N_1+N_2}\left\{\delta x_i^T\left[-\bar{\lambda}_i - \frac{\Delta t_2}{2}\left(\frac{\partial \bar{f}_2}{\partial \bar{x}}\right)_i^T \bar{\lambda}_i\right] + \delta\lambda_i^T\left[\bar{x}_i - \frac{\Delta t_2}{2}\left(\bar{f}_2\right)_i\right]\right.\\
&\quad +\delta x_{i+1}^T\left[\bar{\lambda}_i - \frac{\Delta t_2}{2}\left(\frac{\partial \bar{f}_2}{\partial \bar{x}}\right)_i^T \bar{\lambda}_i\right] - \delta\lambda_{i+1}^T\left[\bar{x}_i + \frac{\Delta t_2}{2}\left(\bar{f}_2\right)_i\right]\\
&\quad \left.-\delta\bar{u}_i^T\left[\Delta t_2\left(\frac{\partial \bar{f}_2}{\partial \bar{u}}\right)_i^T \bar{\lambda}_i\right]\right\} - \delta t_f\left[\hat{H}(t_f)\right] - \delta\nu_2\left(\hat{E}_f + 12500000\right)\\
&\quad +\delta\lambda_{N_1+N_2+1}^T(\hat{x}_f) - \delta x_{N_1+N_2+1}^T(\hat{\lambda}_f) = 0
\end{aligned}
\tag{25}
$$

From Eqs. (8) and (23) it is seen that $\hat{\lambda}_f$ is given by $\lfloor 1 \quad 0 \quad \nu_2 \quad 0 \rfloor^T$. Also, we note that the only jumps are in the mass state and the mass costate and these jumps are

$$
\begin{aligned}
\hat{m}(t_s^-) - \hat{m}(t_s^+) &= 51000\\
\hat{\lambda}_m(t_s^-) - \hat{\lambda}_m(t_s^+) &= \nu_1
\end{aligned}
\tag{26}
$$

The finite element equations are solved using the method of Levenberg-Marquardt as coded in the IMSL subroutine ZXSSQ [11]. Running a case for a few elements generates a good approximation for larger numbers of elements. Initial guesses do not need to be very accurate, but the method is not nearly as computationally efficient as a Newton-Raphson procedure where sparsity in the Jacobian could be exploited.

In Figs. 5 and 6 numerical results for the ALS model are given for 4 and 8 elements in each time interval. (The number of elements in each interval is completely arbitrary.) In Fig. 5 the altitude profile is shown and the control history is shown in Fig. 6. From past experience with the one-stage model, we believe the $N_1 = 8$ and $N_2 = 8$ to be a converged result. Furthermore, we see that even the $N_1 = 4$ and $N_2 = 4$ result gives a reasonable approximation to the solution. It should be noted that these results are not realistic because of the absence of state constraints, and because we have large angles of attack (more than 30° at some points) even though we assumed small angles in the state equations. However, they do suffice to illustrate the power of the method.

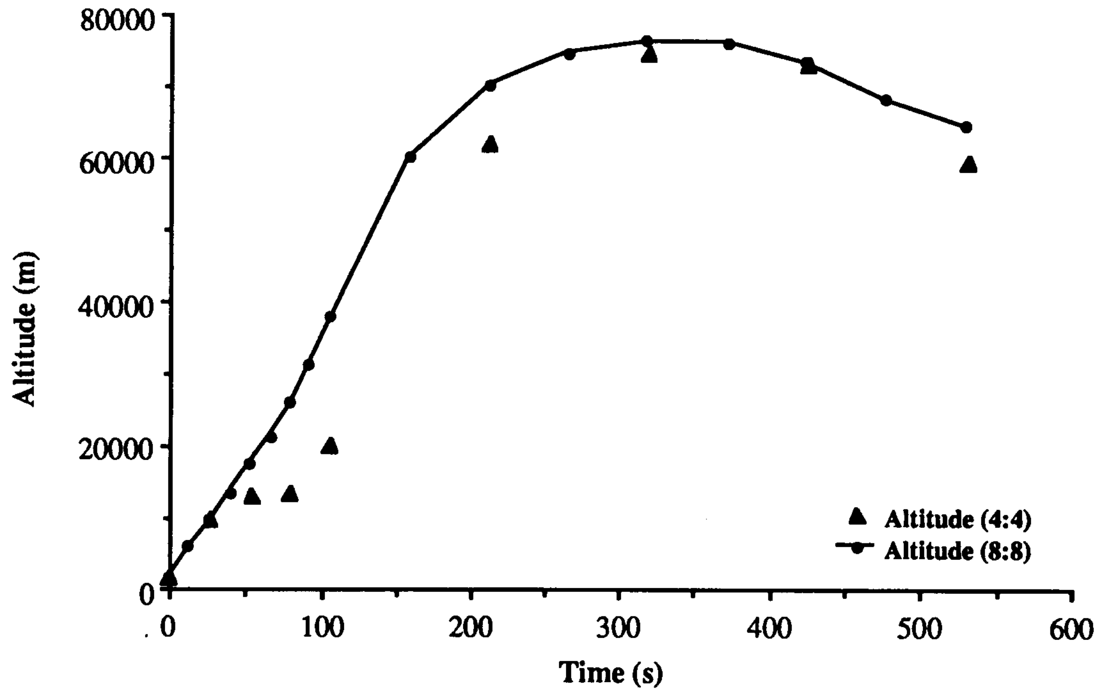

Fig. 5: Altitude profile versus time

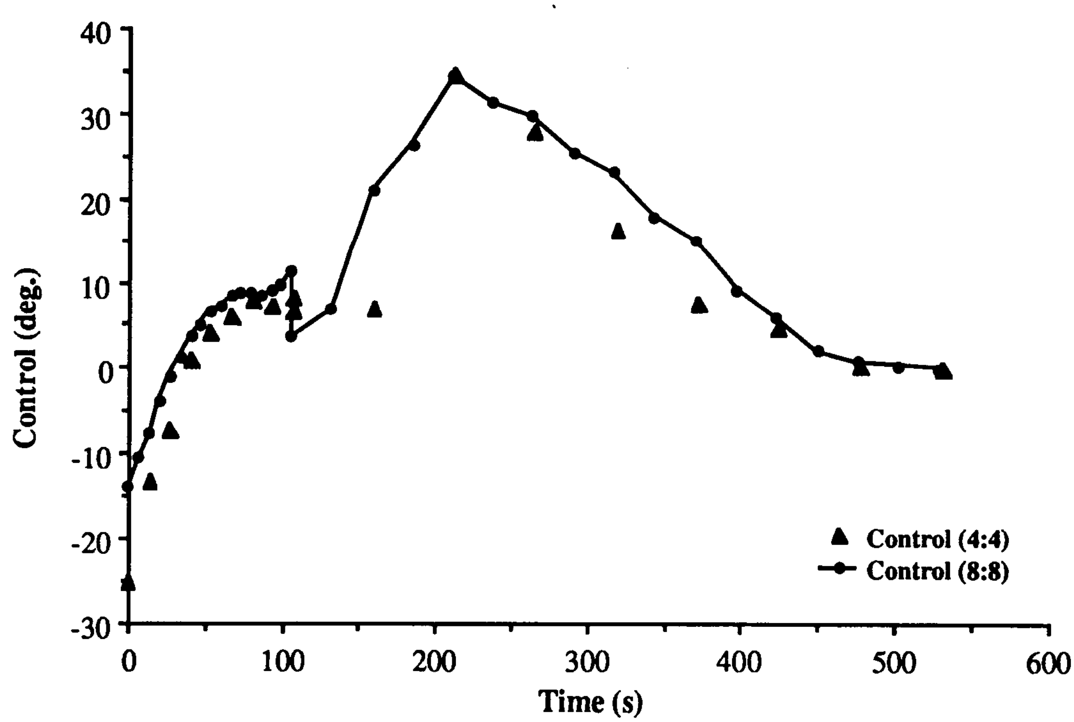

Fig. 6: Angle of attack profile versus time

As an indication of the accuracy of the method in a global sense, the Hamiltonian was observed to converge to zero (the exact answer) all along the trajectory. The finite element results are converging to the exact solution as N increases.

Conclusion

In this paper we present a weak Hamiltonian formulation for optimal control problems. Results are presented based on the weak Hamiltonian finite element formulation for a simple optimal control problem which show the method is reliable, efficient, and accurate. For this and other problems it is easily programmed with a self-starting algorithm.

To address the future needs of real-time guidance for future space transportation systems, we present a four-state model for the dynamics of mass, altitude, energy, and flight-path angle. The angle of attack is the control. The results show the power and efficacy of the present approach. It has several advantages for applications in real-time guidance. It is not only reliable and efficient but has excellent self-starting capability. Furthermore, initial guesses of the costates are not needed, and the method exhibits a graceful degradation in performance with reduction in number of elements.

For future research we intend to concentrate on improving computational efficiency by exploitation of the relatively significant level of sparsity in the Jacobian. We also will incorporate inequality constraints, and begin to address the avoidance of atmospheric anomalies.

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