# A DECOUPLED RECURSIVE APPROACH FOR CONSTRAINED FLEXIBLE MULTIBODY SYSTEM DYNAMICS

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## Abstract

A variational-vector calculus approach is employed to derive a recursive formulation for dynamic analysis of flexible multibody systems. Kinematic relationships for adjacent flexible bodies are derived in a companion paper [7], using a state vector notation that represents translational and rotational components simultaneously. Cartesian generalized coordinates are assigned for all body and joint reference frames, to explicitly formulate deformation kinematics under small deformation assumptions. Relative coordinate kinematics for joints are decoupled from deformation kinematics and an efficient flexible dynamics recursive algorithm is developed. Dynamic analysis of a closed loop robot is performed to illustrate efficiency of the algorithm.

## 1. Introduction

A recursive dynamics formulation was proposed by Armstrong [1] to analyze a robot manipulator, beginning with Cartesian equations of motion in a joint reference frame. Reaction forces were introduced as unknown forces into the equations of motion. These unknown forces were then eliminated to obtain recursion formulas for calculation of reduced equations of motion. The method was reformulated by Featherstone [2] and used to analyze a robot arm that consists of revolute and/or translational joints. He used a spatial notation to relieve notational complexity and introduced a new "articulated inertia" terminology that reflects inertia effects of all outboard bodies in a kinematic chain. Neither method considered the effect of flexibility of components.

Variational approaches have dominated structural analysis for the last decade. The variational method has recently been combined with vector calculus, to permit systematical transformation of the equations of motion from Cartesian space to joint coordinate space [3]. The same variational approach was used to derive a recursive formulation for constrained rigid body mechanical system dynamics in Ref. 4.

A variational equation of motion for constrained flexible systems was derived in Ref. 5, using Cartesian coordinates. The variational approach was applied to extend the rigid body recursive formulation to flexible body systems by Kim [6]. Kinematic relationships between reference frames for a pair of bodies that are connected by a joint are expressed in terms of joint relative coordinates and modal deformation coordinates of bodies. As a result, joint and modal coordinate equations of motion are coupled and must be solved simultaneously. This requires inversion of a moderately large matrix, for coupled modal and joint coordinates.

In order to enhance graph theoretic analysis of deformation characteristics, kinematics of flexible multibody systems is represented in a companion paper [7]. Based on this kinematic analysis, a recursive formulation for dynamic analysis is represented in this paper that decouples relative joint and deformation coordinates, to improve computational efficiency. The proposed formulation can be used with a rigid body formulation by eliminating terms related to modal coordinates, due to its decoupled treatment of gross motion and deformation.

State vector representations and kinematics of flexible multibody systems, defined in Refs. 8 and 7 respectively, are summarized in Section 2. The equation of motion for a flexible body is transformed from the Cartesian space to a state space setting in Section 3. System topology is defined in Section 4 and recursive equations of motion for a single closed loop subsystem are derived in Section 5. Cut joint constraint acceleration equations that are needed in the equations of motion are derived in Section 6. The base body equation of motion is defined in Section 7. Numerical examples and results are presented in Section 8.

## 2. Decoupled Recursive Relationships for Flexible Bodies

To derive the variational equations of motion, state vector notation and decoupled recursive relationships for adjacent reference frames [7, 8] are briefly reviewed here. A matrix representation of the Cartesian velocity of a reference frame with origin at point P, as shown in Fig. 1, is given as $Y_p = [\dot{r}_p^T \ \omega_p^T]^T$, where $\dot{r}_p^T$ is the velocity of point P and $\omega_p$ is the angular velocity of the $x_p'$-$y_p'$-$z_p'$ body reference frame. A generalized velocity state vector $\hat{Y}_p$, based on screw and motor algebra [8, 9], is defined here as

$$\hat{Y}_p \equiv \begin{bmatrix} \dot{r}_p + \tilde{r}_p \omega_p \\ \omega_p \end{bmatrix} = T_p Y_p \equiv T_p \begin{bmatrix} \dot{r}_p \\ \omega_p \end{bmatrix}$$

(2.1)

where the 6x6 nonsingular matrix $T_p$ is defined as

$$T_p = \begin{bmatrix} I & \tilde{r}_p \\ 0 & I \end{bmatrix}$$

(2.2)

The tilde operator is used here to define a skew symmetric matrix as

$$\tilde{r} \equiv \begin{bmatrix} 0 & -r_z & r_y \\ r_x & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

(2.3)

that is associated with a vector $r = [r_x, r_y, r_z]^T$.

The Cartesian virtual displacement $\delta Z_p$ is defined as

$$\delta Z_p \equiv \begin{bmatrix} \delta r_p \\ \delta \pi_p \end{bmatrix}$$

(2.4)

where $\delta r_p$ is virtual displacement of point P and $\delta \pi_p$ is virtual rotation of the $x_p'$-$y_p'$-$z_p'$ frame.

The state variation can be obtained by replacing $\dot{r}_p$ and $\omega_p$ by $\delta r_p$ and $\delta \pi_p$, respectively, in Eq. 2.1; i.e.,

$$\delta \hat{Z}_p \equiv \begin{bmatrix} \delta r_p + \tilde{r}_p \delta \pi_p \\ \delta \pi_p \end{bmatrix} = T_p \delta Z_p$$

(2.5)

The acceleration state vector $\hat{\dot{Y}}_p$ is defined as the time derivative of velocity state $\hat{Y}_p$ of Eq. 2.1; i.e.,

$$\hat{\dot{Y}}_p = \begin{bmatrix} \ddot{r}_p + \tilde{r}_p \dot{\omega}_p + \dot{\tilde{r}}_p \omega_p \\ \dot{\omega}_p \end{bmatrix} = T_p \dot{Y}_p + X_p$$

(2.6)

where $X_p$ is the 6x1 vector

$$X_p = \begin{bmatrix} \dot{r}_p \omega_p \\ 0 \end{bmatrix}$$

(2.7)

The inverse relationships between Cartesian and state vector quantities can be derived from Eqs. 2.1, 2.5, and 2.6 as

$$Y_p = T_p^{-1} \hat{Y}_p$$

(2.8)

$$\delta Z_p = T_p^{-1} \delta \hat{Z}_p$$

(2.9)

$$\dot{Y}_p = T_p^{-1} \dot{\hat{Y}}_p - X_p$$

(2.10)

where the 6x6 inverse matrix $T_p^{-1}$ of the matrix $T_p$ is simply

$$T_p^{-1} = \begin{bmatrix} I & -\tilde{r}_p \\ 0 & I \end{bmatrix}$$

(2.11)

Three flexible bodies, with their body and joint reference frames, are shown in Fig. 2. The x-y-z frame is the global reference frame, denoted as F. Two joint reference frames are attached to a body i at each joint definition point $P_{ij}$. The $x'''_{ij}$-$y'''_{ij}$-$z'''_{ij}$ frame, denoted as $F'''_{ij}$, is fixed to body i and is parallel to the $x'_i$-$y'_i$-$z'_i$ frame, denoted as $F'_i$, in the undeformed state. The $x_{ij}$-$y_{ij}$-$z_{ij}$ frame, denoted as $F_{ij}$, is body fixed and has fixed orientation, relative to the $F'''_{ij}$ frame, since both are fixed to the body at the same joint definition point where the body is assumed to be very stiff.

Recursive relationships between reference frames in a joint, for example, between the $x_{ij}$-$y_{ij}$-$z_{ij}$ and $x_{ji}$-$y_{ji}$-$z_{ji}$ frames of joint (i,j), are

$$\hat{Y}_{ji} = \hat{Y}_{ij} + \Pi_{ij} \dot{q}_{ij}$$

(2.12)

$$\delta \hat{Z}_{ji} = \delta \hat{Z}_{ij} + \Pi_{ij} \delta q_{ij}$$

(2.13)

$$\dot{\hat{Y}}_{ji} = \dot{\hat{Y}}_{ij} + \Pi_{ij} \ddot{q}_{ij} + \Theta_{ij}$$

(2.14)

where $\hat{Y}$, $\dot{\hat{Y}}$, and $\delta \hat{Z}$ are state representations of velocity, acceleration, and virtual displacement, respectively, and $q_{ij}$ is a vector of joint relative coordinates [7].

Recursive relationships between inboard and outboard joint reference frames of a flexible body are

$$\hat{Y}_{ij} = \hat{Y}_{il} + \Gamma_{ij} \dot{a}_i$$

(2.15)

$$\delta Z_{ij} = \delta Z_{il} + \Gamma_{ij} \delta a_i$$

(2.16)

$$\dot{\hat{Y}}_{ij} = \dot{\hat{Y}}_{il} + \Gamma_{ij} \ddot{a}_i + \Delta_{ij}$$

(2.17)

where a is the deformation modal coordinate vector of the flexible body [5-7].

The recursive relationships between frames $F'''_{ij}$ and $F'_i$ of a flexible body are

$$\hat{Y}_{ij} = \hat{Y}_i + \Lambda_{ij}\hat{a}_i \tag{2.18}$$

$$\delta\hat{Z}_{ij} = \delta\hat{Z}_i + \Lambda_{ij}\hat{a}_i \tag{2.19}$$

$$\dot{\hat{Y}}_{ij} = \dot{\hat{Y}}_i + \Lambda_{ij}\ddot{a}_i + \Xi_{ij} \tag{2.20}$$

Detailed expressions for matrices $\Gamma_{ij}$, $\Delta_{ij}$, $\Pi_{ij}$, $\Theta_{ij}$, $\Lambda_{ij}$, and $\Xi_{ij}$ may be found in Ref. 7.

## 3. Equation of Motion for a Flexible Body

The variational Cartesian equations of motion for a flexible multibody system are derived in Ref. 5. They can be written for a typical body i, using the notations defined in Section 2, as

$$\begin{bmatrix} \delta Z_i^T & \delta a_i^T \end{bmatrix} \left\{ M_i \begin{bmatrix} \dot{Y}_i \\ \ddot{a}_i \end{bmatrix} + S_i + V_i - Q_i \right\} = 0 \tag{3.1}$$

which must hold for all kinematically admissible $\delta Z_i$ and $\delta a_i$. The mass matrix $M_i$ is a function of the generalized coordinates, $S_i$ is a collection of quadratic velocity terms, $V_i$ is the elastic generalized force, and $Q_i$ is the applied generalized force.

The equations of motion in Cartesian space are transformed to state vector form by substituting kinematic relationships between the spaces. The state variation and acceleration relationships of Eqs. 2.9 and 2.10 are substituted into Eq. 3.1, to yield

$$\begin{bmatrix} \delta\hat{Z}_i^T & \delta a_i^T \end{bmatrix} \left\{ \hat{M}_i \begin{bmatrix} \dot{\hat{Y}}_i \\ \ddot{a}_i \end{bmatrix} - \hat{Q}_i \right\} = 0 \tag{3.2}$$

where the state representation $\hat{M}_i$ of the mass matrix is partitioned into 4 submatrices, based on state and modal coordinates,

$$\hat{M}_i = \begin{bmatrix} \bar{M}_i^{mm} & \bar{M}_i^{ma} \\ \bar{M}_i^{ma} & \bar{M}_i^{aa} \end{bmatrix} \equiv \begin{bmatrix} T_i^{-1} & 0 \\ 0 & I \end{bmatrix}^T M_i \begin{bmatrix} T_i^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{3.3}$$

Similarly, the state representation of $\hat{Q}_i$, which accounts for generalized force and coupling terms, is divided into two subvectors as

$$\hat{Q}_i = \begin{bmatrix} \bar{Q}_i^z \\ \bar{Q}_i^a \end{bmatrix} \equiv \begin{bmatrix} T_i^{-1} & 0 \\ 0 & I \end{bmatrix}^T \left( M_i \begin{bmatrix} X_i \\ 0 \end{bmatrix} - S_i - V_i + Q_i \right) \tag{3.4}$$

The equations of motion in Eq. 3.2 can be rewritten, using the notations of Eqs. 3.3 and 3.4, as

$$\delta\hat{Z}_i^T(\bar{M}_i^{mm}\dot{\hat{Y}}_i + \bar{M}_i^{ma}\ddot{a}_i - \bar{Q}_i^z) + \delta a_i^T(\bar{M}_j^{am}\dot{\hat{Y}}_i + \bar{M}_i^{aa}\ddot{a}_i - \bar{Q}_i^a) = 0 \tag{3.5}$$

where $\delta\hat{Z}_i$ and $\delta a_i$ must be consistent with all constraints that act on body i.

## 4. System Topology

An extended flexible multibody graph model, in which nodes represent reference frames and edges represent transformations between frames, is presented in Ref. 7. The

corresponding graph for a single closed loop system is shown in Fig. 3. Body $I$ is the junction body, at which chains 1 and 2 of the spanning tree of Fig. 4 meet. If joint J(n,n+1) between bodies n and n+1 is cut, both bodies n and n+1 are treated as tree end bodies.

## 5. Equations of Motion of a Single Closed Loop

The variational equation of motion for the system shown in Fig. 3 is

$$\sum_{i=I}^{m} \{\delta\hat{Z}_i^T(\bar{M}_i^{mm}\dot{Y}_i + \bar{M}_i^{ma}\ddot{a}_i - Q_i^z) + \delta a_i^T(\bar{M}_i^{am}\dot{Y}_i + \bar{M}_i^{aa}\ddot{a}_i - Q_i^a)\} = 0$$

(5.1)

which must hold for all kinematically admissible virtual displacements that satisfy joint and deformation constraints in Fig. 3.

State variation and accelerations of each body reference frame are expressed in corresponding joint terms and modal coordinates from Eqs. 2.19 and 2.20 and substituted into Eq. 5.1. The resulting equations of motion are as follows:

$$\sum_{i=I}^{m} \{\delta\hat{Z}_{i(i-1)}^T(M_i^{mm}\dot{Y}_{i(i-1)} + M_i^{ma}\ddot{a}_i - Q_i^z) + \delta a_i^T(M_i^{am}\dot{Y}_{i(i-1)} + M_i^{aa}\ddot{a}_i - Q_i^a)\}$$

$$= EQM(1) + EQM(2) = 0$$

(5.2)

which must hold for all kinematically admissible virtual displacements. Terms arising in Eq. 5.2 are as follows:

$$M_i^{mm} = \bar{M}_i^{mm}$$

$$M_i^{ma} = \bar{M}_i^{ma} - \bar{M}_i^{mm}\Lambda_{i(i-1)}$$

$$M_i^{am} = M_i^{ma}{}^T$$

$$M_i^{aa} = \bar{M}_i^{aa} - \bar{M}_i^{am}\Lambda_{i(i-1)} + \Lambda_{i(i-1)}^T(\bar{M}_i^{mm}\Lambda_{i(i-1)} - \bar{M}_i^{ma})$$

$$Q_i^z = \bar{Q}_i^z + \bar{M}_i^{mm}\Xi_{i(i-1)}$$

$$Q_i^a = \bar{Q}_i^a + \bar{M}_i^{am}\Xi_{i(i-1)} - \Lambda_{i(i-1)}^T(\bar{Q}_i^z + \bar{M}_i^{mm}\Xi_{i(i-1)})$$

(5.3)

$$EQM(1) = \sum_{i=I}^{n} \{\delta\hat{Z}_{i(i-1)}^T(M_i^{mm}\dot{Y}_{i(i-1)}+M_i^{ma}\ddot{a}_i-Q_i^z)+ \delta a_i^T(M_i^{ma}\dot{Y}_{i(i-1)}+M_i^{aa}\ddot{a}_i-Q_i^a)\}$$

(5.4)

$$EQM(2) = \sum_{i=n+1}^{m} \{\delta\hat{Z}_{i(i-1)}^T(M_i^{mm}\dot{Y}_{i(i-1)}+M_i^{ma}\ddot{a}_i-Q_i^z)+ \delta a_i^T(M_i^{ma}\dot{Y}_{i(i-1)}+M_i^{aa}\ddot{a}_i-Q_i^a)\}$$

(5.5)

The Jacobian matrix of the cut joint constraint function $\Phi^{J(n,n+1)}$ is obtained by differentiation as

$$\delta\Phi^{J(n,n+1)} = \Phi_{\hat{Z}_{n(n+1)}} \delta\hat{Z}_{n(n+1)} + \Phi_{\hat{Z}_{(n+1)n}} \delta\hat{Z}_{(n+1)n} = 0$$

(5.6)

where state variations $\delta\hat{Z}_{n(n+1)}$ and $\delta\hat{Z}_{(n+1)n}$ are obtained by employing the state vector representation of Cartesian virtual displacements. There exists a Lagrange multiplier vector such that

$$EQM(2) + \sum_{i=I}^{n} \{\delta\hat{Z}_{i(i-1)}^{T}(M_{i}^{mm}\dot{Y}_{i(i-1)}+M_{i}^{ma}\ddot{a}_{i}-Q_{i}^{z})$$

$$+ \delta a_{i}^{T}(M_{i}^{am}\dot{Y}_{i(i-1)}+M_{i}^{aa}\ddot{a}_{i}-Q_{i}^{a})\} + \delta\hat{Z}_{n(n+1)}^{T}\Phi_{Z_{n(n+1)}}^{T}\lambda^{n(n+1)} = 0 \tag{5.7}$$

where the virtual displacements need only be consistent with kinematic admissibility conditions for all tree structure joints and deformation constraints. Similarly, the equation of motion for chain 2 is

$$EQM(2) \equiv \sum_{i=n+1}^{m} \{\delta\hat{Z}_{i(i-1)}^{T}(M_{i}^{mm}\dot{Y}_{i(i-1)}+M_{i}^{ma}\ddot{a}_{i}-Q_{i}^{z})$$

$$+ \delta a_{i}^{T}(M_{i}^{am}\dot{Y}_{i(i-1)}+M_{i}^{aa}\ddot{a}_{i}-Q_{i}^{a})\} + \delta\hat{Z}_{(n+1)n}^{T}\Phi_{Z_{(n+1)n}}^{T}\lambda^{n(n+1)} \tag{5.8}$$

The virtual displacement $\delta\hat{Z}_{n(n+1)}$ may be expressed in terms of $\delta\hat{Z}_{n(n-1)}$ and $\delta a_{n}$ from Eq. 2.16. Substituting this relationship into Eq. 5.7, to obtain

$$EQM(2) + \sum_{i=I}^{n-1} \{\delta\hat{Z}_{i(i-1)}^{T}(M_{i}^{mm}\dot{Y}_{i(i-1)}+M_{i}^{ma}\ddot{a}_{i}-Q_{i}^{z})+ \delta a_{i}^{T}(M_{i}^{am}\dot{Y}_{i(i-1)}+M_{i}^{aa}\ddot{a}_{i}-Q_{i}^{a})\}$$

$$+ \delta\hat{Z}_{n(n-1)}^{T}(M_{n}^{mm}\dot{Y}_{n(n-1)}+M_{n}^{ma}\ddot{a}_{n}-Q_{n}^{z}+\Phi_{Z_{n(n+1)}}^{T}\lambda^{n(n+1)})$$

$$+ \delta a_{n}^{T}(M_{n}^{am}\dot{Y}_{n(n-1)}+M_{n}^{aa}\ddot{a}_{n}+\Gamma_{n(n+1)}^{T}\Phi_{Z_{n(n+1)}}^{T}\lambda^{n(n+1)}) = 0 \tag{5.9}$$

which must hold for all virtual displacements that are consistent with tree structure joints and deformation constraints in Fig. 4. Since $\delta a_{n}$ is arbitrary, the coefficient of $\delta a_{n}^{T}$ in Eq. 5.9 must be zero. As a result, the following expression for $\ddot{a}_{n}$ is obtained:

$$\ddot{a}_{n} = R_{n(n-1)}^{z}\dot{Y}_{n(n-1)} + R_{n(n-1)}^{a} + R_{n(n-1)}^{c}\lambda \tag{5.10}$$

where

$$R_{n(n-1)}^{z} = -M_{n}^{aa^{-1}}M_{n}^{am}$$

$$R_{n(n-1)}^{a} = M_{n}^{aa^{-1}}Q_{n}^{a}$$

$$R_{n(n-1)}^{c^{T}} = -M_{n}^{aa^{-1}}\Gamma_{n(n+1)}^{T}\Phi_{Z_{n(n+1)}}^{T} \tag{5.11}$$

Note that superscript $n(n+1)$ for the Lagrange multiplier vector has been dropped, for notational convenience.

Substituting the modal acceleration of body n from Eq. 5.10 into Eq. 5.9,

$$\text{EQM(2)} + \sum_{i=l}^{n-1} \{\delta \hat{Z}_{i(i-1)}^T (M_i^{mm}\dot{Y}_{i(i-1)} + M_i^{ma}\ddot{a}_i - Q_i^z) + \delta a_i^T(M_i^{am}\dot{Y}_{i(i-1)} + M_i^{aa}\ddot{a}_i - Q_i^a)\}$$

$$+ \delta \hat{Z}_{n(n-1)}^T(G_{n(n-1)}^z \dot{Y}_{n(n-1)} - G_{n(n-1)}^q + G_{n(n-1)}^{c^T}\lambda) = 0 \tag{5.12}$$

where

$$G_{n(n-1)}^z = M_n^{mm} + M_n^{ma}R_{n(n-1)}^z$$

$$G_{n(n-1)}^q = Q_n^z - M_n^{ma}R_{n(n-1)}^a$$

$$G_{n(n-1)}^{c^T} = \Phi_{Z_{n(n+1)}}^T + M_n^{ma}R_{n(n-1)}^{c^T} \tag{5.13}$$

and Eqs. 5.9 and 5.12 have the same kinematic admissibility conditions.

The variational equations of motion can be reduced further by substituting $\delta \hat{Z}_{n(n-1)}$ and

$\dot{Y}_{n(n-1)}$ in terms of $\delta \hat{Z}_{(n-1)n}$, $\delta q_{(n-1)n}$, $\dot{Y}_{(n-1)n}$, and $\ddot{q}_{(n-1)n}$, employing Eqs. 2.13 and 2.14. Equation 5.12 thus becomes

$$\text{EQM(2)} + \sum_{i=l}^{n-1} \{\delta \hat{Z}_{i(i-1)}^T(M_i^{mm}\dot{Y}_{i(i-1)} + M_i^{ma}\ddot{a}_i - Q_i^z) + \delta a_i^T(M_i^{am}\dot{Y}_{i(i-1)} + M_i^{aa}\ddot{a}_i - Q_i^a)\}$$

$$+ \delta \hat{Z}_{(n-1)n}^T\{G_{n(n-1)}^z \dot{Y}_{(n-1)n} + G_{n(n-1)}^z \Pi_{(n-1)n}\ddot{q}_{(n-1)n}$$

$$+ G_{n(n-1)}^z \Theta_{(n-1)n} - G_{n(n-1)}^q + G_{n(n-1)}^{c^T}\lambda\}$$

$$+ \delta q_{(n-1)n}^T \Pi_{(n-1)n}^T\{G_{n(n-1)}^z \dot{Y}_{(n-1)n} + G_{n(n-1)}^z \Pi_{(n-1)n}\ddot{q}_{(n-1)n}$$

$$+ G_{n(n-1)}^z \Theta_{(n-1)n} - G_{n(n-1)}^q + G_{n(n+1)}^{c^T}\lambda\} = 0 \tag{5.14}$$

which must hold for all virtual displacements that satisfy constraints inboard of body n-1. Since the kinematic relationship for joint (n-1,n) has been substituted into the equations of motion, $\delta q_{(n-1)n}$ is arbitrary; i.e., the coefficient of $\delta q_{(n-1)n}$ in Eq. 5.14 must be zero, which gives

$$\ddot{q}_{(n-1)n} = R_{(n-1)n}^z \dot{Y}_{(n-1)n} + R_{(n-1)n}^a + R_{(n-1)n}^{c^T}\lambda \tag{5.15}$$

where

$$R_{(n-1)n}^z = -(\Pi_{(n-1)n}^T G_{n(n-1)}^z \Pi_{(n-1)n})^{-1}\Pi_{(n-1)n}^T G_{n(n-1)}^z$$

$$R_{(n-1)n}^a = -(\Pi_{(n-1)n}^T G_{n(n-1)}^z \Pi_{(n-1)n})^{-1}\Pi_{(n-1)n}^T(G_{n(n-1)}^z \Theta_{n(n-1)} - G_{n(n-1)}^q)$$

$$R_{(n-1)n}^{c^T} = -(\Pi_{(n-1)n}^T G_{n(n-1)}^z \Pi_{(n-1)n})^{-1}\Pi_{(n-1)n}^T G_{n(n-1)}^{c^T} \tag{5.16}$$

where existence of $(\Pi_{(n-1)n}^T G_{n(n-1)}^z \Pi_{(n-1)n})^{-1}$ is proved in Ref. 11. Note that the subscripts of $R^z$, $R^a$, and $R^c$ in Eq. 5.16 are in ascending order which are different from Eq. 5.11.

Substituting the relative joint acceleration of Eq. 5.15 into Eq. 5.14,

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$$\text{EQM(2)} + \sum_{i=l}^{n-1}\{\delta\hat{Z}_{i(i-1)}^T(M_i^{mm}\dot{\hat{Y}}_{i(i-1)}+M_i^{ma}\ddot{a}_i-Q_i^z)+\delta a_i^T(M_i^{am}\dot{\hat{Y}}_{i(i-1)}+M_i^{aa}\ddot{a}_i-Q_i^a)\}$$

$$+ \delta\hat{Z}_{(n-1)n}^T\{G_{(n-1)n}^z\dot{\hat{Y}}_{(n-1)n}-G_{(n-1)n}^q+G_{(n-1)n}^{c^T}\lambda\} = 0 \tag{5.17}$$

where

$$G_{(n-1)n}^z = G_{n(n-1)}^z + G_{n(n-1)}^z\Pi_{(n-1)n}R_{(n-1)n}^z$$

$$G_{(n-1)n}^q = G_{n(n-1)}^q - G_{n(n-1)}^z\Theta_{(n-1)n} - G_{n(n-1)}^z\Pi_{(n-1)n}R_{(n-1)n}^a$$

$$G_{(n-1)n}^{c^T} = G_{n(n-1)}^{c^T} + G_{n(n-1)}^z\Pi_{(n-1)n}R_{(n-1)n}^{c^T} \tag{5.18}$$

which must hold for all virtual displacements that satisfy the same kinematic admissibility conditions as Eq. 5.14. Note that the subscripts of $G^z$, $G^q$, and $G^c$ in Eq. 5.18 are in ascending order which are different from Eq. 5. 13.

By employing the recursive relationships between inboard and outboard joint frames of Eqs. 2.16 and 2.17, the variational equations of motion can be reduced to

$$\text{EQM(2)} + \sum_{i=l}^{n-2}\{\delta\hat{Z}_{i(i-1)}^T(M_i^{mm}\dot{\hat{Y}}_{i(i-1)}+M_i^{am}\ddot{a}_i-Q_i^z)+\delta a_i^T(M_i^{am}\dot{\hat{Y}}_{i(i-1)}+M_i^{aa}\ddot{a}_i-Q_i^a)\}$$

$$+ \delta\hat{Z}_{(n-1)(n-2)}^T\{G_{(n-1)n}^z(\dot{\hat{Y}}_{(n-1)(n-2)}+\Gamma_{(n-1)n}\ddot{a}_{n-1}+\Delta_{(n-1)n})$$

$$-G_{(n-1)n}^q+G_{(n-1)n}^{c^T}\lambda+M_{n-1}^{mm}\dot{\hat{Y}}_{(n-1)(n-2)}+M_{n-1}^{ma}\ddot{a}_{n-1}-Q_{n-1}^z\}$$

$$+ \delta a_{n-1}^T[\Gamma_{(n-1)n}^T\{G_{(n-1)n}^z(\dot{\hat{Y}}_{(n-1)(n-2)}+\Gamma_{(n-1)n}\ddot{a}_{n-1}+\Delta_{(n-1)n})$$

$$-G_{(n-1)n}^q+G_{(n-1)n}^{c^T}\lambda\}+M_{n-1}^{am}\dot{\hat{Y}}_{(n-1)(n-2)}+M_{n-1}^{aa}\ddot{a}_{n-1}-Q_{n-1}^a] = 0 \tag{5.19}$$

which must hold for all virtual displacements that are consistent with tree structure constraints inboard of joint (n-2,n-1). Since $\delta a_{n-1}$ must be arbitrary, the coefficient of $\delta a_{n-1}^T$ must be zero, which yields

$$\ddot{a}_{n-1} = R_{(n-1)(n-2)}^z\dot{\hat{Y}}_{(n-1)(n-2)}+R_{(n-1)(n-2)}^a+R_{(n-1)(n-2)}^{c^T}\lambda \tag{5.20}$$

where

$$R_{(n-1)(n-2)}^z = -(M_{n-1}^{aa}+\Gamma_{(n-1)n}^TG_{(n-1)n}^z\Gamma_{(n-1)n})^{-1}(\Gamma_{(n-1)n}^TG_{(n-1)n}^z+M_{n-1}^{am})$$

$$R_{(n-1)(n-2)}^a = -(M_{n-1}^{aa}+\Gamma_{(n-1)n}^TG_{(n-1)n}^z\Gamma_{(n-1)n})^{-1}\{\Gamma_{(n-1)n}^T(G_{(n-1)n}^z\Delta_{(n-1)n}+G_{(n-1)n}^q)-Q_{n-1}^a\}$$

$$R_{(n-1)(n-2)}^{c^T} = -(M_{n-1}^{aa}+\Gamma_{(n-1)n}^TG_{(n-1)n}^z\Gamma_{(n-1)n})^{-1}(\Gamma_{(n-1)n}^TG_{(n-1)n}^{c^T}) \tag{5.21}$$

where existence of $(M_{n-1}^{aa}+\Gamma_{(n-1)n}^TG_{(n-1)n}^z\Gamma_{(n-1)n})^{-1}$ is proved in Ref. 11.

Substituting the modal acceleration of body n-1 from Eq. 5.20 into Eq. 5.19,

$$\text{EQM(2)} + \sum_{i=l}^{n-2}\{\delta\hat{Z}_{i(i-1)}^T(M_i^{mm}\dot{Y}_{i(i-1)}+M_i^{am}\ddot{a}_i-Q_i^z)+\delta a_i^T(M_i^{am}\dot{Y}_{i(i-1)}+M_i^{aa}\ddot{a}_i-Q_i^a)\}$$

$$+\delta\hat{Z}_{(n-1)(n-2)}^T\{G_{(n-1)(n-2)}^z\dot{Y}_{(n-1)(n-2)}-G_{(n-1)(n-2)}^q+G_{(n-1)(n-2)}^{c^T}\lambda\}=0 \qquad (5.22)$$

where

$$G_{(n-1)(n-2)}^z = G_{(n-1)n}^z+M_{n-1}^{mm}+(G_{(n-1)n}^z\Gamma_{(n-1)n}+M_{n-1}^{ma})R_{(n-1)(n-2)}^z$$

$$G_{(n-1)(n-2)}^q = G_{(n-1)n}^q+Q_{n-1}^z-(G_{(n-1)n}^z\Gamma_{(n-1)n}+M_{n-1}^{ma})R_{(n-1)(n-2)}^a$$

$$G_{(n-1)(n-2)}^{c^T} = G_{(n-1)n}^{c^T}+(G_{(n-1)n}^z\Gamma_{(n-1)n}+M_{n-1}^{ma})R_{(n-1)(n-2)}^{c^T} \qquad (5.23)$$

If the reduction procedure is continued to the junction body $l$ for chains 1 and 2, the following reduced variational equation of motions is obtained:

$$\delta\hat{Z}_{l(l-1)}^T\{(G_{l(l+1)}^z+G_{lm}^z)\dot{Y}_{l(l-1)}+(G_{l(l+1)}^z\Gamma_{l(l+1)}+G_{lm}^z\Gamma_{lm})\ddot{a}_l+(G_{l(l+1)}^z\Delta_{l(l+1)}+G_{lm}^z\Delta_{lm})$$

$$-(G_{l(l+1)}^q+G_{lm}^q)+(G_{l(l+1)}^{c^T}+G_{lm}^{c^T})+M_l^{mm}\dot{Y}_{l(l-1)}+M_l^{ma}\ddot{a}_l-Q_l^z\}$$

$$+\delta a_l^T\{(\Gamma_{l(l+1)}^T G_{l(l+1)}^z+\Gamma_{lm}^T G_{lm}^z)\dot{Y}_{l(l-1)}+(\Gamma_{l(l+1)}^T G_{l(l+1)}^z\Gamma_{l(l+1)}+\Gamma_{lm}^T G_{lm}^z\Gamma_{lm})\ddot{a}_l$$

$$+(\Gamma_{l(l+1)}^T G_{l(l+1)}^z\Delta_{l(l+1)}+\Gamma_{lm}^T G_{lm}^z\Delta_{lm})-(\Gamma_{l(l+1)}^T G_{l(l+1)}^q+\Gamma_{lm}^T G_{lm}^q)+(\Gamma_{l(l+1)}^T G_{l(l-1)}^{c^T})$$

$$\Gamma_{lm}^T G_{lm}^{c^T})\lambda+M_l^{am}\dot{Y}_{l(l-1)}+M_l^{aa}\ddot{a}_l-Q_l^a\}=0 \qquad (5.24)$$

Since $\delta a_l$ is arbitrary, the modal acceleration of junction body $l$ can be determined as

$$\ddot{a}_l = R_{l(l-1)}^z\dot{Y}_{l(l-1)} + R_{l(l-1)}^a + R_{l(l-1)}^{c^T}\lambda \qquad (5.25)$$

where

$$R_{l(l-1)}^z = -(M_l^{aa}+\Gamma_{l(l+1)}^T G_{l(l+1)}^z\Gamma_{l(l+1)}+\Gamma_{lm}^T G_{lm}^z\Gamma_{lm})^{-1}(M_l^{am}+\Gamma_{l(l+1)}^T G_{l(l+1)}^z+\Gamma_{lm}^T G_{lm}^z)$$

$$R_{l(l-1)}^a = -(M_l^{aa}+\Gamma_{l(l+1)}^T G_{l(l+1)}^z\Gamma_{l(l+1)}+\Gamma_{lm}^T G_{lm}^z\Gamma_{lm})^{-1}[\Gamma_{l(l+1)}^T\{G_{l(l+1)}^z\Delta_{l(l+1)}+G_{l(l+1)}^q\}$$

$$+\Gamma_{l(l+1)}^T(G_{lm}^z\Delta_{lm}+G_{lm}^q)-Q_l^a]$$

$$R_{l(l-1)}^{c^T} = -(M_l^{aa}+\Gamma_{l(l+1)}^T G_{l(l+1)}^z\Gamma_{l(l+1)}+\Gamma_{lm}^T G_{lm}^z\Gamma_{lm})^{-1}(\Gamma_{l(l+1)}^T G_{l(l+1)}^{c^T}+\Gamma_{lm}^T G_{lm}^{c^T}) \qquad (5.26)$$

Substituting the modal acceleration of the junction body from Eq. 5.25 into Eq. 5.24,

$$\delta\hat{Z}_{l(l-1)}^T(G_{l(l-1)}^z\dot{Y}_{l(l-1)} - G_{l(l-1)}^q + G_{l(l-1)}^{c^T}\lambda) = 0 \qquad (5.27)$$

which must hold for all kinematic constraints acting on junction body $l$. Terms arising in Eq. 5.27 are as follows:

$$G_{l(l-1)}^z = (G_{l(l+1)}^z+G_{lm}^z)+M_l^{mm}+(G_{l(l+1)}^z\Gamma_{l(l+1)}+G_{lm}^z\Gamma_{lm}+M_l^{ma})R_{l(l-1)}^z$$

$$G_{l(l-1)}^q = (\Pi_{l(l+1)}+\Pi_{lm})+Q_l^z+(G_{l(l+1)}^z\Gamma_{l(l+1)}+G_{lm}^z\Gamma_{lm}+M_l^{ma})R_{l(l-1)}^a$$

$$G_{l(l-1)}^{c^T} = (G_{l(l+1)}^{c^T}+G_{lm}^{c^T})+(G_{l(l+1)}^z\Gamma_{l(l+1)}+G_{lm}^z\Gamma_{lm}+M_l^{ma})R_{l(l-1)}^{c^T} \qquad (5.28)$$

500

## 6. Cut Joint Constraint Acceleration Equations

In addition to the equations of motion, cut joint constraints must be used to obtain the same number of equations as unknown accelerations and multipliers. Cut joint constraints may be differentiated twice to obtain the constraint acceleration equation,

$$\ddot{\Phi}^{(n,n+1)} = \Phi_{Z_{n(n+1)}}^T \dot{Y}_{n(n+1)} + \Phi_{Z_{(n+1)n}}^T \dot{Y}_{(n+1)n} - \gamma = 0 \tag{6.1}$$

where $\gamma$ is the collection of all terms that do not include $\dot{Y}_{n(n+1)}$ and $\dot{Y}_{(n+1)n}$. Superscript $(n,n+1)$, which has been used to denote the cut constraint between bodies n and n+1, is omitted for notational convenience in the derivation. Accelerations $\dot{Y}_{n(n+1)}$ and $\dot{Y}_{(n+1)n}$ from Eq. 2.17 are substituted into Eq. 6.1, to yield

$$\Phi_{Z_{n(n+1)}}^T (\dot{Y}_{n(n-1)} + \Gamma_{n(n+1)} \ddot{a}_n + \Delta_{n(n+1)})$$

$$+ \Phi_{Z_{(n+1)n}}^T (\dot{Y}_{(n+1)(n+2)} + \Gamma_{(n+1)(n+2)} \ddot{a}_{n+1} + \Gamma_{(n+1)(n+2)}) - \gamma = 0 \tag{6.2}$$

Substituting $\ddot{a}_n$ from Eq. 5.10 and $\ddot{a}_{n+1}$, obtained by advancing subscripts in Eq. 5.10, into Eq. 6.2 yields

$$G_{n(n-1)n}^c \dot{Y}_{n(n-1)} + G_{(n+1)(n+2)}^c \dot{Y}_{(n+1)(n+2)} + (L_{n(n-1)}^T + L_{(n+1)(n+2)}^T)\lambda$$

$$- N_{n(n-1)} - N_{(n+1)(n+2)} - \gamma = 0 \tag{6.3}$$

where

$$L_{n(n-1)}^T = \Phi_{Z_{n(n+1)}}^T \Gamma_{n(n+1)} R_{n(n-1)}^{c^T}$$

$$N_{n(n-1)} = -\Phi_{Z_{n(n+1)}}^T (\Delta_{n(n+1)} + \Gamma_{n(n+1)} R_{n(n-1)}^a) \tag{6.4}$$

and $L_{(n+1)(n+2)}$ and $N_{(n+1)(n+2)}$ are obtained by substituting appropriate subscripts into Eq. 6.4.

The recursive relationship between triple primed frames of joints (n-1,n) and (n+2,n+1), obtained from Eq. 2.16, are next substituted into Eq. 6.3, to give

$$G_{n(n-1)}(\dot{Y}_{(n-1)n} + \Pi_{(n-1)n} \ddot{q}_{(n-1)n} + \Theta_{(n-1)n}) + G_{(n+1)(n+2)}^c (\dot{Y}_{(n+2)(n+1)} + \Pi_{(n+2)(n+1)} \ddot{q}_{(n+2)(n+1)}$$

$$+ \Theta_{(n+2)(n+1)}) + (L_{n(n-1)}^T + L_{(n+1)(n+2)}^T)\lambda - N_{n(n-1)} - N_{(n+1)(n+2)} - \gamma = 0 \tag{6.5}$$

If the relative joint accelerations $\ddot{q}_{(n-1)n}$ from Eq. 5.15 and $\ddot{q}_{(n+2)(n+1)}$, obtained by replacing subscripts of Eq. 5.15, are substituted into Eq. 6.5,

$$G_{(n-1)n} \dot{Y}_{(n-1)n} + G_{(n+2)(n+1)} \dot{Y}_{(n+2)(n+1)} + (L_{n(n-1)}^T + L_{(n+2)(n+1)}^T)\lambda - N_{(n-1)n} - N_{(n+2)(n+1)} = 0 \tag{6.6}$$

where

$$N_{(n-1)n} = N_{n(n-1)} + G^c_{n(n-1)}(\Theta_{(n-1)n} + \Pi_{(n-1)n}R^a_{(n-1)n})$$

$$L^T_{(n-1)n} = L^T_{n(n-1)} + G_{n(n-1)}\Pi_{(n-1)n}R^{c^T}_{(n-1)n} \qquad (6.7)$$

and $L_{(n+2)(n+1)}$ and $N_{(n+2)(n+1)}$ are obtained by replacing $(n-1,n)$ by $(n+2,n+1)$ in Eq. 6.7.

If this sequence of elimination of modal and relative joint accelerations is repeated to junction body $I$, the following reduced constraint acceleration equations are obtained as

$$G_{J(I-1)}\dot{Y}_{J(I-1)} + L^T_{J(I-1)}\lambda - N_{J(I-1)} - \gamma = 0 \qquad (6.8)$$

where

$$L^T_{J(I-1)} = (L^T_{J(I+1)} + L^T_{Jm}) + (G_{J(I+1)}\Gamma_{J(I+1)} + G_{Jm}\Gamma_{Jm})R^{c^T}_{J(I-1)}$$

$$N_{J(I-1)} = (N_{J(I+1)} + N_{Jm}) - (G_{J(I+1)}\Gamma_{J(I+1)} + G_{Jm}\Gamma_{Jm})R^a_{J(I-1)} - (G_{J(I+1)}\Delta_{J(I+1)} + G_{Jm}\Delta_{Jm}) \qquad (6.9)$$

## 7. Base Body Equation of Motion

A single closed loop subsystem is used to derive decoupled recursive equations of motion in Sections 5 and 6. The variational equations of motion were reduced to the inboard joint reference frame of the junction body. Since the base body does not have an inboard joint, the inboard joint reference frame of the base body is assumed to coincide with the base body reference frame. If the reduction procedures that have been carried out with this subsystem are repeated along all chains of a system to the base body, the base body equation of motion is obtained as

$$\delta\hat{Z}^T_b(G^z_{bb}\dot{Y}_{bb} - G^q_{bb} + G^{c^T}_{bb}\lambda) = 0 \qquad (7.1)$$

where $\delta\hat{Z}_b$ is arbitrary for a floating base body, which yields

$$G^z_{bb}\dot{Y}_{bb} - G^q_{bb} + G^{c^T}_{bb}\lambda = 0 \qquad (7.2)$$

Reduction of cut joint constraint acceleration equations to the base body yields the reduced constraint acceleration equations as

$$G^{c^T}_{bb}\dot{Y}_b + L^T_{bb}\lambda - N_{bb} = 0 \qquad (7.3)$$

Equation 7.2 may be combined with Eq. 7.3, to form the augmented base body equations of motion,

$$\begin{bmatrix} G^z_{bb} & G^{c^T}_{bb} \\ G^c_{bb} & L^T_{bb} \end{bmatrix} \begin{bmatrix} \dot{Y}_{bb} \\ \lambda \end{bmatrix} = \begin{bmatrix} G^q_{bb} \\ N_{bb} \end{bmatrix} \qquad (7.4)$$

In the case of a constrained base body, a Lagrange multiplier vector $\lambda^b$ and corresponding constraint accelerations are introduced into Eq. 7.4. The resulting augmented equation of motion is

$$
\begin{bmatrix} G_{bb}^z & G_{bb}^{c^T} & \Phi_{Z_{bb}}^{b^T} \\ G_{bb}^c & L_{bb}^T & 0 \\ \Phi_{Z_{bb}}^b & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{Y}_{bb} \\ \lambda \\ \lambda^b \end{bmatrix} = \begin{bmatrix} G_{bb}^q \\ N_{bb} \\ \gamma^b \end{bmatrix}
$$

$(7.5)$

where $\Phi^b$ is constraint equations acting on the base body and $\gamma^b$ is the collection of all terms

that do not include $\dot{Y}_b$ in $\ddot{\Phi}^b$.

Equation 7.4 or 7.5 is solved for the base body state acceleration vector and the Lagrange multiplier vector. Detailed computational algorithm is presented in Ref. 11.

## 8. Example Problem

A closed loop spatial robot that consists of two flexible and three rigid bodies is shown in Fig. 5. Bodies 3 and 5 are flexible beams with rectangular cross sections. All other bodies are treated as rigid bodies. Body 1 is connected with ground, which is designated as the base body, by a revolute joint. A lumped mass is attached to body 3 at point P to represent a payload for this robot. Joints 1, 2, 3, 4, and 5 are revolute joints. The connection between bodies 3 and 5 is a spherical joint (joint 6).

One generalized coordinate is assigned for each revolute joint, and three deformation modes have been chosen for each flexible. Joint 6 is defined as the cut joint to form a tree structure.

Inertial properties and geometric data are given in Table 1. For deformation mode computation, flexible beams are discretized in to 10 equal length 3 diemensional beam elements.

Simulation is carried out for 0.5 sec., with the following actuator torques applied at joints 1, 2, and 4, respectively:

$n_1 = 5.0E9 \cdot \sin(0.2\pi t)$

$n_2 = 9.0E7 - 8.0E7 \cdot t$

$n_4 = 8.5E9 - 3.0E9 \cdot t$ \qquad\qquad (8.1)

Results of the simulation have been verified using the three dimensional dynamic analysis program DADS [18], which employes a Cartesian coordinate formulation [13]. In the Cartesian coordinate formulation, 48 generalized coordinates and 40 constraint equations are needed to represent the system. However, only 11 generalized coordinates and 3 constraint equations are required for the recursive formulation presented here. The y coordinate, velocity, and acceleration of the origin of the body reference frame for body 3 are shown in Fig. 6. Both the DADS and recursive formulations yield the same results when implemented on a VAX 11/780 serial computer, which cannot exploit parallelism in the recursive algorithm. Table 2 shows the CPU time required for both methods and the ratio of CPU time between the two methods.

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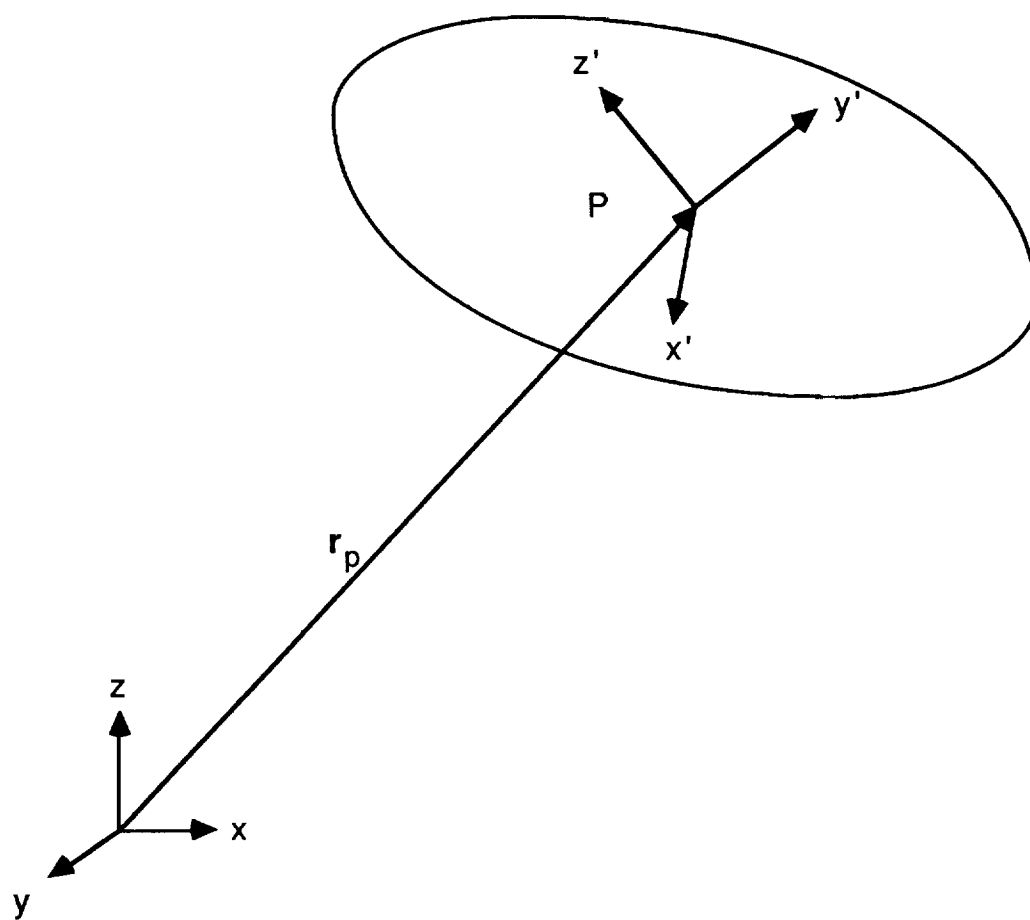18. _DADS User's Manual_, CADSI, P.O. Box 203, Oakdale, Iowa, 52319.
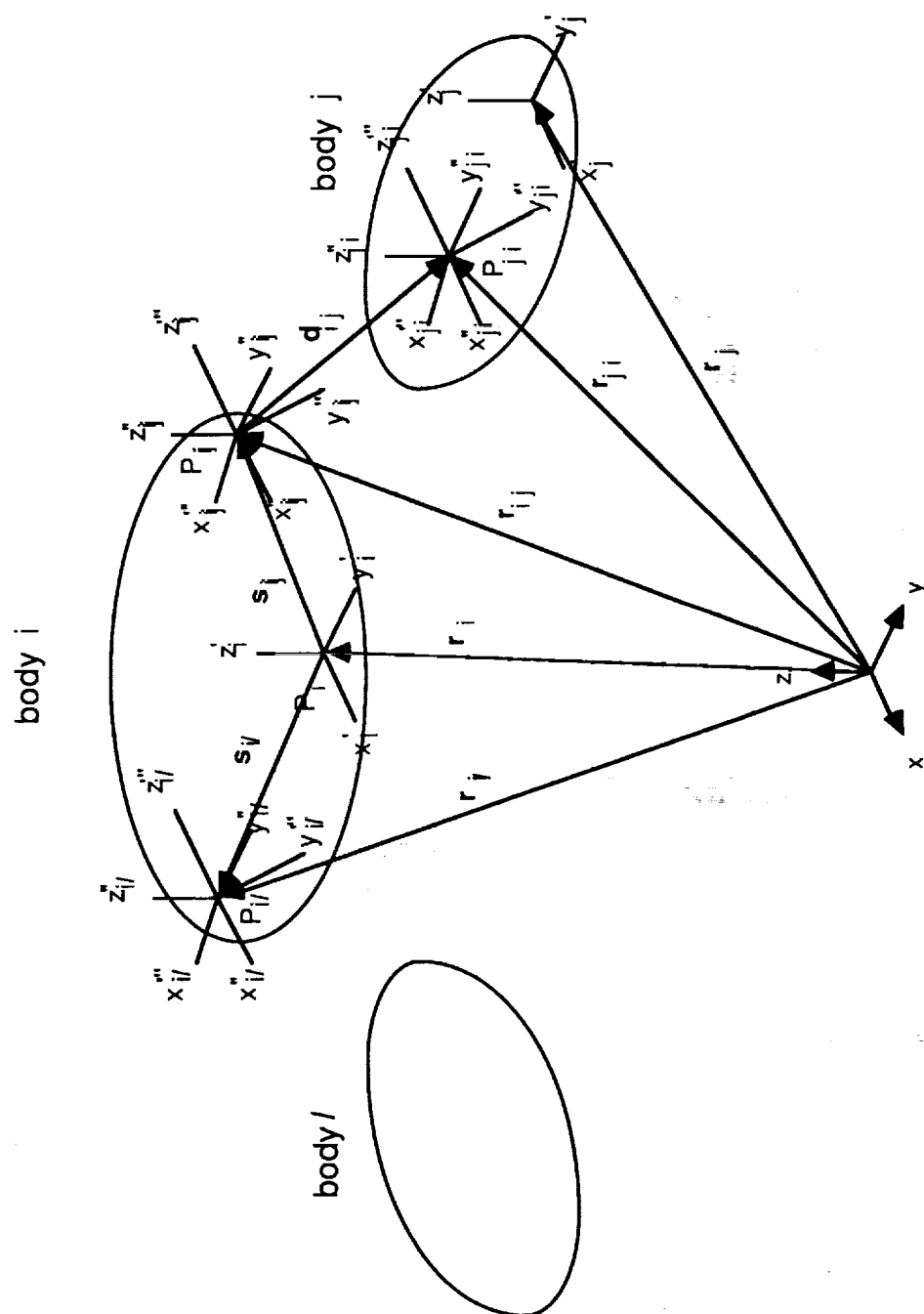
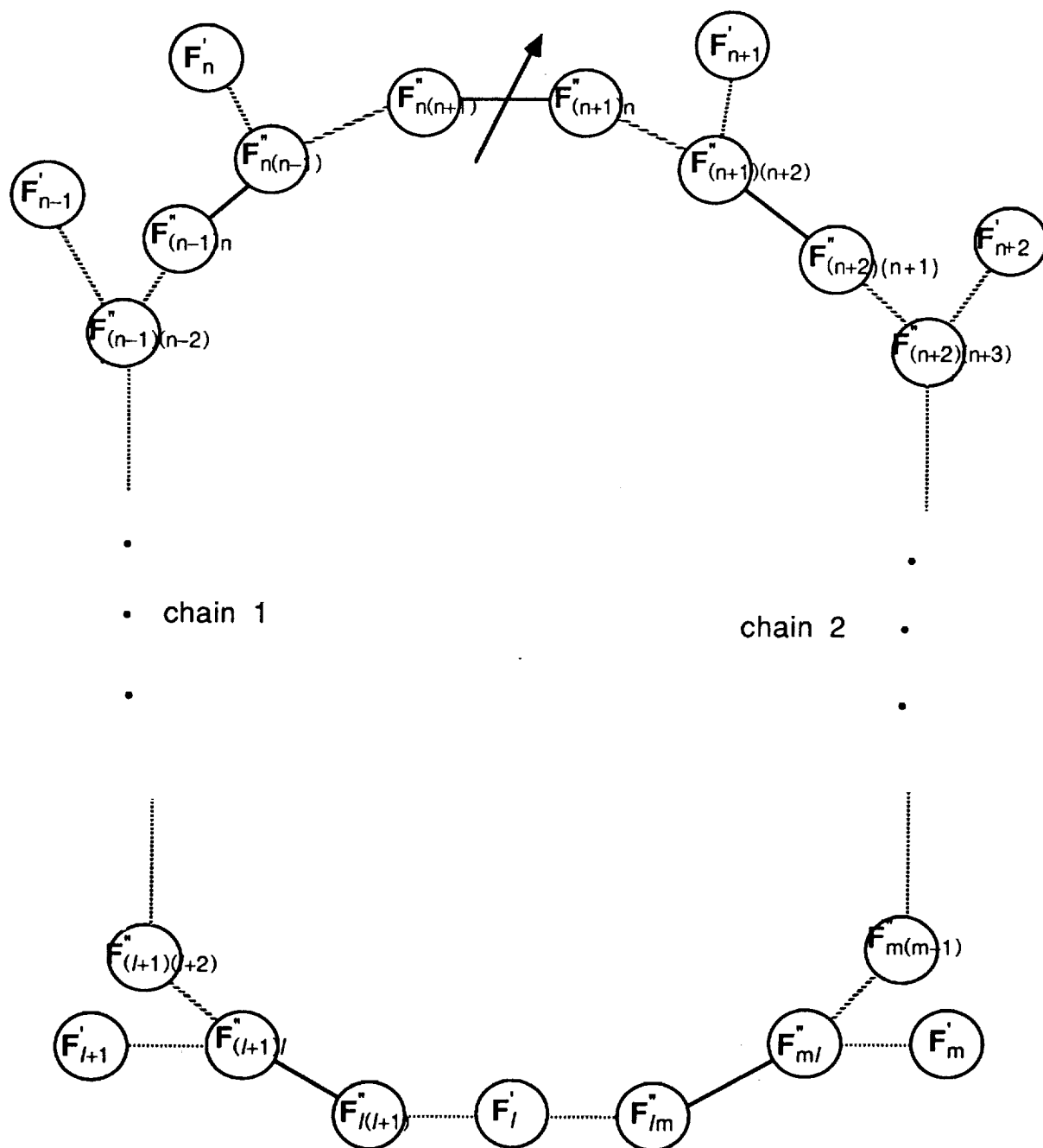Figure 1.  A Reference Frame

Figure. 2. Adjacent Flexible Bodies

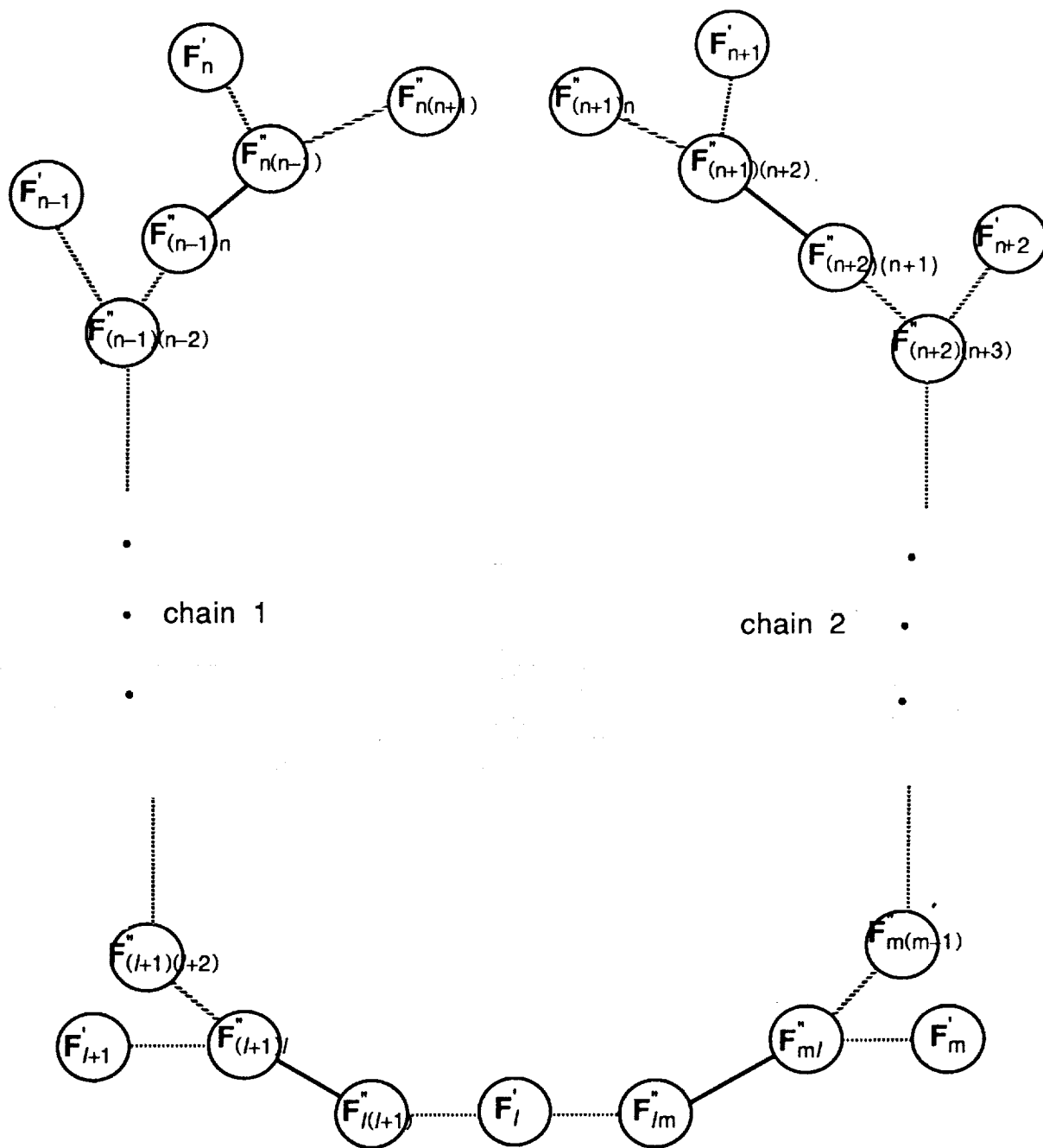Figure 3. A Closed Loop Subsystem

chain 1

chain 2
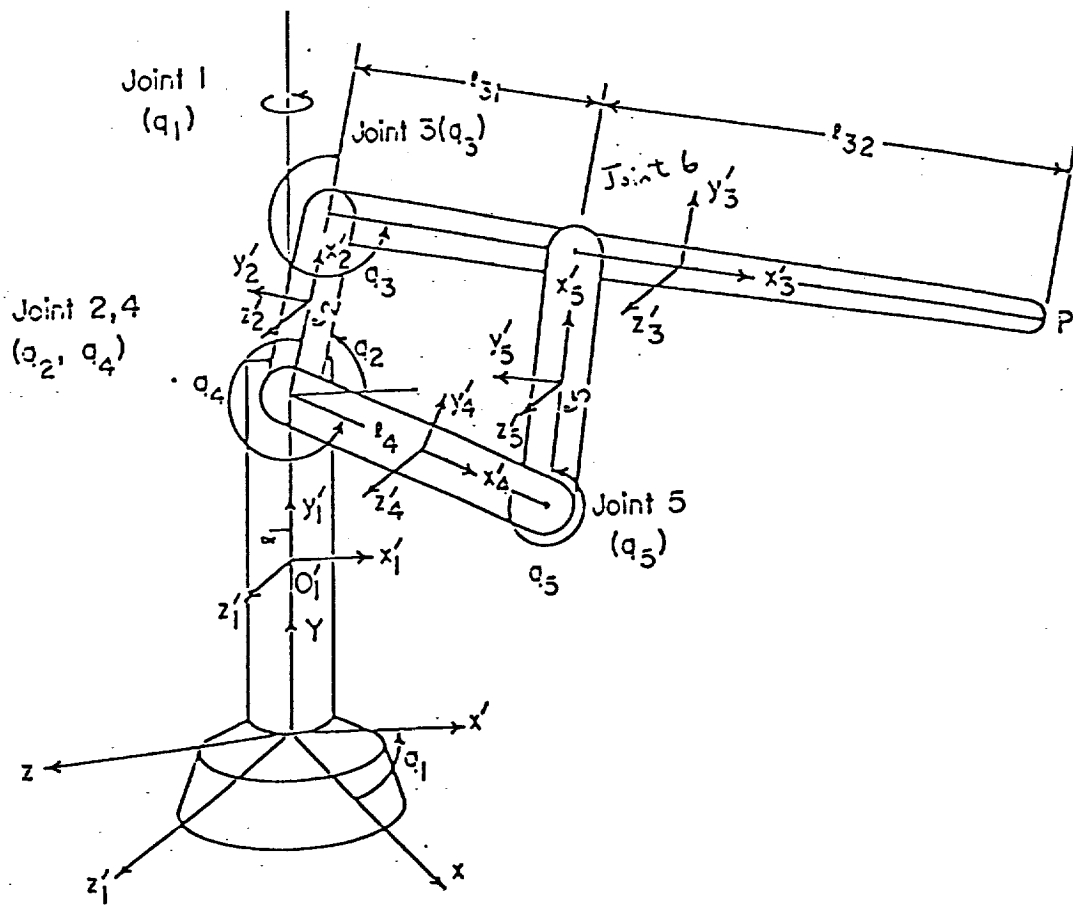
Figure 4. Spanning Tree for the Closed Loop Subsystem
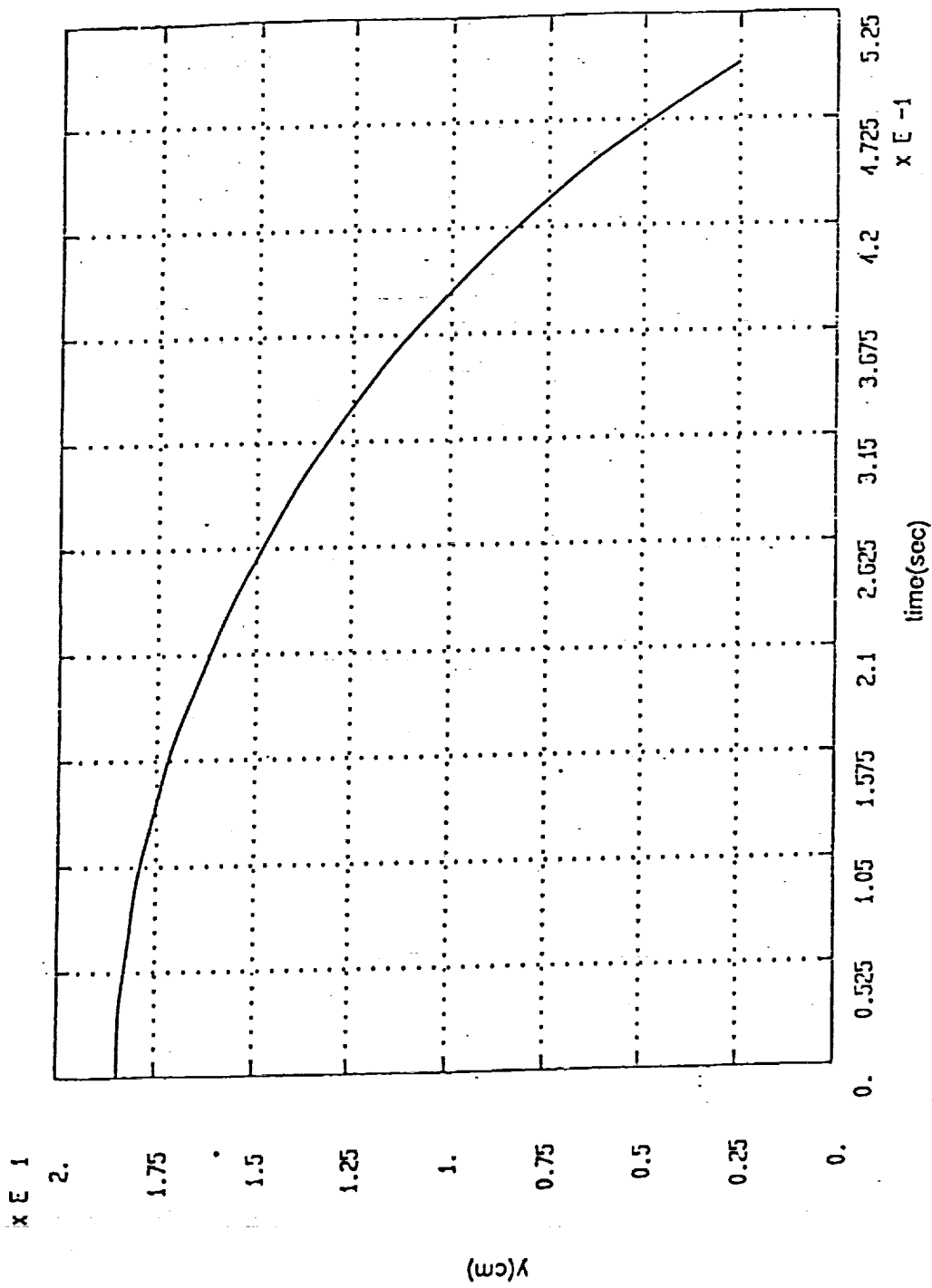
Figure 5. A Closed Loop Manipulator

Figure 6. Y of Body 3

Table 1. Dimension and Inertia of the Closed Loop Manipulator Bodies

| Body | Length(cm) | m(g) | $I_{xx}$ | $I_{yy}$ | $I_{zz}$ | $I_{xy}$ | $I_{xz}$ | $I_{yz}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $\ell_1 = 100.0$ | $6.1497 \times 10^4$ | $5.1632 \times 10^7$ | $7.6071 \times 10^5$ | $5.1632 \times 10^7$ | 0.0 | 0.0 | 0.0 |
| 2 | $\ell_2 = 37.5$ | $1.1745 \times 10^4$ | $0.7100 \times 10^4$ | $1.4000 \times 10^6$ | $1.4390 \times 10^6$ | 0.0 | 0.0 | 0.0 |
| 3 | $\ell_{31} = 77.5$ $\ell_{32} = 57.5$ | $4.2202 \times 10^4$ | $3.1359 \times 10^5$ | $6.4304 \times 10^7$ | $6.4441 \times 10^7$ | 0.0 | 0.0 | 0.0 |
| 4 | $\ell_4 = 132.5$ | $4.1499 \times 10^4$ | $3.0778 \times 10^5$ | $6.0000 \times 10^7$ | $6.0935 \times 10^7$ | 0.0 | 0.0 | 0.0 |
| 5 | $\ell_5 = 100.0$ | $3.1320 \times 10^4$ | $2.3229 \times 10^5$ | $2.6165 \times 10^7$ | $2.6267 \times 10^7$ | 0.0 | 0.0 | 0.0 |

Table 2.  CPU Comparison

| CPU | | Ratio |
|---|---|---|
| DADS | Recursive Method | |
| 8450 sec. | 762 sec. | 11.09 |