COMPONENT MODEL REDUCTION
VIA THE PROJECTION AND ASSEMBLY METHOD

Douglas E. Bernard*

Jet Propulsion Laboratory
California Institute of Technology
4800 Oak Grove Drive
Pasadena, California 91109

ABSTRACT

The problem of acquiring a simple but sufficiently accurate model of a dynamic system is made more difficult when the dynamic system of interest is a multibody system comprised of several components. A low order system model may be created by reducing the order of the component models and making use of various available multibody dynamics programs to assemble them into a system model. The difficulty is in choosing the reduced order component models to meet system level requirements. The projection and assembly method, proposed originally by Eke, solves this difficulty by forming the full order system model, performing model reduction at the system level using system level requirements, and then projecting the desired modes onto the components for component level model reduction. In this paper, the projection and assembly method is analyzed to show the conditions under which the desired modes are captured exactly–to the numerical precision of the algorithm.

INTRODUCTION

The problem to be solved is that of simulating the dynamics of a multibody system. A multibody system is comprised of two or more bodies or components connected at hinges. In general, the bodies may be rigid or flexible, and the hinges may have from one to six independent degrees of freedom. Often all deformations of each body from its reference condition are in the linear range, although the resulting system dynamics is nonlinear. In this case, nonlinear system models may be constructed using linear dynamic models for each component, but allowing large angle motion between components. This is the approach used in a number of existing multibody software tools.

The problem is that system models constructed in this manner may be too large for use in control system design and simulation trades. Model reduction is needed to bring the model down to manageable size. If the system model is available in linear form, system model reduction can be applied directly. For the class of multibody problems discussed above, only the component models are available in linear form, and existing multibody software can be used if we reduce the component models before assembly into the system model. A multibody system is inherently a geometrically nonlinear system because of the time-varying, large-angle articulation between bodies.

Component model reduction is typically done to some level anyway if the source of the model is a finite element program. This first level of model reduction often uses some simple criterion such as "keep all cantilever modes below 40 Hz." The challenge is to reduce the component model further in some manner that preserves how the component behaves when connected to the complete system; how the component affects system level requirements. The projection and assembly method described in this paper attempts to do this.

Model reduction for linear systems has been addressed by a number of researchers, resulting in a variety of suggested linear system model reduction methods[1234]. Fig. 1 gives a high-level view of how these methods work. Less attention has been paid to the problem of model reduction for components of multibody systems. Component modal synthesis methods[5678] have the capability of producing reduced order component models, but typically do so based on component-level rather than system-level criteria. When only one body in a multibody system is flexible, Macala[9] captures desired system modes exactly by augmenting the flexible body by the mass and inertia of the rigid

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*Member, Technical Staff, Guidance and Control Section

body. A subset of the free-free modes of this augmented body are then used as the flexible body component modes. Eke and Man[10] extend this capability to systems of more than one flexible body with a method that involves choosing system modes of interest, projecting the mode shapes of these desired modes onto each flexible component, reducing the order of each component accordingly, and assembling the components into a system model. Upon assembly, each of the original desired system modes is recovered exactly (to the numerical precision of the algorithm.) As can be seen in Fig. 2, this approach is conceptually more complicated than that shown in Fig 1., but allows the introduction of system level requirements.
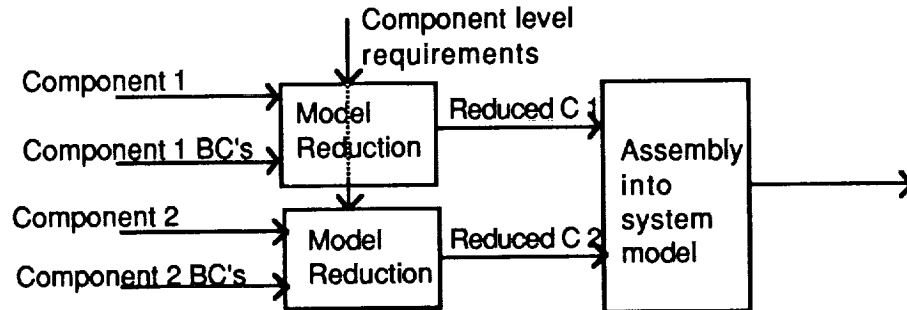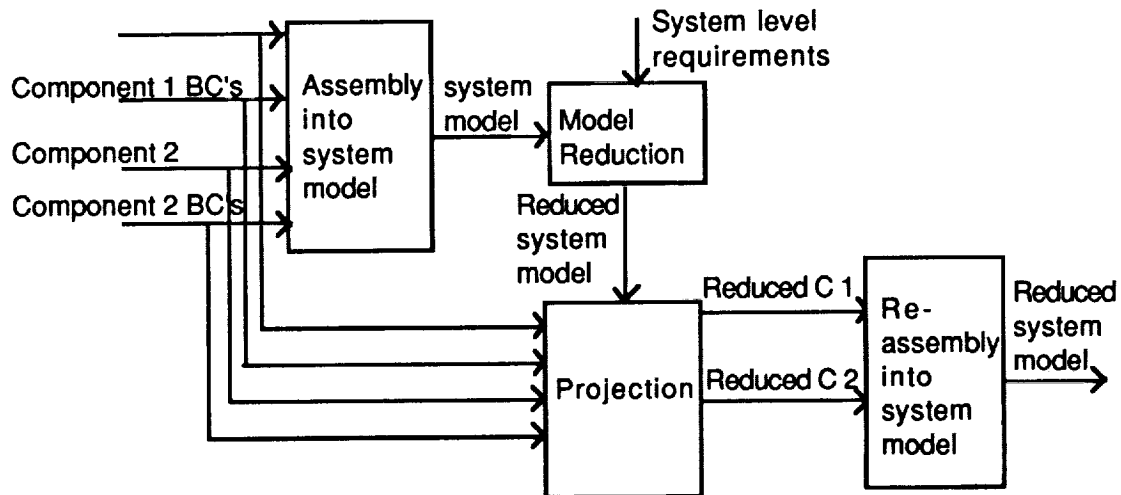
Fig. 1. Conventional Model Reduction

Fig. 2. Projection and Assembly Method

This paper analyzes the method outlined in Ref. 10 to show why the desired modes are returned exactly, presents necessary conditions for the success of the procedure, and proposes an extension to the method to handle situations when these necessary conditions are not met. Simple examples are presented to demonstrate the workings of the algorithm. The name "Projection and Assembly Method" is used to describe this component model reduction method.

DESCRIPTION OF METHOD

The idea of the projection and assembly method is to decide what system modes are important and to choose component models which, when assembled, capture those important system modes. The projection and assembly method is described in detail in Ref. 10. It works as follows:

- Acquire component models
- Synthesize the system model in some configuration of interest
- Apply any system level model reduction desired to choose which system free-free modes to retain
- Project the mode shapes of these retained modes onto each component.
- Choose new component states such that only these projected modes are admissible motions
- Transform the component models into reduced order component models using these new states
- Assemble the reduced order component models into a reduced order system model

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The concept is made clearer by considering a simple qualitative example. Consider the planar motion of a system consisting of a rigid hub with two identical beam appendages, one on each side. Ignoring motions along the beam axes, the first five modes of this system are sketched in Fig. 3.
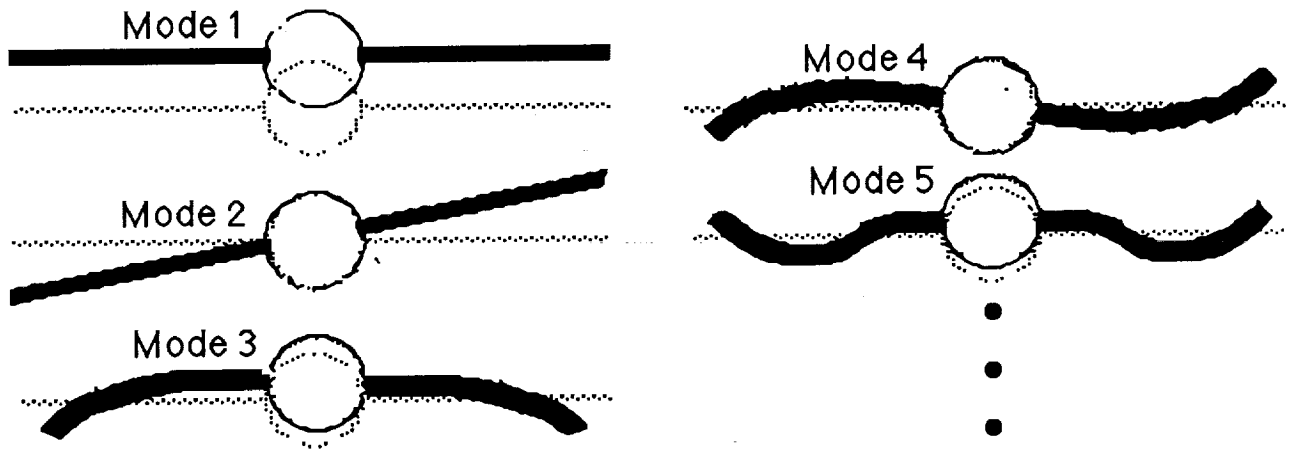

Fig. 3. Qualitative Beam Example

In this example, the lowest three system modes are chosen to be retained in the reduced order model. The projection step is illustrated in Fig. 4; these system modes are projected onto each of the two components. The resulting projections are used as generalized mode shapes for component models.
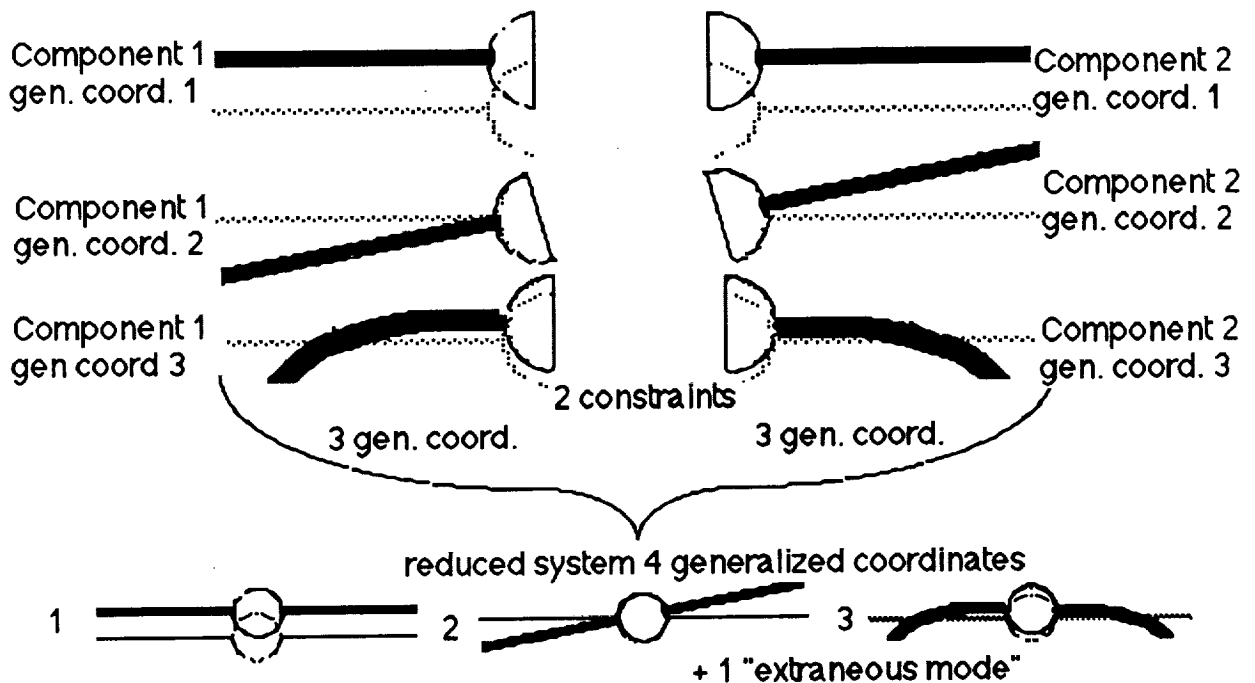

Fig. 4. Example of Projection onto Components

One side effect of the method is apparent by doing a little arithmetic. The projection and assembly method will project three modes onto each body. Assembling the components into a system gives two constraint relations (to match the halves of the rigid body together in angle and offset). When the reduced order component models are assembled, the reduced order system model will have four modes (3+3-2). These four include the three desired modes plus one "extraneous mode."

ANALYSIS
Component Equations of Motion

Assume we have n_b bodies or components. The unconstrained equations of motion of each may be expressed as:

$$M_i \ddot{x}_i + K_i x_i = G_i u, \qquad i = 1, \cdots, n_b \qquad\qquad 1.$$

where

i is the body index,

x_i is a set of generalized coordinates describing the motion of body i as a free body in inertial space. This set of coordinates can be anything from geometric coordinates to free-free normal modes to cantilever modes augmented by six rigid body modes for the fixed end,

M_i is the generalized mass matrix for body i,

K_i is the generalized stiffness matrix for body i,

u is the set of control inputs,

G_i is the control distribution matrix for body i, and

n_b is the number of bodies.

System Equations of Motion

A multibody system is created by constraining the components to share certain common motions and by adding flexible connections between bodies. Assume that the constraints can be described in the form:

$$AX = 0 \qquad\qquad 2.$$

where

$$A = [\, A_1 \; A_2 \cdots A_{n_b} \,], \qquad X^T = [\, x_1^T x_2^T \cdots x_{n_b}^T \,]$$

Let n_c be the number of constraint equations in Eq. 2. The constraints may be introduced into the equations of motion using a vector, Λ, of Lagrange multipliers. The constrained system is:

$$M_i \ddot{x}_i + K_i x_i = G_i u + A_i^T \Lambda, \qquad i = 1, \cdots, n_b \qquad\qquad 3.$$

$$A\dot{X} = 0. \qquad\qquad 4.$$

Let P be any full rank matrix mapping a minimal system state, x, into X:

$$X = Px, \qquad \text{or} \qquad x_i = P_i x, \quad i = 1, \cdots, n_b \qquad\qquad 5.$$

The constraint equation becomes:

$$AP\dot{x} = 0. \qquad\qquad 6.$$

Since the states x are independent, $AP=0$. Once P is chosen so that Eq. 6 is satisfied, the constraint equation (Eq. 4) is automatically satisfied. Inserting Eq. 5 into Eqs. 3 and pre-multiplying by P_i^T gives:

$$P_i^T M_i P_i \ddot{x} + P_i^T K_i P_i x = P_i^T G_i u + P_i^T A_i^T \Lambda, \qquad i = 1, \cdots, n_b \qquad \text{7.}$$

Summing over i:

$$M \ddot{x} + Kx = Gu \qquad \text{8.}$$

Where

$$M = \sum_{i=1}^{n_b} P_i^T M_i P_i \qquad\qquad K = \sum_{i=1}^{n_b} P_i^T K_i P_i \qquad \text{9.}$$

$$G = \sum_{i=1}^{n_b} P_i^T G_i \qquad \text{10.}$$

Equation 8 is the system equation of motion incorporating all constraints. Converting Eq. 8 to modal form:

$$x = \Phi q \qquad \text{11.}$$

$$\ddot{q} + \Omega^2 q = \Phi^T Gu . \qquad \text{12.}$$

<u>System Model Reduction</u>

Assume we choose some model reduction method which yields as its output a set of n_R modes, q_R, to be retained with the remaining set of n_Z modes, q_Z, to be zeroed. Then we can partition Φ:

$$x = [\Phi_R \ \Phi_Z] \begin{bmatrix} q_R \\ q_Z \end{bmatrix}. \qquad \text{13.}$$

Or, setting $q_Z = 0$, the reduced order system model is:

$$\ddot{q}_R + \Omega_R^2 q_R = \Phi_R^T Gu \qquad \text{14.}$$

$$x = \Phi_R q_R \qquad \text{15.}$$

If $\Omega^2 = \text{diag}(\omega_j^2)$ then a homogeneous solution to Eq. 12, and therefore also a solution to the system of equations

3 & 4, is $q = e_j \cos(\omega_j t)$. Each of x_i, $\ddot{x}_i$ and Λ will similarly be described by sinusoids:

$$x_i = P_i \Phi_R e_j \cos(\omega_j t), \qquad \ddot{x}_i = P_i \Phi_R e_j (-\omega_j^2)\cos(\omega_j t), \qquad \Lambda = \Lambda_{oj} \cos(\omega_j t).$$

Inserting the above into Eq. 3 for $u=0$ gives a relation which will be needed in a later derivation:

$$\left[M_i(-\omega_j^2) + K_i \right] P_i \Phi_R e_j = A_i^T \Lambda_{oj} \qquad i = 1, \cdots, n_b \qquad \text{16.}$$

Component Model Reduction

None of the above is unique to the projection and assembly method, which uses the above as a starting point. The concept is as follows: Cause each component to have, as an allowable motion, the mode shape of each retained mode projected onto the component. When the system is reassembled from reduced order components, the retained mode will still be an admissible motion of the reduced order system. In the following, it will be shown that in addition to being an admissible motion of the reduced order system, it is a mode of the reduced order system.

Consider the projection of q_R onto component i. Using Eqs. 5 and 15:

$$x_i = P_i \Phi_R q_R$$

17.

In general, q_R should be of lower order than x_i. Where before, component i had n_i degrees of freedom, Eq. 15 restricts the motion to n_R degrees of freedom. Let x_{Ri} be a set of component i modes that span the space of component motions allowed by Eq. 15. In Ref. 10, the choice: $x_{Ri} = q_R$ is made, so:

$$x_i = P_i \Phi_R x_{Ri}.$$

18.

Implicit in this choice is the assumption that the matrix $P_i \Phi_R$ is of full column rank. This assumption is violated in a number of situations. The most obvious case is when one component has fewer degrees of freedom than the number of modes in Φ_R. Other examples arise when the projections of the modes are linearly dependant within the subspace of a particular component. In a later section of this paper, an alternative choice for x_{Ri} is explored for situations where $P_i \Phi_R$ is not of full rank. Writing the component equations of motion (Eq. 3) and constraint relation (Eq. 4) in terms of the x_{Ri}:

$$M_{Ri} \ddot{x}_{Ri} + K_{Ri} x_{Ri} = G_{Ri} u + A_{Ri}^T \lambda, \qquad i = 1, \cdots, n_b$$

19.

$$A_R \dot{X}_R = \sum_{i=1}^{n_b} A_{Ri} \dot{x}_{Ri} = 0,$$

20.

where

$$M_{Ri} = \Phi_R^T P_i^T M_i P_i \Phi_R \qquad\qquad K_{Ri} = \Phi_R^T P_i^T K_i P_i \Phi_R$$

21.

$$G_{Ri} = \Phi_R^T P_i^T G_i \qquad\qquad A_{Ri} = A_i P_i \Phi_R$$

22.

and A_R and X_R are defined in the same manner as A and X. This system of equations in x_{Ri} and λ may be formulated in terms of a minimal set of states, X_R, with some mapping P_R.

$$X_R = P_R x_R.$$

23.

With this choice, Eq. 20 becomes:

$$A_R P_R x_R = 0.$$

24.

Since the x_R are independent:

$$A_R P_R = 0.$$
<div align="right">25.</div>

In actual practice, P_R has a specific form, but to understand the behavior of the reduced order system, we can consider any P_R which is of full rank and satisfies Eq. 25. If $X_{R1} = X_{R2} = \cdots = X_{Rn_b}$, as will be the case for the desired retained modes, then Eq. 20 becomes:

$$\left(\sum_{i=1}^{n_b} A_i P_i \right) \Phi_R{}^I X_{R1} = 0$$

which is automatically satisfied in view of Eq. 6. This suggests that a partial choice for P_R is the column: $[I \, I \, \cdots \, I \,]^{\mathrm{T}}$. One full rank P_R which satisfies Eq. 20 may be created by taking the singular value decomposition of a portion of A_R:

$$[A_{R2} \cdots A_{Rn_b}] = U_A \Sigma_A V_A^T = U_A [\Sigma_{A1} \ 0] \begin{bmatrix} V_{A1}^T \\ V_{A2}^T \end{bmatrix} = U_A \Sigma_{A1} V_{A1}^T$$
<div align="right">26.</div>

and choosing:

$$P_R^T = \begin{bmatrix} I & I & \cdots & I \\ 0 & & V_{A2}^T & \end{bmatrix}$$
<div align="right">27.</div>

so

$$A_R P_R = \left[\ \left(\sum_{i=1}^{n_b} A_{Ri} \right) \quad [A_{R2} \cdots A_{Rn_b}] V_{A2} \ \right]$$

$$= \left[\ \left(\sum_{i=1}^{n_b} A_i P_i \right) \Phi_R \quad U_A \Sigma_{A1} V_{A1}^T V_{A2} \ \right] = 0$$

as desired. Furthermore, P_R is of full rank by construction.

Starting from Eq. 19, the equations of motion in terms of X_R are:

$$\sum_{i=1}^{n_b} P_{Ri}^T M_{Ri} P_{Ri} \ddot{X}_R + \sum_{i=1}^{n_b} P_{Ri}^T K_{Ri} P_{Ri} X_R = \sum_{i=1}^{n_b} P_{Ri}^T G_{Ri} u + P_R^T A_R^T \Lambda.$$
<div align="right">28.</div>

The form of P_R suggests a partitioning of X_R and P_R into desired and extra states:

$$X_R = \begin{bmatrix} x_D \\ x_E \end{bmatrix}, \qquad P_R = [P_{RD} \ \ P_{RE}]$$

where $P_{RDi} = I$ and $P_{RE} = \begin{bmatrix} 0 \\ V_{A2} \end{bmatrix}$. In partitioned form, Eq. 28 is:

$$\begin{bmatrix} \sum_{i=1}^{n_b} M_{Ri} & \sum_{i=1}^{n_b} M_{Ri} P_{REi} \\ \sum_{i=1}^{n_b} P_{REi}^T M_{Ri} & \sum_{i=1}^{n_b} P_{REi}^T M_{Ri} P_{REi} \end{bmatrix} \begin{bmatrix} \ddot{x}_D \\ \ddot{x}_E \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n_b} K_{Ri} & \sum_{i=1}^{n_b} K_{Ri} P_{REi} \\ \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} & \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} P_{REi} \end{bmatrix} \begin{bmatrix} x_D \\ x_E \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n_b} G_{Ri} \\ \sum_{i=1}^{n_b} P_{REi}^T G_{Ri} \end{bmatrix} u. \qquad\qquad 29.$$

By construction, the system is capable of taking the shape of any of the n_R desired modes. It remains to be shown that the x_D are free-free normal modes of the reduced order system. To show that they are requires only that

$x_R = \begin{bmatrix} e_j \\ 0 \end{bmatrix} \cos\omega_j t$ be a solution of Eq. 29 with $u = 0$. Assume $x_R = \begin{bmatrix} e_j \\ 0 \end{bmatrix} \cos\omega_j t$, then: $\ddot{x}_R = (-\omega_j^2) x_R$ and

Eq. 29 becomes two equations:

$$\begin{bmatrix} \sum_{i=1}^{n_b} M_{Ri}(-\omega_j^2) + \sum_{i=1}^{n_b} K_{Ri} \end{bmatrix} e_j \cos\omega_j t \overset{?}{=} 0 \qquad\qquad 30.$$

$$\begin{bmatrix} \sum_{i=1}^{n_b} P_{REi}^T M_{Ri}(-\omega_j^2) + \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} \end{bmatrix} e_j \cos\omega_j t \overset{?}{=} 0. \qquad\qquad 31.$$

If both left-hand sides in the above equations evaluate to zero, then the desired modes are modes of the reduced order system. Consider ΣM_{Ri} and ΣK_{Ri}:

$$\sum_{i=1}^{n_b} M_{Ri} \underset{(21)}{=} \sum_{i=1}^{n_b} \Phi_R^T P_i^T M_i P_i \Phi_R = \Phi_R^T \sum_{i=1}^{n_b} P_i^T M_i P_i \Phi_R \underset{(9)}{=} \Phi_R^T M \Phi_R = I.$$

Similarly,

$$\sum_{i=1}^{n_b} K_{Ri} = \Omega^2.$$

Eq. 30 becomes:

$$((-\omega_j^2)I + \Omega^2)e_j = (\omega_j^2 - \omega_j^2)e_j = 0$$

and so is satisfied. Consider Eq. 31:

$$\left[\sum_{i=1}^{n_b} P_{REi}^T M_{Ri}(-\omega_j^2) + \sum_{i=1}^{n_b} P_{REi}^T K_{Ri}\right] e_j \underset{(21)}{=} \left[\sum_{i=1}^{n_b} P_{REi}^T \Phi_R^T P_i^T \left(M_i(-\omega_j^2) + K_i\right) P_i \Phi_R\right] e_j$$

$$\underset{(16)}{=} \sum_{i=1}^{n_b} P_{REi}^T \Phi_R^T P_i^T A_i^T \Lambda_{oj} \underset{(22)}{=} \sum_{i=1}^{n_b} P_{REi}^T A_{Ri}^T \Lambda_{oj} = [0 \ \ V_{A2}^T] A_R^T \Lambda_{oj} \underset{(26)}{=} 0 \Lambda_{oj} = 0.$$

Therefore the desired mode shapes and frequencies satisfy Eq. 29 and thus are normal modes of the reassembled reduced order system.

Component Model Reduction—Extended Method

As mentioned above, the choice: $x_{Ri} = q_R$ depends on the matrix $P_i \Phi_R$ being of full column rank. When this is not the case, the method can be extended to allow model reduction to proceed. Consider the singular value decomposition of $P_i \Phi_R$, suppressing the index i on the products in the SVD, let r be the rank of $P_i \Phi_R$, let n_R be the rank of Φ_R, and let n_i be the rank of P_i (and the number of states in x_i). If $r = n_R > n_i$ (Ref. 10)

$$P_i \Phi_R = U \Sigma V^T = [U_1 \ \ U_2] \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} V^T = U_1 \Sigma_1 V^T.$$

If $r = n_i > n_R$ (body i has few DOF)

$$P_i \Phi_R = U \Sigma V^T = U[\Sigma_1 \ \ 0] \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U \Sigma_1 V_1^T.$$

If $r < n_i$, $r < n_R$ (linear dependant projected modes)

$$P_i \Phi_R = U \Sigma V^T = [U_1 \ \ U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_1 V_1^T$$

To ensure that the set x_{Ri} is an independent set spanning the space of component motions, choose $x_{Ri} = \Sigma_1(i) V_1^T(i) q_R$. In the event that $r(i) = n_R$, $V_1^T(i)$ becomes $V(i)$. This choice of x_{Ri} gives for Eq. 18:

$$x_i = U_1(i) x_{Ri} \qquad\qquad\qquad \text{18A.}$$

In the event that $r(i) = n_i$, $U_1(i)$ becomes $U(i)$. Define $Q_i = U_1(i)$. Eqs. 21–22 now take the form:

$$M_{Ri} = Q_i^T M_i Q_i \qquad\qquad K_{Ri} = Q_i^T K_i Q_i \qquad\qquad \text{21A.}$$

$$G_{Ri} = Q_i^T G_i \qquad\qquad A_{Ri} = A_i Q_i \qquad\qquad \text{22A.}$$

Choosing:

$$P_R^T = \begin{bmatrix} V_1(1)\Sigma_1(1) & V_1(2)\Sigma_1(2) & \cdots & V_1(n_b)\Sigma_1(n_b) \\ 0 & & V_{A2}^T & \end{bmatrix} \qquad \text{27A.}$$

gives as desired, $A_R P_R = 0$. Moreover, P_R is again full rank by construction. Partition P_R as before:

$$P_R = \begin{bmatrix} P_{RD} & P_{RE} \end{bmatrix}, \text{ where } P_{RDi} = \Sigma_1(i) V_1^T(i) \text{ and } P_{RE} = \begin{bmatrix} 0 \\ V_{A2} \end{bmatrix}. \text{ Eq. 29 becomes:}$$

$$\begin{bmatrix} \sum_{i=1}^{n_b} P_{RDi}^T M_{Ri} P_{RDi} & \sum_{i=1}^{n_b} P_{RDi}^T M_{Ri} P_{REi} \\ \sum_{i=1}^{n_b} P_{REi}^T M_{Ri} P_{RDi} & \sum_{i=1}^{n_b} P_{REi}^T M_{Ri} P_{REi} \end{bmatrix} \begin{bmatrix} \ddot{x}_D \\ \ddot{x}_E \end{bmatrix}$$

$$+ \begin{bmatrix} \sum_{i=1}^{n_b} P_{RDi}^T K_{Ri} P_{RDi} & \sum_{i=1}^{n_b} P_{RDi}^T K_{Ri} P_{REi} \\ \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} P_{RDi} & \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} P_{REi} \end{bmatrix} \begin{bmatrix} x_D \\ x_E \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n_b} P_{RDi}^T G_{Ri} \\ \sum_{i=1}^{n_b} P_{REi}^T G_{Ri} \end{bmatrix} u \qquad \text{29A.}$$

and the proof that the desired modes are normal modes of the reassembled reduced order system proceeds exactly as before, using the above definition of P_{RDi} and Eqs. 21A and 22A.

SIMPLE EXAMPLES
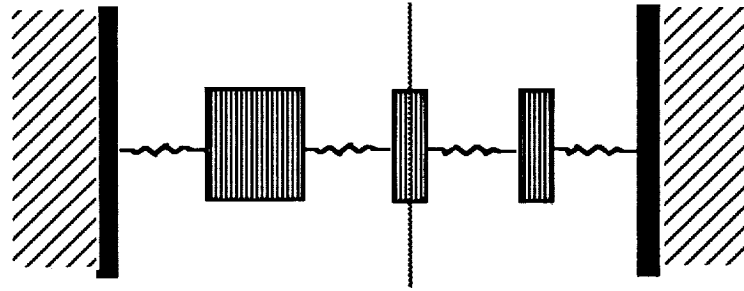### One Dimensional Three Disk Example

Fig. 5. One Dimensional Three Disk Example

Consider Fig. 5. In this example, there are three disks, with rotational displacements (from left to right) y_1, y_2, and y_3 and inertias $4J$, J, and J connected to ground and each other by torsion rods of equal spring constant, k. We choose to consider this simple system as being composed of two simpler subsystems of components. We divide the middle disk in half and allocate one half to each subsystem. Subsystem 1 contains the large disk and the left half of the middle disk. Take $x_1 = [y_1, y_2]^T$. Subsystem 2 contains the rest of the system. Take $x_2 = [y_2, y_3]^T$. Choosing units to make J and k equal to unity, the mass and stiffness matrices for each component are:

$$M_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

The constraint relation that connects the subsystems is that $X_1(2) = X_2(1)$. Expressed in terms of a constraint matrix, A:

$$A = [\ A_1 \quad A_2\ ], \quad [\ A_1 \quad A_2\ ]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

with

$$A_1 = [\ 0 \quad 1\ ], \quad A_2 = [-1 \quad 0\ ].$$

One choice of P which reduces this to minimal form is:

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0.7071 & 0 \\ 0 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system mass and stiffness matrices are:

$$M = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & 0.7071 & 0 \\ 0.7071 & 1 & -0.7071 \\ 0 & -0.7071 & 2 \end{bmatrix}.$$

The eigenvalue and eigenvector matrices for this system are:

$$\Omega^2 = \begin{bmatrix} 0.2803 & 0 & 0 \\ 0 & 1.1694 & 0 \\ 0 & 0 & 3.0502 \end{bmatrix}; \quad \Phi = \begin{bmatrix} 0.4457 & -0.2153 & 0.0703 \\ -0.5539 & -0.8155 & 1.0140 \\ -0.2277 & -0.6943 & -0.6827 \end{bmatrix}.$$

System model reduction: Assume we wish to capture only the lowest frequency system mode ($\Omega^2 = 0.2803$), then

$$\Phi_R = \begin{bmatrix} 0.4457 \\ -0.5539 \\ -0.2277 \end{bmatrix}.$$

Component model reduction: choose $X_{R1} = X_{R2} = q_R$, so

$$x_1 = P_1\Phi_R x_{R1} = \begin{bmatrix} -0.4457 \\ -0.3917 \end{bmatrix} x_{R1} \qquad x_2 = P_2\Phi_R x_{R2} = \begin{bmatrix} -0.3917 \\ -0.2277 \end{bmatrix} x_{R2}$$

and the reduced order component mass and stiffness matrices are:

$$M_{R1} = [0.8714], \quad K_{R1} = [0.2016], \quad M_{R2} = [0.1286], \quad K_{R2} = [0.0787].$$

The reduced order constraint matrix is:

$$A_R = [\ -0.3916 \quad 0.3916\ ].$$

Choose P_R:

$$P_R = \begin{bmatrix} -0.7071 \\ -0.7071 \end{bmatrix}.$$

This gives the reduced order system:

$$[0.5]\ddot{x}_R + [0.1402]x_R = 0.$$

Which has a single eigenvalue at $\Omega^2 = 0.2803$. In this case, no extra modes are created because it happens that ($2n_R$ - # constraints) = n_R. This is not true in general. The next example produces extra modes.
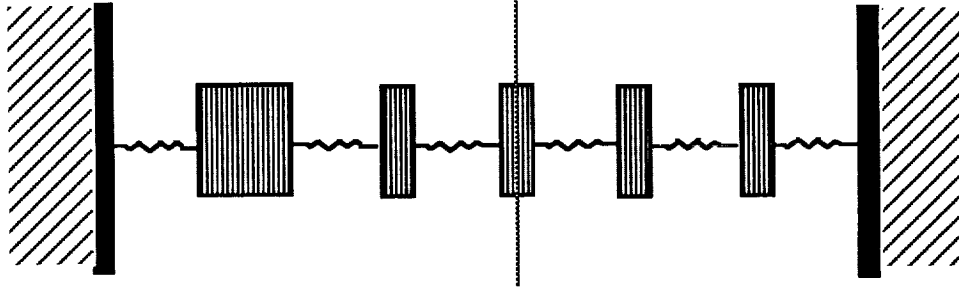
One Dimensional Five Disk Example

Fig. 6. One Dimensional Five Disk Example

Consider Fig. 6. In this example, there are five disks, with displacements (from left to right) y_1, y_2, y_3, y_4, and y_5 and inertias $4J$, J, J, J, and J connected to ground and each other by torsion rods of equal spring constant, k. We choose to consider this simple system as being composed of two simpler subsystems of components. We divide the middle disk in half and allocate one half to each subsystem. Subsystem 1 contains the large disk through the left half of the middle disk. Take $x_1 = [y_1, y_2, y_3]^T$. Subsystem 2 contains the rest of the system. Take $x_2 = [y_3, y_4, y_5]^T$. Choosing units to make J and k equal to unity, the mass and stiffness matrices for each component are:

$$
M_1 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad
M_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad
K_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad
K_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.
$$

The constraint relation that connects the subsystems is that $x_1(3) = x_2(1)$. Expressed in terms of a constraint matrix, A:

$$
A = [\, A_1 \quad A_2 \,], \quad [\, A_1 \quad A_2 \,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,
$$

with

$$
A_1 = [0 \ 0 \ 1], \quad A_2 = [-1 \ 0 \ 0].
$$

One choice of P which reduces this to minimal form is:

$$
P = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -0.7071 & 0.7071 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

The system mass and stiffness matrices are:

$$
M = \begin{bmatrix}
4 & 0 & 0 & 0 & 0 \\
0 & 0.75 & -0.25 & 0 & 0 \\
0 & -0.25 & 0.75 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad
K = \begin{bmatrix}
2 & -0.7071 & 0.71 & 0 & 0 \\
-0.7071 & 2.2071 & -0.5 & -0.5 & 0 \\
0.7071 & -0.5 & 0.7929 & -0.5 & 0 \\
0 & -0.5 & -0.5 & 2 & -1 \\
0 & 0 & 0 & -1 & 2
\end{bmatrix}.
$$

The eigenvalue and eigenvector matrices for this system are:

$$\Omega^2 = \begin{bmatrix} 0.1933 & 0 & 0 & 0 & 0 \\ 0 & 0.5466 & 0 & 0 & 0 \\ 0 & 0 & 1.4696 & 0 & 0 \\ 0 & 0 & 0 & 2.6609 & 0 \\ 0 & 0 & 0 & 0 & 3.6296 \end{bmatrix},$$

$$\Phi = \begin{bmatrix} -0.3489 & 0.3208 & -0.1392 & -0.0704 & -0.0310 \\ 0.1217 & 0.3655 & -0.0438 & 0.7626 & 0.8765 \\ 0.7270 & 0.4501 & -0.8075 & -0.0986 & 0.3272 \\ 0.3386 & 0.5329 & 0.3142 & 0.3895 & -0.5924 \\ 0.1874 & 0.3666 & 0.5924 & -0.5894 & 0.3635 \end{bmatrix}.$$

System model reduction: Assume we wish to capture only the two lowest frequency system modes ($\Omega^2 = 0.1933$, 0.5466), then

$$\Phi_R = \begin{bmatrix} -0.3489 & 0.3208 \\ 0.1217 & 0.3655 \\ 0.7270 & 0.4501 \\ 0.3386 & 0.5329 \\ 0.1874 & 0.3666 \end{bmatrix}.$$

Component model reduction: choose $x_{R1} = x_{R2} = q_R$, so

$$x_1 = P_1 \Phi_R x_{R1} = \begin{bmatrix} 0.3489 & -0.3208 \\ 0.4280 & 0.0598 \\ 0.4244 & 0.4078 \end{bmatrix} x_{R1}, \quad x_2 = P_2 \Phi_R x_{R2} = \begin{bmatrix} 0.4244 & 0.4078 \\ 0.3386 & 0.5329 \\ 0.1874 & 0.3666 \end{bmatrix} x_{R2}.$$

and the reduced order component mass and stiffness matrices are:

$$M_{R1} = \begin{bmatrix} 0.7602 & -0.3357 \\ -0.3357 & 0.4985 \end{bmatrix}, \quad K_{R1} = \begin{bmatrix} 0.1280 & -0.0831 \\ -0.0831 & 0.3689 \end{bmatrix},$$

$$M_{R2} = \begin{bmatrix} 0.2398 & 0.3357 \\ 0.3357 & 0.5015 \end{bmatrix}, \quad K_{R2} = \begin{bmatrix} 0.0653 & 0.0831 \\ 0.0831 & 0.1777 \end{bmatrix}.$$

The reduced order constraint matrix is:

$$A_R = [\, 0.4243 \quad 0.4078 \quad -0.4243 \quad -0.4078 \,].$$

Choose P_R:

$$P_R = \begin{bmatrix} -0.8603 & 0 & 0 \\ 0.2904 & 0.5926 & 0.5696 \\ -0.3021 & 0.7762 & -0.2151 \\ -0.2904 & -0.2151 & 0.7933 \end{bmatrix}.$$

This gives the reduced order system:

$$\begin{bmatrix} 0.8954 & 0.1781 & 0.0875 \\ 0.1781 & 0.2307 & 0.2649 \\ 0.0875 & 0.2649 & 0.3739 \end{bmatrix} \ddot{x}_R + \begin{bmatrix} 0.2029 & 0.0883 & 0.0503 \\ 0.0883 & 0.1494 & 0.1383 \\ 0.0503 & 0.1383 & 0.2062 \end{bmatrix} x_R = 0,$$

which has three eigenvalues at $\Omega^2 = (0.1933, 0.5466, 1.6873)$. The first two are the desired system modes, while the third does not match any of the original system modes; it is an "extraneous mode."

SUMMARY

In this paper the model reduction method described by Eke and Man in Ref. 10 has been analyzed to demonstrate why the desired modes are returned exactly. An explicit set of necessary conditions involving the rank of the projection matrix has been presented, and an extension to the method has been proposed which removes those conditions. The method was demonstrated using two simple examples.

Future work will address extending the method to handle variable configuration systems such as those with multiple articulation angles, better characterizing the "extraneous" modes which are a by-product of this method, and examining scaling issues which will arise when relative sizes of singular values are used to determine how many independent modes are projected onto a component.

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