Kharitonov's Theorem: Generalizations and Algorithms

Abstract

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In 1978, the Russian mathematician V. Kharitonov published a remarkably simple necessary and sufficient condition in order that a rectangular parallelpiped of polynomials be a stable set. Here, "stable" is taken to mean that the polynomials have no roots in the closed right-half of the complex plane.

The possibility of generalizing this result has been studied by numerous authors. We are given a set, Q, of polynomials and we seek a necessary and sufficient condition that the set be stable. Perhaps the most general result is due to Barmish who takes for Q a polytope and proceeds to construct a complicated non-linear function, H, of the points in Q. With the notion of stability we have adopted, Barmish asks that we "sweep the boundary" of the closed right-half plane, that is we consider the set $G = \{j\omega | -\infty < \omega < \infty \}$, and for each $j\delta \in G$, require $H(\delta) > 0$.

Barmish's scheme has the merit that it describes a true generalization of Kharitonov's theorem. On the other hand, even when Q is a polyhedron, the definition of H requires that one do an optimization over the entire set of vertices, and then a subsequent optimization over an auxiliary parameter.

In the present work, we consider only case where Q is a polyhedron and use the standard definition of stability described above. There are straightforward generalizations of the method to the case of discrete stability or to cases where certain root positions are deemed desirable. The cases where Q is non-polyhderal are less certain as candidates for the method. Essentially, we apply a method of geometric programming to problem of finding maximum and minimum angular displacements of points in the "Nyquist locus", $\{Q(j\omega) | -\infty < \omega < \infty \}$. There is an obvious connection here with the boundary sweeping requirement of Barmish.

Presuming that we have a polygonal set of real polynomials, we can begin by looking at {Q(0)}, a line interval, J. Necessary for the stability of Q is that J = [a,b], where a >0. We therefore do the linear programming problems which minimize and maximize Re(Q(0)). (Actually only the minimization would be necessary). Using these points as starting values, we perturb the frequency, say to $\omega = .1$, and look for the vertices in Q whose image under eval(jω) have largest and smallest values. For stability, both must be positive. We can employ an LP-like technique to solve this problem.

Begin, respectively with vertices from the $\omega = 0$ problem for largest and smallest constants. Call these V_1 and V_2. Draw the two vectors in the complex plane from 0 to V_1 and V_2 respectively, and then construct two normals one pointing counterclockwise, and one clockwise. By pulling these two normals back to polynomial space by the transpose of the evaluation matrix,

$$\begin{bmatrix} 1 & 0 & -\omega^2 & 0 & \omega^4 \dots \\ 0 & \omega & 0 & -\omega^3 & 0 \dots \end{bmatrix}$$

we can obtain "objective functions" for use in an LP step for, respectively, maximum and minium angle of rotation. Note that the objective function changes from vertex to vertex.

Since the image of the polyhedron changes smoothly as the frequency is swept along the imaginary axis, there should be only occasional changes in vertex, and in general, we should see only single pivots required when there is a vertex change needed. Finally, we simply test for the presence of a "0-penetration" of the convex set Q(jω) by testing the size of the angular opening:

Max(angle) - min(angle) < π

is necessary and sufficient for global stability of Q.

References

[1] Barmish, B.R. "A generalization of Kharitonov's four-polynomial concept for robust..." *IEEE Trans. A. C.* Vol 34, pp 157-165. 1989
[2] Kharitonov, V.L. "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations." *Differential'nye Uraveniya,* Vol 14. ppp 1483-1485, 1978.