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Forbidden Tangential Orbit Transfers Between Intersecting Keplerian Orbits

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LIST OF SYMBOLS

Latin Symbols

a_i	Semi-major axis of orbit No. i
e_i	Eccentricity of orbit No. i
f	Generic function
g	Generic function
i	Dummy subscript of the range 1 to 3
p_1, p_2, p_3	Functions defined by equations (111), (112), and (113), respectively
q	Quantity defined by either equation (38) or by equation (86) depending upon the case under consideration
Q_i	Quantity defined by equation (11)
r_i	Magnitude of radius vector or orbit No. i
r	Substitution made for algebraic convenience
R	Substitution made for algebraic convenience via equation (125)
x	Cartesian coordinate

Greek Symbols

α	Function defined by equation (119); the angular measure of the crossing points of the intersecting orbits.
γ	Quantity defined by equation (95)
ε	A positive real number which may be chosen to be arbitrarily small
ζ	Angular measure function defined by equation (138)
ϑ	Rotation angle of the apogee of the target orbit
θ	Generic angle
ξ	Angular measure function defined by equation (150)

ρ	Rotation angle of the apogee of the transfer orbit
σ	Angular measure function defined by equation (156)
τ	Angular measure function defined by equation (157)
ϕ_i	True anomaly of orbit No. i
ω	Angular measure function defined by equation (40)

Special Symbols

ex	(Subscript) indicates an extremal value
$+$	(Superscript) indicates the value of a quantity just after the point in question
$-$	(Superscript) indicates the value of a quantity just before the point in question
$\rightarrow$	Indicates a limiting value
$\Longrightarrow$	Implication
mod	Modulus

TECHNICAL PAPER

FORBIDDEN TANGENTIAL ORBIT TRANSFERS BETWEEN INTERSECTING KEPLERIAN ORBITS

INTRODUCTION

Although the subject of impulsive transfers between Keplerian orbits has been treated in many papers (e.g. [1,2]), there has been no demonstration that certain transfers are not allowed. In this paper, it is shown that, for planar intersecting orbits, the tangential impulse transfer cannot always be utilized. The treatment is formulated as purely geometric in nature and no consideration is given to optimality.

For the two-impulse transfer between two coplanar elliptical orbits, there exists a triple infinity of elliptic orbits connecting the given orbits. In the case of realistic space transfer maneuvers, the optimum impulses are contained in a small useful angle between the tangential direction and the local horizontal direction as seen on the transfer orbit [3]. As a practical application, we may search for the optimum cotangential transfer as a sub-optimum solution.

The tangency conditions reduce the triple infinity of solutions to a single infinity of solutions so we are at liberty to parameterize these solutions by one convenient variable; the true anomaly of the initial orbit is chosen. It is then demonstrated that the descriptors of the transfer orbit can either be determined directly in terms of this parameter or eliminated from the discussion. Previous studies relied on a series of sequential calculations, and this obscured the fact that singularities can sometimes occur in the semi-major axis and eccentricities of the transfer orbit.

The arguments which exclude certain transfers for intersecting orbits are shown to be inapplicable for the case of non-intersecting orbits; singularities occur only for transfers between intersecting orbits.

I thank the (anonymous) reviewer who pointed out an interesting geometric argument based on the work of Professor Busemann [4] for Reference 5. The comments that were returned to me used a Busemann space construction to show that if the initial and final orbits do not intersect then the transfer solution is always available; in the case of intersecting orbits, forbidden transfer regions can occur. This entirely agrees with the results presented here.

The exact conditions for the occurrence of forbidden transfers are presented in this paper. Their existence provides a new result in a classical subject.

PROBLEM FORMULATION

In the context of the (planar) Keplerian two body problem, define a cartesian coordinate system with the attracting primary at the origin. The initial orbit is oriented on this coordinate system in such a way that its perigee occurs along the x-axis. The initial orbit will be taken to have an eccentricity e_1 and a semi-major axis a_1. The final orbit has corresponding parameters e_3 and a_3 and is oriented in such a way that its semi-major axis is rotated clockwise through an angle ϑ from the negative x-axis. See Figure 1.

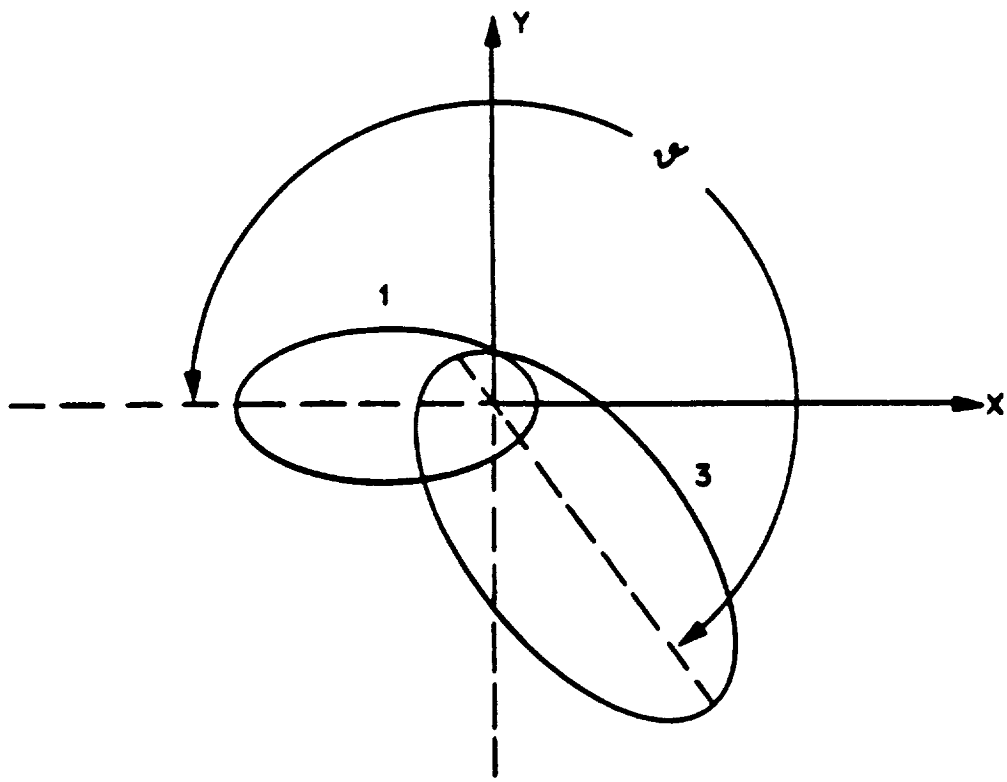

Figure 1

An orbit transfer from orbit No. 1 to orbit No. 3 can occur in many ways but only a subset of general transfers will be considered here, namely tangential transfers. The geometry of a transfer orbit can now be taken as equivalent to constructing a confocal conic section from some point, ϕ_1, on the initial orbit to a point ϕ_2 on the target orbit.

Four constraints exist, i.e., the radius of the transfer orbit must match at the initial and final points and the velocities must be colinear at the same points. The unknowns of the problem include the semi-major axis and eccentricity of the transfer orbit, the rotation (phase) angle of the transfer orbit, the angle ϕ_1 (the angular leave point) and the angle ϕ_2 (the angular arrival point). Since there are five unknowns and only four constraints, it is apparent that one free parameter exists and it is reasonable to parameterize with respect to the leave angle, ϕ_1. Figure 2 shows a typical transfer orbit.

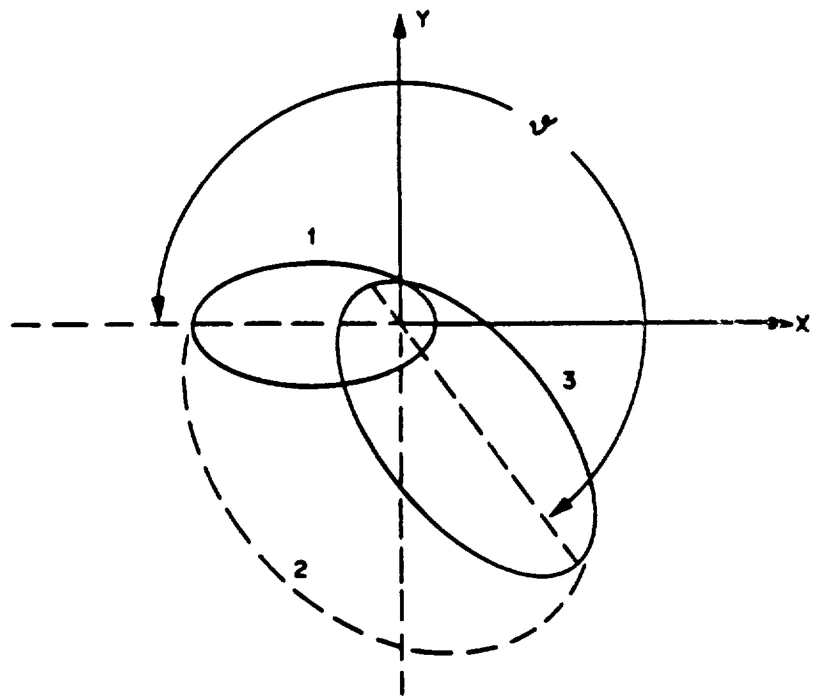

Figure 2.

To proceed with a quantitative formulation, the three orbits can be characterized by

$$r_1 = \frac{a_1\,(1 - e_1^2)}{1 + e_1 \cos \phi} \tag{1}$$

$$r_2 = \frac{a_2\,(1 - e_2^2)}{1 + e_2 \cos(\phi + \rho)} \tag{2}$$

$$r_3 = \frac{a_3\,(1 - e_3^2)}{1 + e_3 \cos(\phi + \vartheta)} \tag{3}$$

where r_1, r_2, r_3 are the radii of the three orbits, a_1, a_2, a_3 and e_1, e_2, e_3 are the semi-major axes and eccentricities of those orbits. The angle ρ is the rotation angle of the apogee of the transfer orbit, ϑ is the rotation angle of the apogee of the final orbit, and ϕ is the true anomaly of the initial orbit.

From these

$$\frac{\partial r_1}{\partial \phi} = \frac{r_1^2\, e_1 \sin \phi}{a_1\,(1 - e_1^2)} \tag{4}$$

$$\frac{\partial r_2}{\partial \phi} = \frac{r_2^2\, e_2 \sin(\phi + \rho)}{a_2\,(1 - e_2^2)} \tag{5}$$

$$\frac{\partial r_3}{\partial \phi} = \frac{r_3^2\, e_3 \sin(\phi + \vartheta)}{a_3\,(1 - e_3^2)} \tag{6}$$

Equating the radii and requiring tangency at a generic point ϕ_1 gives the first two constraint equations

$$\frac{a_1\,(1 - e_1^2)}{1 + e_1 \cos \phi_1} = \frac{a_1\,(1 - e_2^2)}{1 + e_2 \cos(\phi_1 + \rho)} \tag{7}$$

$$\frac{r_1^2\, e_1 \sin \phi_1}{a_1(1 - e_1^2)} = \frac{r_2^2\, e_2 \sin(\phi_1 + \rho)}{a_2(1 - e_2^2)} \tag{8}$$

and the second two constaint equations come from a similar generic point ϕ_2, i.e.,

$$\frac{a_2(1 - e_2^2)}{1 + e_2 \cos(\phi_2 + \rho)} = \frac{a_3(1 - e_3^2)}{1 + e_3 \cos(\phi_2 + \vartheta)} \tag{9}$$

$$\frac{r_2^2\, e_2 \sin(\phi_2 + \rho)}{a_2(1 - e_2^2)} = \frac{r_3^2\, e_3 \sin(\phi_2 + \vartheta)}{a_3(1 - e_3^2)} \tag{10}$$

Since $r_1(\phi_1) = r_2(\phi_1)$ and $r_2(\phi_2) = r_3(\phi_2)$ and abbreviating

$$Q_i = \frac{1}{a_i(1 - e_i^2)} \tag{11}$$

equations (7) through (10) can be written as

$$Q_1(1 + e_1 \cos \phi_1) = Q_2[1 + e_2 \cos(\phi_1 + \rho)] \tag{12}$$

$$Q_1 e_1 \sin\phi_1 = Q_2 e_2 \sin(\phi_1 + \rho) \tag{13}$$

$$Q_2[1 + e_2 \cos(\phi_2 + \rho)] = Q_3[1 + e_3 \cos(\phi_2 + \vartheta)] \tag{14}$$

$$Q_2 e_2 \sin(\phi_2 + \rho) = Q_3 e_3 \sin(\phi_2 + \vartheta) \tag{15}$$

It should be noted that equations (12) through (15) constitute four equations in five unknowns, i.e., Q_2, e_2, ρ, ϕ_2 and the parameter ϕ_1.

The first solution to those equations which were obtained was the most general case, i.e., $e_1 \neq 0$ and $e_2 \neq 0$. Subsequently, it was found that the singular cases where either e_1 or e_3 (or e_1 and e_3) vanished should be treated separately. Since the derivation for singular cases is much easier than the general case, the order in which the solutions were obtained will be reversed, i.e., the presentation will start with $e_1 = e_3 = 0$ (Hohmann transfer), then the two cases where either $e_1 = 0$ or $e_3 = 0$, and finally the general solution. It will be seen, later, that the general solution does not readily reduce to the special cases.

First Singular Case: Hohmann Transfer

Setting $e_1 = e_3 = 0$, and, obviously, $\vartheta = 0$, in equations (12) through (15) gives

$$Q_1 = Q_2[1 + e_2 \cos(\phi_1 + \rho)] \tag{16}$$

$$O = Q_2 e_2 \sin(\phi_1 + \rho) \tag{17}$$

$$Q_2[1 + e_2 \cos(\phi_2 + \rho)] = Q_3 \tag{18}$$

$$Q_2 e_2 \sin(\phi_2 + \rho) = 0 \quad . \tag{19}$$

Excluding $Q_2 = 0$, $e_2 = 0$, and $Q_1 = Q_3$ immediately shows that, from (17) and (19)

$$\cos(\phi_1 + \rho) = \pm 1$$

$$\cos(\phi_2 + \rho) = \pm 1$$

which leaves the two equations

$$Q_1 = Q_2 (1 \pm e_2)$$

$$Q_3 = Q_2 (1 \pm e_2)$$

Since $Q_1 \neq Q_3$ the immediate conclusion is that there are two cases. Either

$$Q_1 = Q_2 (1 + e_2)$$

and

$$Q_3 = Q_2 (1 - e_2)$$

or

$$Q_1 = Q_2 (1 - e_2)$$

and

$$Q_3 = Q_2 (1 + e_2)$$

From the first case comes the solution

$$e_2 = \frac{Q_1 - Q_3}{Q_3 + Q_1} \qquad (20)$$

$$Q_2 = \frac{Q_1 + Q_3}{2} \qquad (21)$$

Since $e_2 > 0$, this solution applies whenever $Q_1 > Q_3$. From the second case,

$$e_2 = \frac{Q_3 - Q_1}{Q_3 + Q_1} \qquad (22)$$

$$Q_2 = \frac{Q_1 + Q_3}{2} \tag{23}$$

which should be used when $Q_3 > Q_1$.

ϕ_1 is arbitrary, and for the transfer to be meaningful, $\rho = \pm\pi$. Then, $\phi_2 = \pm\pi$.

Second Singular Case: Initial Eccentricity = 0, Final Eccentricity ≠ 0

Since $e_1 = 0$ the only thing which vitiates the spatial isotropy is the eccentricity of the target orbit. The rotation angle of the apogee of the target orbit can then be chosen to be equal to 0 (i.e., $\vartheta = 0$) and equations (12) through (15) can be written as

$$Q_1 = Q_2[1 + e_2\cos(\phi_1 + \rho)] \tag{24}$$

$$0 = Q_2 e_2 \sin(\phi_1 + \rho) \tag{25}$$

$$Q_2[1 + e_2\cos(\phi_2 + \rho)] = Q_3[1 + e_3\cos\phi_2] \tag{26}$$

$$Q_2 e_2 \sin(\phi_2 + \rho) = Q_3 e_3 \sin\phi_2 \tag{27}$$

From equation (25), either

$$\rho = -\phi_1 \tag{28}$$

or

$$\rho = \pi - \phi_1 \tag{29}$$

so two choices exist for ρ. Using a "parallel" derivation to reduce space produces the following.

If $\rho = -\phi_1$ then

$$Q_1 = Q_2(1+e_2) \tag{30}$$

$$Q_2[1+e_2\cos(\phi_2-\phi_1)] = Q_3[1+e_3\cos\phi_2] \tag{31}$$

$$Q_2 e_2 \sin(\phi_2-\phi_1) = Q_3 e_3 \sin\phi_2 \tag{32}$$

Dividing equation (32) by equation (31) yields

$$\frac{e_2\sin(\phi_2-\phi_1)}{1+e_2\cos(\phi_2-\phi_1)} = \frac{e_3\sin\phi_2}{1+e_3\cos\phi_2} \tag{33}$$

Solving for e_2 yields

$$e_2 = \frac{e_3\sin\phi_2}{\sin(\phi_2-\phi_1) - e_3\sin\phi_1} \tag{34}$$

Then

$$1+e_2 = \frac{\sin(\phi_2-\phi_1) + e_3(\sin\phi_2-\sin\phi_1)}{\sin(\phi_2-\phi_1) - e_3\sin\phi_1} \tag{35}$$

From equation (30)

$$Q_2 = Q_1\left[\frac{\sin(\phi_2-\phi_1) - e_3\sin\phi_1}{\sin(\phi_2-\phi_1) + e_3(\sin\phi_2-\sin\phi_1)}\right] \tag{36}$$

Inserting e_2 and Q_2 into (32) and clearing gives

$$q\sin(\phi_2-\phi_1) - \sin\phi_2 = -\sin\phi_1 \tag{37}$$

If $\rho = \pi - \phi_1$ then

$$Q_1 = Q_2(1-e_2) \tag{30}$$

$$Q_2[1-e_2\cos(\phi_2-\phi_1)] = Q_3[1+e_3\cos\phi_2] \tag{31}$$

$$-Q_2 e_2 \sin(\phi_2-\phi_1) = Q_3 e_3 \sin\phi_2 \tag{32}$$

Dividing equation (32) by equation (31) yields

$$\frac{-e_2\sin(\phi_2-\phi_1)}{1-e_2\cos(\phi_2-\phi_1)} = \frac{e_3\sin\phi_2}{1+e_3\cos\phi_2} \tag{33}$$

Solving for e_2 yields

$$e_2 = \frac{-e_3\sin\phi_2}{\sin(\phi_2-\phi_1) - e_3\sin\phi_1} \tag{34}$$

Then

$$1-e_2 = \frac{\sin(\phi_2-\phi_1) + e_3(\sin\phi_2-\sin\phi_1)}{\sin(\phi_2-\phi_1) - e_3\sin\phi_1} \tag{35}$$

From equation (30)

$$Q_2 = Q_1\left[\frac{\sin(\phi_2-\phi_1) - e_3\sin\phi_1}{\sin(\phi_2-\phi_1) + e_3(\sin\phi_2-\sin\phi_1)}\right] \tag{36}$$

Inserting e_2 and Q_2 into (32) and clearing gives

$$q\sin(\phi_2-\phi_1) - \sin\phi_2 = -\sin\phi_1 \tag{37}$$

where

$$q = \frac{Q_1 - Q_3}{Q_3\, e_3} \tag{38}$$

From (either) equation (37)

$$(1 - q\cos\phi_1)\sin\phi_2 + (q\sin\phi_1)\cos\phi_2 = \sin\phi_1 \tag{39}$$

Notice that equation (39) is satisfied by $\phi_1 = \phi_2$, a solution specifically excluded by equation (36).

Setting

$$\tan\omega = \frac{1 - q\cos\phi_1}{q\sin\phi_1} \tag{40}$$

so that

$$\cos\omega = \frac{q\sin\phi_1}{\sqrt{q^2 - 2q\cos\phi_1 + 1}} \tag{41}$$

$$\sin\omega = \frac{1 - q\cos\phi_1}{\sqrt{q^2 - 2q\cos\phi_1 + 1}} \tag{42}$$

allows equation (39) to be written as

$$\cos\omega\cos\phi_2 + \sin\omega\sin\phi_2 = \frac{\sin\phi_1}{\sqrt{q^2 - 2q\cos\phi_1 + 1}}$$

or

$$\cos(\phi_2 - \omega) = \frac{\sin\phi_1}{\sqrt{q^2 - 2q\cos\phi_1 + 1}} \tag{43}$$

Since, for any angle θ,

$$\cos\theta = \cos(2\pi - \theta)$$

equation (43) has the two solutions

$$\phi_2 = \omega + \cos^{-1}\frac{\sin\phi_1}{\sqrt{q^2 - 2q\cos\phi_1 + 1}} \tag{44}$$

or

$$\phi_2 = 2\pi + \omega - \cos^{-1}\frac{\sin\phi_1}{\sqrt{q^2 - 2q\cos\phi_1 + 1}} \tag{45}$$

From equation (44), using equations (41) and (42) comes

$$\sin\phi_2 = \frac{(1 - q\cos\phi_1)\sin\phi_1 + q\sin\phi_1\sqrt{(q-\cos\phi_1)^2}}{q^2 - 2q\cos\phi_1 + 1} \tag{46}$$

and

$$\cos\phi_2 = \frac{q\sin^2\phi_1 - (1 - q\cos\phi_1)\sqrt{(q-\cos\phi_1)^2}}{q^2 - 2q\cos\phi_1 + 1} \tag{47}$$

Suppose $q > \cos\phi_1$, then equations (46) and (47) give

$$\sin\phi_2 = \sin\phi_1 \tag{48}$$

$$\cos\phi_2 = \cos\phi_1 \tag{49}$$

But $\phi_2 = \phi_1$ is specifically excluded by equation (36).

Suppose that $q < \cos\phi_1$, equations (46) and (47) give

$$\sin\phi_2 = \frac{(1 - q^2)\sin\phi_1}{q^2 - 2q\cos\phi_1 + 1} \tag{50}$$

$$\cos\phi_2 = \frac{2q - (1 + q^2)\cos\phi_1}{q^2 - 2q\cos\phi_1 + 1} \tag{51}$$

The conclusion is, then, that if

$$q < \cos \phi_1$$

then

$$\phi_2 = \omega + \cos^{-1} \frac{\sin \phi_1}{\sqrt{q^2 - 2q \cos \phi_1 + 1}}$$

From equation (45)

$$\sin \phi_2 = \frac{(1 - q \cos \phi_1) \sin \phi_1 - q \sin \phi_1 \sqrt{(q - \cos \phi_1)^2}}{q^2 - 2q \cos \phi_1 + 1} \tag{52}$$

$$\cos \phi_2 = \frac{q \sin^2 \phi_1 + (1 - q \cos \phi_1) \sqrt{(q - \cos \phi_1)^2}}{q^2 - 2q \cos \phi_1 + 1} \tag{53}$$

Suppose $q > \cos \phi_1$, then

$$\sin \phi_2 = \frac{(1 - q^2) \sin \phi_1}{q^2 - 2q \cos \phi_1 + 1} \tag{54}$$

and

$$\cos \phi_2 = \frac{2q - (1 + q^2) \cos \phi_1}{q^2 - 2q \cos \phi_1 + 1} \tag{55}$$

Now, suppose $\cos \phi_1 > q$, then

$$\sin \phi_2 = \sin \phi_1 \tag{56}$$

and

$$\cos \phi_2 = \cos \phi_1 \tag{57}$$

And the conclusion is that if

$$\cos \phi_1 > q$$

then

$$\phi_2 = 2\pi + \omega - \cos^{-1} \frac{\sin \phi_1}{\sqrt{q^2 - 2q \cos \phi_1 + 1}}$$

since $\phi_1 = \phi_2$ is not allowable.

The term "allowable" is understood in the context that $\phi_1 = \phi_2$ introduces a singularity in equation (36) so that Q_2 diverges to infinity. This is, physically, allowed because it corresponds to a parabolic transfer. The case of $\phi_2 \equiv \phi_1$ is rejected, however.

Equations (50) and (51) [or (54) and (55)] are always valid. Using these, the term $\sin(\phi_2 - \phi_1)$ which appears in equations (34) and (36) can be written as

$$\sin(\phi_2 - \phi_1) = \frac{2 \sin \phi_1 (\cos \phi_1 - q)}{q^2 - 2q \cos \phi_1 + 1} \tag{58}$$

so that either of equations (34) can be combined into

$$e_2 = \frac{\pm e_3(1 - q^2)}{2(\cos \phi_1 - q) - e_3(q^2 - 2q \cos \phi_1 + 1)} \tag{59}$$

Using equation (37) to define $\sin \phi_2$ and equation (58) to define $\sin(\phi_2 - \phi_1)$ equation (36), for Q_2, becomes

$$Q_2 = \left(\frac{Q_1}{1 + e_3 q}\right) \left\{1 - e_3 \left[\frac{q^2 - 2q \cos \phi_1 + 1}{2(\cos \phi_1 - q)}\right]\right\} \tag{60}$$

However, using the definition of q from equation (38) reduces this to

$$Q_2 = Q_3 \left\{ 1 - \frac{e_3(q^2 - 2q \cos \phi_1 + 1)}{2(\cos \phi_1 - q)} \right\} \tag{61}$$

The sign ambiguity present in equation (59) can now be resolved as follows. If the initial and final orbits intersect then the perigee of the final orbit must be less than the radius of the first or the apogee of the final orbit must be greater than the radius of the first. In the first case

$$a_1 > a_3(1-e_3)$$

or

$$\frac{1}{a_3(1-e_3)} > \frac{1}{a_1}$$

or

$$\frac{1+e_3}{a_3(1-e_3^2)} > \frac{1}{a_1}$$

or

$$Q_3(1+e_3) > Q_1$$

so that

$$Q_3e_3 > Q_1 - Q_3$$

or

$$1 > q$$

In the second case,

$$a_3(1+e_3) > a_1$$

which leads to

$$q > -1$$

Thus, the criterion for intersecting orbits is that

$$-1 < q < 1 \tag{62}$$

The numerator of e_2 is, then, certainly positive. Consider the denominator of e_2 in equation (59). Temporarily defining

$$f(\phi_1) = 2(\cos\phi_1 - q) - e_3(q^2 - 2q\cos\phi_1 + 1) \tag{63}$$

the extrema of f occur at 0 or π so that

$$f(0) = (1-q)\,[2 - e_3(1-q)] > 0$$

$$f(\pi) = -(1+q)\,[2 + e_3(1+q)] < 0$$

and

$$\frac{\partial f}{\partial \phi_1} = -2\sin\phi_1\,(1 + e_3 q)$$

For $|q| < 1$, $e_3 < 1$,

$$\frac{\partial f}{\partial \phi_1} \leqslant 0 \qquad (0 \leqslant \phi_1 \leqslant \pi) \tag{64}$$

$$\frac{\partial f}{\partial \phi_1} \geqslant 0 \qquad (\pi \leqslant \phi_1 \leqslant 2\pi) \tag{65}$$

Also, f will vanish (i.e., e_2 will experience a singularity) whenever

$$\cos\phi_1 = \frac{2q + e_3(1+q^2)}{2(1+qe_3)} \tag{66}$$

i.e.,

$$\phi_1 = \cos^{-1}\left[\frac{2q + e_3(1+q^2)}{2(1+qe_3)}\right] \tag{67}$$

or

$$\phi_1 = 2\pi - \cos^{-1}\left[\frac{2q + e_3(1+q^2)}{2(1+qe_3)}\right] \tag{68}$$

The singularities of Q_2 and e_2 are not coincident. They may be ordered as follows:

$$q^2 < 1 \Rightarrow q^2 e_3 < e_3 \quad \Rightarrow \quad 2q^2 e_3 < e_3 + q^2 e_3 = e_3(1+q^2)$$

Thus

$$2q + 2q^2 e_3 < 2q + e_3(1+q^2) \Leftrightarrow 2q(1+qe_3) < 2q + e_3(1+q^2)$$

so that

$$q < \frac{2q + e_3\,(1+q^2)}{2(1+q\,e_3)} \tag{69}$$

Thus, if ϕ_1 begins at zero and progresses through increasing values toward π, and if $q > 0$ the first singularity will be encountered at a value given by equation (65) and, subsequently, the singularity $\phi_1 = \cos^{-1} q$ will be encountered. (In physical terms, e_2 will diverge before Q_2 diverges.) If we proceed from 2π through negative values toward π, the same sequence occurs.

Let the upper half of the polar orbital plane ($0 \leq \phi_1 \leq \pi$) be divided into three regions, labeled I, II, and III, defined as follows:

I. $\phi_1 < \cos^{-1}\left[\frac{2q + e_3\,(1+q^2)}{2(1+e_3\,q)}\right]$ and $\phi_1 < \cos^{-1} q$

II. $\phi_1 > \cos^{-1}\left[\frac{2q + e_3\,(1+q^2)}{2(1+e_3\,q)}\right]$ and $\phi_1 < \cos^{-1} q$

III. $\phi_1 > \cos^{-1}\left[\frac{2q + e_3\,(1+q^2)}{2(1+e_3\,q)}\right]$ and $\phi_1 > \cos^{-1} q$

The goal is to define physically realizable orbits over as many of these regions as the constraints allow. From $f(0) > 0$ and equations (64) and (67), e_2 in region I must be positive, so choose the sign in equation (59) such that

$$e_2 = \frac{+e_3(1 - q^2)}{2(\cos\phi_1 - q) - e_3(q^2 - 2q\cos\phi_1 + 1)} \tag{70}$$

and $\rho = -\phi_1$. Also, for $|q| < 1$, $Q_2(0) > 0$. Inserting (66) into (61) gives

$$Q_2\left\{\cos^{-1}\left[\frac{2q + e_3(1+q^2)}{2(1+q\,e_3)}\right]\right\} = 0 \tag{71}$$

(i.e., parabolic transfer). Thus, region I is an area of valid orbital transfers, and similarly for the mirror image in the lower half of the plane.

In region II, Q_2 decreases from 0 to negative infinity, always staying negative. This region, then, does not correspond to physically realizable transfers since Q_2 must be positive.

Region III again finds Q_2 to be positive since

$$Q_2(\pi) > 0$$

and

$$\frac{\partial Q_2}{\partial Q\phi_1} < 0 \qquad (0 \leq \phi_1 \leq \pi\ ;\ \phi_1 \neq \cos^{-1} q)\quad .$$

Since Q_2 undergoes an infinite discontinuity at $\phi_1 = \cos^{-1} q$, these results are consistent.

The only remaining question is the behavior of e_2 in region III. From (59), certainly

$$q^2 - 2q\cos\phi_1 + 1 > 0$$

and since $\phi_1 > \cos^{-1} q$,

$$\cos\phi_1 - q < 0$$

so that

$$2(\cos \phi_1 - q) - e_3(q^2 - 2q \cos \phi_1 + 1) < 0$$

Since $1 - q^2 > 0$, the conclusion is that e_2 will be positive if

$$e_2 = \frac{-e_3(1-q^2)}{2(\cos \phi_1 - q) - e_3(q^2 - 2q \cos \phi_1 + 1)} \tag{72}$$

from which the second of equations (34) shows that $\rho = \pi - \phi$.

The information gathered on the regions can now be summarized pictorially as follows:

e_2 from eq. (70) Q_2 from eq. (61) $\rho = -\phi_1$	Forbidden zone No physically realizable tangential transfers	e_2 from eq. (72) Q_2 from eq. (61) $\rho = \pi - \phi_1$

$\phi_1 = 0$ $e_2 = \dfrac{e_3(1+q)}{2 - e_3(1-q)}$ $Q_2 = Q_3\left[1 - \dfrac{e_3(1-q)}{2}\right]$	$\phi_1 = \cos^{-1}\left[\dfrac{2q + e_3(1+q^2)}{2(1+q e_3)}\right]$ $e_2 = \infty$ $Q_2 = 0$	$\phi_1 = \cos^{-1} q$ $e_2 = 1$ $Q_2 = \infty$	$\phi_1 = \pi$ $e_2 = \dfrac{e_3(1-q)}{2 + e_3(1+q)}$ $Q_2 = Q_3\left[1 + \dfrac{e_3(1+q)}{2}\right]$

These are valid for any intersecting orbits with $e_1 = 0$ and $e_3 \neq 0$, but no discussion has yet been offered for non-intersecting orbits ($|q| > 1$). That case is considerably easier to deal with than the intersecting case. The equation for e_2, equation (59), is valid but now, since $|q| > 1$, there can be no singularity in e_2. This is because equation (66) cannot be fulfilled ($|\cos \phi_1| \leq 1$).

From the bounds on f, following equation (63),

$$(1 - q)\,[2 - e_3(1 - q)] \geqslant 2(\cos \phi_1 - q) - e_3(q^2 - 2q \cos \phi_1 + 1) > -(1+q)[2+e_3(1+q)]$$

If $q > 1$, then $f < 0$. If $q < -1$ then $f > 0$. This yields a simple rule since $1 - q^2 < 0$.

If $q > 1$, choose

$$e_2 = \frac{e_3(1 - q^2)}{2(\cos \phi_1 - q) - e_3(q^2 - 2q \cos \phi_1 + 1)}$$

If $q < -1$, choose

$$e_2 = \frac{-e_3(1 - q^2)}{2(\cos \phi_1 - q) - e_3(q^2 - 2q \cos \phi_1 + 1)}$$

The calculations on Q_2 are more delicate. From equation (61), if $q > 1$ it is obvious that $Q_2 > 0$ for all ϕ_2 and e_3. If $q < -1$ then additional considerations are needed. From equations (38) and (11) comes the relationship

$$q = \frac{Q_1 - Q_3}{Q_3\, e_3} = \frac{\dfrac{1}{a_1} - \dfrac{1}{a_3(1 - e_3^2)}}{\dfrac{e_3}{a_3(1 - e_3^2)}} = \frac{a_3(1 - e_3^2) - a_1}{a_1\, e_3} \tag{73}$$

so that

$$a_1 = \frac{a_3(1 - e_3^2)}{1 + e_3\, q} \tag{74}$$

or

$$a_3 = \frac{a_1(1 + e_3\, q)}{1 - e_3^2} \tag{75}$$

In each case both a_1 and a_3 must be positive along with $0 \leqslant e_3 < 1$. But combinations can be chosen which violate this physical constraint. (There is no treatment here of the initial or final orbits being parabolas or hyperbolae.) The requirement, then, is that

$$1 + e_3 \, q > 0$$

or

$$-1 > q > -1/e_3 \quad , \quad 0 < e_3 < 1 \quad .$$

Let ϵ be a real positive number defined according to

$$q = \epsilon^2 - 1/e_3 \quad .$$

From the extremal values on Q_2 comes the inequality

$$Q_3 \left[1 - \frac{e_3(1-q)}{} \right] \leqslant Q_2 \leqslant Q_3 \left[1 + \frac{e_3(1+q)}{2} \right]$$

or

$$(Q_3/2) \; [1 - e_3(1 - \epsilon^2)] \leqslant Q_2 \leqslant (Q_3/2) \; [1 + e_3(1 + \epsilon^2)]$$

so that Q_2 will be positive.

The conclusion that is reached is that, for the present case, there are no restrictions on tangential orbit transfers if the orbits do not intersect.

Third Singular Case: Initial Eccentricity $\neq$ 0, Final Eccentricity = 0

This case is not identical to the previously treated case. The analogues of equations (24) through (27) are

$$Q_1(1 + e_1 \cos \phi_1) = Q_2[1 + e_2 \cos(\phi_1 + \rho)] \tag{77}$$

$$Q_1 e_1 \sin \phi_1 = Q_2 e_2 \sin(\phi_1 + \rho) \tag{78}$$

$$Q_2[1 + e_2 \cos(\phi_2 + \rho)] = Q_3 \tag{79}$$

$$Q_2\ e_2\ \sin(\phi_2 + \rho) = 0$$

from which comes, immediately,

$$\rho = -\phi_2 \tag{81}$$

or

$$\rho = \pi - \phi_2 \quad . \tag{82}$$

In a manner exactly parallel to prior developments the following equations are obtained.

If $\rho = -\phi_2$

$$e_2 = \frac{e_1 \sin \phi_1}{\sin(\phi_1 - \phi_2) - e_1 \sin \phi_2} \tag{83}$$

If $\rho = \pi - \phi_2$

$$e_2 = \frac{-e_1 \sin \phi_1}{\sin(\phi_1 - \phi_2) - e_1 \sin \phi_2} \tag{84}$$

In either case

$$Q_2 = Q_3 \left[\frac{\sin(\phi_1 - \phi_2) - e_1 \sin \phi_2}{\sin(\phi_1 - \phi_2) + e_1 (\sin \phi_1 - \sin \phi_2)} \right] \tag{85}$$

The definition of q is changed, for this case, to read

$$q = \frac{Q_3 - Q_1}{Q_1 e_1} \tag{86}$$

then

$$(1 - q \cos \phi_1) \sin \phi_2 + (q \sin \phi_1) \cos \phi_2 = \sin \phi_1 \quad .$$

The value of ω from equation (40) has the same functional form so that equations (44) and (45) are still valid for ϕ_2; similarly, for the values of $\sin \phi_2$ and $\cos \phi_2$ as given by, say, equations (50) and (51).

Since the functional form of e_2 differs, the former equation cannot be expected to hold true. Now

$$\sin(\phi_2 - \phi_1) = \frac{2 \sin \phi_1 (q - \cos \phi_1)}{q^2 - 2q \cos \phi_1 + 1} \tag{87}$$

so that from equations (83) and (84)

$$e_2 = \frac{\pm e_1 (q^2 - 2q \cos \phi_1 + 1)}{2(q - \cos \phi_1) - e_1 (1 - q^2)} \tag{88}$$

The equation for Q_2, (85), becomes

$$Q_2 = Q_1 \left[1 - \frac{e_1(1 - q^2)}{2(q - \cos \phi_1)} \right] \tag{89}$$

Next, the sign of e_2 must be determined. This will be done in the same manner as before, first for the case of intersecting orbits.

Assume that the perigee of the initial orbit is less than the radius of the final orbit

$$a_1(1 - e_1) < a_3$$

$$\frac{1}{a_3} < \frac{1}{a_1(1 - e_1)}$$

$$\frac{1}{a_3} < \frac{1 + e_1}{a_1(1 - e_1^2)}$$

$$Q_3 < Q_1\ (1 + e_1)$$

$$\frac{Q_3 - Q_1}{Q_1\ e_1} < 1$$

Therefore

$$q < 1$$

Next, assume that the apogee of the initial orbit is greater than the radius of the final orbit so that

$$a_1(1 + e_1) > a_3$$

$$\frac{1}{a_3} > \frac{1}{a_1(1 + e_1)}$$

$$\frac{1}{a_3} > \frac{1 - e_1}{a_1(1 - e_1^2)}$$

$$Q_3 > Q_1\ (1 - e_1)$$

$$\frac{Q_3 - Q_1}{Q_1\ e_1} > -1$$

so that the familiar bound is still

$$-1 < q < 1 \tag{90}$$

From equation (88), e_2 will experience local minima at $\phi_1 = 0$ and $\phi_1 = \pi$. Furthermore, e_2 has a singularity at

$$\phi_1 = \cos^{-1}\left[q - \frac{e_1(1-q^2)}{2}\right] \tag{91}$$

At $\phi_1 = 0$,

$$e_2 = \frac{\pm e_1(q-1)}{2+e_1(1+q)}$$

So to ensure a positive e_2, the negative sign is chosen. The same sign choice continues until e_2 diverges. At $\phi_1 = \pi$

$$e_2 = \frac{\pm e_1(q+1)}{2+e_1(q-1)} \tag{92}$$

so that in the locality of π, the positive sign is the proper choice.

Q_2 [eq. (89)] will experience a singularity at

$$\phi_1 = \cos^{-1} q$$

The singularities may be ordered as follows:

$$1 - q^2 > 0$$

$$\frac{e_1}{2(1-q^2)}$$

$$0 > \frac{-e_1}{2(1-q^2)}$$

$$q > q - \frac{e_1}{2(1-q^2)}$$

Thus, if ϕ_1 begins at 0 and increases toward π, the first singularity that will be encountered occurs in Q_2. At that point Q_2 diverges and becomes negative. It continues as negative until the singularity in e_2 is encountered. Thus, the "forbidden" region is determined by

$$\cos^{-1} q \leq \phi_1 \leq \cos^{-1} \frac{q}{q - \dfrac{e_1}{2(1-q^2)}} \tag{93}$$

e_2: choose the negative sign in eq. (88) Q_2: from eq. (89) $\rho = -\phi_2$	Forbidden Zone No physically realizable tangential transfers	e_2: choose the positive sign in eq. (88) Q_2: from eq. (89) $\rho = \pi - \phi_2$

$\phi_1 = 0$ $e_2 = \dfrac{e_1(1-q)}{2+e_1(q+q)}$ $Q_2 = Q_1 \left[1 + \dfrac{e_1(1+q)}{2}\right]$	$\phi_1 = \cos^{-1} q$ $e_2 = 1$ $Q_2 = \infty$	$\phi_1 = \cos^{-1} [q - (e_1/2)(1-q^2)]$ $e_2 = \infty$ $Q_2 = 0$	$\phi_1 = \pi$ $e_2 = \dfrac{e_1(1+q)}{2 \mp e_1(1-q)}$ $Q_2 = Q_1 \left[1 - \dfrac{e_1(1-q)}{2}\right]$

As before, there are no restrictions on the value of ϕ_1 for transfer if the orbits do not intersect.

The General Case: Initial Eccentricity $\neq$ 0, Final Eccentricity $\neq$ 0

The prior three sections assumed that either or both of the eccentricities of the initial and final orbits vanished and this yielded much simplification. The general case will now be treated. Equations (12) through (15) constitute the natural starting point for the development.

Dividing equation (13) by equation (15) yields

$$\frac{\sin(\phi_1 + \rho)}{\sin(\phi_2 + \rho)} = \frac{Q_1\, e_1 \sin \phi_1}{Q_3\, e_3 \sin(\phi_2 + \vartheta)} \tag{94}$$

Defining

$$\gamma = \frac{Q_1 e_1}{Q_3 e_3} \tag{95}$$

allows equation (94) to be written as

$$\sin(\phi_1 + \rho)\sin(\phi_2 + \vartheta) = \gamma \sin \phi_1 \sin(\phi_2 + \rho) \tag{96}$$

Expanding the functions which involve ρ, gathering, and solving for $\tan \rho$ gives

$$\tan \rho = \frac{\gamma \sin \phi_1 \sin \phi_2 - \sin \phi_1 \sin(\phi_2 + \vartheta)}{\cos \phi_1 \sin(\phi_2 + \vartheta) - \gamma \sin \phi_1 \cos \phi_2} = \sin \phi_1 \left\{ \frac{(\gamma - \cos \vartheta) \tan \phi_2 - \sin \vartheta}{\cos \phi_1 \cos \vartheta \tan \phi_2 + \cos \phi_1 \sin \vartheta - \gamma \sin \phi_1} \right\} \tag{97}$$

From equations (12) and (13) comes

$$\frac{e_2 \sin(\phi_1 + \rho)}{1 + e_2 \cos(\phi_1 + \rho)} = \frac{e_1 \sin \phi_1}{1 + e_1 \cos \phi_1}$$

Solving for e_2

$$e_2 = \frac{e_1 \sin \phi_1}{\sin(\phi_1 + \rho) + e_1 \sin \rho} \tag{98}$$

Multiplying equation (12) by $\sin(\phi_1 + \rho)$, equation (13) by $\cos(\phi_1 + \rho)$, and adding gives

$$Q_2 = \frac{Q_1[\sin(\phi_1 + \rho) + e_1 \sin \rho]}{\sin(\phi_1 + \rho)} \tag{99}$$

Equations (98) and (99) can now be used to eliminate e_2 and Q_2 from equation (14). This yields

$$\frac{Q_1[e_1 \sin \rho + \sin(\phi_1 + \rho)]}{\sin(\phi_1 + \rho)} \left[1 + \frac{e_1 \sin \phi_1 \cos(\phi_2 + \rho)}{\sin(\phi_1 + \rho) + e_1 \sin \rho}\right] = Q_3[1 + e_3 \cos(\phi_2 + \vartheta)] \tag{100}$$

Clearing shows that either

$$e_1 \sin\rho + \sin(\phi_1+\rho) = 0 \tag{101}$$

or

$$Q_1\{\sin(\phi_1+\rho) + e_1[\sin\rho + \sin\phi_1\cos(\phi_2+\rho)]\} = Q_3[1+e_3\cos(\phi_2+\vartheta)]\sin(\phi_1+\rho) \tag{102}$$

Comparison between equations (100) and (98) leaves only equation (102) as a valid alternative.

Now using the identity

$$\sin(\phi_1+\rho) = \cos\rho\,[\sin\phi_1 + \cos\phi_1\tan\rho] \tag{103}$$

and substituting for tan ρ from equation (97) into equation (103) gives

$$\sin(\phi_1+\rho) = \frac{\gamma\sin\phi_1\sin(\phi_2-\phi_1)\cos\rho}{\cos\phi_2\,[\cos\phi_1\cos\vartheta\tan\phi_2 + \cos\phi_1\sin\vartheta - \gamma\sin\phi_1]} \tag{104}$$

Similarly, since

$$\begin{aligned}\cos(\phi_2+\rho) &= \cos\rho\,[\cos\phi_2 - \sin\phi_2\tan\rho] \\ &= \frac{[-\gamma\sin\phi_1 + \cos(\phi_1-\phi_2)\sin(\phi_2+\vartheta)]\cos\rho}{\cos\phi_2\,(\cos\phi_1\cos\rho\tan\phi_2 + \cos\phi_1\sin\vartheta - \gamma\sin\phi_1)}\end{aligned} \tag{105}$$

A portion of the second member of equation (102) can now be constructed as

$$\sin\rho + \sin\phi_1\cos(\phi_2+\rho) = \cos\rho\left\{\tan\rho + \frac{-\gamma\sin\phi_1 + \cos(\phi_1-\phi_2)\sin(\phi_2+\vartheta)}{\cos\phi_2\,(\cos\phi_1\cos\vartheta\tan\phi_2 + \cos\phi_1\sin\vartheta - \gamma\sin\phi_1)}\right\} \tag{106}$$

Again substituting for tan ϑ from equation (97) and combining fractions gives

$$\sin\rho + \sin\phi_1 \cos(\phi_2+\rho) = \frac{\cos\rho \sin\phi_1 \{\gamma(\sin\phi_2 - \sin\phi_1) - \sin(\phi_2+\vartheta)[1 - \cos(\phi_1 - \phi_2)]\}}{\cos\phi_2(\cos\phi_1 \cos\vartheta \tan\phi_2 + \cos\phi_1 \sin\vartheta - \gamma \sin\phi_1)} \tag{107}$$

Next, substituting from equations (103) and (107) into equation (102) and cross-multiplying by $\cos\phi_2(\cos\phi_1 \cos\vartheta \tan\phi_2 + \cos\phi_1 \sin\vartheta - \gamma\sin\phi_1)$ gives

$$Q_1 e_1 \cos\rho \sin\phi_1 \{\gamma(\sin\phi_2 - \sin\phi_1) - \sin(\phi_2+\vartheta)[1 - \cos(\phi_1 - \phi_2)] + \frac{\gamma \sin\phi_1 \sin(\phi_2-\phi_1)\cos\rho}{\cos\phi_2[\cos\phi_1 \cos\vartheta \tan\phi_2 + \cos\phi_1 \sin\vartheta - \gamma\sin\phi_1)} \tag{108}$$

$$= Q_3 [1 + e_3 \cos(\phi_2+\vartheta)] \gamma \sin\phi_1 \sin(\phi_2 - \phi_1) \cos\rho$$

Require that none of $\cos\rho$, $\sin\phi_1$, or $\cos\phi_2$ vanish. Then substitute for γ from equation (95) on the right side of equation (108), then expand equation (108) to obtain

$$e_1 e_3\{\gamma(\sin\phi_2 - \sin\phi_1) - \sin(\phi_2+\vartheta)[1 - \cos(\phi_1 - \phi_2)]\} + e_3 \gamma \sin(\phi_2 - \phi_1)$$
$$= e_1 \sin(\phi_2 - \phi_1) + e_1 e_3 \cos(\phi_2+\vartheta) \sin(\phi_2 - \phi_1) \tag{109}$$

Eliminate the remaining γ terms from equation (109), expand the multi-angle expressions and gather on $\sin\phi_2$, $\cos\phi_2$ to obtain

$$[Q_1 e_1 - Q_3 e_3 \cos\vartheta + (Q_1 - Q_3)\cos\phi_1] \sin\phi_2 - [Q_3 e_3 \sin\vartheta + (Q_1 - Q_3)\sin\phi_1] \cos\phi_2$$
$$= Q_1 e_1 \sin\phi_1 - Q_3 e_3 \sin(\phi_1+\vartheta) \tag{110}$$

For convenience, define

$$p_1 = Q_1 e_1 - Q_3 e_3 \cos\vartheta + (Q_1 - Q_3)\cos\phi_1 = Q_3 e_3 (\gamma - \cos\vartheta + q\cos\phi_1) \tag{111}$$

$$p_2 = Q_3 e_3 \sin\vartheta + (Q_1 - Q_3)\sin\phi_1 = Q_3 e_3 [\sin\vartheta + q\sin\phi_1] \tag{112}$$

* Letting $\sin\phi_1 = 0$ can provide a valid transfer.

$$p_3 = Q_1 e_1 \sin \phi_1 - Q_3 e_3 \sin(\phi_1 + \vartheta) \tag{113}$$

where q is given by equation (38), *not* by equation (86).

Equations (111) through (113) are not independent as can be seen from

$$p_3 = p_1 \sin \phi_1 - p_2 \cos \phi_1 \quad , \tag{114}$$

so that equation (110) can be written as

$$p_1 \sin \phi_2 - p_2 \cos \phi_2 = p_1 \sin \phi_1 - p_2 \cos \phi_1 \tag{115}$$

Dividing equation (115) by $\sqrt{p_1^2 + p_2^2}$ and defining a new ω by

$$\omega = \tan^{-1}(p_1/p_2) \tag{116}$$

allows equation (115) to be written as

$$\cos \phi_2 \cos \omega - \sin \phi_2 \sin \omega = \cos \phi_1 \cos \omega - \sin \phi_1 \sin \omega$$

or

$$\cos(\phi_2 + \omega) = \cos(\phi_1 + \omega) \tag{117}$$

Equation (117) has two solutions. The trivial solution is

$$\phi_2 = \phi_1$$

but the desired solution comes via the identity

$$\cos(\phi_2 + \omega) = \cos(2\pi - \phi_2 - \omega)$$

Then

$$2\pi - \phi_1 - \omega = \phi_1 + \omega$$

or

$$\phi_2 = 2(\pi - \omega) - \phi_1 = 2[\pi - \tan^{-1}(p_1/p_2)] - \phi_1 \tag{118}$$

Equation (118) is the generalized form of either equations (44) and (45).

The next step is to utilize equation (118) to eliminate ϕ_2 from equation (97) in order to obtain an expression for ρ which depends only upon ϕ_1.*

Defining the variables

$$\alpha = \tan^{-1}\left(\frac{\sin\vartheta}{\gamma - \cos\vartheta}\right) \tag{119}$$

$$r = \sqrt{\gamma^2 - 2\gamma\cos\vartheta + 1} \tag{120}$$

naturally leads to

$$\sin\vartheta = r\sin\alpha \tag{121}$$

$$\gamma - \cos\vartheta = r\cos\alpha \tag{122}$$

Then equation (97) may be expanded as

* The author wishes to thank J. R. Redus of Marshall Space Flight Center, Huntsville, Alabama, for pointing out the following transformation.

$$\tan \rho = \sin \phi_1 \left[\frac{\gamma \sin \phi_2 - \sin \phi_2 \cos \vartheta - \cos \phi_2 \sin \vartheta}{\cos \phi_1 \sin \phi_2 \cos \vartheta + \cos \phi_1 \cos \phi_2 \sin \vartheta - \gamma \sin \phi_1 \cos \phi_2} \right] \tag{123}$$

$$= \frac{r \sin \phi_1 \sin(\phi_2 - \alpha)}{\gamma \sin(\phi_2 - \phi_1) - r \cos(\phi_2 - \alpha)}$$

Now, substituting for ϕ_2 from equation (118) yields

$$\tan \rho = \frac{r \sin \phi_1 \sin[2\pi - 2\omega - \phi_1 - \alpha]}{\gamma \sin[2\pi - 2\omega - 2\phi_1] - r \cos \phi_1 \sin[2\pi - 2\omega - \phi_1 - \alpha]}$$

$$= \frac{-r \sin \phi_1 \sin(2\omega + \phi_1 + \alpha)}{-\gamma \sin[2(\omega + \phi_1)] + r \cos \phi_1 \sin(2\omega + \phi_1 + \alpha)}$$

For convenience, define

$$R = \sqrt{r^2 + 2rq \cos(\phi_1 - \alpha) + q^2} \tag{125}$$

Then

$$\cos \omega = \frac{r \sin \alpha + q \sin \phi_1}{R}$$

$$\sin \omega = \frac{r \cos \alpha + q \cos \phi_1}{R}$$

so that

$$\cos(\omega + \phi_1) = \frac{-r \sin(\phi_1 - \alpha)}{R}$$

$$\sin(\omega + \phi_1) = \frac{r\cos(\phi_1 - \alpha) + q}{R}$$

$$\cos(\omega + \alpha) = \frac{q\sin(\phi_1 - \alpha)}{R}$$

$$\sin(\omega + \alpha) = \frac{r + q\cos(\phi_1 - \alpha)}{R}$$

Thus,

$$\sin(2\omega + \phi_1 + \alpha) = \sin(\omega + \phi_1)\cos(\omega + \alpha) + \cos(\omega + \phi_1)\sin(\omega + \alpha) = \frac{(q^2 - r^2)\sin(\phi_1 - \alpha)}{R^2} \tag{126}$$

$$\sin[2(\omega + \phi_1)] = 2\sin(\omega + \phi_1)\cos(\omega + \phi_1) = \frac{-2r\sin(\phi_1 - \alpha)\,[r\cos(\phi_1 - \alpha) + q]}{R^2} \tag{127}$$

Inserting equations (126) and (127) into equation (124) gives

$$\tan\rho = \frac{[\sin(\phi_1 - \alpha)/R^2]}{[\sin(\phi - \alpha)/R^2]}\left[\frac{-r(q^2 - r^2)\sin\phi_1}{2r\gamma[r\cos(\phi_1 - \alpha) + q] + r\cos\phi_1\,(q^2 - r^2)}\right] \tag{128}$$

Although cancelling $1/R^2$ in the numerator and denominator of equation (128) is valid, cancelling $\sin(\phi_1 - \alpha)$ is a different matter. The reason for this is that $\sin(\phi_1 - \alpha)$ can change sign and due to the fact that the arctangent function is multi-valued it will not yield the same value for

$$\tan^{-1}\frac{f(x)}{g(x)}$$

as it does for

$$\tan^{-1}\frac{-f(x)}{-g(x)}$$

Equation (128) can safely be written as

$$\tan\rho = \frac{-r(q^2 - r^2)\sin\phi_1 \operatorname{sign}[\sin(\phi_1 - \alpha)]}{\{2r\gamma[r\cos(\phi_1 - \alpha) + q] + r\cos\phi_1(q^2 - r^2)\}\operatorname{sign}[\sin(\phi_1 - \alpha)]} \tag{129}$$

Eliminating α and r from equation (129) gives the final form of equation (129) as

$$\tan\rho = \frac{(\gamma^2 - 2\gamma\cos\vartheta + 1 - q^2)\sin\phi_1 \operatorname{sign}[\sin(\phi_1 - \alpha)]}{[(\gamma^2 + q^2 - 1)\cos\phi_1 + 2\gamma(q + \sin\vartheta\sin\phi_1)]\operatorname{sign}[\sin(\phi_1 - \alpha)]} \tag{130}$$

$$= \frac{(\gamma^2 - 2\gamma\cos\vartheta + 1 - q^2)\sin\phi_1 \operatorname{sign}[\gamma\sin\phi_1 - \cos(\phi_1 - \vartheta)]}{[(\gamma^2 + q^2 - 1)\cos\phi_1 + 2\gamma(q + \sin\vartheta\sin\phi_1)]\operatorname{sign}[\gamma\sin\phi_1 - \cos(\phi_1 - \vartheta)]}$$

The machinery is at hand to obtain an expression for Q_2 in terms of fundamental variables. From equation (99) comes

$$\begin{aligned} Q_2 &= Q_1\left\{1 + \frac{e_1\sin\rho}{\sin\phi_1\cos\rho + \cos\phi_1\sin\rho}\right\} \\ &= Q_1\left\{1 + \frac{e_1\tan\rho}{\sin\phi_1 + \cos\phi_1\tan\rho}\right\} \end{aligned} \tag{131}$$

Using equation (130) (and ignoring the sign terms which make no difference here) yields the expression

$$\sin\phi_1 + \cos\phi_1\tan\rho = 2\gamma\sin\phi_1\left[\frac{\gamma\cos\phi_1 + q - \cos(\phi_1 + \vartheta)}{(\gamma^2 + q^2 - 1)\cos\phi_1 + 2\gamma(q + \sin\vartheta\sin\phi_1)}\right] \tag{132}$$

Then

$$Q_2 = Q_1\left\{1 + \frac{e_1(\gamma^2 - 2\gamma\cos\vartheta + 1 - q^2)}{2\gamma[\gamma\cos\phi_1 + q - \cos(\phi_1 + \vartheta)]}\right\} \tag{133}$$

is a fundamental expression for Q_2.

The final equation which is needed is the expression for e_2 which involves only fundamental variables, i.e., a non-sequential calculation. I have been unable to derive an expression which is aesthetically satisfactory, but such an expression is not required in what follows.

The "forbidden regions" which constitute the focus of this paper exist in the general case and can be analytically defined by enforcing the following conic section restrictions:

$$
\begin{aligned}
& [(0 < e_2 < 1) \text{ and } (a_2 > 0)] \\
\text{or} \quad & [(e_2 > 1) \text{ and } (a_2 < 0)] \\
\text{or} \quad & [(e_2 = 0) \text{ and } (a_2 > 0)] \\
\text{or} \quad & [(e_2 = 1) \text{ and } (a_2 = \infty)]
\end{aligned}
\tag{134}
$$

It will be shown that e_2 and a_2 can take on any value on the extended reals and the regions which violate equation (134) give rise to forbidden regions.

As in the special cases, the apparent dependency of sign(e_2) on sign[sin(ϕ_1)] is illusory [equation (98)]. Consider these points at which e_2 will experience a singularity, i.e., the solutions of

$$\sin(\phi_1 + \rho) + e_1 \sin \rho = 0 \tag{135}$$

which expands to give

$$
\begin{aligned}
\sin(\phi_1 + \rho) + e_1 \sin \rho = {} & [(\gamma^2 + q^2 - 1) \cos \phi_1 + 2\gamma(q + \sin \vartheta \sin \phi_1)] \sin \phi_1 \\
& + [(\gamma^2 - 2\gamma \cos \vartheta + 1 - q^2)] \cos \phi_1 \sin \phi_1 + e_1 (\gamma^2 - 2\gamma \cos \vartheta + 1 - q^2) \sin \phi_1 = 0
\end{aligned}
\tag{136}
$$

Since $\sin \phi_1$ is assumed not to vanish, gather on $\sin \phi_1$, $\cos \phi_1$, to yield

$$(\gamma - \cos \vartheta) \cos \phi_1 + \sin \vartheta \sin \phi_1 = \left\{ \frac{e_1[2\gamma \cos \vartheta + q^2 - \gamma^2 - 1] - 2\gamma q}{2\gamma} \right\} \tag{137}$$

Utilizing the definitions of α from equation (119) and defining

$$\zeta = \cos^{-1} \left\{ \frac{e_1[2\gamma \cos \vartheta + q^2 - \gamma^2 - 1] - 2\gamma q}{2\gamma \sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1}} \right\} \tag{138}$$

then equation (137) has two solutions

$$\begin{aligned}\phi_1 &= \alpha + \zeta \\ \phi_1 &= 2\pi + \alpha - \zeta\end{aligned} \tag{139}$$

as singular points of e_2.

Another complicating factor to the orbit transfer arises when the numerator and denominator of equation (130) vanish simultaneously. These points can be isolated as follows.

Assume that the numerator of equation (97) vanishes [this is more convenient than working with equation (130)]. Then

$$(\gamma - \cos\vartheta)\tan\phi_2 - \sin\vartheta = 0 \tag{140}$$

so that

$$\tan\phi_2 = \tan\alpha \tag{141}$$

If the denominator of equation (97) also vanishes

$$\cos\phi_1\cos\vartheta\tan\phi_2 + \cos\phi_1\sin\vartheta - \gamma\sin\phi_1 = 0 \tag{142}$$

Eliminating ϕ_2 between equations (141) and (142) gives

$$\tan\phi_1 = \frac{\sin\vartheta}{\gamma - \cos\vartheta} = \tan\alpha \tag{143}$$

Since

$$\tan\phi_1 = \tan(\pi + \phi_1)$$

the two solutions of equation (143) are

$$\begin{aligned} \phi_1 &= \phi_2 = \alpha \\ \phi_1 &= \phi_2 = \pi + \alpha \end{aligned} \tag{144}$$

which are also the points at which $\sin(\phi_1 - \alpha) = 0$. Thus, when the numerator and denominator of equation (97) [or (130)] vanish, sign[$\sin(\phi_1 - \alpha)$] loses meaning since zero is unsigned.

Both the numerator and denominator are continuous through the points given by equation (144) and the angle ρ will experience a jump discontinuity of π. The behavior of e_2 at this point is demonstrated as follows. Let ϕ^- and ϕ^+ represent ϕ to the left and right of the solution given by equation (144). Then

$$e_2(\phi^-) = \frac{e_1 \sin \phi^-}{\sin(\phi^- + \rho) + e_1 \sin \rho}$$

$$e_2(\phi^+) = \frac{e_1 \sin \phi^+}{\sin(\phi^+ + \rho \pm \pi) + e_1 \sin(\rho \pm \pi)}$$

$$\begin{aligned} \sin(\phi + \rho \pm \pi) + e_1 \sin(\rho \pm \pi) &= \cos(\pm\pi) [\sin(\phi + \rho) + e_1 \sin \rho] \\ &= -[\sin(\phi + \rho) + e_1 \sin \rho] \end{aligned}$$

$$|e_2(\phi^-)| = -|e_2(\phi^+)|$$

This far, e_2 has exhibited two unbounded discontinuities [see equations (139)] and two bounded discontinuities [see equations (144)]. Additional information on e_2 comes from the observation that if

$$\phi_1 + \rho = \pi \tag{145}$$

$$e_2 = 1 \tag{146}$$

(i.e., parabolic transfer) and if

$$\phi_1 + \rho = 0 \tag{147}$$

then

$$e_2 = -1 \tag{148}$$

To isolate the value of ϕ_1 which corresponds to equation (145), observe that, necessarily

$$\tan \rho = -\tan \phi_1$$

Inserting this result into equation (130) gives

$$(\gamma^2 + q^2 - 1)\cos \phi_1 + 2\gamma(q + \sin \vartheta \sin \phi_1) = -(\gamma^2 - 2\gamma \cos \vartheta + 1 - q^2)\cos \phi_1 \tag{149}$$

Gathering, utilizing α once more and defining

$$\xi = \cos^{-1} \frac{-q}{\sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1}} \tag{150}$$

then equation (149) has the two solutions

$$\begin{aligned} \phi_1 &= \alpha + \xi \\ \phi_1 &= 2\pi + \alpha - \xi \end{aligned} \tag{151}$$

The physical identification of the angles given by equation (151) is worthwhile. Suppose that equation (118) is valid and, simultaneously, the trivial solution is valid so that

$$\phi_1 = \phi_2$$

This corresponds, of course,* to orbital intersection. Then equation (118) requires that

$$\tan \phi_1 = -p_1/p_2$$

* The same results can be obtained by equating the left member of equation (12) to the right member of equation (14).

Inserting p_1 and p_2 from equations (111) and (112) yields

$$(\gamma - \cos\vartheta)\cos\phi_1 + \sin\vartheta\sin\phi_1 = -q \tag{153}$$

which is the reduced form of equation (149). Thus, at an orbital intersection, $e_2 = \pm 1$.

The orbital transfer plane is cleaved into two regions of positive e_2 and two regions of negative e_2 so it is reasonable to suspect that there may be four places where $|e_2| = 1$. Thus far, only two such points have been isolated. Two others can be obtained as follows. Setting $|e_2| = 1$ in equations (12) and (13) yields the pair

$$Q_1\,(1 + e_1\cos\phi_1) = Q_2\,[1 \pm \cos(\phi_1 + \rho)]$$

$$Q_1\,e_1\sin\phi_1 = \pm\,Q_2\sin(\phi_1 + \rho)$$

Isolating $\sin(\phi_1 + \rho)$ and $\cos(\phi_1 + \rho)$, then squaring and adding the resulting equations gives

$$Q_1(1 + 2e_1\cos\phi_1 + e_1^2) = 2\,Q_2(1 + e_1\cos\phi_1) \tag{154}$$

Inserting Q_2 from equation (133) and cancelling like terms gives

$$\begin{aligned}[e_1^2(q^2 + \gamma\cos\vartheta - 1) - \gamma(\gamma - \cos\vartheta)]\cos\phi_1 + \gamma(e_1^2 - 1)\sin\vartheta\sin\phi_1 \\ = e_1(\gamma^2 - 2\gamma\cos\vartheta + 1 - q^2) - q\,\gamma(e_1^2 - 1)]\end{aligned} \tag{155}$$

Defining

$$\sigma = \tan^{-1}\frac{\gamma(e_1^2 - 1)\sin\vartheta}{e_1^2\,(q^2 + \gamma\cos\vartheta - 1) - \gamma\,(\gamma - \cos\vartheta)} \tag{156}$$

and

$$\tau = \cos^{-1}\frac{e_1\,(\gamma^2 - 2\gamma\cos\vartheta + 1 - q^2) - q\gamma(e_1^2 - 1)}{\sqrt{[e_1^2(q^2 + \gamma\cos\vartheta - 1) - \gamma(\gamma - \cos\vartheta)]^2 + [\gamma(e_1^2 - 1)\sin\vartheta]^2}} \tag{157}$$

then the other two points of unity for e_2 are given by the solution of equation (155) as

$$\phi_1 = \sigma + \tau$$
$$\phi_1 = 2\pi + \sigma - \tau \quad . \tag{158}$$

Within this section the behavior of $e_2(\phi_1)$ has been a focal point while $Q_2(\phi_1)$ has received less attention. The next step is to examine $Q_2(\phi_1)$ at each point which has proven to be of interest in $e_2(\phi_1)$.

The first such point pair is given by equations (139). The easiest way to make the substitution is by writing equation (137) as

$$\frac{e_1(\gamma^2 - 2\gamma \cos \vartheta + 1 - q^2)}{2\gamma} = -q-(\gamma-\cos \vartheta)\cos \phi_1 - \sin \vartheta \sin \phi_1 \tag{159}$$

and substituting the left side of equation (159) into equation (133) which immediately yields

$$Q_2 = 0 \tag{160}$$

at this point. This value then shows that a hyperbola of infinite eccentricity has resulted, i.e., a straight line.

The next important point pair comes from equations (144), the point of finite jump discontinuity in e_2. From equation (144) comes

$$\sin \phi_1 = \frac{\pm \sin \vartheta}{\sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1}}$$

$$\cos \phi_1 = \frac{\pm (\gamma - \cos \vartheta)}{\sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1}}$$

so that equation (133) becomes

$$Q_2 = Q_1 \{1 \pm (e_1/2\gamma) [\sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1} \mp q]\} \tag{161}$$

where the positive sign corresponds to the $\phi_1 = \alpha$ and the negative sign corresponds to $\phi_1 = \pi + \alpha$.

The first case where $|e_2| = 1$ comes from equations (145) and (147). In either case, equation (99) shows a divergence of Q_2.

The remaining case occurs when ϕ_1 is given by either of equations (158). The actual value of Q_2 at this point is not difficult to determine, numerically. But a single equation which gives that value is quite messy. Since the coefficient of Q_2 in equation (154) cannot vanish, this point is a regular point.

The following table, Table I, summarizes the information presented here on the behavior of $e_2(\phi_1)$ and $Q_2(\phi_1)$.

TABLE I

Governing Equations	Value of ϕ_1	Value of $e_2(\phi_1)$	Value of $Q_2(\phi_1)$
139	$\alpha + \zeta$	Diverges	0
	$2\pi + \alpha - \zeta$	Diverges	0
144	α	$\dfrac{\pm e_1 \sin \phi_1}{\sin(\phi_1 + \rho) + e_1 \sin \rho}$	$Q_1 \{1 + (e_1/2\gamma) [\sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1} - q]\}$
	$\pi + \alpha$	(Jump Discontinuity)	$Q_1 \{1 - (e_1/2\gamma) [\sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1} + q]\}$
151	$\alpha + \xi$	± 1	Diverges (orbital intersections)
	$2\pi + \alpha - \xi$	± 1	Diverges
158	$\sigma + \tau$	± 1	Regular Point
	$2\pi + \sigma - \tau$	± 1	Regular Point

It is very convenient to order the angles which are presented in Table I. That is, for a given input of the primitive variables, predict the order in which the important points of e_2 will occur. It must be realized that certain conventions are inherent in the derivation of the prior equations – i.e., all angles must be chosen in the range $[0,2\pi]$, the principle value of $\cos^{-1}(\ )$ will lie in the range $[0,\pi]$, etc. These conventions follow from standard computer implementations of the derived equations and we make use of them in the following discussion.

Since all angles are calculated in a modulo 2π system, any angle in column 2 of Table I can exceed any other angle. Arbitrarily, but without loss of generality, choose $[0 < \alpha < \pi]$. Then

$$\alpha + \pi > \alpha > 0 \quad .$$

Because of the principle value condition, equation (138) will yield

$$\pi > \zeta \geqslant 0 \quad ,$$

so that

$$2\pi > 2\pi - \zeta > \pi \quad ,$$

and

$$2\pi + \alpha > 2\pi + \alpha - \zeta > \pi + \alpha \quad ,$$

since $2\pi + \alpha = \alpha$ and $\pi + \alpha = \alpha - \pi$

$$\alpha > 2\pi + \alpha - \zeta > \pi + \alpha = \alpha - \pi \quad .$$

The next ordering occurs between the angles ξ and ζ. For ξ to be real, equation (150) shows that

$$-q \leqslant \sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1} \quad .$$

For convenience, abbreviate, as before,

$$r = \sqrt{\gamma^2 - 2\gamma \cos \vartheta + 1} \quad .$$

Then

$$q^2 \leqslant r^2 \quad ,$$

so that

$$e_1(q^2 - r^2) \leqslant 0 \quad ,$$

and

$$e_1(q^2 - r^2) - 2\gamma q \leqslant -2\gamma q \quad ,$$

or

$$\frac{e_1(q^2 - r^2) - 2\gamma q}{2\gamma r} \leqslant -\frac{q}{r}$$

so that

$$\cos^{-1}(-q/r) > \cos^{-1} \frac{e_1(q^2 - r^2) - 2\gamma q}{2\gamma r}$$

From equations (150) and (138) comes

$$\xi \leqslant \zeta$$

$$\xi + \alpha \leqslant \zeta + \alpha$$

$$2\pi + \alpha - \zeta \leqslant 2\pi + \alpha - \xi \quad .$$

The angle ordering, thus far, appears as

$$\alpha < \alpha + \xi < \alpha + \zeta < \alpha + \pi < 2\pi + \alpha - \zeta < 2\pi + \alpha - \xi \quad .$$

By construction, the remaining two important angles $\sigma + \tau$ and $2\pi + \sigma - \tau$ can only occur in one of two locations. $\sigma + \tau$ or $2\pi + \sigma - \tau$ must occur between $2 + \pi$ and $2\pi + \alpha - \zeta$ or between $\alpha + \zeta$ and $\alpha + \pi$. Whichever angle is established for the first slot determines that the second angle will occupy the second slot. Thus, we need only establish a single inequality for a chosen configuration to determine a complete ordering. Choose $\alpha = \pi$ so that $\alpha - \pi = 0$. From equation (119), $\vartheta = 0$. If $116 = 0$, then (156) shows that $\sigma = 0$. Now because of principle value considerations, $\pi \geqslant \tau$ so that

$$2\pi \geqslant 2\tau$$

or

$$2\pi - \tau \geqslant \tau$$

and

$$2\pi + \sigma - \tau \geqslant \sigma + \tau \quad .$$

Thus, the cyclic ordering can be written as follows:

$$(\alpha + \pi < \sigma + \tau < 2\pi + \alpha - \zeta < 2\pi + \alpha - \xi < \alpha < \alpha + \xi < \alpha + \zeta < 2\pi + \sigma - \tau)_{\text{mod } 2\pi}$$

The computation algorithm is quite simple. Choose the minimum value of any of the important points and arrange the rest in accord with the above ordering.

The above theory has shown that certain regions of intersecting orbits preclude tangential transfers. It is intuitively obvious, however, that these forbidden regions cannot occur in the case of non-intersecting orbits. It is important to demonstrate that if the orbits do not intersect then there are no discontinuities in $e_2(\phi_1)$.

Consider the following, since Q_1 and Q_3 are both positive

$$Q_1 > Q_1 - Q_3$$

so that

$$\frac{Q_1\, e_1}{Q_3\, e_3} > \frac{(Q_1 - Q_3)\, e_1}{Q_3\, e_3}$$

i.e.,

$$\gamma > e_1\, q$$

$$2\gamma > 2e_1\, q = e_1\, q + e_1\, q \quad .$$

Now suppose that an intersection does not occur. Then equation (150) shows that ξ will not be real, i.e.,

$$(-q > r) \text{ or } (-q < -r) \quad .$$

Thus, if $-q > r$,

$$2\gamma > e_1\, q - e_1\, r = e_1\, (q - r)$$

and

$$0 > (r + q)$$

so that

$$e_1(q - r)(q + r) > 2\gamma(r + q)$$

or

$$\frac{e_1(q^2 - r^2) - 2\gamma q}{2\gamma r} > 1$$

Conversely, if $-q < -r$ then $q - r >$ so that

$$\frac{e_1(q^2 - r^2) - 2\gamma q}{2\gamma r} < -1$$

From (138) it is now apparent that e_2 has no infinite discontinuities in the real plane. It is also easy to demonstrate that the finite discontinuities are precluded for non-intersecting orbits. In equation (130), the numerator cannot generally vanish via $r^2 = q^2$ so, if the only way it could vanish is via $\sin \phi_1 = 0$. If this is the case then the denominator will become

$$f = \pm(\gamma^2 + q^2 - 1) + 2\gamma q$$

which can be written as

$$f = \pm(\gamma \pm q)^2 \mp 1$$

Suppose that f does vanish. Substituting for γ and q in terms of primitive values shows that one of the four conditions

$$a_1(1 + e_1) = a_3(1 + e_3)$$

$$a_1(1 - e_1) = a_3(1 + e_3)$$

$$a_1(1 + e_1) = a_3(1 - e_3)$$

$$a_1(1 - e_1) = a_3(1 - e_3)$$

which correspond to osculating contacts between the orbits, i.e., single impulse transfers.

With respect to $Q_2(\phi_1)$, this function will diverge if the denominator of equation (133) vanishes. This leads immediately to the requirement that equation (150) can be satisfied which is contrary to hypothesis. It remains only to be shown that $Q_2(\phi_1)$ is everywhere positive if there is no intersection.

Again using the generic function f, let

$$f = \gamma \cos \phi_1 + q - \cos(\phi_1 + \vartheta)$$

so that extrema of f occur when $\phi_1 = \alpha$, i.e.,

$$\cos\phi_1 = \pm\frac{(\gamma-\cos\vartheta)}{r}$$

$$\sin\phi = \pm\frac{\sin\vartheta}{r}$$

Then the extrema of f are given by

$$f_{ex} = \pm r + q$$

so that, from (133), the extrema of $Q_2(\phi_1)$ are

$$Q_{2,ex} = Q_1\left[1 + \frac{e_1(r-q)(r+q)}{2\gamma(\pm r+q)}\right]$$

Suppose, first, that

$$Q_{2,ex} = Q_1\left[1 + \frac{e_1(x-q)}{2\gamma}\right]$$

and if $q = r+\varepsilon^2$, then

$$Q_{2,ex} = Q_1\left[1 - \frac{e_1\varepsilon^2}{2\gamma}\right]\rightarrow Q_1$$

as ε becomes small, while if $-q = r+\varepsilon^2$ then

$$Q_{2,ex} = Q_1\left[1 + \frac{e_1(-2q-\varepsilon^2)}{2\gamma}\right]\rightarrow Q_1\left[1 - \frac{e_1 q}{\gamma}\right] = Q_1\left[1 - \frac{e_1(Q_1-Q_3)/(Q_3e_3)}{(e_1Q_1)/(Q_3e_3)}\right] = Q_3$$

as ε becomes small.

Next, let

$$Q_{2,ex} = Q_1 \left[1 - \frac{e_1(r+q)}{2\gamma} \right]$$

and $q = r + \varepsilon^2$ then

$$Q_{2,ex} = Q_1 \left[1 - \frac{e_1(2q-\varepsilon^2)}{2\gamma} \right] \rightarrow Q_1 \left[1 - \frac{e_1 q}{\gamma} \right] = Q_1 \left[1 - \frac{e_1(Q_1 - Q_3)/(Q_3 e_3)}{(Q_1 e_1)/(Q_3 e_3)} \right] = Q_3$$

as ε becomes small, while if $-q = r + \varepsilon^2$ then

$$Q_{2,ex} = Q_1 \left[1 + \frac{e_1 \varepsilon^2}{2\gamma} \right] \rightarrow Q_1$$

as ε becomes small.

The conclusion is, then, that if the orbits do not intersect, Q_2 is bounded between the (positive) values of Q_1 and Q_3. Since non-intersecting orbits have no singularities in the $e_2(\phi_1)$ function and positive $Q_2(\phi_1)$ values, we conclude that tangential transfer is possible between any two non-intersecting orbits.

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16. ABSTRACT

The classical problem of tangential impulse transfer between coplanar Keplerian orbits is addressed. A completely analytic solution which does not rely on sequential calculation is obtained and this solution is used to demonstrate that certain initially chosen angles can produce singularities in the parameters of the transfer orbit. A necessary and sufficient condition for such singularities is that the initial and final orbits intersect.

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