

# **Modeling of Composite Beams and Plates for Static and Dynamic Analysis**

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## Introduction

The purpose of this research is to develop a rigorous theory and corresponding computational algorithms for a variety of problems regarding the analysis of composite beams and plates. The modeling approach is intended to be applicable to both static and dynamic analysis of generally anisotropic, nonhomogeneous beams and plates. The major part of the effort during this first reporting period has been devoted to development of a theory for analysis of the local deformation of plates. In addition, some work has been performed on global deformation of beams. Because of the strong parallel between beams and plates, we will treat the two together as *thin bodies*, especially where we believe it will aid in clarifying to the reader the meaning of certain terminology and the motivation behind certain mathematical operations.

## Background

The static and dynamic analysis of beams and plates is of fundamental importance in many engineering problems. In design and analysis of modern aerospace systems, subsystems which consist of laminated plates or composite beams are often encountered such as wing, fuselage, and rotor blade structures. Classical beam and plate theories are known to be adequate for many applications. For beams or plates with anisotropic and nonhomogeneous construction, however, these theories suffer from several sources of inaccuracy. For example, composite beams – even those which are slender – can be very flexible in shear. When such beams are analyzed by classical theories, which generally ignore shear deformation, certain kinematical quantities and associated constitutive couplings are absent which are known to be important. Rehfield, Atilgan, and Hodges (1990) concluded that such phenomena can influence the global deformation in thin-walled beams designed for extension-twist coupling by as much as a factor of two! One of the chief mechanisms for this large effect is a coupling between bending and shear deformation due to anisotropic materials (see, Rehfield and Atilgan, 1989).

This conclusion also holds for beams which are not necessarily thin-walled; however, the cross sectional analysis of general nonhomogeneous, anisotropic beams cannot be treated without finite elements. Unlike classical beam or plate theories, theories for composite beams or plates must contain some means of calculating the elastic properties. These properties are not merely material moduli multiplied by certain integrals over the section of the beam. A review by Hodges (1990a) covers most of the literature dealing with modeling of beams prior to 1988. A similar treatment of composite plates, in which the determination of properties is carried out as a separate analysis, appears to be missing from the literature.

The analysis of deformation for composite thin bodies normally must be done as an iterative process. There are local deformation analyses, which determine the elastic constants and details of local deformation in terms of global deformation parameters. Also there are global analyses, which determine the global (beam- or plate-like) deformation such as bending, extension, torsion, and shear) deformation. One could proceed as follows: (1) calculate the local deformation and elastic constants for the undeformed structure; (2) use

the resulting elastic constants in a global deformation analysis; (3) recalculate the local deformation for the deformed structure; (4) stop the process if the elastic constants do not change more than some tolerance or else go back to step 2. Most plates and many beams do not exhibit a sufficiently large local deformation (which we will call warping) to require more than one calculation of properties. In other words, elastic constants which are determined based on the undeformed state may be sufficient even for geometrically nonlinear analyses.

There is another aspect of research which we originally proposed, that being the use of space-time finite elements to treat the global dynamics of beams. In the time between the submittal of our proposal to the Aerostructures Directorate and its being funded, another of our proposals was funded. It had been under review at the National Science Foundation for a long time; among other things, it concerns the dynamics of beams by space-time finite elements. Since both of these grants have this common subject matter as subsets of their programs, we intend to treat this portion of the research as jointly funded. Prof. David A. Peters and Dr. Weiyu Zhou have contributed to the NSF work, and thus, indirectly to this project.

Most finite element procedures for time-dependent phenomena are based on semi-discretizations; finite elements are used in space to reduce to a system of ordinary differential equations. This kind of procedure is widely used in practice and fairly well understood. Space-time finite elements have been rarely used in solutions of engineering dynamic problems. Classic time integration methods are usually included in computational procedures. Recent developments of the space-time finite element method allow application of approximation techniques to the spatial and temporal domains. Special schemes lead to highly efficient algorithms that reduce both memory requirements and number of arithmetical operations.

The use of space-time finite elements presents yet another duality. The static and frequency-domain global deformation analyses for plates are solved on the 2-D domain of the plate; the space-time dynamics of beams can be cast as a 2-D problem with simultaneous spatial and temporal discretizations.

### **Previous Work Related to Beams**

Although there has been much work published in the literature on beams and plates, a unified approach such as we are undertaking has not received very much attention. Furthermore, most of what does exist in the literature applies to beams only. Thus, before getting into plate references, we will describe the analogous type of analysis for beams with the hope of clarifying what we have begun to do for plates.

Berdichevsky (1981) was the first to prove, from variational-asymptotic considerations, that nonlinear analysis of beams can be split into two separate problems: the local deformation is a linear 2-D problem, and the global deformation is a nonlinear 1-D problem. The global deformation analysis of beams can be undertaken by use of 6 displacement variables associated with a cross section of a beam. The variables describe displacement of a point in the cross section and rotation of the cross section as a rigid body. There are

also 6 generalized strains (extension and two shearing strains at the reference line, twist, and two bending "curvatures") and 6 stress resultants (axial force and two components of shear force, twisting moment, and two components of bending moment); these must be related by elastic constants. This way of describing the global deformation requires at most 21 elastic constants (a symmetric  $6 \times 6$  matrix).

**Beam Local Deformation** To find these constants, all possible local deformations (*i.e.*, warping) of the cross section must be taken into account, although they can be assumed small. Here warping refers not only to out-of-plane distortion of the cross section during torsional deformation as in classical theories, but in-plane and out-of-plane deformation; these are fully coupled when the beam is nonhomogeneous and anisotropic.

As pointed out by Hodges (1990a), the work by Giavotto *et al.* (1983) is the most general of all the published works. Giavotto *et al.* (1983) developed a finite element approach for determining the elastic constants for an arbitrarily nonhomogeneous, anisotropic beam. This approach makes use of a two-dimensional finite element mesh representative of the beam cross section geometry and material properties. The results, in addition to the matrix of elastic constants, include the distribution of warping displacements per unit values of each of the stress resultants (section forces and moments in the cross section basis) and the three-dimensional stress and strain values throughout the cross section per unit values of the stress resultants.

Because of the generality of the work of Giavotto *et al.* (1983), we have adapted a version of the code developed by Giavotto, Borri, and their associates for obtaining the elastic constants for anisotropic beams. Its only shortcoming is its rather long computer times for calculating properties of realistic composite beams. For this reason, under Army Research Office sponsorship, we are also developing a theory based on the variational-asymptotic method as formulated and applied to nonlinear analysis of isotropic shells by Berdichevsky (1978, 1979). This method has some promise of yielding a more computationally efficient algorithm for extracting the properties.

**Beam Global Deformation** These elastic constants can then be used to find either linear or nonlinear global deformation, free-vibration modes and frequencies (see, Hodges and Atilgan *et al.*, 1989, 1990), and buckling behavior (see, Rehfield and Atilgan, 1989). Various modal, direct numerical integration, and finite element methods exist for this purpose. Because of their computational efficiency and modeling flexibility, finite element methods are quite popular. Displacement finite element methods for geometrically nonlinear behavior of beams, however, require numerical quadrature of highly nonlinear functions of the beam deformation. This tends to make the numerical solution procedure quite inefficient. On the other hand, with a mixed method such as that of Hodges (1990b), numerical element quadrature can be avoided (as long as applied load terms that are explicit in the axial coordinate are integrable in closed form). Such a nonlinear global analysis gives the beam displacements, rotations, extensional strain, shear strains, twist, bending curvatures, and sectional forces and moments to a comparable level of accuracy. It should be noted that

one can then use these results for forces and moments to find pointwise stress or strain levels throughout the cross section using the cross sectional finite element mesh.

Beam behavior in the time domain can be calculated by finite elements. For the dynamics of beams, recent work in the field of space-time finite elements applied to structural dynamics are described and reviewed by Bajer and Bonthoux (1988). There have been several developments in this area for time domain analysis of simple linear oscillators and rigid bladed helicopter rotors; for example Borri *et al.* (1985, 1988), Borri (1986), and Peters and Izadpanah (1988). The results indicate that finite elements in time provide a way of determining the dynamic behavior of a deformable body undergoing time-dependent loading. We have concentrated our work on the determination of the dynamic response of flexible structures by simultaneous discretization of the spatial and temporal domains. The main purpose is to determine if this methodology can be made feasible.

### Previous Work Related to Plates

Reissner (1985) departed from a three-dimensional statement of the problem recalling the fact that in the absence of the adjective *thin*, plate theory would be no more than a class of boundary value problems in three-dimensional elasticity. He gave an excellent sociological-historical survey with his interpretations of the nature of approximations and their consequences. He also touched upon the qualitative differences in the modeling of laminated plates. Since we believe that a *consistent* plate theory should include all possible deformations, our starting point *is*, naturally, the three-dimensional kinematics.

The analysis of laminated plates has attracted an enormous amount of attention. A description of the plate problem, analogous to the beam, has not been published to the best of our knowledge. Although the review articles by Bert (1984) and Noor and Burton (1989) contain hundreds of references which treat the laminated plate problem, all of them use some sort of explicit, analytic, through-the-thickness assumptions for displacement. While this is not incorrect, simple, low-order theories of this type will not yield correct stresses and strains through the thickness. The reason for this is rather obvious: in a laminated plate where each layer has distinct material properties, the actual displacement is not and cannot be analytic. When the assumed displacement is differentiated to yield the strain, the approximate strain will also be analytic while the actual strain may be discontinuous. Higher-order theories will be more accurate, but at a potentially high cost.

There are alternatives. One is to use a family of approaches as outlined and reviewed by Librescu and Reddy (1989) in which breaking the plate into finite elements through the thickness is advocated. This will yield the correct answer, but, again, at a potentially high computational cost. Another alternative is to derive a local deformation theory similar to that for the beam described above.

### Present Approach

Plate Modeling In this research, we are developing such a computational method for determining the elastic constants for laminated plates. The approach is very similar to

that described above for beams. Yet, to the best of our knowledge, the present approach has never been attempted. We originally set out to solve the interior St.-Venant solution for the plate by finite elements in a manner that is strictly analogous to the approach of Giavotto *et al.* (1983) for the beam problem. After applying the finite element method, a set of linear equations were obtained that were very similar to those of Giavotto *et al.* (1983). Solution of these equations in a manner similar to the earlier ones, however, turned out to be difficult. Thus, we turned to an approximate solution based on the variational-asymptotic method.

The domain of the local deformation problem for the beam is planar (2-D), just as the global deformation problem for the plate is. On the other hand, the global deformation problem for the beam is solved along a line (1-D), just as the local deformation problem for the plate is. For the local deformation of the plate, instead of an arbitrary interior cross section as with the beam, we work with an arbitrary interior normal line element of the plate (a line of material points normal to the reference surface of the plate). The tractions acting on this line element are used to obtain a variational principle governing the local stress resultants and deformation of the line element. The variational principle leads to a symmetric  $8 \times 8$  matrix of elastic constants (for a total of 36) based on the linear relation between the 8 stress resultants and 8 generalized strains.

As with the beam problem, the elastic constants will be determined from a finite element code that is linear. This code will enable us to calculate the elastic constants for an arbitrary laminated plate. In this report, after presentation of the theory, we present some preliminary results. These results, which essentially duplicate classical theory, were obtained to check out the methodology and the code.

**Dynamics of Beams** An important step towards obtaining a general and consistent form of beam elastodynamic equations was taken by Hodges (1990b). Therein, geometrically nonlinear beam elastodynamic equations as derived from Hamilton's Principle; also, using appropriate Lagrange multipliers a mixed variational formulation suitable for space-time finite element discretization was developed. In order to exploit the usefulness of space-time finite elements we decided to start from the very basic linear equations for longitudinal dynamics of a beam, a special case of Hodges (1990b), and their solutions. In this report, after a brief treatment of the theory, there are a few preliminary results presented.

The report closes with a description of work to be undertaken during the next reporting period.

## Unified Variational Formulation for Anisotropic Plates

Our starting point is the nonlinear kinematics of deformation for plates. We will develop the three-dimensional Biot strain field based upon the kinematical development of Danielson (1989). After this, we will formulate the strain energy. Finally, approximations of this strain energy function will be discussed and asymptotically correct solutions for the through-the-thickness analysis of isotropic plates, as well as the corresponding elastic constants, will be given.

## Kinematics

A plate is a flexible body in which matter is distributed about a planar surface so that one dimension is significantly smaller than the other two. (Much of our analysis can be easily extended to treat shells, but herein we will consider only plates.) The reference surface is an arbitrary planar surface, not necessarily the mid-surface of the plate. Throughout the analysis, Greek indices assume values 1 or 2, Latin indices assume values 1, 2, and 3 and repeated indices are summed over their ranges. Let us establish a Cartesian coordinate system  $x_i$  so that  $x_\alpha$  denote lengths along orthogonal lines in the reference surface and  $x_3$  is the distance the normal to the reference surface. Let  $\mathbf{b}_i$  denote an orthogonal reference triad along the undeformed coordinate lines. The position vector to an arbitrary point along the normal line is

$$\mathbf{r}^*(x_1, x_2, x_3) = \mathbf{r}(x_1, x_2) + x_3 \mathbf{b}_3 = x_i \mathbf{b}_i \quad (1)$$

Covariant and contravariant undeformed base vectors are defined as, respectively,

$$\begin{aligned} \mathbf{g}_i &= \frac{\partial \mathbf{r}^*}{\partial x_i} \\ \mathbf{g}^i &= \frac{1}{2\sqrt{g}} \epsilon_{ijk} \frac{\partial \mathbf{r}^*}{\partial x_j} \times \frac{\partial \mathbf{r}^*}{\partial x_k} \end{aligned} \quad (2)$$

where  $g = \det(\mathbf{g}_i \cdot \mathbf{g}_j)$ . For this analysis, both reduce to

$$\mathbf{g}_i = \mathbf{g}^i = \mathbf{b}_i \quad (3)$$

In a similar manner, consider the deformed state configuration. The particle which had position vector  $\mathbf{r}^*(x_1, x_2, x_3)$  in the undeformed plate now has position vector  $\mathbf{R}^*(x_1, x_2, x_3)$  relative to the same point, which can be represented by

$$\mathbf{R}^*(x_1, x_2, x_3) = \mathbf{R}(x_1, x_2) + x_3 \mathbf{B}_3(x_1, x_2) + w_i(x_1, x_2, x_3) \mathbf{B}_i(x_1, x_2) \quad (4)$$

where  $\mathbf{R}(x_1, x_2) = \mathbf{r}(x_1, x_2) + \mathbf{u}(x_1, x_2)$  and  $\mathbf{u} = u_i \mathbf{b}_i$  is the displacement vector of the points on the reference surface and  $w_i(x_1, x_2, x_3)$  is the general local (*i.e.*, warping) displacement of an arbitrary point on the normal line, consisting of both in- and out-of-plane components, so that all possible deformations are considered (Figs. 1, 2). The relationship between  $\mathbf{B}_i$  and  $\mathbf{b}_i$  is given by

$$\mathbf{B}_i(x_1, x_2) = C_{ij}(x_1, x_2) \mathbf{b}_j \quad (5)$$

where  $C(x_1, x_2)$  is the matrix of direction cosines. Covariant deformed base vectors

$$\mathbf{G}_i = \frac{\partial \mathbf{R}^*}{\partial x_i} \quad (6)$$

can be obtained by standard means. It should be noted, however, the measure numbers  $w_i$  provide redundant information since the normal line undergoes rigid-body displacement due to  $u_i$  and rigid-body rotation due to  $C$ . Therefore, some means of removing this redundancy must be introduced. For a plate, we can choose the unit vectors  $B_i$  so that  $w_i$  is small, at least in some sense. Setting an appropriate number of weighted average displacements to zero is one way to remove the redundancy. This can be conveniently done by the finite element method and will be dealt with below.

Here we restrict ourselves to the case when strain and local rotation are small so that the three-dimensional Biot strain can be expressed as

$$\Gamma^* = \frac{A + A^T}{2} - I \quad (7)$$

where  $A_{ij} = B_i \cdot G_k g^k \cdot b_j$  is the deformation gradient matrix and  $I$  is the  $3 \times 3$  identity matrix. Here  $\Gamma^*$  is a  $3 \times 3$  symmetric matrix. Introducing the column matrix  $w$  with components  $w_i$ , one can express the three-dimensional strain field as a  $6 \times 1$  column matrix  $\Gamma = [\Gamma_{11} \quad 2\Gamma_{12} \quad \Gamma_{22} \quad 2\Gamma_{13} \quad 2\Gamma_{23} \quad \Gamma_{33}]^T$  so that

$$\Gamma = \mathcal{H}\epsilon + I_3 w_{,3} + I_1 w_{,1} + I_2 w_{,2} \quad (8)$$

where  $( )_{,i}$  denotes the partial differentiation with respect to  $x_i$  and

$$\begin{aligned} \mathcal{H} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ I_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (9)$$

and  $\epsilon = [\gamma \quad \kappa]^T$ , the intrinsic strain measure which is function of only  $x_\alpha$ . Also  $\gamma = [\gamma_{11} \quad 2\gamma_{12} \quad \gamma_{22} \quad 2\gamma_{13} \quad 2\gamma_{23}]^T$  and  $\kappa = [\kappa_{11} \quad 2\kappa_{12} \quad \kappa_{22}]^T$  are the so-called force and moment strain measures. Note that  $\gamma_{11}$  and  $\gamma_{22}$  are the extensional strains of the reference surface,  $\gamma_{12}$  is the shear strain in the plane of the reference surface,  $\gamma_{\alpha 3}$  are the transverse shear strains of the normal line element,  $\kappa_{11}$  and  $\kappa_{22}$  are the elastic components of the bending curvature, and  $\kappa_{12}$  is the elastic twist. The force and moment strain measures are so

designated because they are conjugate to the actual running stress and moment resultants, respectively.

Since warping displacements are supposed to be quite small, the few nonlinear terms in the strain field, which couple  $w$  and  $\epsilon$ , have been neglected in Eq. (8). The form of the strain field is of great importance because it is now linear in  $\gamma$ ,  $\kappa$ , and  $w$  and its derivatives. If the top and the bottom of the line element through the thickness of the plate are free of tractions, application of the principle of virtual work to an infinitesimal line element, would lead to a system of *linear* equations over the one-dimensional line element governing  $w$ . The warping could then be determined in terms of these intrinsic strain measures or stress resultants as in Giavotto *et al.* (1983). This would lead to a unique two-dimensional strain energy function  $U(\gamma, \kappa)$ . The elastic law could then be put in a form

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \quad (10)$$

where  $F$  and  $M$  are column matrices  $F = [F_{11} \ F_{12} \ F_{22} \ F_{13} \ F_{23}]^T$  and  $M = [M_{11} \ M_{12} \ M_{22}]^T$ . Here  $F_{11}$  and  $F_{22}$  are the in-plane stretching stress resultants,  $F_{12}$  is the in-plane shear stress resultant,  $F_{\alpha 3}$ , are the transverse shear stress resultants;  $M_{11}$  and  $M_{22}$  are the bending moment resultants and  $M_{12}$  is the twisting moment resultant, all expressed in the  $B_i$  basis. The elastic stiffness matrix relating  $F$  and  $M$  to  $\gamma$  and  $\kappa$  is  $8 \times 8$ . The matrix  $A$  is  $5 \times 5$ ,  $D$  is  $3 \times 3$ , and  $B$  is  $5 \times 3$ .

Even though this methodology is successful for beam formulations (for linear analysis see Giavotto *et al.*, 1983, and for nonlinear analysis see Atilgan and Hodges, 1990), it has not been completely resolved for plate analysis. (This analysis is outlined in the Appendix as far as we have been able to take it.) Therefore, we have changed our methodology from direct to asymptotical analysis.

## Strain Energy and Approximations

One of the most consistent ways to obtain the constitutive law for thin-body (*i.e.* beam and plate/shell) analysis is the use of asymptotical analysis. Literature for the successful analysis of asymptotical methods for beams can be found in Hodges (1987) and Atilgan and Hodges (1990) and for plates in Noor and Burton (1989). In addition to the *direct asymptotical* analysis, which is applied to *differential equations*, Berdichevsky (1978, 1979) developed the *variational-asymptotical* analysis, which is applied to *functionals*. Berdichevsky and his co-workers applied this method successfully to beams and shells for static and dynamic analysis. An outline and a simple application of this method to beam analysis can be found in Berdichevsky (1980). In what follows we will apply this method to non-homogeneous and anisotropic plates to obtain the most consistent approximations of the constitutive law.

### Three-Dimensional Strain Energy for Anisotropic Plates

The three-dimensional strain energy for an anisotropic plate can be written as

$$U = \frac{1}{2} \int_A \int_h \Gamma^T D \Gamma dx_3 dA \quad (11)$$

where  $D$  is the  $6 \times 6$  symmetric material stiffness matrix which relates the three-dimensional Biot strain  $\Gamma$  to the three-dimensional Jaumann stress  $Z$ . The form of this matrix can be found in Jones (1975) for all possible type of material structures ranging from transversely isotropic case to most general anisotropy. The Jaumann stress is also arranged in a  $6 \times 1$  column matrix form  $Z = [Z_{11} \ Z_{12} \ Z_{22} \ Z_{13} \ Z_{23} \ Z_{33}]^T$  so that

$$Z = D\Gamma \quad (12)$$

Since warping is a three-dimensional function of all coordinates, for most general configurations, it is not possible to deal with it only through the thickness. Therefore, one should discretize the warping as follows

$$w(x_1, x_2, x_3) = N(x_3) W(x_1, x_2) \quad (13)$$

Then, using this together with Eq. (8) in Eq. (11), one can obtain the strain energy as follows

$$\begin{aligned} U = \frac{1}{2} \int_A & \left[ \epsilon^T A \epsilon + \right. \\ & \epsilon^T R^T W + W^T R \epsilon + W^T E W + \\ & \epsilon^T L_\alpha^T W_{,\alpha} + W_{,\alpha}^T L_\alpha \epsilon + W_{,\alpha}^T C_\alpha^T W + W^T C_\alpha W_{,\alpha} + \\ & \left. W_{,\alpha}^T M_{\alpha\beta} W_{,\beta} \right] dA \end{aligned} \quad (14)$$

where

$$\begin{aligned} A &= \int_h \mathcal{H}^T D \mathcal{H} dx_3 & R &= \int_h N'^T I_3^T D \mathcal{H} dx_3 & E &= \int_h N'^T I_3^T D I_3 N' dx_3 \\ L_\alpha &= \int_h N^T I_\alpha^T D \mathcal{H} dx_3 & C_\alpha &= \int_h N'^T I_3^T D I_\alpha N dx_3 & M_{\alpha\beta} &= \int_h N^T I_\alpha D I_\beta N dx_3 \end{aligned} \quad (15)$$

and where  $( )'$  denotes differentiation with respect to  $x_3$ . Because the description of the displacement is 5 times redundant, the rigid-body portion of the warping degrees of freedom must be removed in forming these equations. After this, the matrix  $E$  will be positive definite. The rigid-body portion of  $w$  can be removed by constraining the finite element

nodes. In order to do this, we set  $w_3 = 0$  at the reference surface and  $w_\alpha = 0$  at the upper and lower surfaces.

### Development of Finite Element Approximations for Anisotropic Plates

The *zeroth* approximation of the strain energy can be obtained by neglecting the warping completely. Then, the energy is only due to the global measures of the deformation and the stiffness matrix of the plate is just the matrix  $A$ . It can be shown that the matrix  $A$  gives an upper bound for the stiffnesses.

A first approximation of this functional can be obtained by taking only the first two lines of the strain energy functional. (This approximation cannot be called *the* first approximation until it is proven. We address this below.) These four terms are considered to be the most dominant terms in the strain energy functional. The reason for this is that differentiation with respect to the in-plane coordinates  $x_1$  and  $x_2$  will always result in smaller magnitudes than differentiation with respect to the thickness coordinate  $x_3$ . Therefore, the energy obtained by the remainder terms should be smaller than the first four terms. The first approximation of the strain energy functional then reads

$$U^* = \frac{1}{2} \int_A (\epsilon^T A \epsilon + \epsilon^T R^T W + W^T R \epsilon + W^T E W) dA \quad (16)$$

Since at the beginning we consider warping to be an arbitrary quantity, independent of the global strain measures, the Euler-Lagrange equation associated with  $W$  will be obtained from this functional by taking a straightforward variations of with respect to  $W$  yielding

$$R\epsilon + EW = 0 \quad (17)$$

From this we obtain a relationship between our global strain measure  $\epsilon$  and the warping  $W$  to be

$$W = -E^{-1} R\epsilon \quad (18)$$

In order to prove that this is *the* first approximation we need to find the second approximation. It can be shown by using a parallel development with that of Berdichevsky (1979) that the solution obtained here is the first approximation.

In order to find the stiffness martix, Eq. (18) is used in the first approximation of the strain energy functional, Eq. (16) which gives

$$U^* = \frac{1}{2} \int_A \epsilon^T (A - R^T E^{-1} R) \epsilon dA \quad (19)$$

This is the the first approximation for the strain energy. Derivative of the strain energy with respect to the global strain measure  $\epsilon$  results in the conjugate measures of the section stress resultants

$$Q = [F \quad M]^T = \left( \frac{\partial U^*}{\partial \epsilon} \right)^T \quad (20)$$

This relation suffices the existence of a relationship between stress resultants and the global measure of strains through an elastic law, which we call the first approximation of the *matrix of elastic stiffness constants*,  $S^*$ , as in Eq. (10)

$$Q = S^* \epsilon \quad (21)$$

Following the above operation,  $S^*$  can then be found as

$$S^* = A - R^T E^{-1} R \quad (22)$$

The matrix representing the first-order warping contribution to the stiffness matrix,  $R^T E^{-1} R$ , is positive definite. It can also be shown that a finite element approximation for the first asymptotic approximation  $S^*$  will be an upper bound on the actual stiffnesses.

### The First Approximation for Isotropic Plates

When we reduce our equations to the isotropic case it is possible to obtain an analytical solution for warping and the stiffness matrix. Using Eq. (16) for the isotropic case (for the first approximation for isotropic case, it is not necessary to discretize the warping) gives the warping displacements as follows

$$\begin{aligned} w_1 &= w_2 = 0 \\ w_3 &= -2\nu \left[ (\gamma_{11} + \gamma_{22}) x_3 + (\kappa_{11} + \kappa_{22}) \frac{x_3^2}{2} \right] \end{aligned} \quad (23)$$

where  $\nu$  is Poisson's ratio. The the elastic stiffness constants can be expressed in terms of  $\nu$ , Young's modulus  $E$ , and the shear modulus  $G$ . For the first approximation, the matrix of elastic constants is found to be

$$S^* = \begin{bmatrix} \frac{Eh}{(1-\nu^2)} & 0 & \frac{Eh}{(1-\nu^2)}\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & Gh & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{Eh}{(1-\nu^2)}\nu & 0 & \frac{Eh}{(1-\nu^2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Gh & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Gh & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Eh^3}{12(1-\nu^2)} & 0 & \frac{Eh^3}{12(1-\nu^2)}\nu \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{Eh^3}{6(1+\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Eh^3}{12(1-\nu^2)} & 0 & \frac{Eh^3}{12(1-\nu^2)} \end{bmatrix} \quad (24)$$

It can be seen that for the first approximation our results are exactly the same as those from classical isotropic plate theory. Since classical plate theory makes use of the *plane stress* assumption we see that the first approximation of our general functional also coincides with the plane stress assumption. If we were to neglect the warping starting from the beginning of our analysis then the stiffness values would be overestimated. The important point here is that *classical isotropic plate theory does include warping!*

As a first step in developing a numerical procedure, a finite element code has been written to evaluate the warping and the elastic constants for the isotropic case. Two-noded elements were used with  $C^0$  continuous shape functions. Results which have been obtained coincide almost identically with classical theory, and the agreement becomes much better as more elements are taken. An example output from our finite element program is shown in Fig. 3. The distribution of warping through the thickness due to  $\gamma_{11} + \gamma_{22}$  and  $\kappa_{11} + \kappa_{22}$ , respectively, are plotted in Figs. 4 and 5.

## Elastodynamics of Beams

In this section, we describe a simplified case for longitudinal dynamics of a beam. We begin with the equations of motion and develop a weak form for mixed space-time finite elements. Finally, we present some numerical results.

### Linear Rod Elastodynamic Equations

The equation of motion for a rod is given as

$$F' - \dot{P} + f = 0 \quad (25)$$

where  $F$  is the internal axial force,  $P$  is the linear momentum, and  $f$  is the force applied to the rod. All of these quantities are scalars and are functions of both space and time.

The linear rod kinematics is defined by a single displacement variable  $u$ . The velocity of the rod is obtained by differentiating the displacement with respect to time. Then the velocity of any point along the rod generator is

$$v = \dot{u} \quad (26)$$

The strain of the rod can be obtained by differentiating the displacement with respect to the spatial variable. So that the strain is

$$\epsilon = u' \quad (27)$$

The constitutive laws for the rod, which connect the strain and velocity to axial force and linear momentum, can be simply written as

$$F = \mu\epsilon \quad P = mv \quad (28)$$

The response of the rod can then be obtained by solving these equations simultaneously. Because of the simplicity of the structure, analytical solutions exist for some simple loadings. Thus, some initial and two-point boundary value problems can be used as benchmark cases in which space-time finite element method based upon above equations can be compared with some analytical solutions in the literature.

### Weak Formulation for Space-Time Finite Elements

It is possible to obtain a weak formulation for a rod by just specializing the weak form given by Hodges (1990b). However, since for this simple model we have such a simple constitutive law, one may find it useful to satisfy the constitutive relationships (algebraic equations) strongly. In this way we do not introduce any more unknowns than are necessary. Consequently, the following weak form can be obtained

$$\begin{aligned} & \iint_{A(x,t)} \left( \delta u - \frac{\delta P}{m} \right) P + \left( \frac{\delta F}{\mu} - \delta u' \right) F + u \left( -\delta P + \delta F \right) + \\ & f \delta u + \oint_t (-P \delta u + u \delta P) |_x + \oint_x (F \delta u - u \delta F) |_t \end{aligned} \quad (29)$$

where  $\delta u$ ,  $\delta P$ , and  $\delta F$  are test functions. It can be seen that one of the important properties of this functional is that none of the unknowns are ever differentiated. All the spatial and temporal differentiations are performed over the test functions. Weak forms with this property have been termed as “the weakest possible form” by Atilgan (1989). However, by allowing differentiation only over test functions yields a form such that the field equations now govern the test functions; note that they govern the trial functions in the primitive weak form. Since one may assign any function as a test function, the Green functions of the field equations could be chosen as test functions. The method using this kind of weak form has been called the “boundary element method” in the literature. Therefore, even though the weak forms can be same, selection of different shape functions can lead different solution strategies. This simple example can show that the differences in finite and

boundary element techniques are superficial; both are coming from the same background. More details along these lines and weak forms for theoretical mechanics will be found in a paper under preparation.

## Applications

The initial value problem of a cantilevered rod subjected to a suddenly applied load (Heaviside step function) at the free end is considered. This is a classical wave propagation problem for which the force and linear momentum are discontinuous. This problem was also investigated by Iura *et al.* (1988) by using a different weak form and by Mansur and Brebbia (1984) by using the boundary element method.

Our space-time element is rectangular. With the weak form, Eq. (29), we have chosen the following shape functions. For  $u$ ,  $F$ , and  $P$ , constants were chosen in the element interior. For the boundary,  $u$  is a constant but distinct value from the interior on each of the space and time boundaries. On the other hand,  $F$  and  $P$  are represented by Dirac delta functions at the element corners. The test functions  $\delta F$  and  $\delta P$  are linear in the space and time directions, respectively; and  $\delta u$  is bilinear in space and time directions. Our results are shown in Figs. 6 – 8 for the displacement, force, and linear momentum. Notice the discontinuous quantities are predicted accurately. Similar results for a case with initial displacement are shown in Figs. 9 – 11. In both cases, the results match the exact solution.

## Future Work

In the near future we will develop the second approximation for plate analysis. When applied to the isotropic plate, this will result in a computational method for generating, as a check, the so-called "shear-correction factors." After validation of the code, we will then extend it to treat anisotropic, laminated plate problems.

For the work on beam elastodynamics, we will continue to expand the capability of the analysis to deal with periodic excitation, arbitrary beam deformation, and nonlinear problems.

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## Appendix

Consider a line element through the plate thickness (Fig. 12). The principle of virtual work for this filament can be written as follows

$$\left( \int_h \delta s^T Z_\alpha dx_3 \right)_{,\alpha} = \int_h \delta \Gamma^T Z dx_3 \quad (30)$$

where  $\delta s$  is the virtual displacement of an arbitrary point on the normal line element,  $\delta \Gamma$  is a three-dimensional virtual strain, and  $Z$  is the three-dimensional stress measure conjugate to the strain, arranged in a  $6 \times 1$  column matrix form  $Z = [Z_{11} \ Z_{12} \ Z_{22} \ Z_{13} \ Z_{23} \ Z_{33}]^T$ . The tractions on the lateral surfaces of the line elements are written as

$$Z_1 = [Z_{11} \ Z_{12} \ Z_{13}]^T \quad Z_2 = [Z_{22} \ Z_{12} \ Z_{23}]^T \quad (31)$$

This principle enforces the weak satisfaction of the three-dimensional equilibrium equations, and traction-free upper and lower surfaces. This is clearly analogous to the beam St. Venant problem, except that this is for a one-dimensional line element through the thickness of a plate.

Let us decompose the displacement field into warping and rigid body displacement components

$$s = w + u + x_3 \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \quad (32)$$

where  $w$  and  $u$  are  $3 \times 1$  column matrices for warping and the displacement of the reference surface, respectively. As outlined in the text, it is possible to find the three-dimensional strain as

$$\Gamma = \mathcal{H}\epsilon + I_3 w_{,3} + I_1 w_{,1} + I_2 w_{,2} \quad (33)$$

where matrices  $\mathcal{H}$ ,  $I_3$ ,  $I_1$ , and  $I_2$  are defined in the text. The stress-strain relationship is given as  $Z = D\Gamma$ . Substitution of Eq. (33) into the principle of virtual work, and discretizing warping as  $w = N(x_3)W(x_1, x_2)$  one can obtain the following system of equations

$$\begin{bmatrix} A & B & D & I \\ B^T & C & E & J \\ D^T & E^T & H & K \\ I^T & J^T & K^T & L \end{bmatrix} \begin{Bmatrix} W_{,1} \\ W_{,2} \\ W \\ \epsilon \end{Bmatrix} = \begin{Bmatrix} \mathcal{P}_1 \\ \mathcal{P}_2 \\ \mathcal{P}_{1,1} + \mathcal{P}_{2,2} \\ Q \end{Bmatrix} \quad (34)$$

where

$$\begin{aligned} Q &= [F \ M]^T & \mathcal{P}_\alpha &= \int_h N^T Z_\alpha dx_3 \\ F &= [F_{11} \ F_{12} \ F_{22} \ F_{13} \ F_{23}]^T & M &= [M_{11} \ M_{12} \ M_{22}]^T \end{aligned} \quad (35)$$

and the matrices are defined as

$$\begin{aligned} A &= \int_h N^T I_1^T D I_1 N dx_3 & B &= \int_h N^T I_1^T D I_2 N dx_3 \\ C &= \int_h N^T I_2^T D I_2 N dx_3 & D &= \int_h N^T I_1^T D I_3 N dx_3 & E &= \int_h N^T I_2^T D I_3 N dx_3 \\ H &= \int_h N^T I_3^T D I_3 N dx_3 & I &= \int_h N^T I_1^T D \mathcal{H} dx_3 \\ J &= \int_h N^T I_2^T D \mathcal{H} dx_3 & K &= \int_h N^T I_3^T D \mathcal{H} dx_3 & L &= \int_h \mathcal{H}^T D \mathcal{H} dx_3 \end{aligned} \quad (36)$$

Eq. (34) can be reduced to one matrix equation governing the warping. However, in order to solve that equation, one must find a scalar algebraic equation governed by each stress resultant (each element of  $Q$ ). Since we have not been able to find such an equation, we have changed methods. It may be possible to find a solution for Eq. (34) by using a different approach, and this possibility is still open.

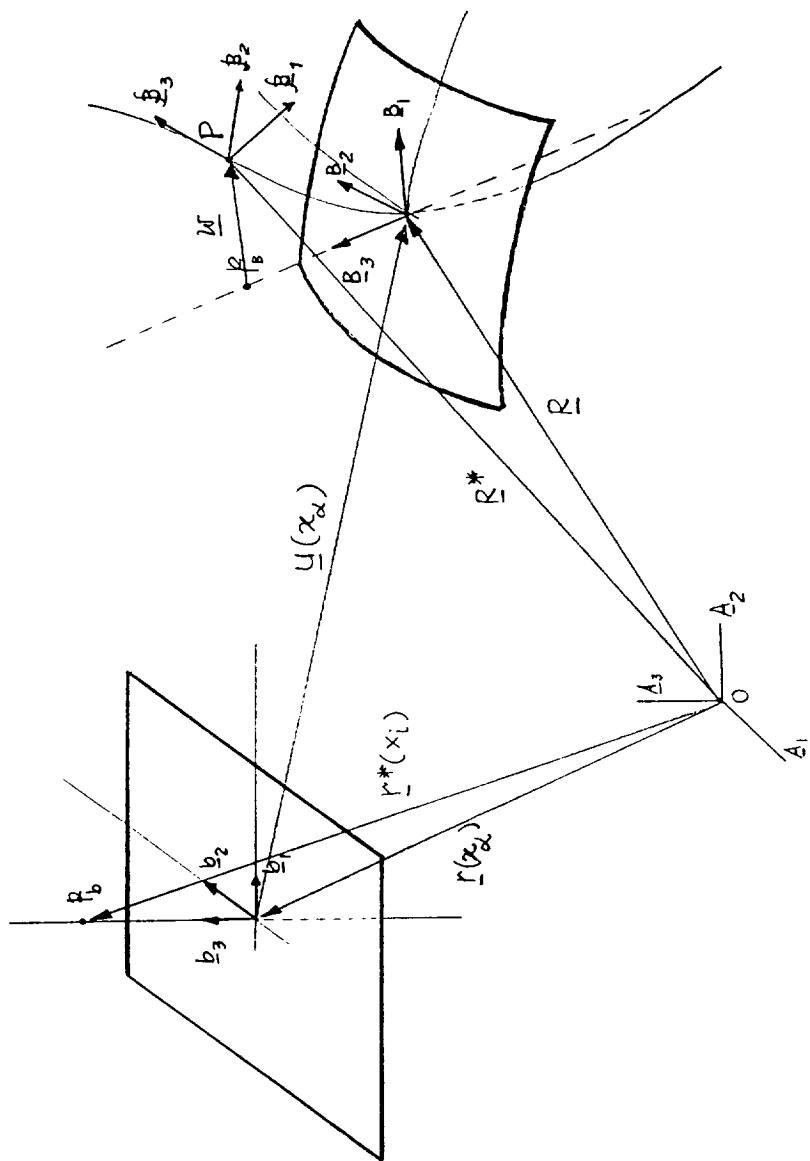


Figure 1: Geometry of plate deformations

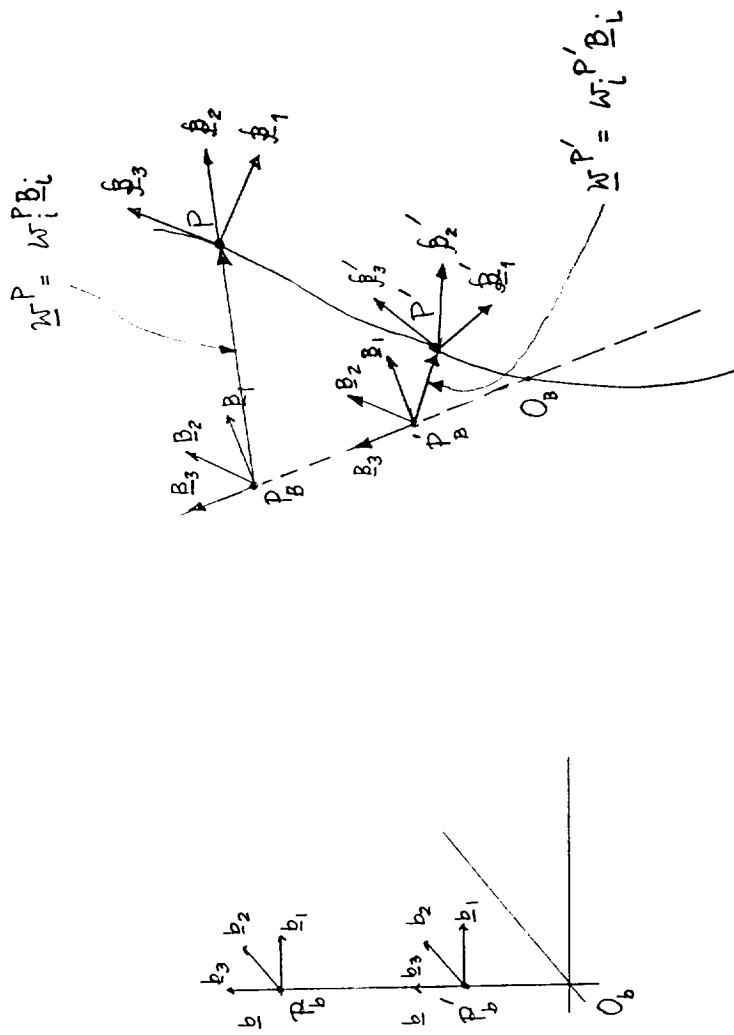


Figure 2: Concept of local rotation

Number of elements 32  
 Thickness (m) 0.00200  
 Young's modulus (N/m<sup>2</sup>) 0.2000D+09  
 Poisson's ratio .25

Constraints : 1 2 51 97 98

#### A matrix (without warping)

0.4800D+06	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.1600D+06	0.0000D+00	0.4800D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00
0.0000D+00	0.1600D+06						

#### R<sup>-1</sup>R (warping effects)

0.5333D+05	0.0000D+00	0.5333D+05	0.0000D+00	0.0000D+00	0.1421D-13	0.0000D+00	0.1421D-13
0.0000D+00							
0.5333D+05	0.0000D+00	0.5333D+05	0.0000D+00	0.0000D+00	0.1421D-13	0.0000D+00	0.1421D-13
0.0000D+00							
0.0000D+00							
0.1776D-13	0.0000D+00	0.1776D-13	0.0000D+00	0.0000D+00	0.1776D-01	0.0000D+00	0.1776D-01
0.0000D+00							
0.1776D-13	0.0000D+00	0.1776D-13	0.0000D+00	0.0000D+00	0.1776D-01	0.0000D+00	0.1776D-01

#### S\* (stiffness matrix)

0.4267D+06	0.0000D+00	0.1067D+06	0.0000D+00	0.0000D+00	-0.1421D-13	0.0000D+00	-0.1421D-13
0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.1067D+06	0.0000D+00	0.4267D+06	0.0000D+00	0.0000D+00	-0.1421D-13	0.0000D+00	-0.1421D-13
0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00
-0.1421D-13	0.0000D+00	-0.1421D-13	0.0000D+00	0.0000D+00	0.1422D+00	0.0000D+00	0.3557D-01
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.5333D-01	0.0000D+00	0.0000D+00
-0.1421D-13	0.0000D+00	-0.1421D-13	0.0000D+00	0.0000D+00	0.3557D-01	0.0000D+00	0.1422D+00

Figure 3: An example output from finite element code

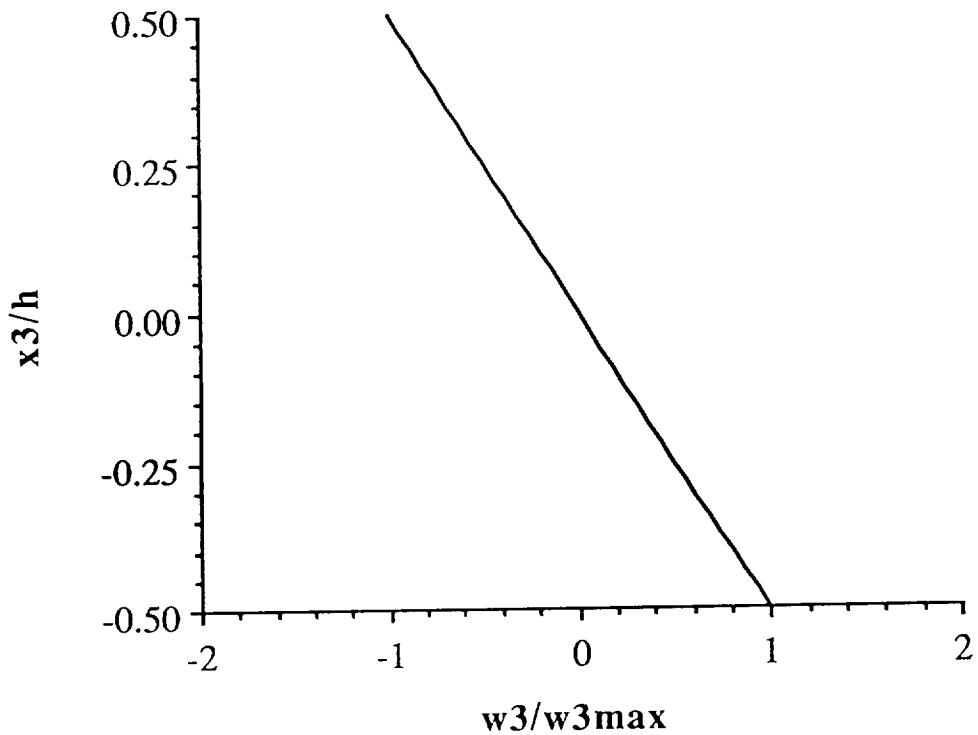


Figure 4: The distribution of warping through the thickness due to  $\gamma_{11} + \gamma_{22}$

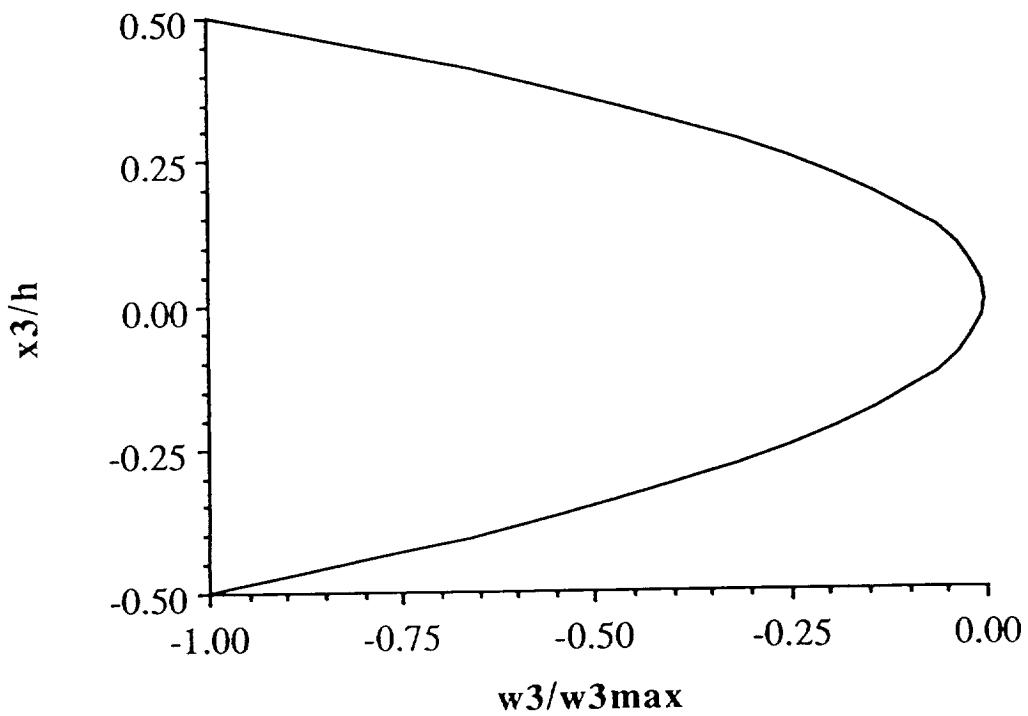


Figure 5: The distribution of warping through the thickness due to  $\kappa_{11} + \kappa_{22}$

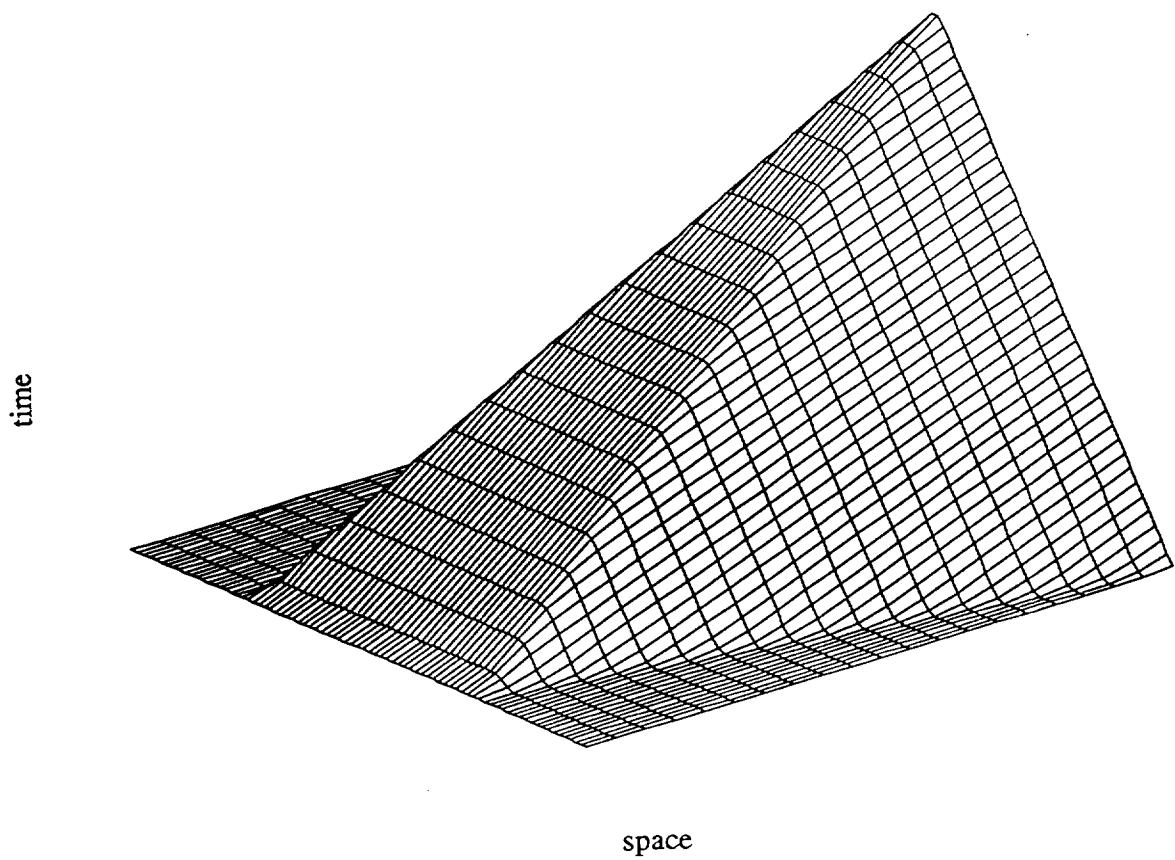


Figure 6: Displacement distribution in space-time domain due to heaviside step function

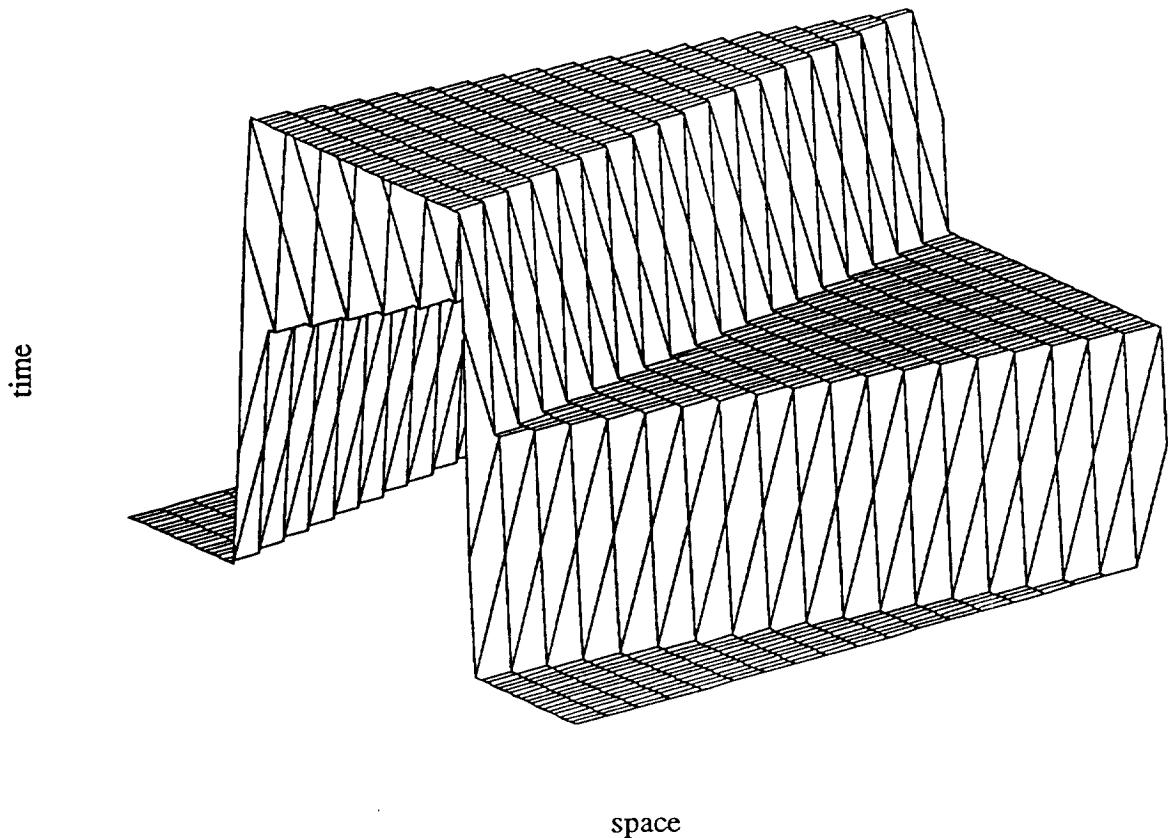


Figure 7: Force distribution in space-time domain due to heaviside step function

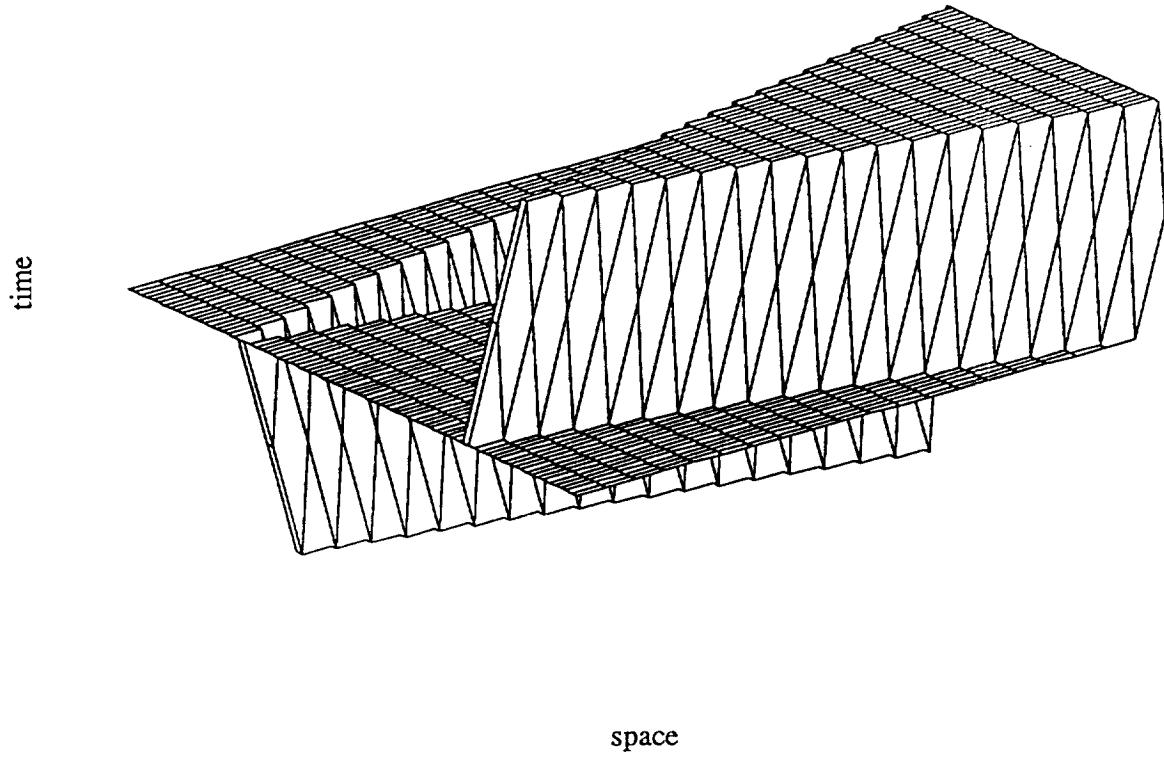


Figure 8: Linear momentum distribution in space-time domain due to heaviside step function

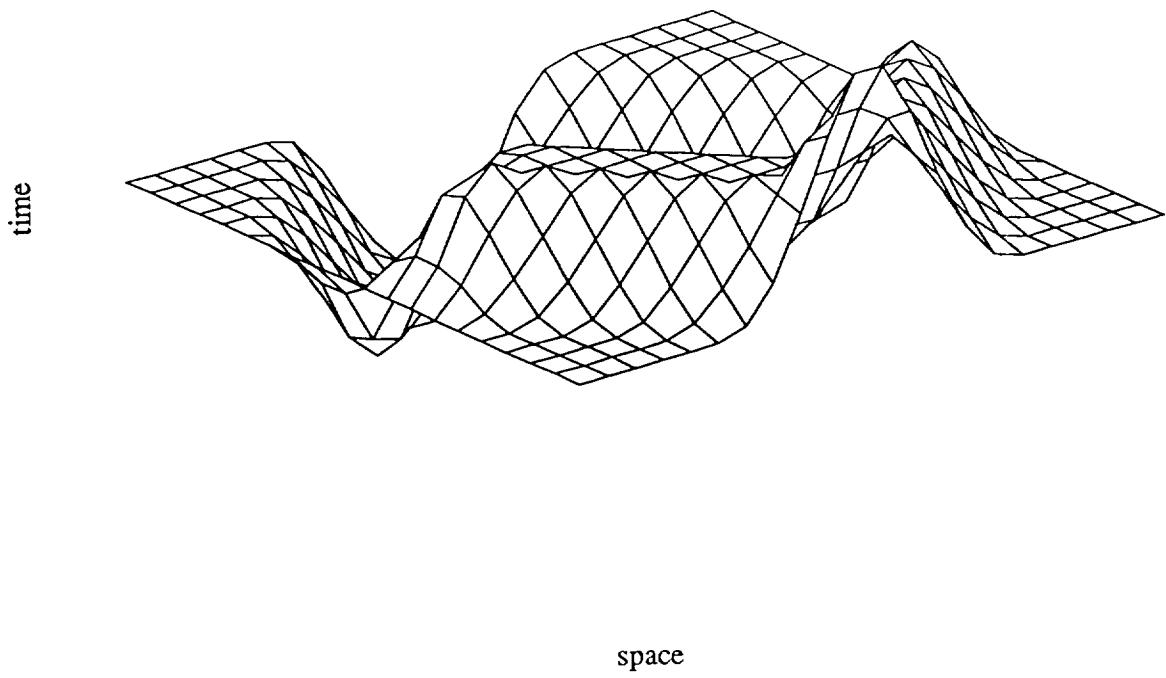


Figure 9: Displacement distribution in space-time domain due to initial displacement

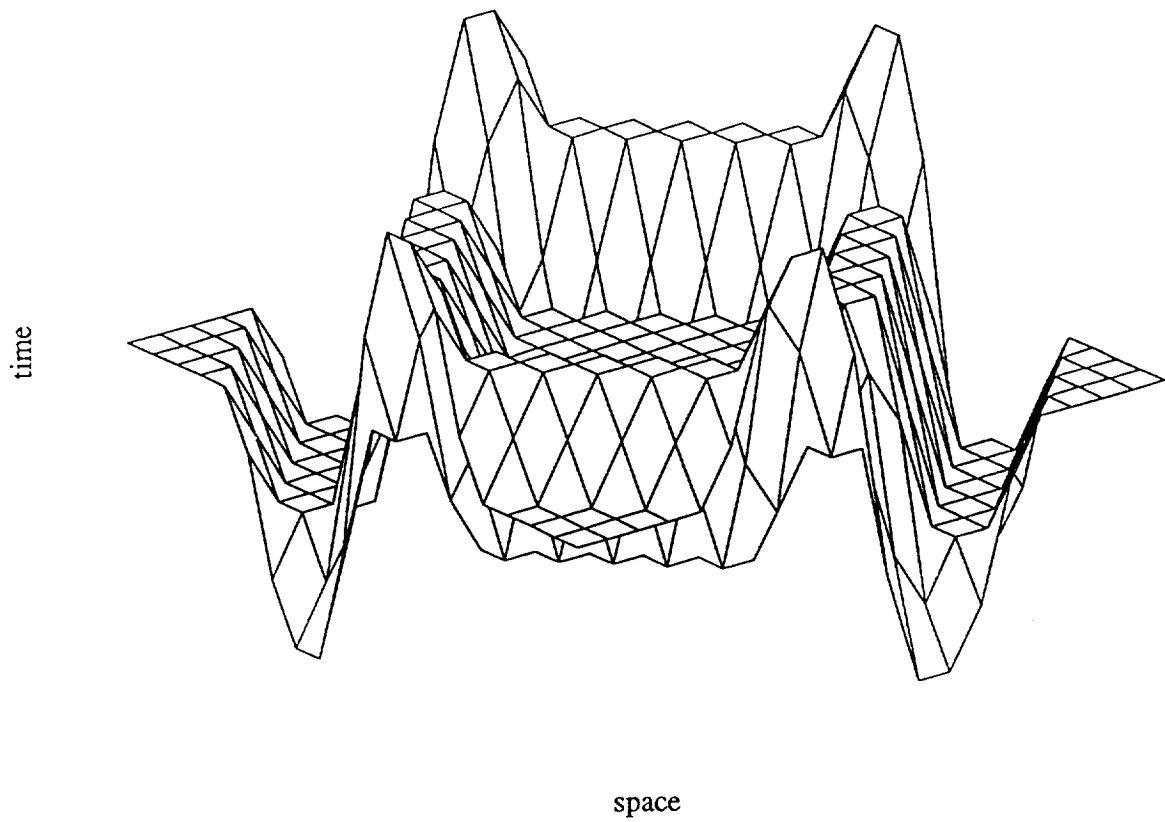


Figure 10: Force distribution in space-time domain due to initial displacement

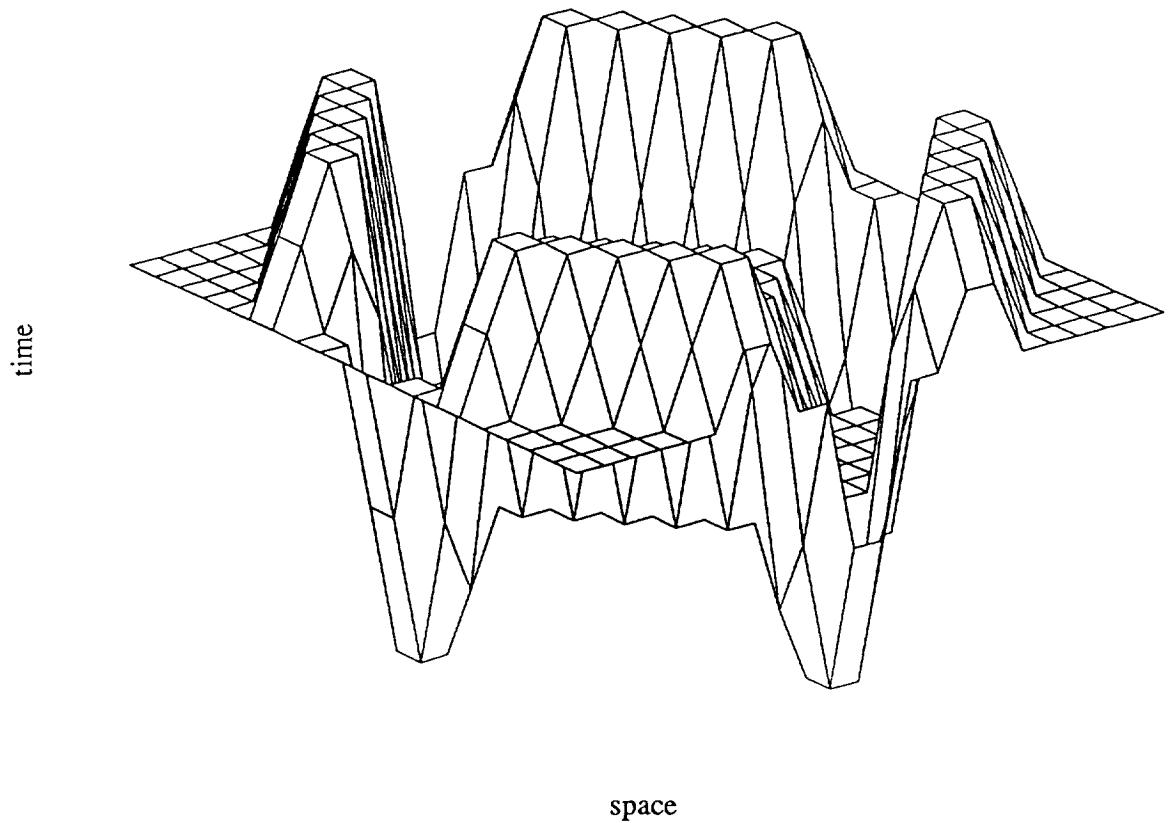


Figure 11: Linear momentum distribution in space-time domain due to initial displacement

Figure 12: An interior normal line element of a plate

