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Summary

Plates can have more than one buckled solution for a fixed set of boundary conditions. The theory for the identification and the computation of multiple solutions in buckled plates is examined in this paper. The theory is used to predict modal interaction, which is also called change in buckle pattern or secondary buckling, in experiments on certain plates with multiple theoretical solutions. A set of coordinate functions for Galerkin's method are defined so that the Von Karman plate equations are reduced to a coupled set of cubic equations in generalized coordinates that are uncoupled in the linear terms. An iterative procedure for solving modal interaction problems is suggested in the paper based on this cubic form.

Introduction

Modal interaction in plates is a nonlinear boundary-value problem. The plates considered here are loaded by edge loads in the plane of the undeformed plate. The edge loads are defined so that they appear in the theory as a linear boundary condition. The nonlinear part of the problem is the set of nonlinear partial differential equations of Von Karman plate theory. Modal interaction is concerned with identifying and computing all the solutions of the plate equations for a fixed set of boundary conditions.

The theory for modal interaction is examined in this paper. The theory shows why problems with multiple postbuckling solutions require more care in computations than problems with unique postbuckling solutions and provides insight into a proposed algorithm for handling such problems.

The theory for unique postbuckling solutions is often presented in the context of perturbation theory (ref. 1). The assumptions implied in expansions in a single perturbation parameter do not necessarily hold true for modal interaction problems. Continuation methods that treat the load parameter as a dependent variable have been implemented in finite-element codes (refs. 2 to 4). The algorithms in these codes assume continuity of the solutions that may lead to poor convergence properties when applied to modal interaction problems.

For discrete mechanical systems, the theory (refs. 5 and 6) is general enough to treat modal interaction problems. In the current paper, a discrete system for the plate problem is derived in generalized coordinates. Newton's method is the connection between the nonlinear partial differential equations of plate theory and the discrete theory. The analysis in the paper starts with the linear form of Newton's method. This form of Newton's method has convergence problems for modal interaction problems. However, the source of the poor convergence in the linear method is clear, and a modification to the linear algorithm is printed in this paper, so that rapid convergence is maintained. The modification is an extension of previous work on postbuckling analysis using Newton's method (ref. 7) for problems with isolated bifurcation points or limit points.

Newton's method is started by reducing the nonlinear problem to a sequence of linear boundary-value problems. Two difficulties arise in the application of Newton's method. First, the linear boundary-value problems do not have closed-form solutions; therefore, the computation requires some kind of an approximate solution. The second difficulty is the lack of convergence of the sequence of linear problems near bifurcation points, where the different nonlinear solutions intersect. Because the linear boundary-value problems must be solved by approximate methods, one approach is to discretize the nonlinear problem from the beginning and then apply Newton's method to the resulting set of nonlinear algebraic equations.

The completely algebraic approach is not followed here because of the second difficulty of ensuring convergence of Newton's method. Convergence is obtained here by converting the linear partial differential equations to a Sturm-Liouville problem with orthogonal eigenfunctions. The nonlinear problem is then discretized by a Galerkin solution with the eigenfunctions as coordinate functions. The resulting set of nonlinear algebraic equations with unknown generalized coordinates is uncoupled in the linear terms. The solution procedure can be summarized as a Galerkin solution of the nonlinear boundary-value problem for the plate. However, the choice of coordinate functions is not arbitrary; the coordinate functions are determined by the problem itself. The second, more subtle, difficulty of convergence is overcome directly before addressing the first difficulty, that of computing accurate approximate solutions for linear boundary-value problems.

The final result in this paper is a set of nonlinear algebraic equations in generalized coordinates. The equations are in the form postulated by Thompson and Hunt (ref. 6) in their studies of stability theory. The coefficients in the algebraic equations are defined explicitly for the plate problem. The process of defining the coefficients suggests algorithms for numerical solutions of the plate problem. Newton's method applied to the continuous formulation of Von Karman plate theory provides a direct connection with the theory for discrete approximations.

Symbols

$A_{11}, A_{12}, A_{22}, A_{66}$	stretching stiffness terms, force per unit length
a_i	coefficient in infinite series for w_0
D	discriminant of cubic equation in q_i
$D_{11}, D_{12}, D_{22}, D_{66}$	bending stiffness terms, force times length
(E_1, E_2, E_3)	residual error in plate equations for current approximation
e_x, e_y, e_{xy}	membrane strains
$< f, g >$	integral of product of functions f and g over area of plate
$\mathbf{i}, \mathbf{j}, \mathbf{n}$	unit vectors
$L_{11}(u), L_{12}(u)$	linear terms in partial derivatives of u
$L_{12}(v), L_{22}(v)$	linear terms in partial derivatives of v
$L_{33}(w)$	linear terms in partial derivatives of w
M_x, M_y, M_{xy}	moment stress resultants, force
N_x, N_y, N_{xy}	membrane stress resultants, force per unit length
$N_1(w,w), N_2(w,w)$	bilinear terms in partial derivatives of w
$N_3(N_x, N_y, N_{xy}, w)$	bilinear terms in stress resultants and partial derivatives of w
q_i	generalized coordinate in Galerkin solution of plate equations
u, v	in-plane components of plate displacements
u_L, v_L	solution of linear in-plane equilibrium equation
(u_0, v_0, w_0)	current approximation for a solution of the plate equations
w	transverse plate displacement
$(\delta u, \delta v, \delta w)$	correction to current approximation of the plate equations
λ	load parameter that multiplies displacements on plate boundary
λ_i	ith eigenvalue of Sturm-Liouville problem
(ξ_i, η_i, ϕ_i)	ith eigensolution of Sturm-Liouville problem

A subscript following a comma indicates partial differentiation with respect to the subscripted variable.

Nonlinear Equations for Plate Problem

The nonlinear theory (ref. 8) is summarized here first, and the linear form of Newton's method is then applied to the three equilibrium equations written in terms of the displacement components u, v, and w. The linear problem is then transformed into a Sturm-Liouville problem that shows why the linear form of Newton's method can fail to converge for modal interaction problems. The next section of the paper contains the modification of the linear form of Newton's method using the eigenfunctions of the Sturm-Liouville problem.

The nonlinear plate equations for a specially orthotropic plate are summarized as follows:

1. Constitutive relations:

$$N_x = A_{11}e_x + A_{12}e_y \tag{1a}$$

$$N_y = A_{12}e_x + A_{22}e_y \tag{1b}$$

$$N_{xy} = A_{66}e_{xy} \tag{1c}$$

$$M_x = D_{11}w_{,xx} + D_{12}w_{,yy} \tag{2a}$$

$$M_y = D_{12}w_{,xx} + D_{22}w_{,yy} \tag{2b}$$

$$M_{xy} = D_{66}w_{,xy} \tag{2c}$$

2. Strain-displacement relations:

$$e_x = u_{,x} + (1/2)w_{,x}^2 \tag{3a}$$

$$e_y = v_{,y} + (1/2)w_{,y}^2 \tag{3b}$$

$$e_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y} \tag{3c}$$

3. Equilibrium equations:

$$N_{x,x} + N_{xy,y} = 0 \tag{4a}$$

$$N_{xy,x} + N_{y,y} = 0 \tag{4b}$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} = (N_x w_{,x} + N_{xy}w_{,y})_{,x} + (N_y w_{,y} + N_{xy}w_{,x})_{,y} \tag{4c}$$

4. Boundary conditions

The plate problems considered here have boundary arcs where u and v are prescribed. The plate may or may not have additional arcs that are free of membrane stresses. The boundary conditions are indicated schematically by the loaded boundary C_1 and the unloaded boundary C_2 in figure 1. In the figure, the displacements on the boundary C_1 are of the form

$$u = \lambda u_L \qquad (\text{on } C_1) \tag{5a}$$

$$v = \lambda v_L \qquad (\text{on } C_1) \tag{5b}$$

where u_L and v_L are functions of arc length on the boundary, and the load factor λ is a scalar multiplier. On the other hand, the stress resultants vanish on the C_2 boundary defined as follows:

$$N_n = N_{ns} = 0 \qquad (\text{on } C_2) \tag{5c}$$

The boundary conditions on the transverse displacement w are homogeneous and correspond to simply supported or clamped conditions in linear-plate bending theory and are of the following form:

$$w = 0 \text{ on } C; \;\; w_{,n} = 0; \text{ or } w_{,nn} = 0 \text{ on } C \tag{5d}$$

The boundary conditions on w could be more general, but the method of analysis is illustrated with less complexity by the choice made here. The load parameter (λ) is also introduced in the in-plane boundary conditions to simplify the analysis.

Substituting the strain-displacement relations (eqs. (3)) into the constitutive relations (eqs. (1)) and the resulting equations into the equilibrium equations (eqs. (4)) allows the equilibrium equations to be written as nonlinear partial differential equations in terms of the displacement components u, v, and w. The equations, written here in operator notation for brevity, are

$$L_{11}(u) + L_{12}(v) + N_1(w,w) = 0 \tag{6a}$$

$$L_{12}(u) + L_{22}(v) + N_2(w,w) = 0 \tag{6b}$$

$$L_{33}(w) - N_3(N_x, N_y, N_{xy}, w) = 0 \tag{6c}$$

The linear operators, labeled L_{ij}, are

$$L_{11}(u) = A_{11}u_{,xx} + A_{66}u_{,yy} \tag{7a}$$

$$L_{12}(f) = (A_{12} + A_{66})f_{,xy} \tag{7b}$$

$$L_{22}(v) = A_{22}v_{,yy} + A_{66}v_{,xx} \tag{7c}$$

$$L_{33}(w) = D_{11}w_{,xxxx} + (2D_{12} + D_{66})w_{,xxyy} + D_{22}w_{,yyyy} \tag{7d}$$

The nonlinear operators are

$$N_1(f,g) = A_{11}f_{,x}g_{,xx} + (A_{12} + A_{66})f_{,y}g_{,xy} + A_{66}f_{,x}g_{,yy} \tag{8a}$$

$$N_2(f,g) = A_{22}f_{,y}g_{,yy} + (A_{12} + A_{66})f_{,x}g_{,xy} + A_{66}f_{,y}g_{,xx} \tag{8b}$$

$$N_3(N_x, N_y, N_{xy}, w) = (N_x w_{,x} + N_{xy}w_{,y})_{,x} + (N_y w_{,y} + N_{xy}w_{,x})_{,y} \tag{8c}$$

Linear Form of Newton's Method for Plate Problem

Linear Form of Newton's Method for Displacement Formulation

Newton's method starts with an approximate solution (u_0, v_0, w_0) for the nonlinear system (eqs. (6)) plus the boundary conditions. The zeroth approximation is corrected by letting

$$u = u_0 + \delta u,\ \ v = v_0 + \delta v,\ \ w = w_0 + \delta w \tag{9}$$

The linear form of Newton's method seeks the correction $(\delta u, \delta v, \delta w)$ by substituting equations (9) into the nonlinear equations (6) and dropping nonlinear terms in the corrections to arrive at the following linear variational equations:

$$L_{11}(\delta u) + L_{12}(\delta v) + N_1(\delta w, w_0) + N_1(w_0, \delta w) = -E_1 \tag{10a}$$

$$L_{12}(\delta u) + L_{22}(\delta v) + N_2(\delta w, w_0) + N_2(w_0, \delta w) = -E_2 \tag{10b}$$

$$L_{33}(\delta w) - N_3(\delta N_x, \delta N_y, \delta N_{xy}, w_0) - N_3(N_{x0}, N_{y0}, N_{xy0}, \delta w) = -E_3 \tag{10c}$$

where the residual-error terms are known functions of the zeroth approximation; that is,

$$E_1 = L_{11}(u_0) + L_{12}(v_0) + N_1(w_0, w_0) \tag{11a}$$

$$E_2 = L_{12}(u_0) + L_{22}(v_0) + N_2(w_0, w_0) \tag{11b}$$

$$E_3 = L_{33}(w_0) - N_3(N_{x0}, N_{y0}, N_{xy0}, w_0) \tag{11c}$$

$$N_{x0} = A_{11}\left[u_{0,x} + (1/2)w_{0,x}^2\right] + A_{12}\left[v_{0,y} + (1/2)w_{0,y}^2\right] \tag{12}$$

with similar expressions for N_{y0} and N_{xy0} in terms of the zeroth approximation for u, v, and w. The corrections to the membrane stress resultants are linear in δu, δv, and δw. For example,

$$\delta N_{xy} = A_{66}\left(\delta u_{,y} + \delta v_{,x} + w_{,x0}\delta w_{,y} + \delta w_{,x} w_{0,y}\right) \tag{13}$$

Linear Sturm-Liouville Theory

The linear variational equations of Newton's method (eqs. (10)) have variable coefficients and, in general, cannot be solved in closed form. However, the linear problem can be reduced to a Sturm-Liouville problem, which can be solved by approximate methods. The reduction to a Sturm-Liouville problem is achieved by defining the zeroth approximation for the in-plane displacements as the sum of two pairs of functions as follows:

$$u_0 = \lambda u_L + u_{w0} \tag{14a}$$

$$v_0 = \lambda v_L + v_{w0} \tag{14b}$$

The sums are defined by requiring that u_L and v_L satisfy the linear boundary-value problem,

$$L_{11}(u_L) + L_{12}(v_L) = 0 \tag{15a}$$

$$L_{12}(u_L) + L_{22}(v_L) = 0 \tag{15b}$$

plus nonhomogeneous boundary conditions. The functions λu_L and λv_L that satisfy equations (15) are also required to satisfy the boundary conditions on u and v in equations (5).

The functions u_{w0} and v_{w0} in the zeroth approximation for u and v (eqs. (14)) are then defined as the solution of the boundary-value problem as follows:

$$L_{11}(u_{w0}) + L_{12}(v_{w0}) + N_1(w_0, w_0) = 0 \tag{16a}$$

$$L_{12}(u_{w0}) + L_{22}(v_{w0}) + N_2(w_0, w_0) = 0 \tag{16b}$$

with the homogeneous boundary conditions

$$u_{w0} = 0 \qquad (\text{on } C_1) \tag{17a}$$

$$v_{w0} = 0 \qquad (\text{on } C_1) \tag{17b}$$

Finally, the definition of the zeroth approximation for u and v as sums of solutions of two boundary-value problems is completed by partitioning the zeroth approximation for the membrane stress resultants (eq. (12) and appendix) as follows:

$$N_{x0} = \lambda N_{xL} + N_{xw0} \tag{18a}$$

$$N_{y0} = \lambda N_{yL} + N_{yw0} \tag{18b}$$

$$N_{xy0} = \lambda N_{xyL} + N_{xyw0} \tag{18c}$$

where the functions multiplied by the load parameter λ and with the subscript L satisfy the boundary conditions on C_2 (eq. (5c)). The functions with the subscript $w0$ also satisfy the conditions on C_2 and are completely independent of the value of λ.

Splitting the zeroth approximation for u and v reduces the linear variational equations (eqs. (10)) to a boundary-value problem with homogeneous boundary conditions and with the load factor λ appearing as a parameter in the following partial differential equations:

$$L_{11}(\delta u) + L_{12}(\delta v) + N_1(\delta w, w_0) + N_1(w_0, \delta w) = 0 \tag{19a}$$

$$L_{12}(\delta u) + L_{22}(\delta v) + N_2(\delta w, w_0) + N_2(w_0, \delta w) = 0 \tag{19b}$$

$$L_{33}(\delta w) - N_3(\delta N_x, \delta N_y, \delta N_{xy}, w_0) - N_3(N_{xw0}, N_{yw0}, N_{xyw0}, \delta w) - \lambda N_3(N_{xL}, N_{yL}, N_{xyL}, \delta w) = -E_3 \tag{19c}$$

Eigenfunctions of Sturm-Liouville Problem

The solution for the linear system (eqs. (19)) can be formally written as an expansion in eigenfunctions ξ_i, η_i, and ϕ_i of the Sturm-Liouville system:

$$L_{11}(\xi_i) + L_{12}(\eta_i) + N_1(\phi_i, w_0) + N_1(w_0, \phi_i) = 0 \tag{20a}$$

$$L_{12}(\xi_i) + L_{22}(\eta_i) + N_2(\phi_i, w_0) + N_2(w_0, \phi_i) = 0 \tag{20b}$$

$$\begin{aligned} L_3(\phi_i) &- N_3(n_{xwi}, n_{ywi}, n_{xywi}, w_0) - N_3(N_{xw0}, N_{yw0}, N_{xyw0}, \phi_i) \\ &- \lambda_i N_3(N_{xL}, N_{yL}, N_{xyL}, \phi_i) = 0 \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \end{aligned} \tag{20c}$$

where the functions n_{xwi}, n_{ywi}, and n_{xywi} are defined in equations (A7a) to (A7c). The notation for equations (20) is somewhat cumbersome, but the final result is simple. The eigenfunctions that are solutions of equations (20) obey the following orthogonality relation that is derived in the appendix:

$$(\lambda_i - \lambda_j) \int \left[\phi_j, N_3\left(N_{xL}, N_{yL}, N_{xyL}, \phi_i\right)\right] dA = 0 \tag{21}$$

The orthogonality relation suggests seeking the solution of the linear variational equations as a modal expansion as follows:

$$\delta u = u_{w1} = \sum_{i=1}^{\infty} q_i \xi_i \tag{22a}$$

$$\delta v = v_{w1} = \sum_{i=1}^{\infty} q_i \eta_i \tag{22b}$$

$$\delta w = \sum_{i=1}^{\infty} q_i \phi_i \tag{22c}$$

$$\delta N_x = N_{xw1} = \sum_{i=1}^{\infty} q_i n_{xwi} \tag{23a}$$

$$\delta N_y = N_{yw1} = \sum_{i=1}^{\infty} q_i n_{ywi} \tag{23b}$$

$$\delta N_{xy} = N_{xyw1} = \sum_{i=1}^{\infty} q_i n_{xywi} \tag{23c}$$

The generalized coordinates q_i are unknown. They are determined by Galerkin's method. The assumed solution satisfies equations (19a) and (19b) term by term. Equation (19c) is satisfied in the least-squares sense by multiplying the equation by each eigenfunction ϕ_j in turn, integrating over the area of the plate, and equating the results from each side of the equation. Because of the orthogonality condition (eq. (21)), the resulting equations take the form

$$q_j = \frac{< -E_3, \phi_j >}{(\lambda - \lambda_j)} \qquad (j = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{24}$$

where $< f, g >$ represents the integral over the area of the product of the functions f and g. That is,

$$< f, g > = \int (fg)\, dA \tag{25}$$

It is also assumed in equations (24) that the eigenfunctions have been normalized so that

$$< \phi_i, N_3(N_{xL}, N_{yL}, N_{xyL}, \phi_i) > = -1 \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{26}$$

Equation (24) is a canonical form for the edge-loaded plate problem. The solution of the linear variational equations of Newton's method is completed by substituting the generalized coordinates q_i from equations (24) into equations (22) to determine the correction $(\delta u, \delta v, \delta w)$. If the correction is small compared with the zeroth approximation (u_0, v_0, w_0), then an iterative solution of the nonlinear plate equations based on the linear form of Newton's method can be expected to converge. The linear iterative sequence is continued by going back to equations (9), letting $w_1 = w_0 + \delta w$, and repeating the analysis through equations (24) with w_1 as the zeroth approximation w_0 for w during the second iteration cycle.

For the iterative procedure in the linear form of Newton's method to converge, the corrections from successive iteration cycles must approach zero as the iteration continues. A quantitative measure of the correction δw from any iteration cycle is the magnitudes of the generalized coordinates q_j, which are modal amplitudes, as determined by equations (24). The current solution for w can be expanded in a least-squares sense as a series in the eigenfunctions of the Sturm-Liouville problem

$$w = \sum_{i=1}^{\infty} a_i \phi_i \tag{27}$$

A measure of convergence is the set of ratios of the absolute values of q_i to the largest coefficient in absolute value in the series for w. Each q_i in turn depends on the numerators and denominators in equations (24). The denominators are a function of the load parameter λ and the eigenvalues λ_i. Obviously, a large value for the corresponding q_i is the result of dividing by a denominator that is zero or small in absolute value, unless the numerator is also small.

The numerators are also linear in λ and can be written as

$$< E_3, \phi_i > = E_{3i} - a_i \lambda \tag{28}$$

The coefficient a_i in equation (28) is the same as the coefficient of ϕ_i in the expansion for $w = w_0$ in equation (27). (See appendix.) Therefore, equations (24) can be rewritten as

$$q_i = (E_{3i} - a_i \lambda)/(\lambda - \lambda_i) \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{29}$$

The load parameter can be prescribed to make one of the generalized coordinates, for example q_I, vanish if the corresponding coefficient a_I is not zero, that is,

$$\bar{\lambda} = E_{3I}/a_I \tag{30}$$

If $\bar{\lambda} - \lambda_I$ is also zero, q_I can still be set to zero arbitrarily during any given iteration cycle. Then, if $\bar{\lambda}$ is not too close to the remaining λ_i, the iteration using the linear form of Newton's method can be continued. If one of the equations is indeterminate, for example,

$$q_k = (E_{3k} - a_k \bar{\lambda})/(\bar{\lambda} - \lambda_k) = (0 + 0\bar{\lambda})/0 \qquad (i \neq k) \tag{31}$$

the iteration can be continued after setting q_k equal to zero. However, the indeterminate form is an indication of modal interaction with other solutions to the nonlinear problem for which q_k is not zero. A bifurcation point or limit point, where a second nonlinear solution intersects the current solution, is a limiting case of the indeterminate form in which the expansion is about an exact solution, so that the residual error function $E_3 = 0$ for $\lambda = \bar{\lambda}$ and $\bar{\lambda} = \lambda_k$.

In cases where some of the q_i determined by equations (29) are not small or are indeterminate, the linear form of Newton's method must be modified either to speed convergence of the current approximation or to obtain convergence for solutions that intersect the current solution at bifurcation points.

Modified Newton Method for Modal Interaction Problems

The analysis of the preceding section shows that the linear form of Newton's method for buckled plates may diverge or miss solutions when the load parameter λ is near one or more eigenvalues λ_i of the Sturm-Liouville problem associated with the linear variational equations. A modification of the iteration that has better convergence properties is derived in this section.

The linear variational equations (eqs. (10)) were derived by dropping nonlinear terms in the correction δw. The exact nonlinear equations that result from substituting equations (9) into the nonlinear boundary-value problem (eqs. (6)) are

$$L_{11}(\delta u) + L_{12}(\delta v) + N_1(\delta w, w_0) + N_1(w_0, \delta w) = -E_1 - N_1(\delta w, \delta w) \tag{32a}$$

$$L_{12}(\delta u) + L_{22}(\delta v) + N_2(\delta w, w_0) + N_2(w_0, \delta w) = -E_2 - N_2(\delta w, \delta w) \tag{32b}$$

$$L_{33}(\delta w) - N_3(\delta N_x, \delta N_y, \delta N_{xy}, w_0) - N_3(N_{x0}, N_{y0}, N_{xy0}, \delta w) = -E_3 - N_3(\delta N_x, \delta N_y, \delta N_{xy}, \delta w) \tag{32c}$$

The nonlinear terms in δw are placed on the right-hand sides of equations (32) to indicate that an iteration sequence can be devised in which the nonlinear terms in δw are based on some current approximation. The form of that iteration sequence is suggested by examining the exact nonlinear equations in the generalized coordinates q_i.

Nonlinear Problem in Generalized Coordinates

In equations (14), the initial approximation for the in-plane displacements (u_0,v_0) was partitioned into two sets of functions. This partitioning remains the same for the complete nonlinear problem. The corrections $(\delta u, \delta v)$ are further partitioned for the nonlinear problem as follows:

$$\delta u = u_{w1} + u_{w2} \tag{33a}$$

$$\delta v = v_{w1} + v_{w2} \tag{33b}$$

where (u_{w1}, v_{w1}) is the first approximation for $(\delta u, \delta v)$ defined by equations (22). The additional terms u_{w2} and v_{w2} in δu and δv are defined by quadratic terms in δw. Formally, they satisfy the differential equations

$$L_{11}(u_{w2}) + L_{12}(v_{w2}) + N_1(\delta w, \delta w) = 0 \tag{34a}$$

$$L_{12}(u_{w2}) + L_{22}(v_{w2}) + N_2(\delta w, \delta w) = 0 \tag{34b}$$

The nonlinear operators $N_1(\delta w, \delta w)$ and $N_2(\delta w, \delta w)$ are quadratic in δw. The series solution for δw in equation (22c) is unchanged in the complete nonlinear formulation, and the generalized coordinates q_i in the series for δw remain to be determined. Equations (34) are solved in terms of the generalized coordinates to obtain

$$u_{w2} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} q_j q_k \xi_{jk} \tag{35a}$$

$$v_{w2} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} q_j q_k \eta_{jk} \tag{35b}$$

where the functions ξ_{jk} and η_{jk} are defined as solutions of the following equations:

$$L_{11}(\xi_{jk}) + L_{12}(\eta_{jk}) + N_1(\phi_j, \phi_k) = 0 \tag{35c}$$

$$L_{12}(\xi_{jk}) + L_{22}(\eta_{jk}) + N_2(\phi_j, \phi_k) = 0 \tag{35d}$$

The complete corrections to the stress resultants in terms of the generalized coordinates are obtained by adding quadratic terms to the linear terms already defined in equations (23). That is,

$$\delta N_x = N_{xw1} + N_{xw2} = \sum_{i=1}^{\infty} q_i n_{xwi} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} q_j q_k n_{xwjk} \tag{36a}$$

$$\delta N_y = N_{yw1} + N_{yw2} = \sum_{i=1}^{\infty} q_i n_{ywi} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} q_j q_k n_{ywjk} \tag{36b}$$

$$\delta N_{xy} = N_{xyw1} + N_{xyw2} = \sum_{i=1}^{\infty} q_i n_{xywi} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} q_j q_k n_{xywjk} \tag{36c}$$

The terms in the double summations are defined explicitly in the appendix.

The final step is to substitute the partitioned form for the stress resultants into the transverse equilibrium equation (eq. (19c)) and to solve the resulting equation by Galerkin's method. The equilibrium equation is

$$\begin{aligned} L_{33}(\delta w) &- N_3(N_{xw1}, N_{yw1}, N_{xyw1}, w_0) \\ &- N_3(N_{xw0}, N_{yw0}, N_{xyw0}, \delta w) \\ &- \lambda N_3(N_{xL}, N_{yL}, N_{xyL}, \delta w) \\ = &- E_3 + N_3(N_{xw1}, N_{yw1}, N_{xyw1}, \delta w) \\ &+ N_3(N_{xw2}, N_{yw2}, N_{xyw2}, w_0) \\ &+ N_3(N_{xw2}, N_{yw2}, N_{xyw2}, \delta w) \end{aligned} \tag{37}$$

The left-hand side of equation (37) is identical to equation (19c). The quadratic and cubic terms in the generalized coordinates have been added to the right-hand side of the equation. The formal Galerkin solution of equation (37) is an infinite set of cubic equations in the q_i as follows:

$$(\lambda - \lambda_i)q_i = C_{0i} + C_{2ijk}q_j q_k + C_{3ijkm}q_j q_k q_m \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{38}$$

In equations (38), repeated subscripts j, k, and m are summed. The integrals that define the coefficients in equations (38) are listed in the appendix. Equations (38) are the final results of reducing the nonlinear plate problem to a nonlinear algebraic problem in the generalized coordinates q_i.

Equations (38) are similar to the equations in generalized coordinates postulated by Thompson and Hunt (ref. 6) for conservative systems. The analysis in the body of this paper and in the appendix gives a precise formulation for determining the coefficients in the cubic equations. The displacement formulation of the Von Karman equations and the choice of boundary conditions allow exact determination of the role of the load parameter λ. The equations for the plate problem are cubic in the generalized coordinates and linear in the load parameter λ.

Only the real roots of the cubic equations correspond to real solutions of the plate equations. The number of real roots is affected by the algebraic signs of the coefficients. Therefore, in the theoretical analysis, much qualitative information is available when it is known how the coefficients vary with the load parameter λ. The equations are uncoupled in the linear terms for the values of q_i with the coefficients $(\lambda - \lambda_i)$. The sign of each coefficient of a linear term depends on whether λ exceeds λ_i, which in turn is a function of w_0. The coefficients C_{ijk} and C_{ijkm} are independent of the load parameter λ for the edge-loaded plate. The residual-error terms C_{0i} are exactly the same terms that appeared in the linear form of Newton's method as follows:

$$C_{0i} = < E_3, \phi_i > = E_{3i} + a_i\lambda \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{39}$$

The form of the complete set of cubic equations in generalized coordinates suggests a modification of the linear form of Newton's method that will improve convergence.

Modification of Linear Form in Generalized Coordinates

The linear form of Newton's method corresponds to dropping quadratic and cubic terms in equations (38) to obtain equations (22). This procedure breaks down when the load factor λ is nearly equal to one of the eigenvalues (e.g., the Ith eigenvalue). A modification of the linear form is to retain only nonlinear terms in the modal amplitude q_I during a given iteration cycle. The modified form of equations (38) is then

$$(\lambda - \lambda_i)\, q_i = (E_{3i} + a_i\lambda) + C_{2iII}q_I^2 + C_{3iIII}q_I^3 \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{40a}$$

The Ith equation of this modified set is the cubic equation

$$(\lambda - \lambda_I)q_I = (E_{3I} + a_I\lambda) + C_{2III}q_I^2 + C_{3IIII}q_I^3 \tag{40b}$$

If the coefficient C_{3III} is not zero, the cubic equation has at least one real solution for q_I. In some cases, there can be three real roots. The discriminant of the cubic equation is

$$D = c^3 + (3b^2/4)c^2 - (3bd/2)c - d(b^3 + d/4) \tag{41}$$

where

$$c = (\lambda - \lambda_I)\,/(3C_{3III}),\ b = C_{2III}/(3C_{3IIII}),\ \text{and}\ d = (E_{3I} - a_I\lambda)/C_{3III}$$

When the discriminant D is positive, the cubic equation has three real roots. The real roots q_I of the cubic equation in q_I are then substituted in the remaining equations (40), which are linear in the remaining q_i. The solution of equations (40) for any root of the cubic equation completes an iteration cycle in the modified form of Newton's method.

This solution of equations (40) can be the basis of a second iteration cycle to compute a solution of the complete set of nonlinear algebraic equations (eqs. (38)). The solution of equations (38) can be assumed as

$$q_i = q_i^{(1)} + \ \delta q_i \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{42}$$

where $q_i^{(1)}$ denotes the q_i from a solution of equations (40) and the values of δq_i are corrections to be determined. Substitution of equations (42) into equations (38) results in a new set of coupled cubic equations in the unknown δq_i. An approximate solution of this set is obtained by truncating the new set of equations; this truncation is accomplished by retaining only nonlinear terms in δq_I to obtain an updated set of equations (40); these equations are solved to complete the second modified iteration cycle. The modified iteration can be continued until an accurate solution or solutions are obtained for q_i in equations (38). When the absolute values of $q_i^{(1)}$ are all small, the iteration can be expected to converge; this convergence is expected, since the residual-error vector for any iteration cycle after the first is equal to the summation of the quadratic and cubic terms dropped in going from the full nonlinear set of equations (eqs. (38)) to the truncated set (eqs. (40)).

This direct iterative solution of equations (38) is straightforward in theory but has disadvantages for actual computations. The large number of coefficients of quadratic and cubic terms are defined by integrals that must be evaluated numerically or in closed form. An alternative approach, which is equivalent to summing the quadratic and cubic terms into an updated error vector, is to update the zeroth approximation for w_0 after solving equations (40) for the first time. The current correction δw is computed from equation (22c). Examining the details of updating w_0 is beyond the scope of the present paper. However, if the numerical analysis is also connected with a Galerkin solution for the linear eigenvalue problem, updating w_0 becomes a part of an iterative procedure whose rapid convergence makes it a practical numerical algorithm. In this procedure, it is also necessary to compute only a few of the eigenvalues λ_i and the corresponding eigenfunctions ϕ_i. An equivalence transformation can be used in the numerical analysis (ref. 9). The equivalence transformation is derived by going back to equations (22) and letting the correction δw be a series of admissible functions g_i, of which only a small number are eigenfunctions ϕ_i.

Modification for Modal Interaction

The modified form of Newton's method can be extended to the cases where two eigenvalues (e.g., λ_1 and λ_2) are close together. In those cases, equations (38) are truncated during an iteration cycle to contain only nonlinear terms in q_1 and q_2. The pair of equations for $i = 1$ and $i = 2$ are simultaneous cubic equations. Real solutions for q_1 and q_2 are then substituted in the remaining linearized equations for the rest of the q_i.

Conclusions

Newton's method has been applied to the nonlinear postbuckling problem for plates. The method reduces the nonlinear partial differential equations of plate theory to a set of simultaneous cubic equations in generalized coordinates. The cubic equations are uncoupled in the linear terms. The uncoupling is achieved by solving the linear variational equations of Newton's method as a Sturm-Liouville problem. The eigenfunctions of the Sturm-Liouville problem are then used in a Galerkin solution of the full nonlinear plate equations to derive the set of cubic equations.

By specifying boundary conditions on displacements, instead of on the in-plane stress resultants, the coefficients in the cubic equations are linear in the load parameter, which is a multiplier of the boundary conditions. The analysis also shows that coefficients of quadratic and cubic terms in the generalized coordinates of the cubic equations are independent of the load parameter.

The special form of the cubic equations suggests a method of solution for modal interaction problems. The method is a modification of the linear form of Newton's method. The solutions of the plate equations derived by the method can be either approximate or very accurate, depending on the number of generalized coordinates retained in the solution.

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Appendix

Definitions of Sums of Functions and Integrals That Appear in the Analysis

Green's Theorem and Properties of Bilinear Operators

The analysis in the body of the paper is an examination of a formal Galerkin solution for the Von Karman plate equations. The integrals in the Galerkin solution make repeated use of Green's theorem for integration in a plane. The form of the theorem used here is

$$\int_A z\,(r_{,x} + s_{,y})\,dA = \int_C z\,(r\mathbf{i} + s\mathbf{j}) \cdot \mathbf{n}\,ds - \int_A (rz_{,x} + sz_{,y})\;dA \tag{A1}$$

The area integrals are over the area of the plate indicated schematically in figure 1, and the line integral is over the boundary of the plate.

The plate equations are nonlinear and contain a number of bilinear operators. The analysis in the paper uses the properties of bilinear operators of the general form $N(f,g)$. The operators $N_1(w,w)$ and $N_2(w,w)$, defined in equations (6), are bilinear operators; each is a sum of products of linear operators. The operator $N_3(N_x, N_{xy}, N_x, w)$ is a sum of bilinear operators. The property of the bilinear operators that is used repeatedly here is that if a, b, c, and d are constants, and if

$$f = af_1 + bf_2 \text{ and } g = cg_1 + dg_2$$

then

$$N(f,g) = acN(f_1,g_1) + adN(f_1,g_2) + bcN(f_2,g_1) + bdN(f_2,g_2) \tag{A2}$$

For example,

$$N_1(w,w) = N_1(w_0 + \delta w, w_0 + \delta w)$$

$$N_1(w,w) = N_1(w_0,w_0) + N_1(w_0,\delta w) + N_1(\delta w,w_0) + N_1(\delta w,\delta w)$$

and

$$N_1(w_0,\delta w) = N_1\left(w_0, \sum_{i=1}^{\infty} q_i\phi_i\right) = \sum_{i=1}^{\infty} q_i N_1(w_0,\phi_i)$$

$$N_1(\delta w,\delta w) = N_1\left(\sum_{i=1}^{\infty} q_i\phi_i,\ \sum_{j=1}^{\infty} q_j\phi_j\right) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} q_i q_j N_1(\phi_i,\phi_j)$$

A bilinear operator on a single sum results in a quadratic form with double subscripts. The double sums in the quadratic forms for the bilinear operators on $w = w_0 + \delta w$ are the main factors in defining the solutions for u and v in the analysis of the plate equations leading to the cubic equations in the generalized coordinates q_i. Since the operators on u and v are linear in the first two equations of equations (6) and the boundary conditions on u and v are also linear, the linear partial differential equations whose solutions determine u and v can be solved by solving equations (15), (16), (20), and (35) separately and superposing the results. The superposition on u and v is summarized in the following section.

Definitions of Sums of Functions

The in-plane displacements are defined in the form

$$u = \lambda u_L + u_{w0} + u_{w1} + u_{w2} \tag{A3a}$$

$$v = \lambda v_L + v_{w0} + v_{w1} + v_{w2} \tag{A3b}$$

The membrane strains are summed as

$$e_x = \lambda e_{xL} + e_{xw0} + e_{xw1} + e_{xw2} \tag{A4a}$$

$$e_y = \lambda e_{yL} + e_{yw0} + e_{yw1} + e_{yw2} \tag{A4b}$$

$$e_{xy} = \lambda e_{xyL} + e_{xyw0} + e_{xyw1} + e_{xyw2} \tag{A4c}$$

where

$$e_{xL} = u_{L,x}$$

$$e_{yL} = v_{L,y}$$

$$e_{xyL} = u_{L,y} + v_{L,x}$$

$$e_{xw0} = u_{w0,x} + (1/2)w_{0,x}^2$$

$$e_{xyw0} = u_{w0,y} + v_{w0,x} + w_{0,x}w_{0,y}$$

$$e_{xw1} = u_{w1,x} + w_{0,x}\delta w_{,x}$$

$$e_{xyw1} = u_{w1,y} + v_{w1,x} + w_{0,x}\,\delta w_{,y} + \delta w_{,x}w_{0,y}$$

Individual strain terms not listed can be derived by permutations of u and v and x and y in the terms listed above.

Membrane stresses follow the same notation pattern as the strains

$$N_x = \lambda N_{xL} + N_{xw0} + N_{xw1} + N_{xw2} \tag{A5a}$$

$$N_y = \lambda N_{yL} + N_{yw0} + N_{yw1} + N_{yw2} \tag{A5b}$$

$$N_{xy} = \lambda N_{xyL} + N_{xyw0} + N_{xyw1} + N_{yw2} \tag{A5c}$$

where

$$N_{xL} = A_{11}e_{xL} + A_{12}e_{yL},\ N_{xwk} = A_{11}e_{xwk} + A_{12}e_{ywk} \qquad (k = 0,\ 1,\ \text{or } 2)$$

$$N_{yL} = A_{11}e_{yL} + A_{12}e_{yL},\ N_{ywk} = A_{12}e_{xwk} + A_{22}e_{ywk} \qquad (k = 0,\ 1,\ \text{or } 2)$$

$$N_{xyL} = A_{66}e_{xyL},\ \text{and } N_{xywk} = A_{66}e_{xywk} \qquad (k = 0,\ 1,\ \text{or } 2)$$

Boundary Conditions

The solution (u_L, v_L) satisfies the linear partial differential equations (eqs. (15)) plus the boundary conditions (eqs. (5)). The functions N_{xL}, N_{yL}, and N_{xyL} satisfy the stress-free boundary conditions on the C_2 arcs.

Since the solution (u_L, v_L) satisfies the nonhomogeneous boundary conditions, the solutions that are added must vanish on the boundary. They are defined here to vanish term by term as follows:

$$u_{wk} = 0 \text{ on arc } C_1 \qquad (k = 0,\ 1,\ \text{or } 2) \tag{A6a}$$

$$v_{wk} = 0 \text{ on arc } C_1 \qquad (k = 0,\ 1,\ \text{or } 2) \tag{A6b}$$

The stress resultants with the subscripts wk satisfy the stress-free boundary conditions on arc C_2. These boundary conditions appear in the derivation of the orthogonality conditions for the Sturm-Liouville problem that is derived from the linear variational equations of Newton's method (eqs. (10)).

Orthogonality Relation for the Sturm-Liouville Problem

The orthogonality relation for the Sturm-Liouville problem (eq. (21)) follows from the boundary conditions, the definition of terms in the subscript $w1$, and Green's theorem. In equations (22), the functions u_{w1} and v_{w1} are written as sums of functions that satisfy the linear partial differential equations (20). From the definition of the strains in equations (A4) and the stress resultants in equations (A5), the terms in the summation in equations (23) are also defined as follows:

$$n_{xwi} = A_{11}(\xi_{i,x} + w_{0,x}\phi_{i,x}) + A_{12}(\eta_{i,y} + w_{0,y}\phi_{i,y}) \tag{A7a}$$

$$n_{ywi} = A_{12}(\xi_{i,x} + w_{0,x}\phi_{i,x}) + A_{22}(\eta_{i,y} + w_{0,y}\phi_{i,y}) \tag{A7b}$$

$$n_{xywi} = A_{66}(\xi_{i,y} + \eta_{i,x} + w_{0,x}\phi_{i,y} + \phi_{i,x}w_{0,y}) \tag{A7c}$$

Equations (20a) and (20b) have the alternate form

$$n_{xwi,x} + n_{xywi,y} = 0 \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{A8a}$$

$$n_{xywi,x} + n_{ywi,y} = 0 \qquad (i = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{A8b}$$

The orthogonality relation is derived by applying Green's theorem to the above equations and to equation (20c). For equations (A8), the result is

$$\int_A \xi_j(n_{xwi,x} + n_{xywi,y})\, dA = \int_C \xi_j(n_{xwi}\mathbf{i} + n_{xywi}\mathbf{j}) \cdot \mathbf{n} ds - \int_A (n_{xwi}\xi_{j,x} + n_{xywi}\xi_{j,y})\, dA = 0 \tag{A9a}$$

$$\int_A \eta_j(n_{xywi,x} + n_{ywi,y})\, dA = \int_C \eta_j(n_{xywi}\mathbf{i} + n_{ywi}\mathbf{j}) \cdot \mathbf{n} ds - \int_A (n_{xywi}\eta_{j,x} + n_{xwi}\eta_{j,y})\, dA = 0 \tag{A9b}$$

The boundary conditions on the solutions of equations (20) are that each eigenfunction ξ_i and η_i must vanish on the boundary C_1 and that the dot products in the line integrals in equations (A9) vanish on the boundary C_2. Therefore, in the notation of equation (25) for integrals of products,

$$\int_A (n_{xwi}\xi_{j,x} + n_{xywi}\xi_{j,y})\, dA = < n_{xwi}, \xi_{j,x} > + < n_{xywi}, \xi_{j,y} > = 0 \tag{A10a}$$

$$< n_{xywi}, \eta_{j,x} > + < n_{xwi}, \eta_{j,y} > = 0 \qquad (i = 1,\ 2,\ 3,\ \ldots, \infty;\ j = 1,\ 2,\ 3,\ \ldots,\ \infty) \tag{A10b}$$

It is assumed that each of the functions ϕ_i in the solutions of equations (20) satisfies the same boundary conditions as w. Green's theorem applied to any N_3 operator in equation (20c) then has the general form

$$< \phi_j, N_3(N_x, N_y, N_{xy}, w) > = - < (N_x w_{,x} + N_{xy} w_{,y}), \phi_{j,x} > - < (N_y w_{,y} + N_{xy} w_{,x}), \phi_{j,y} >$$

where

$$N_3(N_x, N_y, N_{xy}, w) = (N_x w_{,x} + N_{xy} w_{,y})_{,x} + (N_y w_{,y} + N_{xy} w_{,x})_{,y}$$

Specifically,

$$< \phi_j, N_3(n_{xwi}, n_{ywi}, n_{xywi}, w_0) > = - < (n_{xwi} w_{0,x} + n_{xywi} w_{0,y}), \phi_{j,x} > - < (n_{ywi} w_{0,y} + n_{xywi} w_{0,x}), \phi_{j,y} > \tag{A11a}$$

$$< \phi_j, N_3(N_{xw0}, N_{yw0}, N_{xyw0}, \phi_i) > = - < (N_{xw0}\phi_{i,x} + N_{xyw0}\phi_{i,y}), \phi_{j,x} > - < (N_{yw0}\phi_{i,y} + N_{xyw0}\phi_{i,x}), \phi_{j,y} > \tag{A11b}$$

$$< \phi_j, N_3(N_{xL}, N_{yL}, N_{xyL}, \phi_i) > = - < (N_{xL}\phi_{i,x} + N_{xyL}\phi_{i,y}), \phi_{j,x} > - < (N_{yL}\phi_{i,y} + N_{xyL}\phi_{i,x}), \phi_{j,y} > \tag{A11c}$$

The orthogonality relation (eq. (21)) is proven by multiplying the jth equation of equations (20c) by ϕ_i, integrating over the area of the plate, and subtracting the result from the integral of the ith equation multiplied by ϕ_j. Considering one operator at a time in equations (20c), it can be shown first that

$$< \phi_j, L_{33}(\phi_i) > - < \phi_i, L_{33}(\phi_j) > = 0 \tag{A12a}$$

Next, by inspection of the right-hand sides of the last two of equations (A11), it is apparent that

$$< \phi_j, N_3(N_{xw0}, N_{yw0}, N_{xyw0}, \phi_i) > - < \phi_i, N_3(N_{xw0}, N_{yw0}, N_{xyw0}, \phi_j) = 0 \tag{A12b}$$

$$< \phi_j, N_3(N_{xL}, N_{yL}, N_{xyL}, \phi_i) > = < \phi_i, N_3(N_{xL}, N_{yL}, N_{xyL}, \phi_j) > \tag{A12c}$$

If equations (A10) are subtracted from the right-hand side of equation (A11a) and the terms are rearranged, the result is

$$\begin{aligned} < \phi_j, N_3(n_{xwi}, n_{ywi}, n_{xywi}, w_0) > = & - < n_{xwi}, (\xi_{j,x} + w_{0,x}\phi_{j,x}) > - < n_{ywi}, (\eta_{j,y} + w_{0,y}\phi_{j,y}) > \\ & - < n_{xywi}, (\xi_{j,y} + \eta_{j,x} + w_{0,y}\phi_{j,x} + w_{0,x}\phi_{j,y}) > \end{aligned}$$

Finally, comparison of the above equation with equations (A7) shows that the subscripts i and j can be interchanged with the result that

$$< \phi_j, N_3(n_{xwi}, n_{yw}, n_{ywi}, n_{xywi}, w_0) > - < \phi_i, N_3(n_{xwj}, n_{ywj}, n_{xywj}, w_0) > = 0 \tag{A12d}$$

Equations (A12), based on the solutions of equations (20), are sufficient to derive the orthogonality relation (eq. (21)).

Expansion of w in Terms of Eigenfunctions ϕ_i

In equation (27), the current approximation w_0 for w is expressed as a series of eigenfunctions. The coefficients a_i in the series are computed from the orthogonality relation (eq. (21)) and the normalizing equation (eq. (26)) as follows:

$$< \phi_i, N_3(N_{xL}, N_{yL}, N_{xyL}, w_0) > = < \phi_i, N_3\left(N_{xL}, N_{yL}, N_{xyL}, \sum_{j=1}^{\infty} a_j\phi_j\right) > = -a_i \tag{A13}$$

The formal expansion of w_0 as a series of eigenfunctions appears again in the analysis in equations (28) and is repeated in equations (39). Using equations (18), the residual error E_3 is written as

$$E_3 = L_{33}(w_0) - N_3(N_{xw0}, N_{yw0}, N_{xyw0}, w_0) - \lambda N_3(N_{xL}, N_{yL}, N_{xyL}, w_0) \tag{A14}$$

Equations (A14) and (A13) are used to derive equation (28); also, the term

$$E_{3i} = < \phi_i, L_{33}(w_0) > - < \phi_i, N_3(N_{xw0}, N_{yw0}, N_{xyw0}, w_0) >$$

is independent of the load factor λ.

Coefficients of Higher Order Terms in Generalized Coordinates

The coefficients of quadratic and cubic terms in the generalized coordinates that appear in equations (38) are defined as follows:

$$
\begin{aligned}
&< \phi_i, N_3(N_{xw1}, N_{yw1}, N_{xyw1}, \delta w) > + < \phi_i, N_3(N_{xw2}, N_{yw2}, N_{xyw2}, w_0) > \\
&\quad = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} q_j q_k < \phi_i, N_3(n_{xwj}, n_{ywj}, n_{xywj}, \phi_k) > \\
&\quad\quad + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} q_j q_k < \phi_i, N_3(n_{xwjk}, n_{ywjk}, n_{xywjk}, \phi_k) > \\
&\quad = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} C_{2ijk} q_j q_k
\end{aligned} \tag{A15}
$$

$$
\begin{aligned}
&< \phi_i, N_3(N_{xw2}, N_{yw2}, N_{xyw2}, \delta w) > \\
&\quad = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} q_j q_k q_m < \phi_i, N_3(n_{xwjk}, n_{ywjk}, n_{xywjk}, \phi_m) > \\
&\quad = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} q_j q_k q_m C_{3ijkm}
\end{aligned} \tag{A16}
$$

When equations (A15) and (A16) are substituted into equations (38), the summation signs are suppressed to condense the notation.

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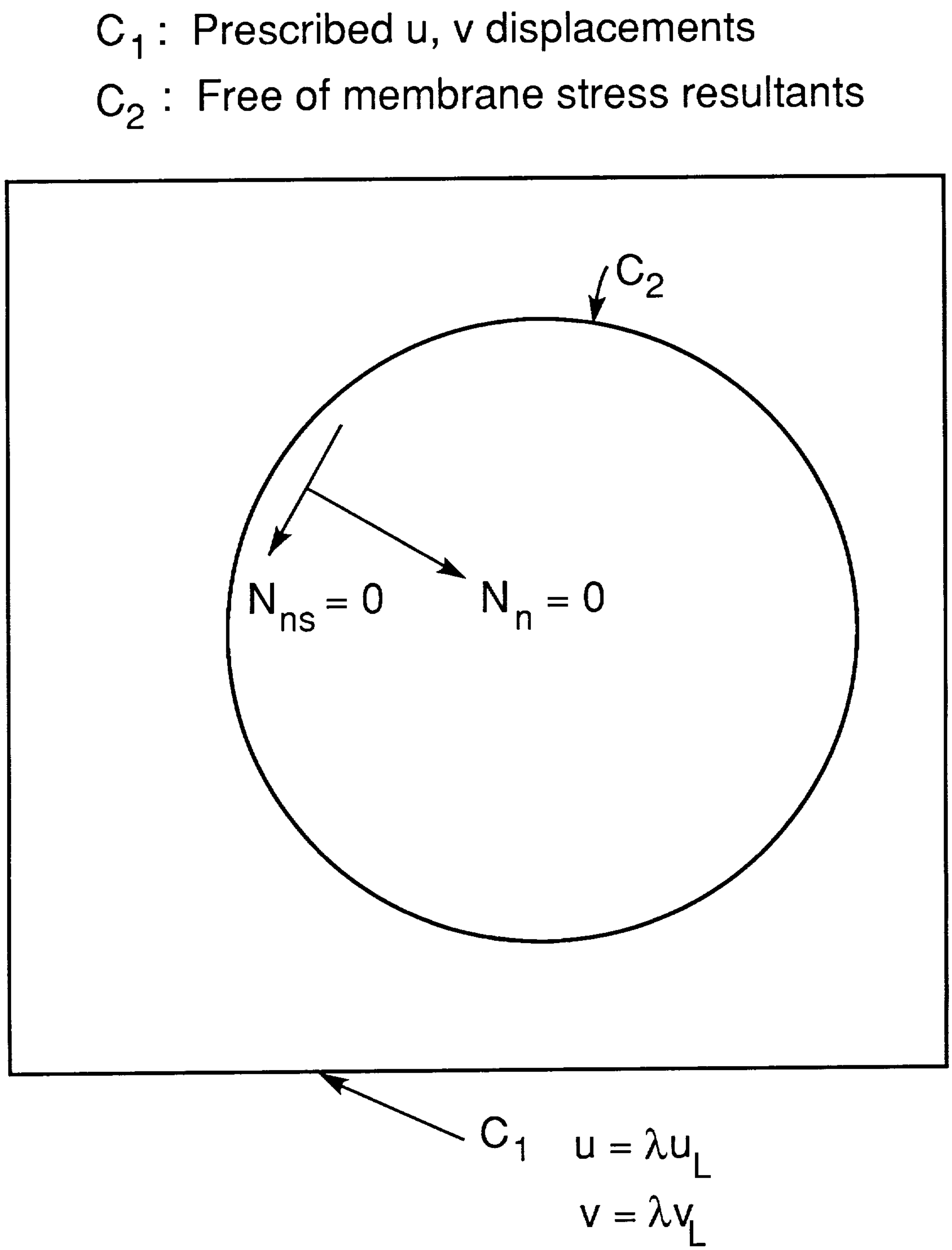

Figure 1. Schematic of C_1 and C_2 boundary arcs of a plate.

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16. Abstract

Plates can have more than one buckled solution for a fixed set of boundary conditions. The theory for the identification and the computation of multiple solutions in buckled plates is examined in this paper. The theory is used to predict modal interaction, which is also called change in buckle pattern or secondary buckling, in experiments on certain plates with multiple theoretical solutions. A set of coordinate functions for Galerkin's method are defined so that the Von Karman plate equations are reduced to a coupled set of cubic equations in generalized coordinates that are uncoupled in the linear terms. An iterative procedure for solving modal interaction problems is suggested in the paper based on this cubic form.

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