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NASA Technical Memorandum

NASA TM - 103511

RELIABILITY GROWTH MODELING ANALYSIS OF THE SPACE SHUTTLE MAIN ENGINES BASED UPON THE WEIBULL PROCESS

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August 1990

(NASA-TM-103511) RELIABILITY GROWTH MODELING ANALYSIS OF THE SPACE SHUTTLE MAIN ENGINES BASED UPON THE WEIBULL PROCESS (NASA) 67 p CSCL 14D

N90-28098

Unclas
G3/38 0303930

National Aeronautics and Space Administration

George C. Marshall Space Flight Center

MSFC - Form 3190 (Rev. May 1983)

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RELIABILITY GROWTH MODELING ANALYSIS OF THE SPACE SHUTTLE MAIN ENGINES BASED UPON THE WEIBULL PROCESS

INTRODUCTION

The purpose of this report is to present (1) some additional data of percentage-point probabilities for use in setting $(1-\alpha)100$-percent two-sided confidence interval estimates on the mean time between failures (MTBF) for several other values of the confidence coefficient for two categories, namely, (a) time-terminated testing and (b) failure-terminated testing; (2) additional data of the critical values for the goodness-of-fit test statistics of Cramér-von Mises, Kolmogorov-Smirnov, and chi-square; and (3) the numerical results based on the Weibull process.

The proposed reliability growth modeling considers the nonhomogeneous Poisson process with the Weibull intensity function

$$u(t) = \lambda\beta t^{\beta-1} \ , \qquad t>0 \ . \tag{1}$$

This modeling method, which is known as the Weibull process developed by the U.S. Army Materiel Systems Analysis Activity using the Duane postulate, is described in more detail in reference 1. The Weibull process is used to model reliability growth within test phases. The Weibull process describes the changes in the intensity function with time from first time interval to second time interval and so forth. The homogeneous process applies to the constant intensity function over the test phase with mean λt. The Duane model hypothesizes that the logarithm of observed cumulative MTBF is a linear function of time.

Integral presentations for cumulative distribution functions have been mathematically formulated to compute the probabilities for two different terminated testing cases. Computer programs, including methods of numerical integration and procedures of iterative root-solving, have been developed in the form of BASIC language to compute the data as a means of evaluating or predicting the reliability potential of a system.

MODELING

The reliability growth analysis uses the Weibull process to obtain the reliability estimates for the operating time of the development test phase, for the rate of reliability growth of the different systems, and for the future predictions of the system failure probabilities.

If a system is being tested until the n-th failure occurs, any improvement may need to be made, based on reliability growth decisions over a period of time. Thus, confidence limits on current system reliability are one of the several possible features that would enhance the reliability growth process. The established statistical inference methods are applied to determine the percentage-point probabilities for the small sample and asymptotic confidence intervals on the MTBF when, for example, the structural test data on the engines or components from the Weibull process are failure- or time-terminated at time within development test phases. Figure 1 shows diagrams illustrating failures for time-terminated testing and failure-terminated testing.

All equations have been formulated for the numerical procedures to integrate for these two types of terminated tests.

Accordingly, the stochastic process $[N(t),\ t>0]$ with the Weibull failure rate function $u(t)$ infers that the probability of a system failure occurring in an infinitesimally small time interval $(t,\ t+\Delta t)$ is approximately $u(t)\ \Delta t$. $N(t)$ is the number of failures during the time interval $(0,t)$ with the successive failures times, $0<X_1<X_2<X_3\ldots<X_n$, on a cumulative time scale for a system undergoing development tests. The probabilistic model for the reliability growth has the mean value function

$$
\begin{aligned}
\theta(t) &= E\,[N(t)] \\
&= \sum_{n=0}^{\infty} n\ P[N(t)=n] \\
&= \sum_{n=0}^{\infty} \frac{n\ (\lambda t^{\beta})^{n}\ e^{-\lambda t^{\beta}}}{n!}
\end{aligned}
$$

Now, let $n=n!/(n-1)!$

$$
\begin{aligned}
\theta(t) &= \sum_{n=1}^{\infty} \frac{(\lambda t^{\beta})^{n}\ e^{-\lambda t^{\beta}}}{(n-1)!} \\
&= \lambda t^{\beta}\ e^{-\lambda t^{\beta}} \sum_{n=1}^{\infty} \frac{(\lambda t^{\beta})^{n-1}}{(n-1)!}
\end{aligned}
$$

Let $k=n-1$, then

$$
\begin{aligned}
\theta(t) &= \lambda t^{\beta}\ e^{-\lambda t^{\beta}} \sum_{k=0}^{\infty} \frac{(\lambda t^{\beta})^{k}}{k!} \\
&= \lambda t^{\beta}\ e^{-\lambda t^{\beta}}\ e^{\lambda t^{\beta}} \\
&= \lambda t^{\beta}\ , \qquad (2)
\end{aligned}
$$

Weibull Process = Nonhomogeneous Poisson Process with Weibull Intensity Function, $u(t) = \lambda \beta t^{B-1}$ $(t > 0, \lambda > 0, \beta > 0)$

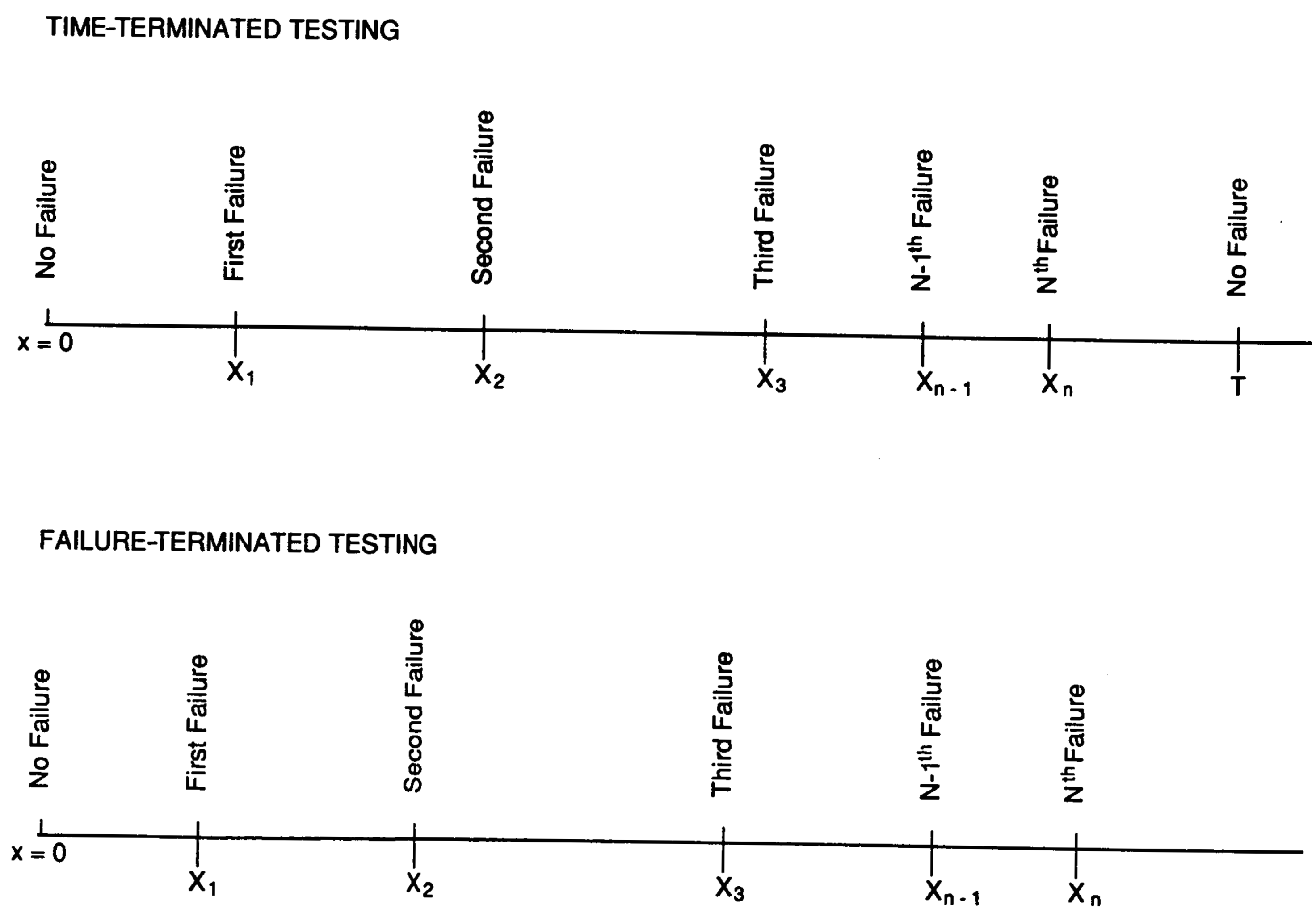

Figure 1. Diagrams illustrating failures for time-terminated testing and failure-terminated testing.

where $E[\cdot]$ is the mathematical expectation of the function $N(t)$; λ is the scale parameter; β is the shape parameter; and t is the cumulative test time. The Weibull intensity function, as derived from equation (2), becomes

$$u(t) = \theta'(t) = dE\,[N(t)]/dt = \lambda\beta t^{\beta-1} \tag{3}$$

The instantaneous MTBF of the system at time t is computed as $M(t) = 1/u(t)$, representing the system reliability growth. With total cumulative time given for the system, $M(t)$ is the achieved MTBF in the present system configuration.

The Poisson distribution probability for all n independent failures to occur in any time interval is

$$\text{Prob}[N(t)=n] = \frac{[\theta(t)]^n\, e^{-\theta(t)}}{n!} \quad , \tag{4}$$

where $\theta(t) = \lambda t^\beta$ from equation (2), which indicates the expected number of failures expressed as a function of test time, and $n = 0,1,2,3,\ldots$.

The criteria for determining the intensity function are as follows:

1. $\beta=1$: If $u(t) = \lambda$, then it is constant for all t, indicating no change in the probability with test time, meaning no reliability growth.

2. $\beta<1$: If $u(t)$ is decreasing $(\lambda_1>\lambda_2>\lambda_3>\lambda_4>\lambda_5 \ldots)$, then the failure probability $u(t)\Delta t$ is decreasing, implying the reliability growth.

3. $\beta>1$: If $u(t)$ is increasing $(\lambda_1<\lambda_2<\lambda_3<\lambda_4<\lambda_5\ldots)$, then the system is deteriorating.

Point Estimation

The method of maximum likelihood computes point estimates of the parameters of the reliability growth process.

Time-Terminated Testing:

The likelihood function for the time-terminated testing is given by

$$L = \lambda^n \beta^n\, e^{-\lambda T^\beta} \prod_{i=1}^{n} x_i^{\beta-1} \quad , \tag{5}$$

where n denotes a sample size. So that

$$ln(L) = n\ ln(\lambda) + n\ ln(\beta) - \lambda T^{\beta} + ln\left(\prod_{i=1}^{n} x_i^{\beta-1}\right) \tag{6}$$

Now,

$$ln\left(\prod_{i=1}^{n} x_i^{\beta-1}\right) = ln\left(x_1^{\beta-1} * x_2^{\beta-1} * x_3^{\beta-1} \ldots x_n^{\beta-1}\right) . \tag{7}$$

Manipulation of equation (7) finally yields

$$ln\left(\prod_{i=1}^{n} x_i^{\beta-1}\right) = (\beta-1) \sum_{i=1}^{n} ln(x_i) . \tag{8}$$

Thus, putting equation (8) into equation (6) yields

$$ln(1) = n\ ln(\lambda) + n\ ln(\beta) - \lambda T^{\beta} + (\beta-1) \sum_{i=1}^{n} ln(x_i) . \tag{9}$$

Differentiation with respect to the scale parameter λ yields

$$\frac{d(L)}{d\lambda} = \frac{n}{\lambda} - x_n^{\beta} . \tag{10}$$

By equating the resulting expression to zero, the maximum likelihood estimate $\hat{\lambda}$, which estimates the scale parameter λ, is obtained:

$$\hat{\lambda} = \frac{n}{T^{\hat{\beta}}} . \tag{11}$$

Subsequently, differentiation with respect to the shape parameter β yields

$$\frac{d(L)}{d\beta} = \frac{n}{\beta} - \lambda(ln(T))\ T^{\beta} + \sum_{i=1}^{n} ln(x_i) \tag{12}$$

Equating to zero yields the maximum likelihood estimate $\hat{\beta}$:

$$0 = \frac{n}{\hat{\beta}} - \lambda(ln(T))\ T^{\hat{\beta}} + \sum_{i=1}^{n} ln(x_i) \quad , \tag{13}$$

or

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} ln\left(\frac{T}{x_i}\right)} \tag{14}$$

For any time t, the intensity function is estimated by $\hat{\rho}(t) = \hat{\lambda}\hat{\beta}\,t^{\hat{\beta}-1}$.

Failure-Terminated Testing:

The likelihood function for the failure-terminated testing is given by

$$L = \lambda^n \beta^n e^{-\lambda x_n^{\beta}} \prod_{i=1}^{n} x_i^{\beta-1} \tag{15}$$

The procedure for finding $\hat{\lambda}$ is similar to the one for the time-terminated testing. Therefore, the maximum likelihood estimate $\hat{\lambda}$, estimating the scale parameter λ, is obtained:

$$\hat{\lambda} = \frac{n}{x_n^{\hat{\beta}}} \tag{16}$$

Now, differentiation with respect to the shape parameter β results in

$$\frac{d(L)}{d\beta} = \frac{n}{\beta} - \lambda\left[\beta x_n^{\beta-1}\frac{dx_n}{d\beta} + \{ln(x_n)\}\ x_n^{\beta}\ \frac{d\beta}{d\beta}\right] + \sum_{i=1}^{n} ln(x_i) \tag{17}$$

Equating equation (17) to zero yields that

$$\hat{\beta} = \frac{n}{n\ ln(x_n) - \sum_{i=1}^{n-1} ln(x_i) - ln(x_n)}$$

$$= \frac{n}{(n-1)\ ln(x_n) - \sum_{i=1}^{n-1} ln(x_i)} \quad , \tag{18}$$

or

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n-1} ln\left(\frac{x_n}{x_i}\right)} \quad , \tag{19}$$

$\hat{\lambda}$ and $\hat{\beta}$ are the maximum likelihood estimators of λ and β.

Interval Estimation

For time- and failure-terminated testings, the confidence intervals are measurements of precision in estimating a parameter regarding the reliability testing. For the reliability growth process, the parameter is the MTBF that the system would show after the production start. The probability distribution of the point estimate of the intensiity function at the end of the test provides a basis for the interval estimate of the true value of the intensity function at that time.

Tables 1A through 1C for the time-terminated testing and tables 2A through 2C for the failure-terminated testing provide the values for computation of confidence interval estimates for the MTBF. The two-sided interval estimates are obtained from the tables for the ratio of the true MTBF to the estimated MTBF for several values of the confidence coefficient. If the number of failures is N and α is the selected confidence coefficient, then the appropriate tabular values are $L_{N,\alpha}$ and $U_{N,\alpha}$.

The interval estimate of MTBF for both terminated testings is

$$\frac{L_{N,\alpha}}{\hat{\rho}(t)} \leqslant \text{MTBF} \leqslant \frac{U_{N,\alpha}}{\hat{\rho}(t)} \quad , \tag{20}$$

where $t=T$ for time-terminated testing or $t=x_n$ for failure-terminated testing.

Goodness-of-Fit

Three different test statistics, namely the Cramér-von Mises, Kolmogorov-Smirnov, and the chi-square, are used to test the null hypothesis that a nonhomogeneous Poisson process with the Weibull intensity function, $u(t)=\lambda\beta t^{\beta-1}$ (equation (1)), properly describes the reliability growth of a particular system. A brief description of each test statistic is given below.

Cramér-von Mises:

For time-terminated testing, the Cramér-von Mises test statistic includes an unbiased estimate of the shape parameter, which is, along with with equation (14),

Table 1A. Confidence intervals for MTBF from time-terminated tests: α = 0.50, 0.55, 0.60, 0.65.

N \ α	0.50 L	0.50 U	0.55 L	0.55 U	0.60 L	0.60 U	0.65 L	0.65 U
2	0.4233	6.6342	0.3960	7.5268	0.3692	8.6415	0.3426	10.0736
3	0.5021	3.2846	0.4745	3.5696	0.4470	3.9057	0.4193	4.3106
4	0.5559	2.5052	0.5286	2.6808	0.5013	2.8838	0.4737	3.1229
5	0.5955	2.1491	0.5689	2.2792	0.5421	2.4280	0.5148	2.6011
6	0.6262	1.9425	0.6003	2.0475	0.5741	2.1667	0.5473	2.3043
7	0.6509	1.8066	0.6257	1.8957	0.6001	1.9961	0.5739	2.1115
8	0.6713	1.7100	0.6468	1.7879	0.6219	1.8754	0.5962	1.9755
9	0.6886	1.6374	0.6647	1.7071	0.6404	1.7852	0.6152	1.8740
10	0.7034	1.5808	0.6801	1.6441	0.6564	1.7148	0.6318	1.7951
11	0.7163	1.5352	0.6936	1.5935	0.6704	1.6584	0.6463	1.7319
12	0.7276	1.4976	0.7055	1.5518	0.6828	1.6119	0.6592	1.6799
13	0.7378	1.4660	0.7161	1.5167	0.6938	1.5730	0.6707	1.6364
14	0.7468	1.4391	0.7256	1.4869	0.7038	1.5398	0.6811	1.5993
15	0.7550	1.4158	0.7343	1.4611	0.7129	1.5111	0.6906	1.5673
16	0.7625	1.3955	0.7421	1.4385	0.7211	1.4860	0.6993	1.5393
17	0.7693	1.3775	0.7493	1.4186	0.7287	1.4639	0.7072	1.5147
18	0.7756	1.3615	0.7560	1.4008	0.7357	1.4442	0.7146	1.4928
19	0.7814	1.3471	0.7621	1.3849	0.7422	1.4266	0.7214	1.4732
20	0.7868	1.3342	0.7678	1.3706	0.7482	1.4107	0.7277	1.4554
21	0.7918	1.3224	0.7731	1.3576	0.7538	1.3962	0.7336	1.4394
22	0.7964	1.3116	0.7781	1.3457	0.7591	1.3830	0.7392	1.4247
23	0.8008	1.3018	0.7827	1.3348	0.7640	1.3710	0.7444	1.4113
24	0.8049	1.2927	0.7871	1.3247	0.7686	1.3598	0.7493	1.3989
25	0.8087	1.2843	0.7912	1.3154	0.7730	1.3496	0.7539	1.3875
26	0.8124	1.2766	0.7951	1.3068	0.7771	1.3400	0.7582	1.3769
27	0.8158	1.2693	0.7987	1.2988	0.7810	1.3312	0.7624	1.3671
28	0.8191	1.2626	0.8022	1.2914	0.7847	1.3229	0.7663	1.3579
29	0.8222	1.2563	0.8055	1.2844	0.7882	1.3151	0.7701	1.3493
30	0.8251	1.2503	0.8087	1.2778	0.7916	1.3079	0.7736	1.3412
35	0.8379	1.2254	0.8224	1.2502	0.8062	1.2773	0.7892	1.3073
40	0.8482	1.2062	0.8335	1.2290	0.8182	1.2538	0.8020	1.2812
45	0.8569	1.1908	0.8428	1.2120	0.8281	1.2350	0.8126	1.2604
50	0.8642	1.1783	0.8507	1.1981	0.8366	1.2196	0.8217	1.2433
60	0.8760	1.1588	0.8634	1.1765	0.8503	1.1957	0.8364	1.2169
70	0.8851	1.1443	0.8734	1.1604	0.8611	1.1779	0.8480	1.1971
80	0.8925	1.1328	0.8814	1.1478	0.8698	1.1639	0.8574	1.1816
90	0.8986	1.1236	0.8881	1.1375	0.8770	1.1526	0.8652	1.1691
100	0.9038	1.1159	0.8937	1.1290	0.8831	1.1432	0.8719	1.1586

Table 1B. Confidence intervals for MTBF from time-terminated tests: α = 0.70, 0.75, 0.80, 0.85.

N \ α	0.70 L	0.70 U	0.75 L	0.75 U	0.80 L	0.80 U	0.85 L	0.85 U
2	0.3161	11.9818	0.2892	14.6517	0.2614	18.6549	0.2321	25.3246
3	0.3914	4.8118	0.3626	5.4555	0.3325	6.3262	0.3001	7.6006
4	0.4454	3.4117	0.4161	3.7720	0.3851	4.2429	0.3513	4.9031
5	0.4867	2.8074	0.4574	3.0610	0.4262	3.3864	0.3918	3.8325
6	0.5196	2.4668	0.4906	2.6645	0.4594	2.9152	0.4250	3.2540
7	0.5467	2.2468	0.5180	2.4101	0.4872	2.6155	0.4528	2.8902
8	0.5695	2.0922	0.5413	2.2323	0.5108	2.4074	0.4767	2.6396
9	0.5981	1.9772	0.5613	2.1006	0.5312	2.2538	0.4974	2.4558
10	0.6061	1.8881	0.5788	1.9987	0.5491	2.1356	0.5157	2.3150
11	0.6211	1.8167	0.5942	1.9174	0.5650	2.0414	0.5320	2.2034
12	0.6344	1.7582	0.6080	1.8508	0.5792	1.9646	0.5466	2.1125
13	0.6464	1.7092	0.6205	1.7952	0.5921	1.9006	0.5599	2.0370
14	0.6573	1.6676	0.6317	1.7480	0.6037	1.8462	0.5719	1.9732
15	0.6672	1.6316	0.6420	1.7073	0.6144	1.7996	0.5830	1.9184
16	0.6762	1.6003	0.6514	1.6719	0.6242	1.7589	0.5932	1.8708
17	0.6845	1.5727	0.6601	1.6407	0.6333	1.7232	0.6026	1.8291
18	0.6922	1.5482	0.6682	1.6130	0.6417	1.6915	0.6114	1.7921
19	0.6994	1.5262	0.6757	1.5882	0.6495	1.6632	0.6196	1.7591
20	0.7060	1.5064	0.6826	1.5658	0.6568	1.6377	0.6273	1.7294
21	0.7122	1.4884	0.6892	1.5456	0.6637	1.6147	0.6344	1.7026
22	0.7181	1.4720	0.6953	1.5272	0.6701	1.5936	0.6412	1.6782
23	0.7236	1.4570	0.7011	1.5103	0.6762	1.5744	0.6476	1.6559
24	0.7287	1.4432	0.7065	1.4948	0.6819	1.5568	0.6536	1.6354
25	0.7336	1.4305	0.7117	1.4804	0.6874	1.5405	0.6594	1.6166
26	0.7382	1.4187	0.7165	1.4672	0.6925	1.5254	0.6648	1.5991
27	0.7426	1.4077	0.7212	1.4548	0.6974	1.5114	0.6699	1.5829
28	0.7468	1.3974	0.7256	1.4433	0.7020	1.4983	0.6749	1.5678
29	0.7507	1.3878	0.7298	1.4325	0.7065	1.4861	0.6796	1.5537
30	0.7545	1.3789	0.7338	1.4224	0.7107	1.4746	0.6841	1.5405
35	0.7711	1.3411	0.7513	1.3801	0.7294	1.4267	0.7039	1.4852
40	0.7846	1.3120	0.7658	1.3476	0.7447	1.3899	0.7202	1.4429
45	0.7960	1.2889	0.7779	1.3216	0.7576	1.3606	0.7341	1.4093
50	0.8057	1.2699	0.7883	1.3004	0.7687	1.3367	0.7460	1.3819
60	0.8215	1.2405	0.8052	1.2676	0.7869	1.2997	0.7655	1.3395
70	0.8340	1.2186	0.8186	1.2431	0.8013	1.2721	0.7810	1.3081
80	0.8441	1.2014	0.8295	1.2240	0.8130	1.2506	0.7937	1.2836
90	0.8525	1.1875	0.8386	1.2084	0.8229	1.2332	0.8044	1.2637
100	0.8597	1.1759	0.8464	1.1955	0.8313	1.2186	0.8135	1.2471

Table 1C. Confidence intervals for MTBF from time-terminated tests: α = 0.90, 0.95, 0.98.

N \ α	0.90		0.95		0.98	
	L	U	L	U	L	U
2	0.1996	38.6608	0.1595	78.6633	0.1242	198.6660
3	0.2633	9.7360	0.2167	14.5520	0.1742	24.1033
4	0.3125	5.9469	0.2622	8.0927	0.2153	11.8110
5	0.3520	4.5174	0.2996	5.8619	0.2499	8.0425
6	0.3846	3.7644	0.3311	4.7378	0.2796	6.2540
7	0.4124	3.2984	0.3582	4.0611	0.3054	5.2156
8	0.4363	2.9810	0.3818	3.6090	0.3282	4.5393
9	0.4572	2.7503	0.4027	3.2853	0.3486	4.0643
10	0.4758	2.5748	0.4214	3.0418	0.3670	3.7124
11	0.4924	2.4364	0.4382	2.8518	0.3837	3.4412
12	0.5074	2.3244	0.4534	2.6992	0.3990	3.2257
13	0.5210	2.2316	0.4674	2.5737	0.4130	3.0501
14	0.5335	2.1535	0.4802	2.4686	0.4260	2.9041
15	0.5450	2.0866	0.4921	2.3792	0.4380	2.7808
16	0.5556	2.0288	0.5031	2.3022	0.4492	2.6752
17	0.5654	1.9781	0.5133	2.2350	0.4597	2.5836
18	0.5745	1.9333	0.5229	2.1759	0.4695	2.5033
19	0.5831	1.8934	0.5319	2.1234	0.4788	2.4324
20	0.5911	1.8576	0.5403	2.0764	0.4875	2.3693
21	0.5987	1.8253	0.5483	2.0342	0.4958	2.3126
22	0.6058	1.7960	0.5558	1.9959	0.5036	2.2615
23	0.6125	1.7692	0.5629	1.9610	0.5110	2.2151
24	0.6189	1.7446	0.5697	1.9292	0.5181	2.1727
25	0.6249	1.7221	0.5761	1.8999	0.5249	2.1340
26	0.6307	1.7012	0.5822	1.8729	0.5313	2.0983
27	0.6362	1.6818	0.5881	1.8478	0.5375	2.0654
28	0.6414	1.6638	0.5937	1.8246	0.5434	2.0348
29	0.6464	1.6469	0.5990	1.8030	0.5490	2.0064
30	0.6511	1.6312	0.6042	1.7828	0.5544	1.9800
35	0.6723	1.5655	0.6270	1.6987	0.5787	1.8704
40	0.6898	1.5153	0.6460	1.6349	0.5991	1.7879
45	0.7047	1.4756	0.6622	1.5846	0.6166	1.7232
50	0.7175	1.4433	0.6763	1.5438	0.6318	1.6710
60	0.7387	1.3935	0.6996	1.4813	0.6572	1.5915
70	0.7556	1.3566	0.7183	1.4353	0.6778	1.5337
80	0.7694	1.3279	0.7338	1.3996	0.6948	1.4891
90	0.7811	1.3047	0.7468	1.3707	0.7092	1.4527
100	0.7911	1.2853	0.7581	1.3466	0.7217	1.4221

Table 2A. Confidence intervals for MTBF from failure-terminated tests:
α = 0.50, 0.55, 0.60, 0.65.

N \ α	0.50		0.55		0.60		0.65	
	L	U	L	U	L	U	L	U
2	1.6080	11.0791	1.4590	12.7185	1.3178	14.7819	1.1834	17.4532
3	1.1443	4.3727	1.0655	4.7876	0.9884	5.2807	0.9126	5.8796
4	1.0169	3.0348	0.9581	3.2621	0.8998	3.5262	0.8416	3.8393
5	0.9612	2.4810	0.9123	2.6388	0.8633	2.8199	0.8139	3.0316
6	0.9318	2.1749	0.8888	2.2974	0.8456	2.4369	0.8017	2.5984
7	0.9146	1.9838	0.8757	2.0849	0.8365	2.1993	0.7964	2.3310
8	0.9039	1.8518	0.8681	1.9387	0.8318	2.0365	0.7946	2.1484
9	0.8969	1.7550	0.8635	1.8317	0.8295	1.9176	0.7946	2.0156
10	0.8924	1.6808	0.8609	1.7497	0.8288	1.8267	0.7957	1.9142
11	0.8894	1.6220	0.8595	1.6848	0.8290	1.7549	0.7974	1.8343
12	0.8875	1.5742	0.8590	1.6321	0.8297	1.6966	0.7995	1.7695
13	0.8863	1.5344	0.8589	1.5883	0.8309	1.6482	0.8018	1.7158
14	0.8856	1.5007	0.8593	1.5513	0.8322	1.6074	0.8041	1.6705
15	0.8853	1.4718	0.8599	1.5196	0.8337	1.5724	0.8066	1.6318
16	0.8853	1.4468	0.8607	1.4920	0.8353	1.5420	0.8090	1.5981
17	0.8855	1.4248	0.8616	1.4678	0.8370	1.5153	0.8113	1.5686
18	0.8858	1.4053	0.8626	1.4464	0.8386	1.4918	0.8137	1.5425
19	0.8863	1.3880	0.8637	1.4274	0.8403	1.4708	0.8160	1.5193
20	0.8869	1.3724	0.8648	1.4103	0.8420	1.4519	0.8182	1.4984
21	0.8875	1.3584	0.8659	1.3949	0.8436	1.4349	0.8204	1.4797
22	0.8881	1.3457	0.8671	1.3809	0.8453	1.4195	0.8225	1.4626
23	0.8888	1.3341	0.8682	1.3681	0.8469	1.4055	0.8245	1.4471
24	0.8896	1.3234	0.8694	1.3564	0.8484	1.3926	0.8265	1.4329
25	0.8903	1.3136	0.8705	1.3457	0.8500	1.3808	0.8284	1.4199
26	0.8911	1.3045	0.8717	1.3357	0.8515	1.3699	0.8303	1.4078
27	0.8919	1.2961	0.8728	1.3265	0.8530	1.3598	0.8321	1.3967
28	0.8927	1.2882	0.8739	1.3178	0.8544	1.3503	0.8339	1.3863
29	0.8934	1.2808	0.8750	1.3098	0.8558	1.3414	0.8356	1.3766
30	0.8942	1.2738	0.8761	1.3022	0.8572	1.3331	0.8373	1.3675
35	0.8977	1.2446	0.8808	1.2700	0.8632	1.2977	0.8446	1.3283
40	0.9011	1.2227	0.8852	1.2459	0.8686	1.2712	0.8511	1.2991
45	0.9043	1.2052	0.8892	1.2267	0.8734	1.2501	0.8568	1.2760
50	0.9072	1.1909	0.8928	1.2110	0.8778	1.2329	0.8619	1.2571
60	0.9122	1.1686	0.8990	1.1865	0.8852	1.2060	0.8705	1.2275
70	0.9164	1.1529	0.9042	1.1692	0.8913	1.1869	0.8776	1.2063
80	0.9202	1.1402	0.9086	1.1553	0.8965	1.1716	0.8836	1.1895
90	0.9234	1.1301	0.9124	1.1441	0.9010	1.1593	0.8888	1.1760
100	0.9262	1.1218	0.9158	1.1349	0.9049	1.1492	0.8932	1.1648

Table 2B. Confidence intervals for MTBF from failure-terminated tests: $\alpha = 0.70, 0.75, 0.80, 0.85$.

N \ α	0.70 L	0.70 U	0.75 L	0.75 U	0.80 L	0.80 U	0.85 L	0.85 U
2	1.0543	21.0397	0.9293	26.0953	0.8065	33.7594	0.6834	46.6522
3	0.8372	6.6278	0.7614	7.5981	0.6840	8.9268	0.6028	10.8910
4	0.7828	4.2198	0.7227	4.6982	0.6601	5.3288	0.5930	6.2216
5	0.7636	3.2851	0.7116	3.5982	0.6568	4.0024	0.5972	4.5608
6	0.7567	2.7898	0.7098	3.0234	0.6600	3.3211	0.6053	3.7299
7	0.7551	2.4858	0.7118	2.6732	0.6656	2.9098	0.6145	3.2275
8	0.7561	2.2793	0.7156	2.4368	0.6720	2.6340	0.6237	2.8965
9	0.7583	2.1296	0.7200	2.2661	0.6787	2.4361	0.6325	2.6606
10	0.7612	2.0157	0.7247	2.1368	0.6852	2.2867	0.6409	2.4837
11	0.7645	1.9261	0.7295	2.0351	0.6915	2.1700	0.6844	2.3458
12	0.7679	1.8535	0.7342	1.9531	0.6975	2.0755	0.6562	2.2350
13	0.7713	1.7936	0.7388	1.8854	0.7033	1.9980	0.6632	2.1440
14	0.7746	1.7430	0.7432	1.8284	0.7087	1.9329	0.6697	2.0680
15	0.7780	1.6998	0.7474	1.7798	0.7139	1.8775	0.6758	2.0033
16	0.7812	1.6623	0.7514	1.7378	0.7188	1.8296	0.6816	1.9476
17	0.7843	1.6295	0.7553	1.7009	0.7234	1.7877	0.6871	1.8991
18	0.7873	1.6005	0.7590	1.6684	0.7278	1.7507	0.6923	1.8562
19	0.7902	1.5746	0.7625	1.6394	0.7320	1.7178	0.6972	1.8182
20	0.7931	1.5514	0.7659	1.6134	0.7360	1.6883	0.7018	1.7840
21	0.7957	1.5305	0.7692	1.5899	0.7398	1.6617	0.7062	1.7533
22	0.7983	1.5116	0.7723	1.5687	0.7435	1.6376	0.7104	1.7254
23	0.8008	1.4944	0.7752	1.5494	0.7469	1.6157	0.7144	1.7001
24	0.8032	1.4786	0.7781	1.5317	0.7503	1.5956	0.7183	1.6769
25	0.8056	1.4641	0.7808	1.5155	0.7535	1.5773	0.7219	1.6556
26	0.8078	1.4508	0.7835	1.5006	0.7565	1.5604	0.7254	1.6361
27	0.8100	1.4384	0.7860	1.4868	0.7594	1.5448	0.7288	1.6181
28	0.8121	1.4270	0.7885	1.4740	0.7623	1.5303	0.7320	1.6014
29	0.8141	1.4162	0.7909	1.4621	0.7650	1.5169	0.7351	1.5860
30	0.8161	1.4062	0.7931	1.4509	0.7676	1.5044	0.7381	1.5716
35	0.8248	1.3628	0.8033	1.4027	0.7794	1.4503	0.7516	1.5101
40	0.8324	1.3306	0.8121	1.3667	0.7894	1.4098	0.7630	1.4638
45	0.8391	1.3050	0.8197	1.3384	0.7980	1.3780	0.7728	1.4274
50	0.8449	1.2841	0.8264	1.3152	0.8056	1.3521	0.7814	1.3980
60	0.8548	1.2516	0.8376	1.2792	0.8184	1.3119	0.7959	1.3525
70	0.8630	1.2280	0.8469	1.2528	0.8288	1.2821	0.8077	1.3184
80	0.8698	1.2095	0.8546	1.2323	0.8375	1.2591	0.8175	1.2924
90	0.8756	1.1945	0.8612	1.2157	0.8449	1.2406	0.8258	1.2715
100	0.8807	1.1821	0.8669	1.2020	0.8514	1.2253	0.8331	1.2542

Table 2C. Confidence intervals for MTBF from failure-terminated tests: $\alpha = 0.90, 0.95, 0.98$.

N \ α	0.90		0.95		0.98	
	L	U	L	U	L	U
2	0.5552	74.6675	0.4099	151.5022	0.2944	389.9034
3	0.5137	14.2397	0.4054	21.9589	0.3119	37.5993
4	0.5174	7.6517	0.4225	10.6462	0.3368	15.9562
5	0.5290	5.4261	0.4415	7.1490	0.3603	9.9966
6	0.5421	4.3387	0.4595	5.5208	0.3814	7.3878
7	0.5548	3.7021	0.4760	4.5953	0.4003	5.9627
8	0.5668	3.2842	0.4910	4.0020	0.4173	5.0738
9	0.5780	2.9892	0.5046	3.5892	0.4327	4.4690
10	0.5883	2.7697	0.5171	3.2859	0.4466	4.0319
11	0.5979	2.5998	0.5285	3.0539	0.4594	3.7016
12	0.6067	2.4640	0.5391	2.8702	0.4712	3.4430
13	0.6150	2.3528	0.5488	2.7210	0.4821	3.2354
14	0.6227	2.2602	0.5579	2.5972	0.4923	3.0642
15	0.6299	2.1818	0.5664	2.4928	0.5017	2.9214
16	0.6367	2.1144	0.5743	2.4036	0.5106	2.7996
17	0.6431	2.0559	0.5818	2.3266	0.5189	2.6948
18	0.6491	2.0045	0.5888	2.2592	0.5267	2.6037
19	0.6547	1.9589	0.5954	2.1998	0.5341	2.5238
20	0.6601	1.9181	0.6016	2.1469	0.5411	2.4531
21	0.6652	1.8813	0.6076	2.0994	0.5478	2.3900
22	0.6701	1.8480	0.6132	2.0565	0.5541	2.3334
23	0.6747	1.8177	0.6186	2.0174	0.5601	2.2821
24	0.6791	1.7900	0.6237	1.9817	0.5659	2.2353
25	0.6833	1.7646	0.6286	1.9489	0.5714	2.1925
26	0.6873	1.7412	0.6333	1.9188	0.5766	2.1531
27	0.6911	1.7196	0.6378	1.8909	0.5817	2.1166
28	0.6948	1.6997	0.6420	1.8651	0.5865	2.0828
29	0.6984	1.6812	0.6462	1.8412	0.5911	2.0514
30	0.7018	1.6641	0.6501	1.8189	0.5956	2.0221
35	0.7173	1.5923	0.6681	1.7288	0.6158	1.9048
40	0.7303	1.5376	0.6832	1.6597	0.6328	1.8162
45	0.7415	1.4948	0.6962	1.6055	0.6476	1.7469
50	0.7513	1.4602	0.7076	1.5620	0.6605	1.6911
60	0.7677	1.4074	0.7268	1.4962	0.6823	1.6071
70	0.7811	1.3674	0.7423	1.4467	0.6999	1.5459
80	0.7922	1.3371	0.7553	1.4092	0.7148	1.4987
90	0.8017	1.3129	0.7663	1.3794	0.7274	1.4614
100	0.8100	1.2929	0.7760	1.3549	0.7385	1.4310

$$\bar{\beta} = \frac{N-1}{N}\hat{\beta} \quad , \tag{21}$$

with n failure occurrences. The goodness-of-fit statistic, from reference 1, becomes

$$C_M^2 = \frac{1}{12M} + \sum_{i=1}^{M}\left[\left(\frac{x_i}{T}\right)^{\bar{\beta}} - \frac{2i-1}{2M}\right]^2 \quad , \tag{22}$$

in which the failure times are arranged in increasing order of magnitude, $0 < x_1 < x_2 < x_3 \ldots < x_n$.

The null hypothesis is rejected if the statistic C_M^2 exceeds the critical value for the selected level of significance. Critical values of the Cramér-von Mises statistic for the 0.03, 0.25, 0.20, 0.15, and 0.10 levels of significance (α) are listed in table 3A and the 0.05, 0.03, 0.01, 0.005, and 0.001 levels of significance in table 3B, with both tables being indexed by the parameter M. For the time-terminated testing, M is equal to n, the number of failures, and to 2(1)20(10)30(30) 60(20)100.

For failure-terminated testing, the null hypothesis that the AMSAA model is appropriate for a particular system can be tested, using the Cramér-von Mises test statistic. An unbiased estimate of the shape parameter along with equation (19) is given by

$$\bar{\beta} = \frac{N-2}{N}\hat{\beta} \quad . \tag{23}$$

The parameter for indexing that statistic is $M = n-1$, where n is the number of failures. The Cramér-von Mises test statistic, from reference 1, is now

$$C_M^2 = \frac{1}{12M} + \sum_{i=1}^{M}\left[\left(\frac{x_i}{x_N}\right)^{\bar{\beta}} - \frac{2i-1}{2M}\right]^2 \tag{24}$$

Tables 3A and 3B provide the critical values for use in the test. The model is deemed inappropriate if the statistic C_M^2 exceeds the critical value for some specified level of significance.

Kolmogorov-Smirnov:

A Kolmogorov-Smirnov method, for testing the one-sample probability that a set of numbers is a sample from a known distribution, consists of comparing the empirical (or sample) cumulative distribution function of the sample, $S(x)$, with the known continuous cumulative distribution function $F(x)$.

Table 3A. Critical values of Cramér-von Mises test statistic: α = 0.30, 0.25, 0.20, 0.15, 0.10.

N \ α	0.30	0.25	0.20	0.15	0.10
2	0.118	0.127	0.137	0.149	0.162
3	0.101	0.109	0.120	0.133	0.152
4	0.102	0.111	0.122	0.136	0.156
5	0.098	0.107	0.118	0.135	0.159
6	0.100	0.110	0.122	0.138	0.162
7	0.101	0.112	0.125	0.142	0.167
8	0.101	0.112	0.125	0.142	0.166
9	0.101	0.111	0.124	0.141	0.166
10	0.102	0.112	0.126	0.143	0.167
11	0.102	0.113	0.125	0.141	0.165
12	0.103	0.114	0.128	0.146	0.171
13	0.102	0.113	0.126	0.144	0.170
14	0.104	0.115	0.129	0.146	0.172
15	0.104	0.116	0.129	0.148	0.174
16	0.103	0.114	0.128	0.146	0.170
17	0.103	0.114	0.127	0.143	0.168
18	0.101	0.112	0.126	0.144	0.170
19	0.102	0.112	0.126	0.142	0.168
20	0.103	0.114	0.128	0.146	0.173
30	0.102	0.113	0.126	0.142	0.167
60	0.105	0.116	0.130	0.148	0.174
80	0.105	0.115	0.129	0.147	0.173
100	0.105	0.116	0.130	0.148	0.173

Table 3B. Critical values of Cramér-von Mises test statistic: α = 0.05, 0.03, 0.01, 0.005, 0.001.

N \ α	0.05	0.03	0.01	0.005	0.001
2	0.175	0.181	0.186	0.188	0.189
3	0.180	0.199	0.230	0.245	0.273
4	0.194	0.220	0.284	0.335	0.414
5	0.199	0.228	0.296	0.334	0.421
6	0.202	0.233	0.308	0.355	0.493
7	0.208	0.241	0.316	0.355	0.486
8	0.208	0.242	0.318	0.372	0.479
9	0.211	0.246	0.325	0.376	0.498
10	0.210	0.246	0.330	0.377	0.531
11	0.208	0.240	0.316	0.376	0.498
12	0.216	0.257	0.326	0.388	0.540
13	0.216	0.249	0.330	0.371	0.470
14	0.216	0.250	0.330	0.388	0.519
15	0.220	0.258	0.348	0.401	0.511
16	0.217	0.250	0.322	0.375	0.494
17	0.214	0.247	0.325	0.371	0.493
18	0.214	0.245	0.332	0.382	0.493
19	0.214	0.246	0.321	0.375	0.513
20	0.220	0.254	0.335	0.399	0.544
30	0.213	0.250	0.324	0.371	0.514
60	0.223	0.256	0.335	0.382	0.507
80	0.222	0.259	0.336	0.390	0.524
100	0.221	0.256	0.340	0.395	0.523

The one-sample test is based on the maximum absolute difference D between the values of the cumulative distribution of a random sample of size n and a specified theoretical distribution. The maximum difference occurs at one of the jump points of $S(x)$. On the graphs, the lower and upper limits are plotted with $S(x)$ as the boundaries for the percent confidence band. The criterion calls for the null hypothesis to be rejected if the D is greater than the critical value at any specified level of significance α $(0<\alpha<1)$.

The critical values for the Kolmogorov-Smirnov test statistic are obtained for tables 4A through 4D by using the computational techniques described in textbooks for α = 0.30, 0.25, 0.20, 0.15, 0.10, 0.05, 0.02, 0.01, 0.005, and 0.001 and N = 1(1)100.

Chi-Square:

A chi-square goodness-of-fit test is used to test the null hypothesis that the reliability growth model adequately represents the grouped data. The total number of failures in the interval between inspections is the sum of the number of failures detected at the time of occurrence and the number of failures found in the inspection. Such totals for each interval can estimate the reliability growth, provided there are at least three intervals.

The maximum likelihood estimate of the shape parameter β is the value satisfying

$$\sum_{i=1}^{K} N_i \left[\frac{t_i^{\hat{\beta}}\, ln(t_i) - t_{i-1}^{\hat{\beta}}\, ln(t_{i-1})}{t_i^{\hat{\beta}} - t_{i-1}^{\hat{\beta}}} - ln(t_K) \right] = 0 \quad , \tag{25}$$

in which t_o log (t_o) is equal to zero. The scale parameter estimate is

$$\hat{\lambda} = \frac{\sum_{i=1}^{K} N_i}{t_K^{\hat{\beta}}} \quad . \tag{26}$$

The expected number of failures in the interval from t_{i-1} to t_i is approximated by

$$e_i = \hat{\lambda}\left(t_i^{\hat{\beta}} - t_{i-1}^{\hat{\beta}}\right) \ . \tag{27}$$

Adjacent intervals may have to be combined so that the expected number of failures in any combined interval is at least five. K is equal to the number of intervals after this combination. The number of failures in the i-th interval is equal to N_i. Let e_i be the expected number of failures in the i-th new interval. Accordingly, the statistic

Table 4A. Critical values of Kolmogorov-Smirnov test statistic: α = 0.30, 0.25, 0.20, 0.15, 0.10; N = 1–50.

N \ α	0.30	0.25	0.20	0.15	0.10
1	0.850000	0.875000	0.900000	0.925000	0.950000
2	0.612702	0.646447	0.683772	0.726139	0.776393
3	0.514660	0.538329	0.564810	0.595819	0.636045
4	0.447301	0.467882	0.492653	0.524754	0.565216
5	0.403486	0.423848	0.446980	0.474389	0.509449
6	0.371673	0.389701	0.410373	0.435261	0.467993
7	0.345684	0.362254	0.381476	0.404976	0.436069
8	0.324425	0.340042	0.358313	0.380633	0.409623
9	0.306816	0.321726	0.339102	0.360072	0.387464
10	0.291963	0.306172	0.322602	0.342508	0.368663
11	0.279119	0.292626	0.308292	0.327348	0.352421
12	0.267801	0.280731	0.295770	0.314092	0.338151
13	0.257759	0.270204	0.284698	0.302345	0.325490
14	0.248786	0.260807	0.274807	0.291826	0.314170
15	0.240711	0.252348	0.265886	0.282341	0.303973
16	0.233391	0.244671	0.257784	0.273737	0.294720
17	0.226711	0.237657	0.250387	0.265885	0.286269
18	0.220578	0.231218	0.243601	0.258681	0.278511
19	0.214920	0.225282	0.237346	0.252037	0.271357
20	0.209680	0.219787	0.231555	0.245885	0.264734
21	0.205063	0.214894	0.226344	0.240296	0.258664
22	0.200505	0.210117	0.221312	0.234954	0.252916
23	0.196242	0.205648	0.216605	0.229956	0.247538
24	0.192243	0.201456	0.212189	0.225267	0.242492
25	0.188481	0.197513	0.208034	0.220857	0.237745
26	0.184934	0.193795	0.204117	0.216698	0.233268
27	0.181582	0.190281	0.200416	0.212767	0.229038
28	0.178408	0.186954	0.196910	0.209045	0.225031
29	0.175396	0.183797	0.193584	0.205513	0.221229
30	0.172534	0.180796	0.190422	0.202156	0.217615
31	0.169808	0.177939	0.187412	0.198960	0.214174
32	0.167209	0.175214	0.184541	0.195911	0.210892
33	0.164727	0.172612	0.181800	0.193000	0.207758
34	0.162353	0.170124	0.179178	0.190216	0.204760
35	0.160080	0.167741	0.176667	0.187550	0.201890
36	0.157901	0.165456	0.174260	0.184994	0.199138
37	0.155809	0.163263	0.171950	0.182540	0.196496
38	0.153799	0.161156	0.169729	0.180182	0.193957
39	0.151865	0.159129	0.167594	0.177914	0.191515
40	0.150003	0.157177	0.165537	0.175730	0.189164
41	0.148209	0.155296	0.163555	0.173625	0.186898
42	0.146478	0.153481	0.161643	0.171595	0.184711
43	0.144807	0.151730	0.159797	0.169635	0.182600
44	0.143192	0.150037	0.158014	0.167740	0.180561
45	0.141630	0.148400	0.156289	0.165909	0.178589
46	0.140119	0.146816	0.154620	0.164136	0.176680
47	0.138656	0.145282	0.153004	0.162420	0.174832
48	0.137238	0.143795	0.151437	0.160756	0.173041
49	0.135862	0.142354	0.149918	0.159143	0.171304
50	0.134528	0.140955	0.148445	0.157578	0.169619

Table 4B. Critical values of Kolmogorov-Smirnov test statistic: α = 0.30, 0.25, 0.20, 0.15, 0.10; N = 51–100.

N \ α	0.30	0.25	0.20	0.15	0.10
51	0.133233	0.139597	0.147014	0.156059	0.167982
52	0.131974	0.138278	0.145624	0.154583	0.166393
53	0.130751	0.136996	0.144273	0.153148	0.164848
54	0.129562	0.135749	0.142959	0.151753	0.163346
55	0.128404	0.134535	0.141681	0.150396	0.161884
56	0.127278	0.133354	0.140437	0.149074	0.160461
57	0.126180	0.132204	0.139225	0.147787	0.159075
58	0.125111	0.131083	0.138044	0.146533	0.157725
59	0.124069	0.129991	0.136893	0.145310	0.156408
60	0.123053	0.128925	0.135770	0.144118	0.155124
61	0.122061	0.127886	0.134675	0.142955	0.153872
62	0.121093	0.126871	0.133606	0.141820	0.152649
63	0.120148	0.125880	0.132562	0.140711	0.151456
64	0.119225	0.124913	0.131543	0.139629	0.150290
65	0.118323	0.123967	0.130546	0.138571	0.149151
66	0.117441	0.123043	0.129573	0.137537	0.148037
67	0.116579	0.122139	0.128620	0.136525	0.146948
68	0.115735	0.121255	0.127689	0.135536	0.145883
69	0.114910	0.120390	0.126778	0.134568	0.144841
70	0.114102	0.119544	0.125886	0.133621	0.143821
71	0.113312	0.118715	0.125013	0.132694	0.142822
72	0.112537	0.117903	0.124157	0.131786	0.141844
73	0.111779	0.117108	0.123319	0.130896	0.140886
74	0.111035	0.116328	0.122499	0.130024	0.139947
75	0.110306	0.115565	0.121694	0.129170	0.139027
76	0.109592	0.114816	0.120905	0.128332	0.138125
77	0.108891	0.114081	0.120131	0.127510	0.137240
78	0.108204	0.113361	0.119372	0.126704	0.136372
79	0.107530	0.112654	0.118627	0.125913	0.135521
80	0.106868	0.111960	0.117897	0.125137	0.134685
81	0.106218	0.111280	0.117179	0.124375	0.133865
82	0.105580	0.110611	0.116475	0.123627	0.133059
83	0.104954	0.109954	0.115783	0.122893	0.132268
84	0.104338	0.109309	0.115103	0.122171	0.131491
85	0.103734	0.108675	0.114436	0.121462	0.130728
86	0.103140	0.108053	0.113780	0.120765	0.129977
87	0.102556	0.107441	0.113135	0.120081	0.129240
88	0.101982	0.106839	0.112501	0.119407	0.128515
89	0.101417	0.106247	0.111877	0.118745	0.127802
90	0.100862	0.105665	0.111264	0.118094	0.127101
91	0.100316	0.105093	0.110661	0.117454	0.126412
92	0.099778	0.104530	0.110068	0.116824	0.125733
93	0.099250	0.103975	0.109484	0.116204	0.125066
94	0.098729	0.103430	0.108910	0.115594	0.124409
95	0.098217	0.102893	0.108344	0.114993	0.123762
96	0.097713	0.102365	0.107787	0.114402	0.123125
97	0.097216	0.101844	0.107239	0.113820	0.122498
98	0.096727	0.101332	0.106699	0.113246	0.121881
99	0.096246	0.100827	0.106167	0.112682	0.121273
100	0.095771	0.100330	0.105643	0.112125	0.120674

Table 4C. Critical values of Kolmogorov-Smirnov test statistic:
α = 0.05, 0.02, 0.01, 0.005, 0.001; N = 1–50.

N \ α	0.05	0.02	0.01	0.005	0.001
1	0.975000	0.990000	0.995000	0.997500	0.999500
2	0.841886	0.900000	0.929289	0.950000	0.977639
3	0.707598	0.784556	0.829002	0.864279	0.920630
4	0.623938	0.688870	0.734238	0.776393	0.850465
5	0.563275	0.627180	0.668531	0.705431	0.781369
6	0.519262	0.577407	0.616607	0.652865	0.724791
7	0.483424	0.538440	0.575812	0.609753	0.679305
8	0.454267	0.506543	0.541792	0.574291	0.640979
9	0.430011	0.479596	0.513317	0.544431	0.608464
10	0.409246	0.456624	0.488932	0.518725	0.580417
11	0.391224	0.436703	0.467702	0.496386	0.555878
12	0.375430	0.419178	0.449045	0.476715	0.534217
13	0.361432	0.403621	0.432473	0.459212	0.514899
14	0.348901	0.389695	0.417616	0.443516	0.497534
15	0.337596	0.377128	0.404200	0.429336	0.481818
16	0.327334	0.365709	0.392007	0.416439	0.467505
17	0.317963	0.355275	0.380862	0.404644	0.454398
18	0.309360	0.345693	0.370622	0.393802	0.442338
19	0.301425	0.336852	0.361170	0.383792	0.431192
20	0.294076	0.328661	0.352411	0.374513	0.420851
21	0.287285	0.321062	0.344279	0.365899	0.411272
22	0.280908	0.313953	0.336673	0.357836	0.402270
23	0.274942	0.307299	0.329552	0.350286	0.393836
24	0.269342	0.301054	0.322868	0.343197	0.385914
25	0.264074	0.295177	0.316577	0.336524	0.378453
26	0.259106	0.289633	0.310642	0.330228	0.371410
27	0.254410	0.284393	0.305031	0.324274	0.364749
28	0.249962	0.279429	0.299715	0.318633	0.358434
29	0.245742	0.274717	0.294669	0.313278	0.352438
30	0.241729	0.270238	0.289871	0.308185	0.346735
31	0.237908	0.265972	0.285301	0.303334	0.341300
32	0.234264	0.261903	0.280942	0.298706	0.336115
33	0.230784	0.258016	0.276777	0.294285	0.331160
34	0.227455	0.254298	0.272794	0.290056	0.326418
35	0.224267	0.250738	0.268979	0.286004	0.321875
36	0.221211	0.247323	0.265320	0.282119	0.317518
37	0.218276	0.244046	0.261807	0.278388	0.313334
38	0.215457	0.240896	0.258431	0.274802	0.309311
39	0.212744	0.237865	0.255183	0.271352	0.305440
40	0.210132	0.234947	0.252055	0.268030	0.301711
41	0.207615	0.232134	0.249040	0.264827	0.298117
42	0.205186	0.229420	0.246131	0.261736	0.294648
43	0.202841	0.226800	0.243322	0.258752	0.291298
44	0.200576	0.224268	0.240607	0.255868	0.288060
45	0.198385	0.221820	0.237982	0.253079	0.284928
46	0.196264	0.219450	0.235441	0.250380	0.281896
47	0.194211	0.217155	0.232981	0.247765	0.278960
48	0.192221	0.214931	0.230596	0.245231	0.276114
49	0.190291	0.212774	0.228284	0.242774	0.273353
50	0.188419	0.210681	0.226039	0.240389	0.270674

Table 4D. Critical values of Kolmogorov-Smirnov test statistic: α = 0.05, 0.02, 0.01, 0.005, 0.001; N = 51–100.

N \ α	0.05	0.02	0.01	0.005	0.001
51	0.186601	0.208649	0.223861	0.238073	0.268073
52	0.184835	0.206675	0.221744	0.235824	0.265545
53	0.183119	0.204756	0.219686	0.233637	0.263088
54	0.181449	0.202890	0.217685	0.231510	0.260698
55	0.179825	0.201075	0.215738	0.229441	0.258372
56	0.178244	0.199307	0.213842	0.227426	0.256107
57	0.176704	0.197586	0.211996	0.225464	0.253902
58	0.175204	0.195908	0.210197	0.223551	0.251752
59	0.173741	0.194273	0.208443	0.221687	0.249656
60	0.172314	0.192678	0.206733	0.219869	0.247612
61	0.170923	0.191122	0.205064	0.218095	0.245618
62	0.169564	0.189603	0.203435	0.216363	0.243671
63	0.168238	0.188120	0.201844	0.214673	0.241770
64	0.166942	0.186672	0.200290	0.213021	0.239913
65	0.165676	0.185256	0.198772	0.211407	0.238098
66	0.164439	0.183873	0.197288	0.209829	0.236324
67	0.163229	0.182520	0.195837	0.208287	0.234589
68	0.162045	0.181197	0.194417	0.206777	0.232891
69	0.160887	0.179902	0.193028	0.205301	0.231231
70	0.159754	0.178634	0.191669	0.203855	0.229605
71	0.158644	0.177393	0.190338	0.202440	0.228013
72	0.157558	0.176178	0.189034	0.201054	0.226454
73	0.156493	0.174988	0.187757	0.199697	0.224927
74	0.155450	0.173821	0.186506	0.198366	0.223430
75	0.154427	0.172678	0.185279	0.197062	0.221963
76	0.153425	0.171557	0.184076	0.195783	0.220525
77	0.152441	0.170457	0.182897	0.194529	0.219114
78	0.151477	0.169379	0.181740	0.193299	0.217730
79	0.150531	0.168320	0.180605	0.192092	0.216372
80	0.149602	0.167282	0.179491	0.190907	0.215039
81	0.148690	0.166262	0.178397	0.189744	0.213730
82	0.147795	0.165261	0.177323	0.188603	0.212446
83	0.146916	0.164278	0.176268	0.187481	0.211184
84	0.146053	0.163313	0.175232	0.186380	0.209944
85	0.145204	0.162364	0.174215	0.185297	0.208726
86	0.144371	0.161432	0.173214	0.184234	0.207529
87	0.143551	0.160515	0.172231	0.183188	0.206353
88	0.142746	0.159614	0.171265	0.182160	0.205196
89	0.141954	0.158728	0.170314	0.181150	0.204059
90	0.141175	0.157857	0.169379	0.180156	0.202940
91	0.140408	0.157000	0.168460	0.179178	0.201840
92	0.139654	0.156157	0.167555	0.178216	0.200757
93	0.138913	0.155328	0.166665	0.177270	0.199692
94	0.138182	0.154511	0.165789	0.176338	0.198644
95	0.137464	0.153707	0.164927	0.175421	0.197611
96	0.136756	0.152916	0.164078	0.174518	0.196595
97	0.136060	0.152137	0.163242	0.173629	0.195595
98	0.135374	0.151370	0.162419	0.172754	0.194609
99	0.134698	0.150614	0.161608	0.171892	0.193639
100	0.134032	0.149869	0.160809	0.171042	0.192683

$$\chi^2 = \sum_{i=1}^{K} \frac{(N_i - e_i)^2}{e_i} \quad , \tag{28}$$

is approximately distributed as a χ^2 random variable with K–2 degrees of freedom. The critical values for equation (28) are found in tables 5A through 5D using the chi-square distribution.

PROBLEM FORMULATION

Time-Terminated Testing

In the time-terminated testing case, the interval estimates provide a measure of uncertainty surrounding testing reliability. The MTBF represents the reliability status of the system at time t after the initiation of production. The reliability analysis would analyze the data from tests either terminated at a predetermined time or in progress with data available through some time. The needed data consist of the cumulative test time on all systems at the occurrence of each failure or the accumulated test time so that the Weibull/Duane postulate can be conformed with. The probability disitribution of the point estimate of the Weibull intensity function at the end of the test produces the basis for the interval estimate of the true value of the intensity function at that time.

To begin with the mathematical formulation of the equations needed for the confidence interval estimation for time-terminated testing, the time T is assumed to be predetermined and $n>1$ failures with time measurements $0<X_1<X_2<X_3 \ldots <X_n$ are observed for the Weibull process during the time interval $(0,T)$, prior to the accumulated test time T. The maximum likelihood estimates of the scale parameter λ and the shape parameter β from these data are, respectively,

$$\hat{\lambda} = n/T^{\hat{\beta}} \quad , \tag{11}$$

and

$$\hat{\beta} = n/\sum_{i=1}^{n} ln(T/X_i) \tag{14}$$

and the ML estimate of $M(T)$, the achieved MTBF at time T, is

$$\hat{M}(T) = 1/\hat{u}(T) = 1/(\hat{\lambda}\hat{\beta}\, T^{\hat{\beta}-1}) = T/n\hat{\beta} \quad . \tag{29}$$

Now, let $N(T) = N$ and from equation (6), let

Table 5A. Critical values of chi-square test statistic:
α = 0.30, 0.25, 0.20, 0.15, 0.10; ν = 1–50.

ν \ α	0.30	0.25	0.20	0.15	0.10
1	1.07419	1.32330	1.64237	2.07225	2.70554
2	2.40795	2.77259	3.21888	3.79424	4.60517
3	3.66487	4.10834	4.64163	5.31705	6.25139
4	4.87843	5.38527	5.98862	6.74488	7.77944
5	6.06443	6.62568	7.28928	8.11520	9.23636
6	7.23114	7.84080	8.55806	9.44610	10.64464
7	8.38343	9.03715	9.80325	10.74790	12.01704
8	9.52446	10.21886	11.03009	12.02707	13.36157
9	10.65637	11.38875	12.24215	13.28804	14.68366
10	11.78072	12.54886	13.44196	14.53394	15.98718
11	12.89867	13.70069	14.63142	15.76710	17.27501
12	14.01110	14.84540	15.81199	16.98931	18.54935
13	15.11872	15.98391	16.98480	18.20198	19.81193
14	16.22210	17.11693	18.15077	19.40624	21.06414
15	17.32169	18.24509	19.31066	20.60301	22.30713
16	18.41789	19.36886	20.46508	21.79306	23.54183
17	19.51102	20.48868	21.61456	22.97703	24.76904
18	20.60135	21.60489	22.75955	24.15547	25.98942
19	21.68913	22.71781	23.90042	25.32885	27.20357
20	22.77454	23.82769	25.03751	26.49758	28.41198
21	23.85779	24.93478	26.17100	27.66201	29.61509
22	24.93902	26.03926	27.30145	28.82245	30.81328
23	26.01836	27.14134	28.42879	29.97919	32.00690
24	27.09596	28.24115	29.55332	31.13246	33.19624
25	28.17192	29.33885	30.67520	32.28249	34.38159
26	29.24633	30.43457	31.79461	33.42947	35.56317
27	30.31929	31.52841	32.91169	34.57358	36.74122
28	31.39088	32.62049	34.02656	35.71498	37.91592
29	32.46117	33.71091	35.13936	36.85383	39.08747
30	33.53023	34.79974	36.25019	37.99025	40.25602
31	34.59813	35.88708	37.35914	39.12437	41.42174
32	35.66492	36.97298	38.46631	40.25630	42.58474
33	36.73065	38.05753	39.57179	41.38614	43.74518
34	37.79538	39.14078	40.67565	42.51399	44.90316
35	38.85914	40.22279	41.77796	43.63994	46.05879
36	39.92198	41.30362	42.87880	44.76407	47.21217
37	40.98394	42.38331	43.97822	45.88645	48.36341
38	42.04505	43.46191	45.07628	47.00717	49.51258
39	43.10535	44.53946	46.17304	48.12628	50.65977
40	44.16487	45.61601	47.26854	49.24385	51.80506
41	45.22363	46.69160	48.36284	50.35994	52.94851
42	46.28168	47.76625	49.45597	51.47459	54.09020
43	47.33902	48.84001	50.54799	52.58787	55.23019
44	48.39569	49.91290	51.63892	53.69982	56.36854
45	49.45171	50.98495	52.72882	54.81048	57.50531
46	50.50711	52.05619	53.81770	55.91991	58.64054
47	51.56189	53.12666	54.90561	57.02814	59.77429
48	52.61609	54.19636	55.99258	58.13520	60.90661
49	53.66972	55.26534	57.07863	59.24114	62.03754
50	54.72279	56.33361	58.16380	60.34599	63.16712

Table 5B. Critical values of chi-square test statistic: α = 0.30, 0.25, 0.20, 0.15, 0.10; ν = 51–100.

ν \ α	0.30	0.25	0.20	0.15	0.10
51	55.77534	57.40118	59.24811	61.44979	64.29540
52	56.82736	58.46809	60.33158	62.55256	65.42241
53	57.87888	59.53435	61.41425	63.65433	66.54820
54	58.92990	60.59998	62.49613	64.75514	67.67279
55	59.98045	61.66500	63.57724	65.85500	68.79621
56	61.03054	62.72942	64.65762	66.95396	69.91851
57	62.08018	63.79326	65.73727	68.05202	71.03971
58	63.12938	64.85654	66.81621	69.14921	72.15984
59	64.17815	65.91927	67.89448	70.24556	73.27893
60	65.22651	66.98146	68.97207	71.34110	74.39701
61	66.27446	68.04313	70.04901	72.43582	75.51409
62	67.32201	69.10429	71.12532	73.52977	76.63021
63	68.36918	70.16496	72.20101	74.62296	77.74538
64	69.41597	71.22514	73.27609	75.71540	78.85964
65	70.46239	72.28485	74.35058	76.80711	79.97300
66	71.50846	73.34409	75.42450	77.89812	81.08549
67	72.55417	74.40289	76.49785	78.98843	82.19711
68	73.59954	75.46124	77.57065	80.07806	83.30790
69	74.64457	76.51916	78.64291	81.16703	84.41787
70	75.68928	77.57666	79.71465	83.25535	85.52704
71	76.73366	78.63374	80.78587	83.34304	86.63543
72	77.77773	79.69042	81.85659	84.43011	87.74305
73	78.82149	80.74670	82.92681	85.51656	88.84992
74	79.86495	81.80260	83.99655	86.60243	89.95605
75	80.90812	82.85812	85.06581	87.68771	91.06146
76	81.95099	83.91326	86.13461	88.77242	92.16617
77	82.99358	84.96804	87.20296	89.85656	93.27018
78	84.03590	86.02246	88.27086	90.94016	94.37352
79	85.07794	87.07653	89.33832	92.02322	95.47619
80	86.11971	88.13026	90.40535	93.10576	96.57820
81	87.16122	89.18365	91.47196	94.18777	97.67958
82	88.20247	90.23670	92.53816	95.26927	98.78033
83	89.24347	91.28944	93.60395	96.35028	99.88046
84	90.28422	92.34185	94.66934	97.43079	100.97999
85	91.32473	93.39395	95.73434	98.51083	102.07892
86	92.36500	94.44574	96.79896	99.59039	103.17726
87	93.40504	95.49723	97.86320	100.66948	104.27504
88	94.44484	96.54842	98.92707	101.74812	105.37225
89	95.48441	97.59932	99.99058	102.82631	106.46890
90	96.52376	98.64993	101.05372	103.90406	107.56501
91	97.56289	99.70026	102.11652	104.98138	108.66058
92	98.60181	100.75031	103.17896	106.05827	109.75563
93	99.64051	101.80009	104.24107	107.13474	110.85015
94	100.67901	102.84960	105.30284	108.21079	111.94417
95	101.71729	103.89884	106.36428	109.28644	113.03769
96	102.75538	104.94783	107.42540	110.36169	114.13071
97	103.79327	105.99656	108.48619	111.43655	115.22324
98	104.83096	107.04503	109.54668	112.51102	116.31530
99	105.86845	108.09326	110.60685	113.58511	117.40688
100	106.90576	109.14124	111.66671	114.65882	118.49800

Table 5C. Critical values of chi-square test statistic:
α = 0.05, 0.025, 0.01, 0.005, 0.001; ν = 1–50.

ν \ α	0.05	0.025	0.01	0.005	0.001
1	3.84146	5.02389	6.63490	7.87944	10.82757
2	5.99146	7.37776	9.21034	10.59664	13.81551
3	7.81473	9.34840	11.34487	12.83816	16.26624
4	9.48773	11.14329	13.27670	14.86026	18.46683
5	11.07050	12.83250	15.08627	16.74960	20.51501
6	12.59159	14.44938	16.81189	18.54758	22.45774
7	14.06714	16.01276	18.47531	20.27774	24.32188
8	15.50731	17.53455	20.09024	21.95496	26.12448
9	16.91898	19.02277	21.66599	23.58935	27.87716
10	18.30704	20.48318	23.20925	25.18818	29.58830
11	19.67514	21.92005	24.72497	26.75685	31.26413
12	21.02607	23.33666	26.21697	28.29952	32.90949
13	22.36203	24.73561	27.68825	29.81947	34.52818
14	23.68479	26.11895	29.14124	31.31935	36.12327
15	24.99579	27.48839	30.57791	32.80132	37.69730
16	26.29623	28.84535	31.99993	34.26719	39.25236
17	27.58711	30.19101	33.40866	35.71847	40.79022
18	28.86930	31.52638	34.80531	37.15645	42.31240
19	30.14353	32.85233	36.19087	38.58226	43.82020
20	31.41043	34.16961	37.56624	39.99685	45.31475
21	32.67057	35.47888	38.93217	41.40106	46.79704
22	33.92444	36.78071	40.28936	42.79566	48.26794
23	35.17246	38.07563	41.63840	44.18128	49.72823
24	36.41503	39.36408	42.97982	45.55851	51.17860
25	37.65248	40.64647	44.31411	46.92789	52.61966
26	38.88514	41.92317	45.64168	48.28988	54.05196
27	40.11327	43.19451	46.96294	49.64492	55.47602
28	41.33714	44.46079	48.27824	50.99338	56.89229
29	42.55697	45.72229	49.58788	52.33562	58.30117
30	43.77297	46.97924	50.89218	53.67196	59.70306
31	44.98534	48.23189	52.19140	55.00270	61.09831
32	46.19426	49.48044	53.48577	56.32812	62.48722
33	47.39988	50.72508	54.77554	57.64845	63.87010
34	48.60237	51.96600	56.06091	58.96393	65.24722
35	49.80185	53.20335	57.34207	60.27477	66.61883
36	50.99846	54.43729	58.61922	61.58118	67.98517
37	52.19232	55.66797	59.89250	62.88334	69.34645
38	53.38354	56.89552	61.16209	64.18141	70.70289
39	54.57223	58.12006	62.42812	65.47557	72.05466
40	55.75848	59.34171	63.69074	66.76596	73.40196
41	56.94239	60.56057	64.95007	68.05273	74.74494
42	58.12404	61.77676	66.20624	69.33600	76.08376
43	59.30351	62.99036	67.45935	70.61590	77.41858
44	60.48089	64.20146	68.70951	71.89255	78.74952
45	61.65623	65.41016	69.95683	73.16606	80.07673
46	62.82962	66.61653	71.20140	74.43654	81.40033
47	64.00111	67.82065	72.44331	75.70407	82.72042
48	65.17077	69.02259	73.68264	76.96877	84.03713
49	66.33865	70.22241	74.91947	78.23071	85.35056
50	67.50481	71.42020	76.15389	79.48998	86.66082

Table 5D. Critical values of chi-square test statistic:
α = 0.05, 0.025, 0.01, 0.005, 0.001; ν = 51–100.

ν \ α	0.05	0.025	0.01	0.005	0.001
51	68.66929	72.61599	77.38596	80.74666	87.96798
52	69.83216	73.80986	78.61576	82.00083	89.27215
53	70.99345	75.00186	79.84334	83.25255	90.57341
54	72.15322	76.19205	81.06877	84.50191	91.87185
55	73.31149	77.38047	82.29212	85.74895	93.16753
56	74.46832	78.56716	83.51343	86.99376	94.46055
57	75.62375	79.75219	84.73277	88.23638	95.75096
58	76.77780	80.93559	85.95018	89.47687	97.03883
59	77.93052	82.11741	87.16571	90.71529	98.32423
60	79.08194	83.29768	88.37942	91.95170	99.60723
61	80.23210	84.47644	89.59134	93.18614	100.88789
62	81.38102	85.65373	90.80153	94.41865	102.16625
63	82.52873	86.82959	92.01002	95.64930	103.44238
64	83.67526	88.00405	93.21686	96.87812	104.71633
65	84.82065	89.17714	94.42208	98.10514	105.98815
66	85.96491	90.34890	95.62572	99.33043	107.25788
67	87.10807	91.51936	96.82782	100.55401	108.52558
68	88.25016	92.68854	98.02840	101.77592	109.79130
69	89.39121	93.85647	99.22752	102.99621	111.05506
70	90.53123	95.02318	100.42518	104.21490	112.31694
71	91.67024	96.18870	101.62144	105.43203	113.57694
72	92.80827	97.35306	102.81631	106.64763	114.83512
73	93.94534	98.51626	104.00984	107.86174	116.09151
74	95.08147	99.67835	105.20203	109.07438	117.34616
75	96.21667	100.83934	106.39292	110.28558	118.59909
76	97.35097	101.99925	107.58254	111.49538	119.85035
77	98.48438	103.15811	108.77092	112.70380	121.09996
78	99.61693	104.31594	109.95807	113.91087	122.34795
79	100.74862	105.47275	111.14402	115.11661	123.59436
80	101.87947	106.62857	112.32879	116.32106	124.83922
81	103.00951	107.78341	113.51241	117.52422	126.08256
82	104.13874	108.93729	114.69490	118.72614	127.32440
83	105.26718	110.09024	115.87627	119.92682	128.56476
84	106.39484	111.24226	117.05654	121.12629	129.80369
85	107.52174	112.39337	118.23575	122.32458	131.04120
86	108.64789	113.54360	119.41390	123.52170	132.27731
87	109.77331	114.69295	120.59102	124.71768	133.51207
88	110.89800	115.84144	121.76711	125.91254	134.74549
89	112.02199	116.98908	122.94221	127.10628	135.97757
90	113.14527	118.13589	124.11632	128.29894	137.20836
91	114.26787	119.28189	125.28946	129.49053	138.43786
92	115.38979	120.42708	126.46166	130.68107	139.66612
93	116.51105	121.57148	127.63291	131.87058	140.89313
94	117.63165	122.71511	128.80325	133.05906	142.11894
95	118.75161	123.85797	129.97268	134.24655	143.34354
96	119.87094	125.00007	131.14122	135.43305	144.56696
97	120.98964	126.14144	132.30888	136.61858	145.78923
98	122.10774	127.28207	133.47567	137.80315	147.01036
99	123.22522	128.42199	134.64162	138.98678	148.23036
100	124.34211	129.56120	135.80672	140.16949	149.44925

$$W = \sum_{i=1}^{N} ln(T/X_i) \quad . \tag{30}$$

The probability density function of the time measurements X_1, X_2, X_3, ..., X_N, following the time-to-failure Weibull process, is given by

$$f(x_1,x_2,x_3,\ldots,x_n,\lambda,\beta) = e^{-\lambda T^\beta}\left(\lambda^n \beta^n \prod_{i=1}^{n} x_i^{\beta-1}\right) \quad . \tag{31}$$

The likelihood function, which is the mathematical expression of the probability of obtaining the observed data, is

$$L = \prod_{i=1}^{n} f(x_1,x_2,x_3,\ldots x_n,\lambda,\beta) \quad , \tag{32}$$

which is acquired by equation (31). The maximum likelihood estimates λ and β can be obtained from equations (11) and (14).

Equation (31) indicates that (N,W) are sufficient statistics for (λ,β) with W from equation (30). Thus, for placing confidence bounds on current and projected failure rates and the MTBF, $M(T)=T/\beta\theta$, where $\theta = \lambda T^\beta$, the joint probability density function of (N,W), given $W>0$, is needed to be determined. Since the number of failures in the time interval is Poisson distributed with mean θ, the mathematical induction from equations (4) and (31) results in that the conditional probability density function of jointly distributed random variables (X_1, X_2, X_3, ..., X_N), under the condition that $N=n$, becomes

$$f(x_1,x_2,x_3,\ldots,x_n,\ \beta|N=n) = n! \prod_{i=1}^{n} \beta\ x_i^{\beta-1}/T^\beta \quad . \tag{33}$$

In equation (11), the ordered times $x_1,x_2,x_3,\ldots,x_n$, conditioned on $N=n$, are distributed as the order statistics for a random sample of size n arising from a distribution having a probability density function

$$H(\cdot) = \beta\ x^{\beta-1}/T^\beta \quad . \tag{34}$$

Equation (12) is derived from equation (33).

Accordingly, if X_1, X_2, X_3, ..., X_N are independent and identically exponential random variables with parameter β, the probability density function of W, given $N=n>1$, follows the gamma probability law given by

$$q(w,\beta|N=n) = \beta(\beta w)^{n-1} e^{-\beta w}/(n-1)! \ , \qquad w>0 \ , \ n=1,2,3,\dots \ . \tag{35}$$

Now, let $\phi = T/M(T)$. The joint probability density function of (N,W) is computed to be, on the condition that $W=0$,

$$p(n,w,\phi,\beta) = \frac{\phi^n w^{n-1} e^{-\beta w-\phi/\beta}}{n!(n-1)! \ (1-e^{-\phi/\beta})} \ , \tag{36}$$

with the conditions of $w>0$, $n=1,2,3,\dots$.

Placing confidence bounds on ϕ and hence on $M(T)$, equation (36) is identified as a member of the multiparameter exponential family, according to the sufficient statistics (N,W)). Determination of confidence bounds on ϕ constitutes the distribution of N, given $W=w>0$. From equation (36), the probability density function of N, given $W=w>0$, is expressed by

$$p(n,\phi|W=w) = \frac{\dfrac{\phi^n w^{n-1} e^{-\beta w-\phi/\beta}}{n!(n-1)!(1-e^{-\phi/\beta})}}{\displaystyle\sum_{j=1}^{\infty} \frac{\phi^j w^{j-1} e^{-\beta w-\phi/\beta}}{j!(j-1)! \ (1-e^{-\phi/\beta})}} = \frac{(w\phi)^{n-1/2}}{n!(n-1)! \ I_1(2\sqrt{w\phi})} \ , \tag{37}$$

where $I_1(2\sqrt{w\phi})$ is the modified bessel function of order 1.

Having considered, from equation (37), the conditions of $N=n$, $W=w>0$, a lower $(1-\alpha)100$-percent confidence bound for the parameter $\phi = T/M(T) = \lambda\beta T^\beta$ is the value ϕ_1 satisfying

$$\sum_{j=n}^{\infty} \frac{(w\phi_1)^{j-1/2}}{j!(j-1)! \ I_1(2\sqrt{w\phi_1})} = \alpha \ . \tag{38}$$

Likewise, an upper $(1-\alpha)100$-percent confidence bound on ϕ is the value ϕ_2 satisfying

$$\sum_{j=1}^{n} \frac{(w\phi_2)^{j-1/2}}{j!(j-1)! \ I_1(2\sqrt{w\phi_2})} = \alpha \ . \tag{39}$$

Equations (38) and (39) have been programmed into the computer programs for the time-terminated testing case.

Failure-Terminated Testing

Now, for the failure-terminated testing case, let cumulative test time $t = X$. In the data synthesis, tests are terminated upon the accumulation of a specified number of failures. The test data emerging from the Weibull process consist of the first n successive failure times $X_1, X_2, X_3, \ldots, X_n$.

As before, the statistical inference procedures include the method of maximum likelihood for computation of point estimates of the β parameter

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n-1} ln\left(\frac{x_n}{x_i}\right)} \quad , \tag{19}$$

and the λ parameter

$$\hat{\lambda} = \frac{n}{x_n^{\hat{\beta}}} \quad . \tag{18}$$

It is noted that equations (18) and (19) are equivalent to the estimates for time-terminated testing (equations (11) and (14)) with the test time equal to the time of last failure occurrence. $M(X_n)$, achieved MTBF of the system, is estimated by

$$\hat{M}(X_n) = \frac{1}{\hat{u}(x_n)} = \frac{1}{\hat{\lambda}\hat{\beta}X_n^{\hat{\beta}-1}} = \frac{X_n}{n\,\hat{\beta}} \quad , \tag{40}$$

where $n = \hat{\lambda} X_n^{\hat{\beta}}$ from equation (18).

If no further improvements are planned at time x_n, the system is assumed to have a constant failure rate which takes the current value of the intensity function $u(x_n)$. Consequently, the system life length has an exponential distribution with mean $u^{-1}(x_n)$, and the current system reliability is represented by $\exp[-u(x_n)t_o]$ for some time interval $(0, t_o)$ of its useful life.

The equation required for computing the percentage-point probabilities for the failure-terminated testing is expressed as a measure of the achieved MTBF to the instantaneous MTBF by

$$n^2\,\hat{M}(X_n)/M(X_n) = V_{n-1} * V_n \quad , \tag{41}$$

where V_{n-1} and V_n are independent random variables, which uses the formula of the gamma probability density function

$$V_r(y) = y^{r-1} \quad e^{-y}/(r-1)! \;, \qquad y>0 \;, \qquad r=1,2,3,\ldots \;. \tag{42}$$

To proceed with computation of equation (41), the analysis objective is for the system failure times to follow a nonhomogeneous Poisson process $\{N(t),\ t>0\}$ with the Weibull intensity $u(t)$. This means that the probability of a failure occurring in an infinitesimally small interval $(t,t+\Delta t)$ is approximately $u(t)\ \Delta t$. In addition, for $t>s$, $N(t)-N(s)$ is equal to the number of failures that have occurred in the time interval (t,s). Thus, the solution of $N(t,s)$ has a Poisson distribution with mean $E[N(t)]-E[N(s)]$. This process inherently possesses independent increments. Therefore, based on these properties, the probability density function for the first n successive X_1, X_2, X_3, ... X_n, letting $x_o=0$, can be written as

$$b(x_1,x_2,x_3,\ldots,x_n) = \prod_{i=1}^{n} u(x_i) \prod_{i=1}^{n} \text{Prob}[N(x_i,x_{i-1}) = 0] \;. \tag{43}$$

Using the method of likelihood function, equation (43), in accord with the Weibull process, becomes

$$b(x_1,x_2,x_3,\ldots,x_n,\lambda_1,\beta) = \left(\lambda^n\ \beta^n \prod_{i=1}^{n} x_i^{\beta-1}\right)\left(e^{-\lambda x_n^\beta}\right) \;. \tag{44}$$

It can be shown that the probability density function of the random variable X is given by

$$h(x_n,\lambda,\beta) = \text{Prob}[N(x) = n-1]\ u(x) = \frac{(\lambda x_n^{\beta})^{n-1}\ e^{-\lambda x_n^\beta}}{(n-1)!}\ (\lambda\beta x_n^{\beta-1}) \;, \tag{45}$$

Now, if $x_1,x_2,x_3,\ldots x_n$ are the values of n random variables, the joint conditional probability density function of these random variables X_1, X_2, X_3, ..., X_n, given that the random variable X has an observed value equal to x, can be expressed as

$$c(x_1,x_2,x_3,\ldots,x_{n-1},\beta|X_n=x_n) = \frac{\text{Equation (44)}}{\text{Equation (45)}} = (n-1)! \prod_{i=1}^{n-1} \beta\ \frac{x_i^{\beta-1}}{x_n^{\beta}} \;. \tag{46}$$

Conditioning on $X_n=x_n$, the times X_1, X_2, X_3,..., X_{n-1}, at which failures occur, considered as unordered random variables, are distributed as $n-1$ independent continuous random variables with common probability density function as obtained from the right-hand side of equation (46):

$$\frac{\beta\ x^{\beta-1}}{x_n^{\beta}} \;. \tag{47}$$

Equation (47) implies that the statistic becoming

$$Z = 2n\beta/\hat{\beta} \quad . \tag{48}$$

is distributed as a chi-square random variable with $2(n-1)$ degrees of freedom with the gamma probability density function defined by

$$g_{n-1}(x) = \frac{x^{n-2}\, e^{-x}}{(n-2)!} \quad , \tag{49}$$

independent of x_n. Also, from equation (25), the statistic becoming

$$B = 2\lambda x_n{}^{\beta} \quad , \tag{50}$$

is the chi-square distribution with $2n$ degrees of freedom with the gamma probability density function defined by

$$g_n(x) = \frac{x^{n-1}\, e^{-x}}{(n-1)!} \quad . \tag{51}$$

Unknown parameters λ and β give rise that the inference procedures can be based on the quantization of equation (48). If the parameter β is known, then the statistic of equation (50) would be used to test hypotheses and construct confidence bounds on the parameter λ.

Since equation (41) can be used to obtain confidence limits on the current system reliability or the MTBF, the resulting ratio of the true MTBF to the estimated MTBF is determined to be

$$R = u(X_n)/\hat{u}(X_n) \quad , \tag{52}$$

where $\hat{u}(X_n) = \hat{\lambda}\hat{\beta}x_n{}^{\hat{\beta}-1}$; $\hat{u}(X_n) = n\hat{\beta}x_n{}^{-1}$ is the maximum likelihood estimate of $u(X_n)$ based on the times $X_1, X_2, X_3, \ldots, X_n$. Combining the two statistics from equations (48) and (50) into equation (52) yields the mathematical representation

$$R = [1/(4n^2)]\; Z\, B \quad . \tag{53}$$

The cumulative distribution function of equation (53) is found by the formula

$$P[R \leq t] = \int_0^\infty G\left(\frac{2n^2 t}{z}\right) g(z)\, dz \quad , \tag{54}$$

which can be computed, using the following relevant relations:

the incomplete gamma function

$$G(x) = \frac{1}{(n-1)!} \int_0^x y^{n-1}\, e^{-y}\, dy \ , \tag{55}$$

and the chi-square density

$$g(z) = \frac{z^{n-2}\, e^{-z/2}}{2^{n-1}\, (n-2)!} \quad . \tag{56}$$

Equation (54) is specialized to the failure-terminated testing case.

NUMERICAL RESULTS

After the tables for confidence interval estimation and goodness-of-fit tests have been calculated and tabulated, the reliability growth assessment takes place for the five groupings of SSME failure data. The five groupings are as follows with designated tables:

Case	SSME Grouping	Table
A	Summary Engine Failures; N=24	6
B	SSME Major Incidents; N=27	7
C	Turbopump Vibration Incidents; N=38	8
D	High Pressure Turbopump Failures; N=56	9
E	Engine Failures by Subsystem; N=56	10

Several algorithms for numerically integrating and iteratively root-solving the equations arising in the percentage-point probability analysis of the engine failure data have been written to augment the program simulation. Although any one of those algorithms can be used, the Pegasus method with an estimated order of convergence superior to a secant method and the 20-point Gaussian quadrature procedure, which is considered more accurate, have been used for numerical solutions.

Table 6. SSME test hisitory: summary engine failures; $N = 24$.

Number	Test Number	Engine	Date
1	901.110	0003	24 March 1977
2	901.136	0004	8 September 1977
3	901.147	0103	1 December 1977
4	901.173	0002	31 March 1978
5	901.183	0005	5 June 1978
6	902.120	0101	18 July 1978
7	902.132	0006	3 October 1978
8	901.222	0007	5 December 1978
9	901.225	2001	27 December 1978
10	750.040	0201	14 May 1979
11	SF6.010	2002	2 July 1979
12	SF6.030	2002	4 November 1979
13	SF10.010	0006	12 July 1980
14	902.198	2004	23 July 1980
15	901.284	0010	30 July 1980
16	901.307	0009	29 January 1981
17	902.244	0204	14 July 1981
18	901.331	2108	15 July 1981
19	750.148	0110	2 September 1981
20	902.249	0204	21 September 1981
21	750.160	0110F	12 February 1982
22	901.364	2013	7 April 1982
23	750.168	0107	15 May 1982
24	750.175	2208	27 August 1982

Table 7. SSME test history: SSME major incidents; $N = 27$.

Number	Test Number	Engine	Date
1	740.007		4 February 1976
2	901.110	0003	24 March 1977
3	901.136	0004	9 September 1977
4	902.095	0002	17 November 1977
5	901.147	0103	1 December 1977
6	901.173	0002	31 March 1978
7	901.183	0005	5 June 1978
8	902.120	0101	18 July 1978
9	902.132	0006	3 October 1978
10	901.222	0007	6 December 1978
11	901.225	2001	27 December 1978
12	750.041	0201	14 May 1979
13	SF006.001	2002	4 July 1979
14	SF006.003	2002	4 November 1979
15	SF010.001	0006	12 July 1980
16	902.198	2004	23 July 1980
17	901.284	0010	30 July 1980
18	901.307	0009	28 January 1981
* 19	901.331	2108	15 July 1981
* 20	750.140	0110	20 June 1981
21	902.249	0204	21 September 1981
22	901.340	0107	15 October 1981
23	901.364	2013	7 April 1982
24	750.165	0107	21 April 1982
25	750.175	2208	27 August 1982
26	901.436	0108	14 February 1984
27	750.259	2308	27 March 1985

*Note discrepancy of ordered dates when the data were received.

Table 8. SSME test history: turbopump vibration incidents; $N = 38$.

Number	Test Number	Engine	Date
1	901.103	0003	18 February 1977
2	901.105	0003	3 March 1977
3	901.110	0003	24 March 1977
4	901.136	0004	8 September 1977
5	901.139	0103	7 November 1977
6	902.095	0002	17 November 1977
7	901.147	0103	1 December 1977
8	902.100	2002	2 February 1978
9	901.160	0002	12 February 1978
10	901.161	0002	14 February 1978
11	901.162	0002	15 February 1978
12	901.163	0002	17 February 1978
13	901.169	0002	21 March 1978
14	902.109	0101	21 April 1978
15	901.182	0005	2 June 1978
16	901.183	0005	5 June 1978
17	902.111	0101	8 June 1978
18	902.114	0101	24 June 1978
19	902.116	0101	30 June 1978
20	902.120	0101	18 July 1978
21	902.127	2002	5 September 1978
22	902.143	2002	3 December 1978
23	902.144	2002	4 December 1978
24	902.145	2002	8 December 1978
25	901.227	2003	3 March 1979
26	901.284	0010	30 July 1980
27	902.249	0204	21 September 1981
28	901.353	0107	14 January 1982
29	901.356	0107	25 January 1982
30	901.364	2013	7 April 1982
31	750.173	2208	18 August 1982
32	750.175	2208	27 August 1982
33	901.393	2012	21 October 1982
34	902.306	2017	13 January 1983
35	901.421	2012	25 September 1983
36	901.436	0108	14 February 1984
37	750.245	2308	23 August 1984
38	750.259	2308	27 March 1985

Table 9. SSME test history: high pressure turbopump failures; $N = 56$.

Number	Test. Number	Engine	Date
1	901.096	0003	11 January 1977
2	901.103	0003	18 February 1977
3	901.105	0003	3 March 1977
4	901.110	0003	24 March 1977
5	902.059	0002	27 April 1977
6	902.062	0002	5 May 1977
7	901.124	0004	25 July 1977
8	901.136	0004	8 September 1977
9	901.139	0103	7 November 1977
10	902.095	0002	17 November 1977
11	901.147	0103	1 December 1977
12	901.150	0002	15 December 1977
13	902.100	2002	2 February 1978
14	901.160	0002	12 February 1978
15	901.161	0002	14 February 1978
16	901.162	0002	15 February 1978
17	901.163	0002	17 February 1978
18	901.169	0002	21 March 1978
19	902.107	0101	16 April 1978
20	902.108	0101	19 April 1978
21	902.109	0101	21 April 1978
22	901.178	0005	13 May 1978
23	901.182	0005	2 June 1978
24	901.183	0005	5 June 1978
25	901.111	0101	8 June 1978
26	902.114	0101	24 June 1978
27	902.116	0101	30 June 1978
28	902.118	0101	10 July 1978
29	902.120	0101	18 July 1978
30	902.189	0005	23 August 1978
31	902.127	2002	5 September 1978
32	901.208	0005	17 October 1978
33	902.143	2002	3 December 1978
34	902.144	2002	4 December 1978
35	902.145	2002	8 December 1978
36	901.227	2003	3 March 1979
37	901.250	0007	11 August 1979
38	SF0603	0006	4 November 1979
39	SF0901	2003	16 April 1980
40	901.284	0010	30 July 1980
41	902.249	0204	21 September 1981
42	901.340	0107	15 October 1981
43	901.353	0107	14 January 1982
44	901.356	0107	25 January 1982
45	901.364	2013	7 April 1982
46	750.173	2208	18 August 1982
47	750.175	2208	27 August 1982
48	902.297	2015	30 September 1982
49	901.393	2012	21 October 1982
50	902.302	2016	15 November 1982
51	902.306	2017	13 January 1983
52	750.194	2308	14 April 1983
53	901.421	2010	25 September 1983
54	901.436	0108	14 February 1984
55	750.245	2308	23 August 1984
56	750.259	2308	27 March 1985

Table 10. SSME test history: engine failures by subsystem; $N = 56$.

Number	Test Number	Engine	Date
1	902.157	2004	10 May 1979
2	902.158	2004	22 May 1979
3	902.162	2004	13 June 1979
4	901.246	2007	12 July 1979
5	902.172	2004	6 September 1979
6	902.173	2004	8 September 1979
7	902.174	2004	17 September 1979
8	901.269	0009	15 February 1980
9	902.198	2004	23 July 1980
10	901.284	0010	30 July 1980
11	902.226	0204	15 April 1981
12	902.227	0204	17 April 1981
13	901.331	2108	15 July 1981
14	750.148	0110	2 September 1981
15	902.249	0204	21 September 1981
16	901.338	0107	9 October 1981
17	901.340	0107	15 October 1981
18	750.151	0110	4 December 1981
19	901.353	0107	14 January 1982
20	901.356	0107	25 January 1982
21	750.160	0110	12 February 1982
22	750.163	0107	25 March 1982
23	901.364	2013	7 April 1982
24	902.292	2010	9 August 1982
25	750.173	2208	18 August 1982
26	750.175	2208	27 August 1982
27	902.297	2015	30 September 1982
28	901.393	2012	21 October 1982
29	902.301	2016	25 October 1982
30	902.302	2016	15 November 1982
31	902.306	2017	13 January 1983
32	750.189	2308	28 January 1983
33	750.194	2308	14 April 1983
34	901.421	2010	25 September 1983
35	901.426	2010	24 October 1983
36	750.228	2308	9 January 1984
37	902.330	2010	4 February 1984
38	901.436	0108	14 February 1984
39	750.235	2308	19 May 1984
40	STS41D	2017	26 June 1984
41	750.245	2308	23 August 1984
42	750.248	2308	18 October 1984
43	901.459	0207	26 October 1984
44	901.465	0207	19 January 1985
45	901.468	0207	4 February 1985
46	750.259	2308	27 March 1985
47	STS51F	2020	12 July 1985
48	901.485	2105	24 July 1985
49	STS51F	2023	29 July 1985
50	902.386	2026	11 December 1985
51	750.266	2025	16 October 1986
52	901.501	2105	6 November 1986
53	750.285	0210	21 May 1987
54	750.288	0210	25 June 1987
55	902.427	2106	26 June 1987
56	902.428	2106	1 July 1987

Equations (38) and (39) have been programmed into the computer programs for computation of the data of percentage-point probabilities for tabulation in tables 1A through 2C for two-sided $(1-\alpha)100$-percent confidence intervals for MTBF from time-terminated testing. First, the solutions to equations (38) and (39) are determined and then $L = n^2/$(solution to equation (39)) and $U = n^2/$(solution to equation (38)) are obtained for each α and n.

For failure-terminated testing, equation (54) is used to compute $P(R<r_\alpha)$, where r_α is the $1-\alpha$ percentile. The output results of

$$\text{Prob}[M(X_n)/\hat{M}(X_n)<u_p] = p \tag{57}$$

are tabulated in tables 1A through 2C. The data for the lower and upper confidence bounds for all tables cover the confidence coefficients of 50(5)95(3)98 percentages and the numbers of degrees of freedom of 2(1)30(5)50(10)100.

The integrand in equation (54) is numerically computed with the values approaching zero at the upper tail of the gamma distribution. The incomplete gamma function is bounded between zero and one. The computation procedure uses two methods of with or without a transformation of $y = z/2n$ as applied to shift the mean of chi-square density function close to one to generate the data of percentage-point probabilities. Comparisons of the values from the tables for confidence coefficients of 80 to 98 with those in the reference tables are favorably accurate.

The personal computers are found to be contaminated by their basic limitations, inducing production of numerical overflows. This problem is remedied by the HP-71B hand computer, which handles the greater capability in computations.

Cases A through E have been categorized from the database of the engine failures and are summarized in tables 6 through 10, using the failure days. The cumulative days have been deduced from the epoch of May 19, 1975, and are used throughout the reliability growth analyses that develop the model to reflect the actual response to the failures. Figures 2 through 6 show the scatterings of the data points indicating linear regression. Also shown on the same figures are the curve-fit data ($y = a + bx$) calculated by the method of least squares. The procedures for improving the curved lines of the curve fit have been used with many attempts to identify the distributions. Since the procedures to determine the origins of zero time obviously do not improve considerably, the efforts have been discontinued. The plots indicate increase in degree of curving with increase in $\hat{\beta}$.

The modified Cramér-von Mises test statistic C_m^2 (equation (22)) has been calculated to produce the critical values using the A Programming Language computer with the Monte Carlo method. The method is used to predict the final consequence of 15,000 samples, each having its own probability. Accordingly, the percentage points of critical-value statistic are given in tables 3A and 3B for M 2(1)20(10)30(30)60(20)100 and α = 0.30(–0.05)0.05(–0.02)0.01(–0.005) 0.005(–0.004)0.001. In addition to the Monte Carlo technique, it is possible to generate the critical

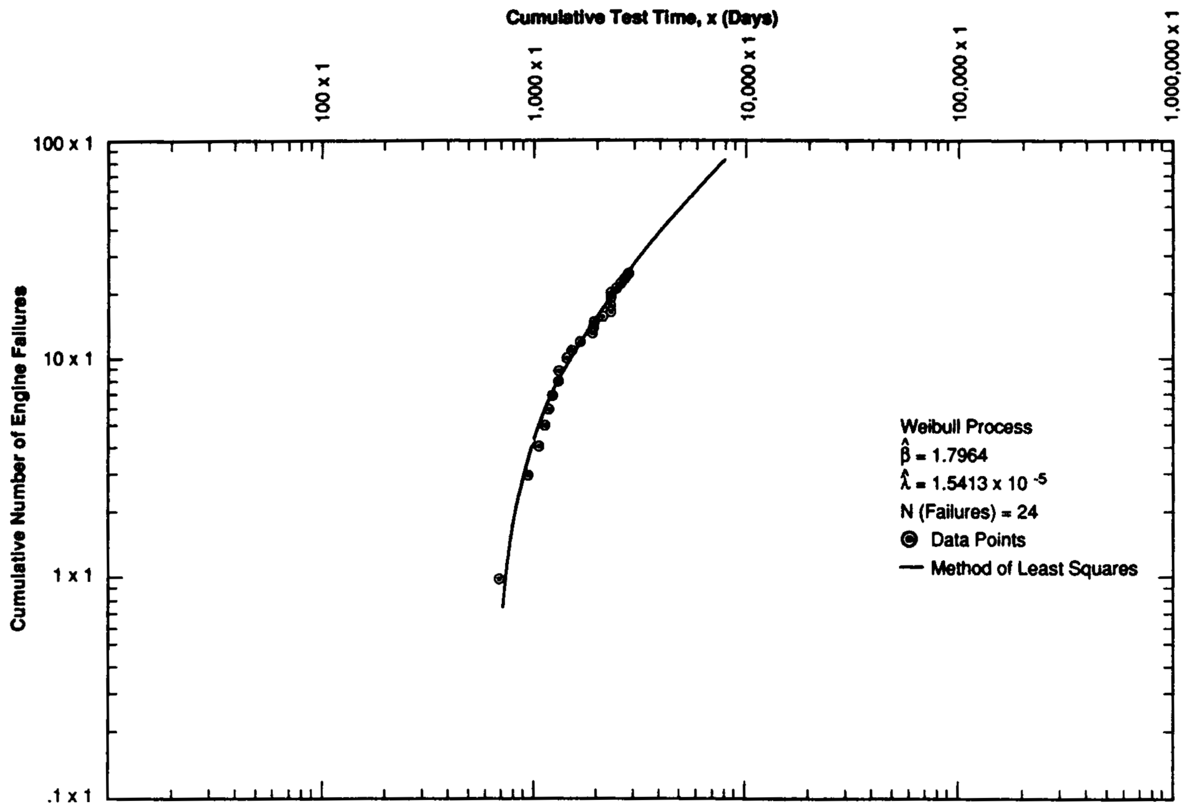

Figure 2. Diagram for least squares criterion for Weibull process for case A ($N=24$).

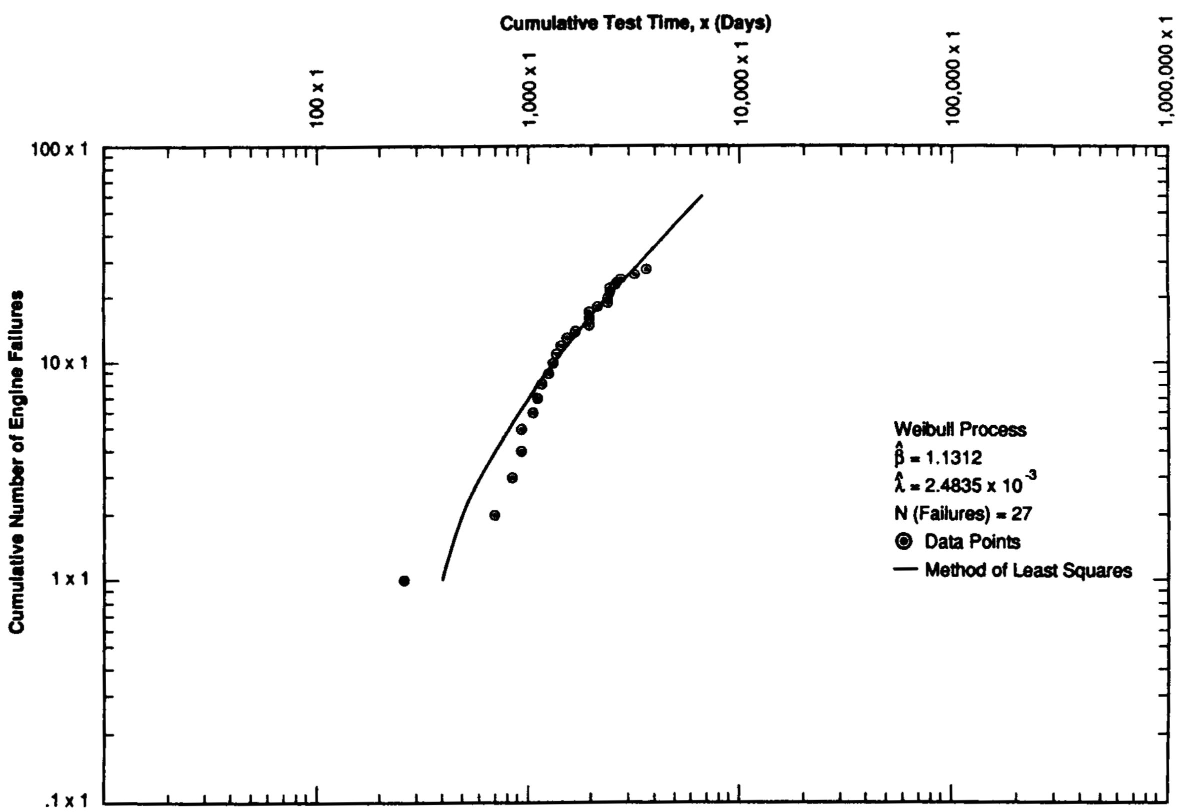

Figure 3. Diagram for least squares criterion for Weibull process for case B ($N=27$).

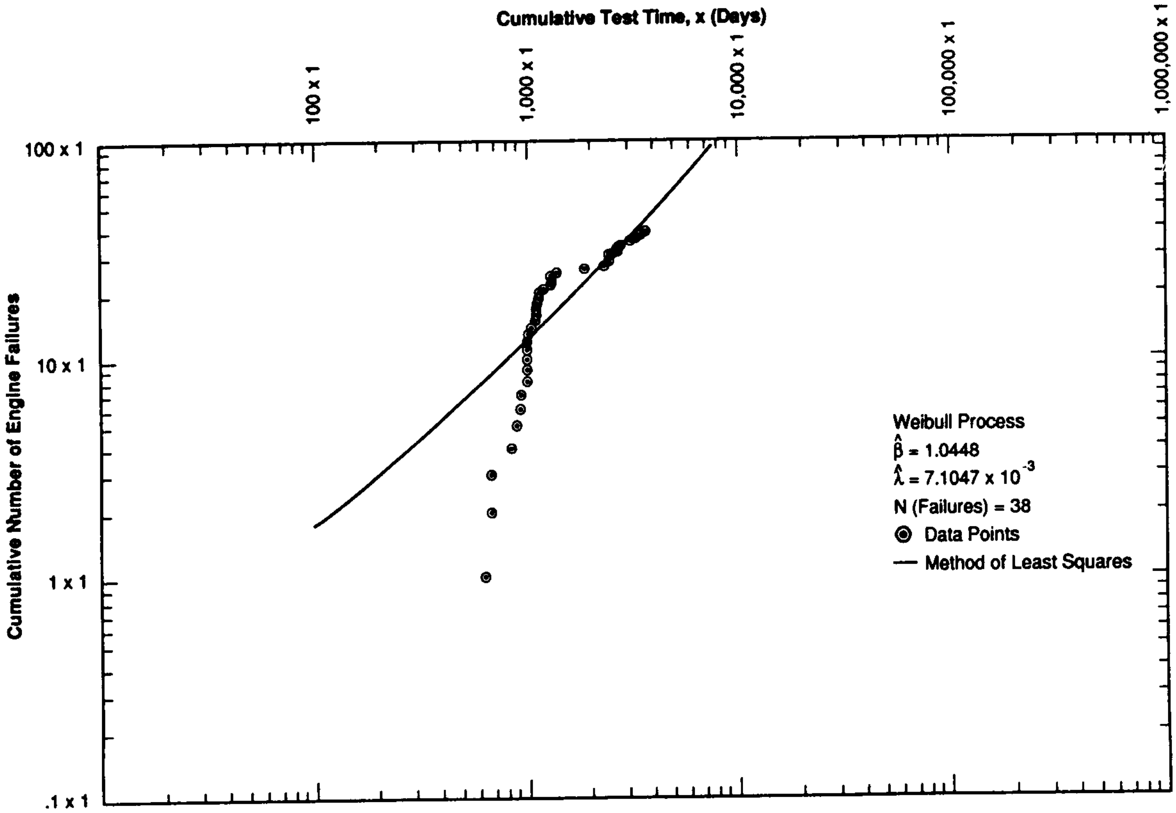

Figure 4. Diagram for least squares criterion for Weibull process for case C ($N=38$).

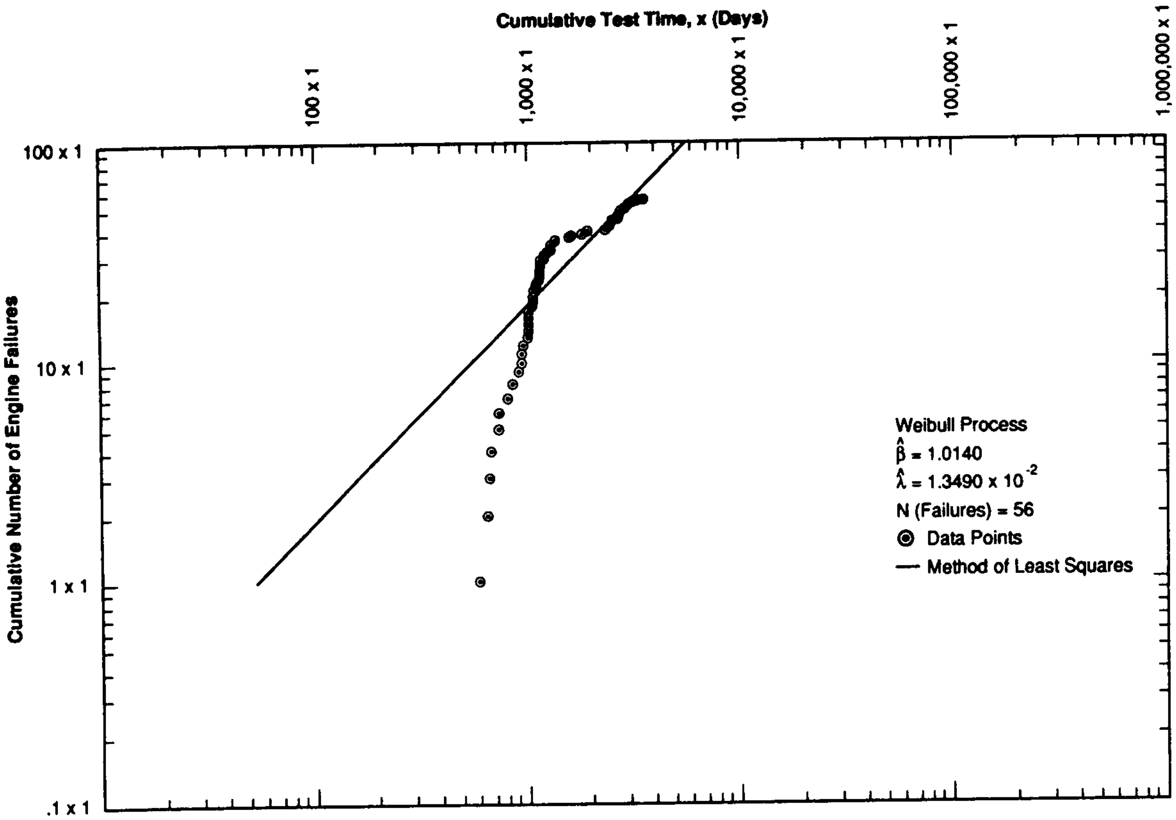

Figure 5. Diagram for least squares criterion for Weibull process for case D ($N=56$).

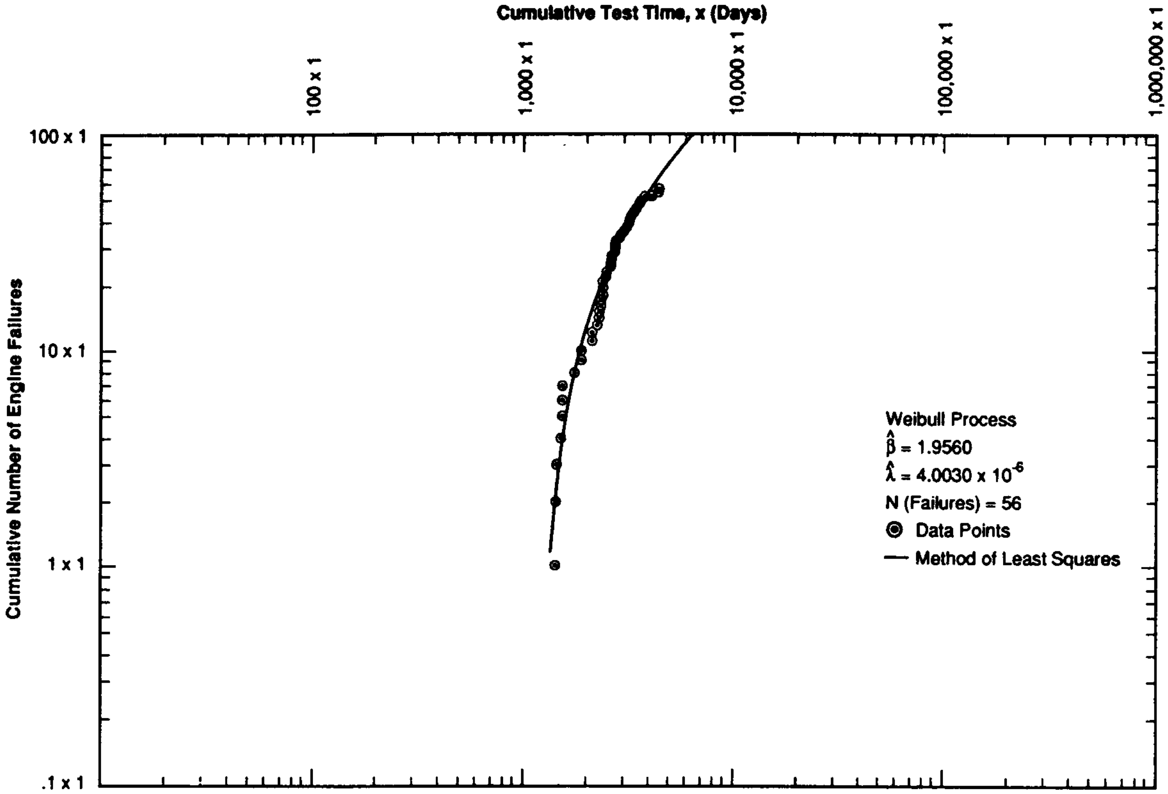

Figure 6. Diagram for least squares criterion for Weibull process for case E ($N = 56$).

values of C_m^2, using some other approaches like repetition and curve fitting using several methods such as Weibull/rank, generalized lambda for total range, log-normal distribution, and Kolomogorov-Smirnov. The programs have been developed using all above methods and the results indicate close comparisons. However, the procedure of the Monte Carlo method with the APL computer is readily adapted for computer usage for presentation of the final results.

Figure 7 shows the data of the critical values of Cramér-von Mises statistic and the asymptotic distribution of the statistic. The overall results using the statistical procedures for cases A through E are summarized in table 11 for time-terminated testing and table 10 for failure-terminated testing.

To illustrate one example of time-terminated testing using the statistical estimation procedures, the data for case A from table 6 for the conditions terminating at $T = 2{,}800$ days are used. Using equation (14), the point estimate of β is calculated to be 1.7964. The model estimates that the reliability for case A is deteriorating substantially. The estimate of the scale parameter with equation (11) is found to be 1.5413×10^{-5}. At the end of the nonfailure 2,800-day test, the estimated intensity function is 0.015397 failures per day. This corresponds to MTBF equal to 64.9463 days if there are no further changes made.

For failure-terminated testing (table 12), the data for case C from table 8 are used to determine the end conditions occurring at the last failure day of 3,384. The point estimate of β is 1.0756, according to calculation of equation (18). This calculated value indicates moderate deterioration in the proximity of no reliability change for case C. The scale parameter estimate, using equation (16), is determined to be 5.6816×10^{-3}. At $X_n = 3{,}384$ days (the last failure), the estimated

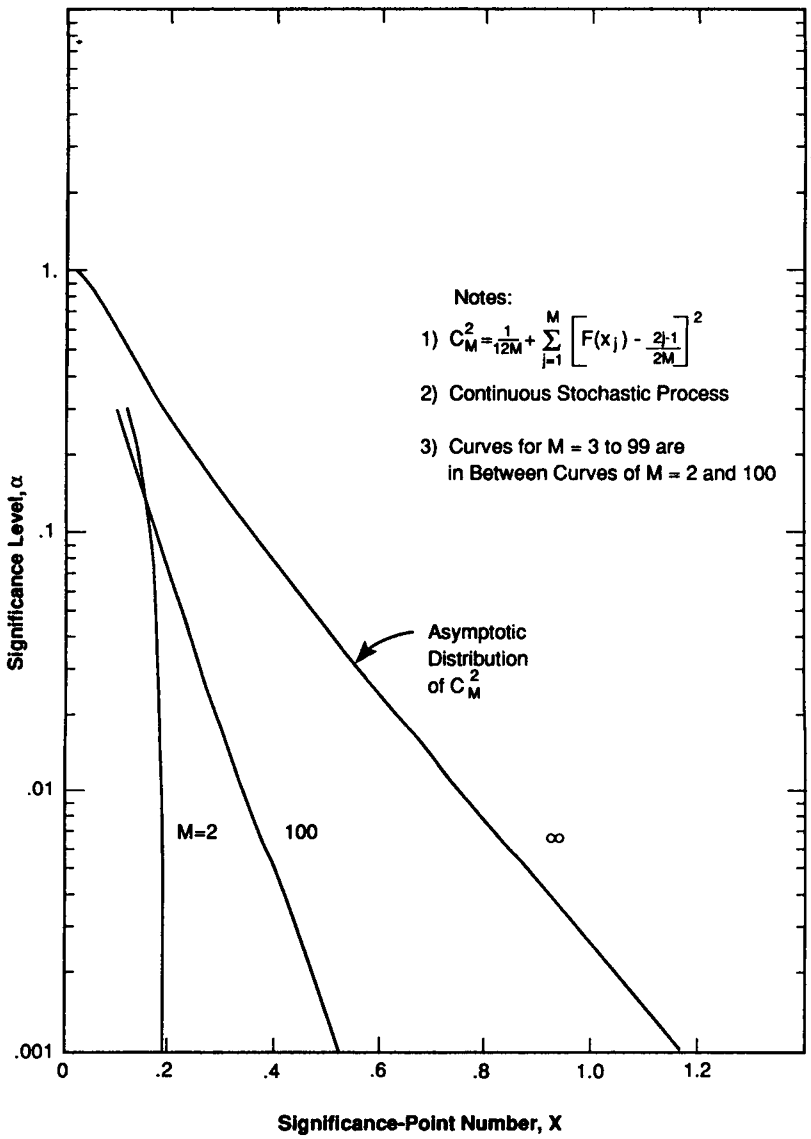

Figure 7. Critical values of asymptotic distribution of Cramér-von Mises test statistic for goodness-of-fit.

Table 11. TTT reliability results.

Time-Terminated Testing (at end of T)

Case	N	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\rho}(t)$	$\hat{m}(t)$
A	24	1.7964	1.5413×10^{-5}	1.5397×10^{-2}	64.9463
B	27	1.1312	2.4835×10^{-3}	8.2546×10^{-3}	121.1447
C	38	1.0448	7.1047×10^{-3}	1.0731×10^{-2}	93.1889
D	56	1.0140	1.3490×10^{-2}	1.5347×10^{-2}	65.1588
E	56	1.9560	4.0030×10^{-6}	2.4342×10^{-2}	41.0817

Table 12. FTT reliability results.

Failure-terminated testing (at end of X_N)

Case	N	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\rho}(t)$	$\hat{m}(t)$
A	24	1.9831	3.8842×10^{-6}	1.7913×10^{-2}	55.8259
B	27	1.1674	1.9048×10^{-3}	8.7552×10^{-3}	114.2174
C	38	1.0756	5.6816×10^{-3}	1.1354×10^{-2}	88.0746
D	56	1.0430	1.0940×10^{-2}	1.6224×10^{-2}	61.6364
E	56	2.0216	2.3846×10^{-6}	2.5578×10^{-2}	39.0956

intensity function is calculated to be 0.011354 failures per day. The point estimate of MTBF at the end of the 3,384-day test is 88.0746 days. Figures 8 through 12 for cases A, B, C, D, and E demonstrate the graphical results of equation (3) superimposed on the average failure frequencies.

The tabular values from tables 1A through 2C for time-terminated and failure-terminated testings are obtained for construction of the confidence intervals for MTBF having the degree of confidence $(1-\alpha)100$ percent. The 50 percent and 95 percent confidence intervals for MTBF are given in table 13.

For case A (time-terminated testing), a random sample of size $N=24$ is given. The interval estimate of MTBF with a 50-percent confidence coefficient is calculated to be $(0.8049/1.5397 \times 10^{-2}, 1.2927/1.5397 \times 10^{-2})$ or 52.2753 to 83.9561 days. There is a 50-percent confidence that this interval contains the MTBF calculated at the end of 2,800 days.

For case C (failure-terminated testing), the interval estimate of MTBF with a 95 percent confidence coefficient for $N=38$ is $(0.8998/1.1354 \times 10^{-2}, 1.2309/1.1354 \times 10^{-2})$ or $72.2495 <$ MTBF (days) < 108.4110. The tabular values are determined by interpolation.

To test the goodness-of-fit that the Weibull process has an appropriate model to represent the engine failure data, the levels of significance used in the analysis are 0.001, 0.005, 0.01, 0.03, 0.05, 0.10, 0.15, 0.20, 0.25, and 0.30. The analysis results are summarized in table 14.

In both cases of time- and failure-terminated testing, case A has statistics that do not exceed the critical values of the test statistic and, as a result, the null hypothesis is accepted that the model for the Weibull process is appropriate to represent the engine failure data. The results, however, indicate deterioration of the reliability for case A and also for the other cases.

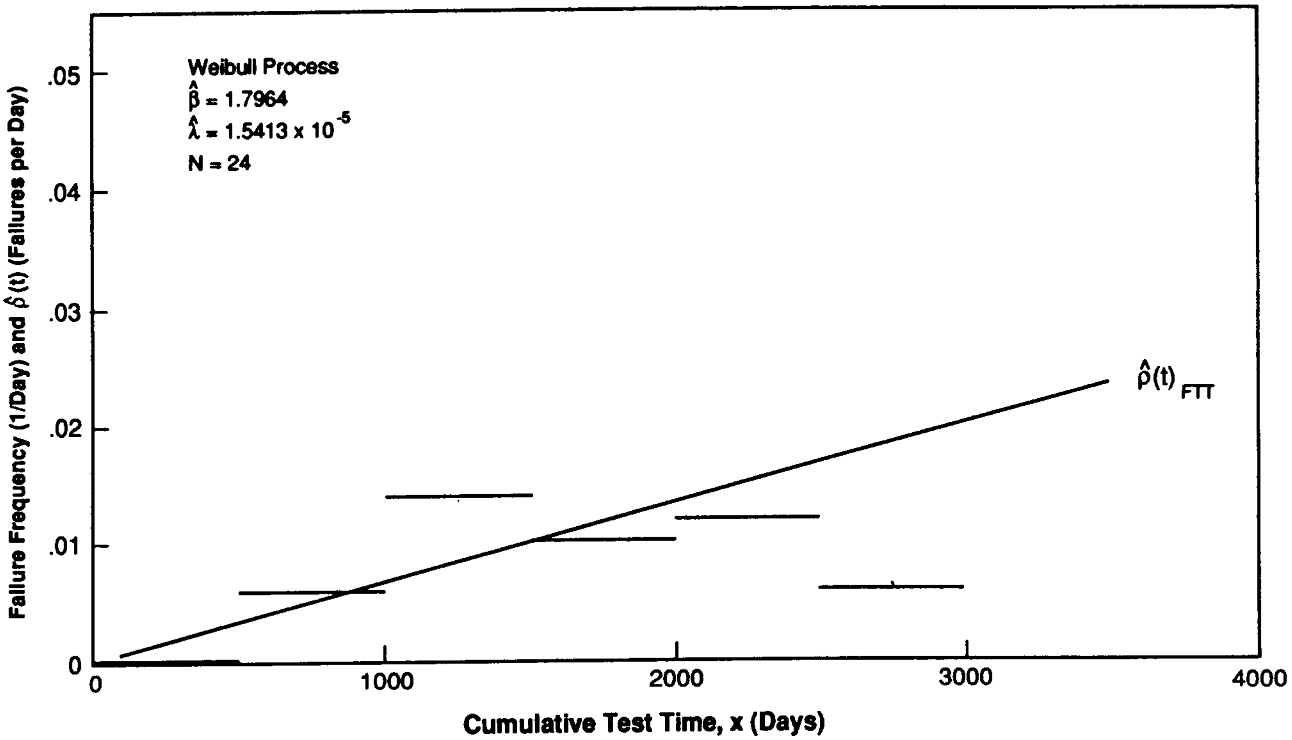

Figure 8. Estimated intensity function for case A ($N=24$).

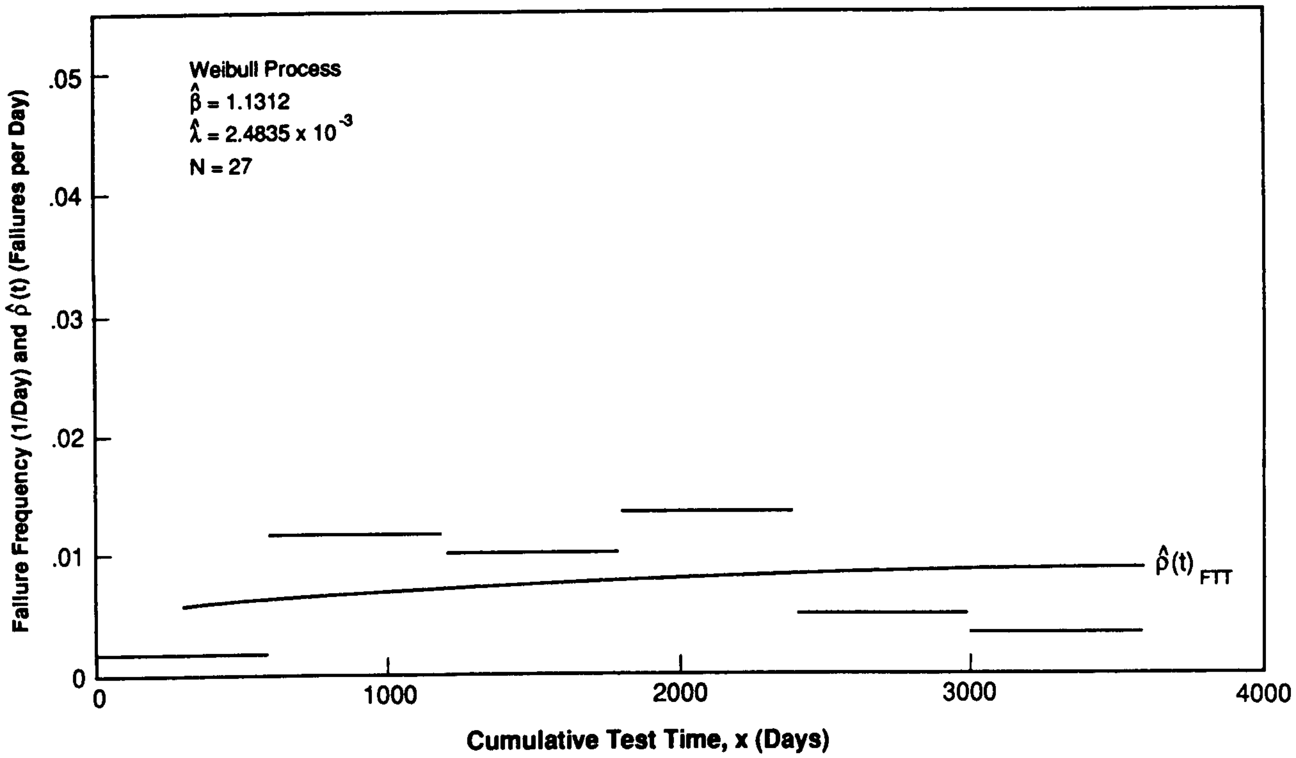

Figure 9. Estimated intensity function for case B ($N=27$).

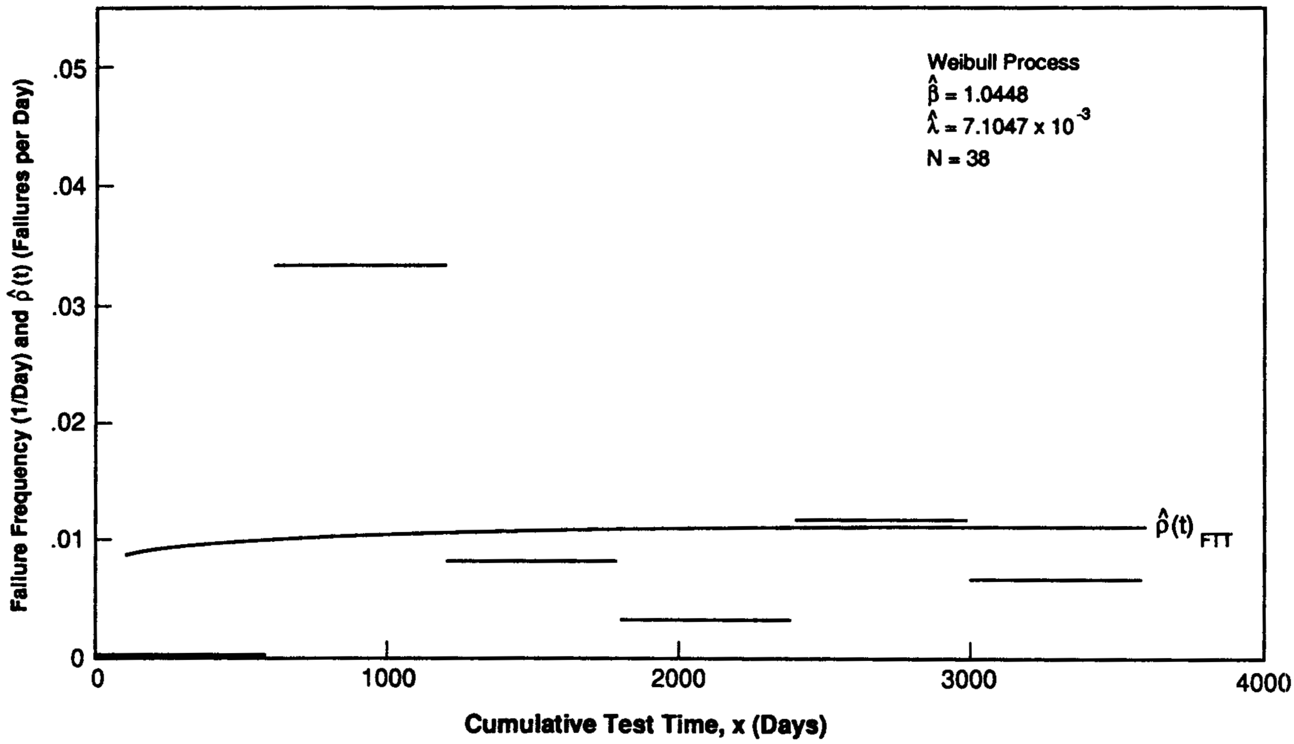

Figure 10. Estimated intensity function for case C ($N=38$).

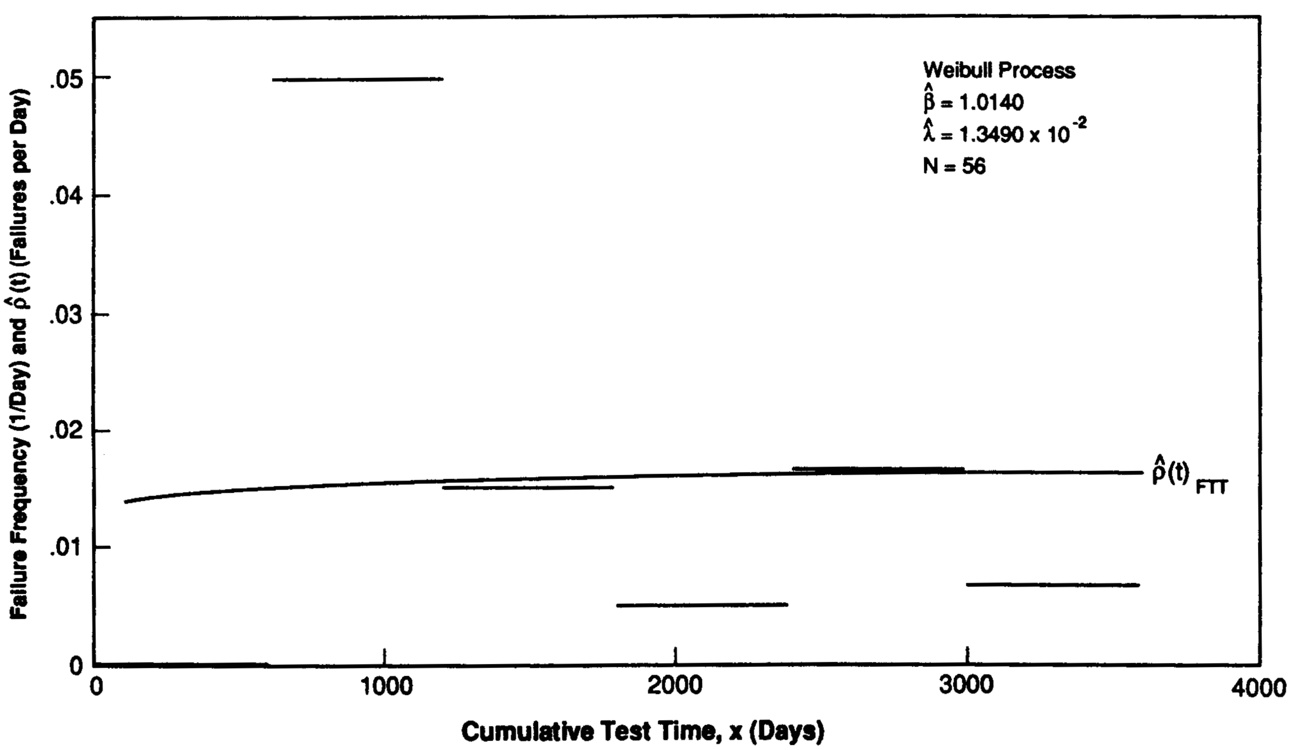

Figure 11. Estimated intensity function for case D ($N=56$).

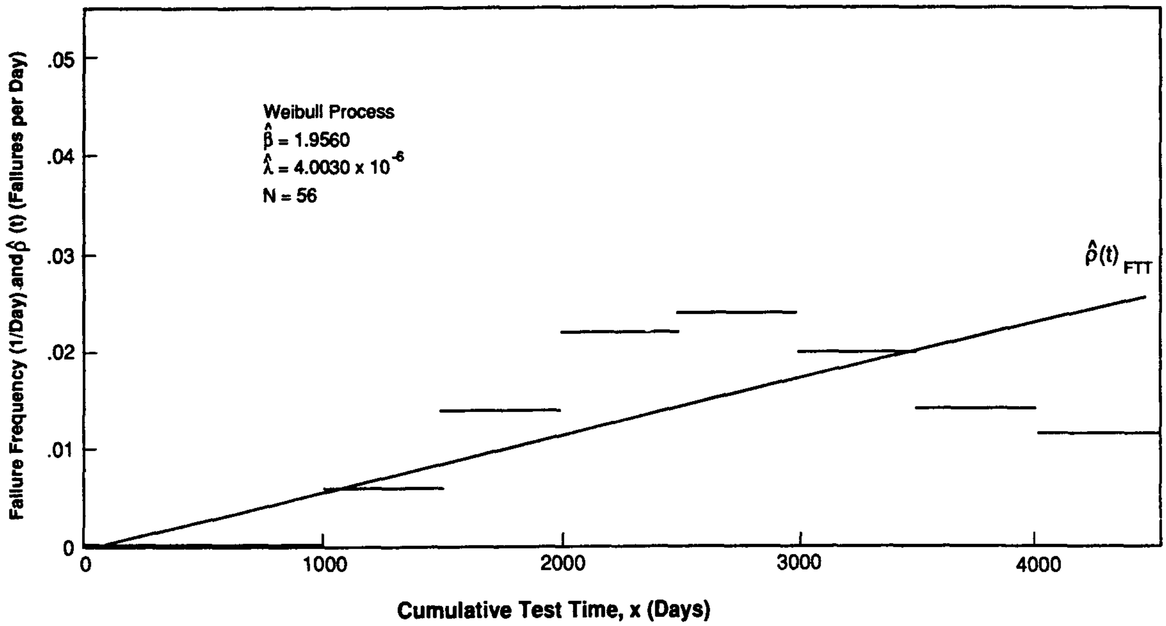

Figure 12. Estimated intensity function for case E ($N=56$).

Another method for goodness-of-fit testing is the Kolmogorov-Smirnov nonparametric method for differences between the cumulative distributions. Only the failure-terminated testing case is used here since the time-terminated testing produces very similar results. The engine failure data consist of a random sample X_1, X_2, X_3, ..., X_n of size n associated with a cumulative distribution function, $F(x) = (x_i/x_n)^{\beta}$, where β is calculated from equation (18). The one-sample test concerns calculations of the maximum absolute differences between the observed cumulative distribution function $S_n(x)$ and the specified continuous $F(x)$. The procedure calls for the plotting of the two cumulative distributions in figures 13 through 17 to determine the greatest differences of ordinal measurement. The graphical results are tabulated in table 15.

For case C, at a 0.10 level of significance, the critical value of Kolmogorov-Smirnov statistic for $N=38$ is 0.193957. In the 10 percent of random samples of size 38, the maximum absolute deviation between the observed cumulative distribution and continuous cumulative distribution is at least 0.193957. The maximum deviation of $F(x)$–$s(x)$, which occurs at $i=25$ step, is 0.3003. The value of $|F(x)–S(x)|$ is greater than the critical value so that the null hypothesis is not accepted at the 10-percent level of significance that the model for Weibull process is appropriate to represent the data. However, at the 0.001 significance point, the null hypothesis is accepted since the critical value is greater than the maximum deviation. Accordingly, for the specified values of level of significance 0.005–0.30, the null hypothesis is rejected.

Also on figures 13 through 17 are the confidence bands of the 0.20 and 0.01 levels of significance. $S_n(x) \pm$ the critical values are plotted as the upper and lower boundaries of the confidence bands that contain the unknown $F(x)$ completely within their boundaries. The maximum limit for ($S_n(x)+$ critical value) is 1.0 and the minimum limit for ($S_n(x)-$ critical value) is 0.

Table 13. TTT and FTT confidence interval estimates.

Time-Terminated Testing

	Case				
$\alpha = 0.50$:	A	B	C	D	E
$L_{N,\alpha}$	0.8049	0.8158	0.8444	0.8716	0.8716
$MTBF_{(L)}$ (days)	52.2753	98.8299	78.6887	56.7924	35.8068
$U_{N,\alpha}$	1.2927	1.2693	1.2132	1.1660	1.1660
$MTBF_{(U)}$ (days)	83.9561	153.7690	113.0567	75.9752	47.9013
$\alpha = 0.95$:					
$L_{N,\alpha}$	0.5697	0.5881	0.6387	0.6908	0.6908
$MTBF_{(L)}$ (days)	36.9999	71.2452	59.5197	45.0117	28.3792
$U_{N,\alpha}$	1.9292	1.8478	1.6588	1.5043	1.5043
$MTBF_{(U)}$ (days)	125.2945	223.8512	154.5817	98.0184	61.7992

Failure-Terminated Testing

$\alpha = 0.50$:	A	B	C	D	E
$L_{N,\alpha}$	0.8896	0.8919	0.8998	0.9103	0.9103
$MTBF_{(L)}$ (days)	49.6627	101.8705	79.2495	56.1076	35.5888
$U_{N,\alpha}$	1.3234	1.2961	1.2309	1.1767	1.1767
$MTBF_{(U)}$ (days)	73.8800	148.0371	108.4110	72.5276	46.0038
$\alpha = 0.95$:					
$L_{N,\alpha}$	0.6237	0.6378	0.6774	0.7196	0.7196
$MTBF_{(L)}$ (days)	34.8186	72.8478	59.6617	44.3536	28.1332
$U_{N,\alpha}$	1.9817	1.8909	1.6856	1.5206	1.5206
$MTBF_{(U)}$ (days)	110.6302	215.9736	148.4585	93.7243	59.4488

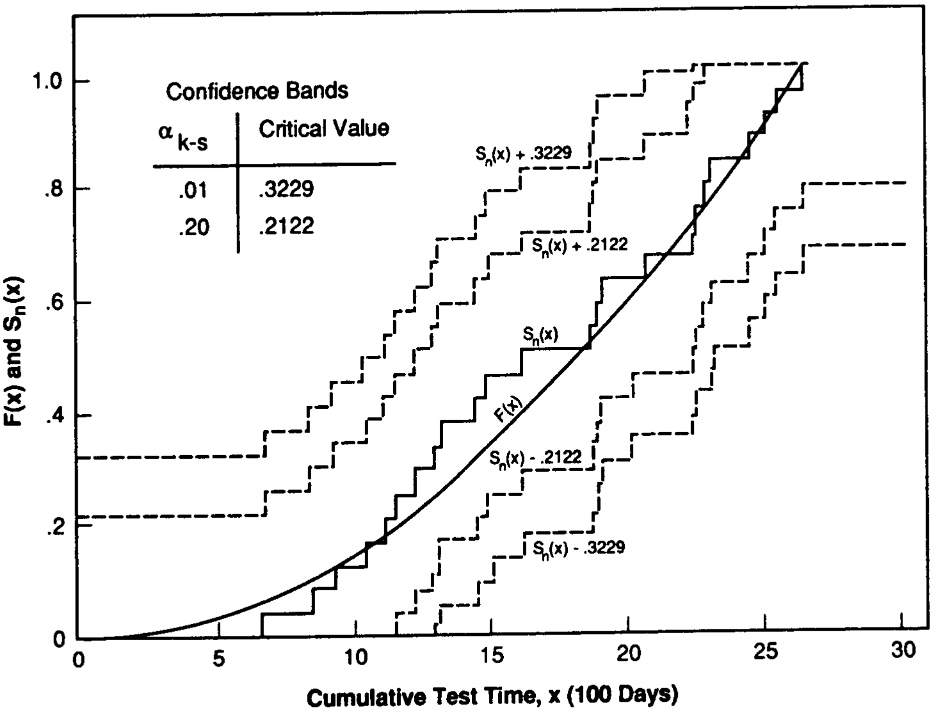

Figure 13. Diagram for Kolmogorov-Smirnov goodness-of-fit test with confidence bands for Weibull process for case A ($N=24$).

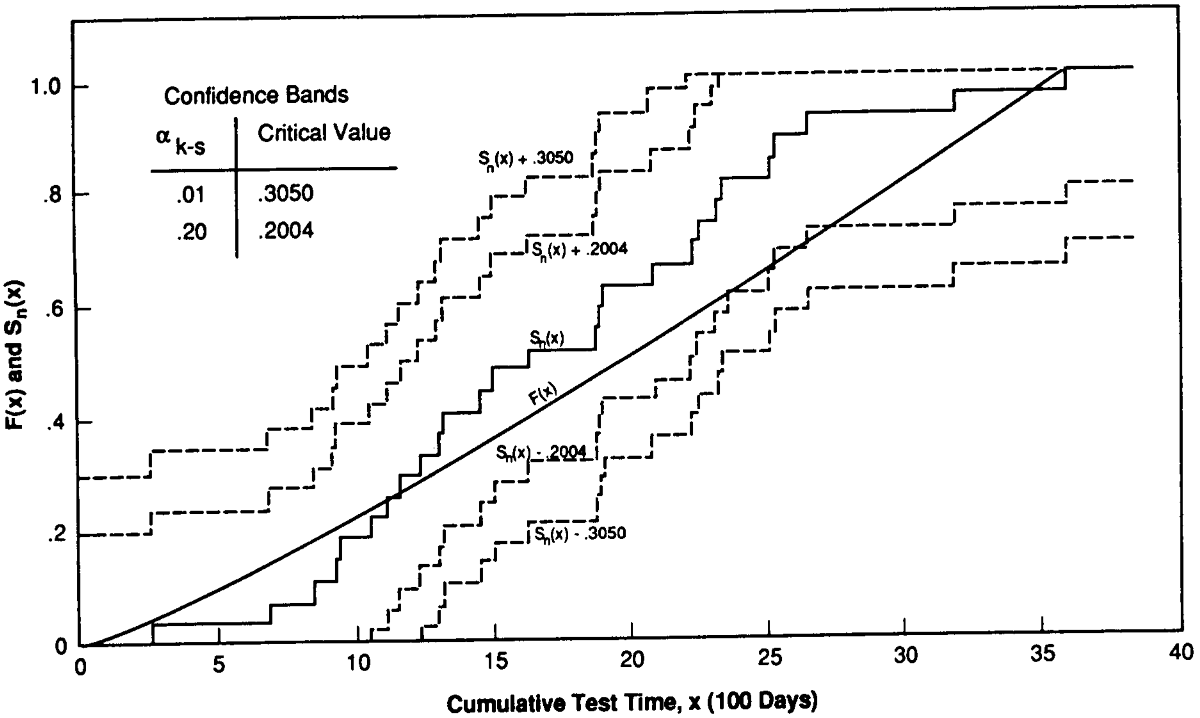

Figure 14. Diagram for Kolmogorov-Smirnov goodness-of-fit test with confidence bands for Weibull process for case B ($N=27$).

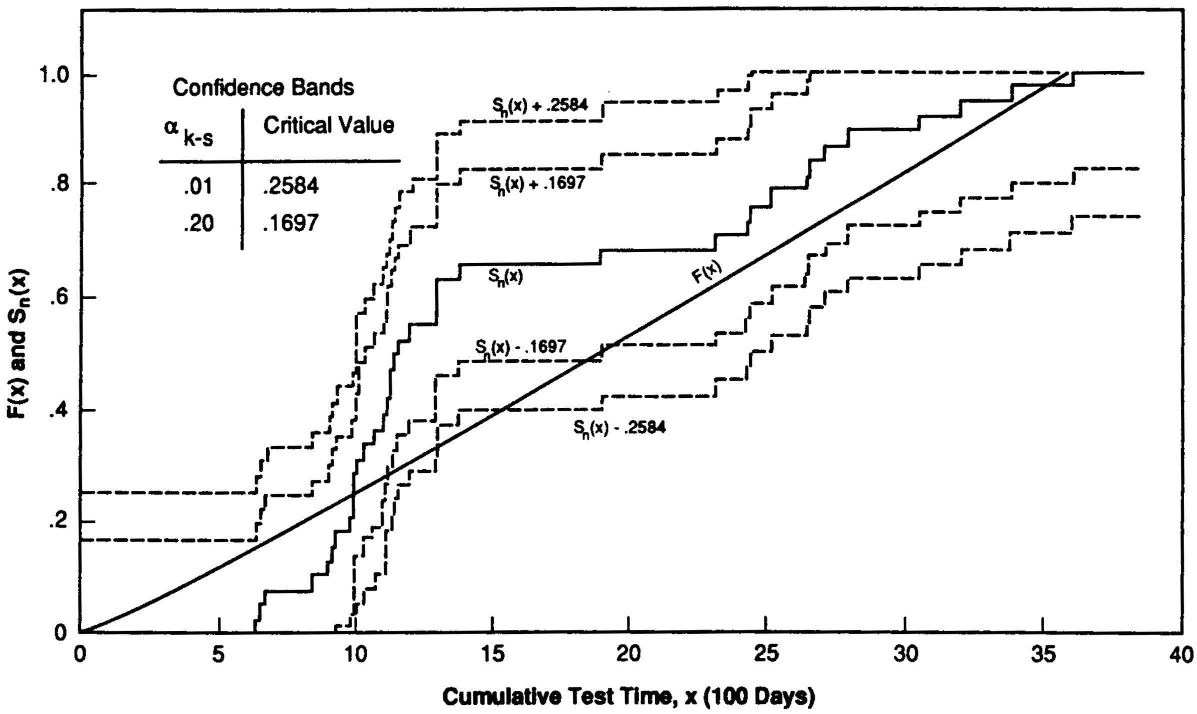

Figure 15. Diagram for Kolmogorov-Smirnov goodness-of-fit test with confidence bands for Weibull process for case C ($N = 38$).

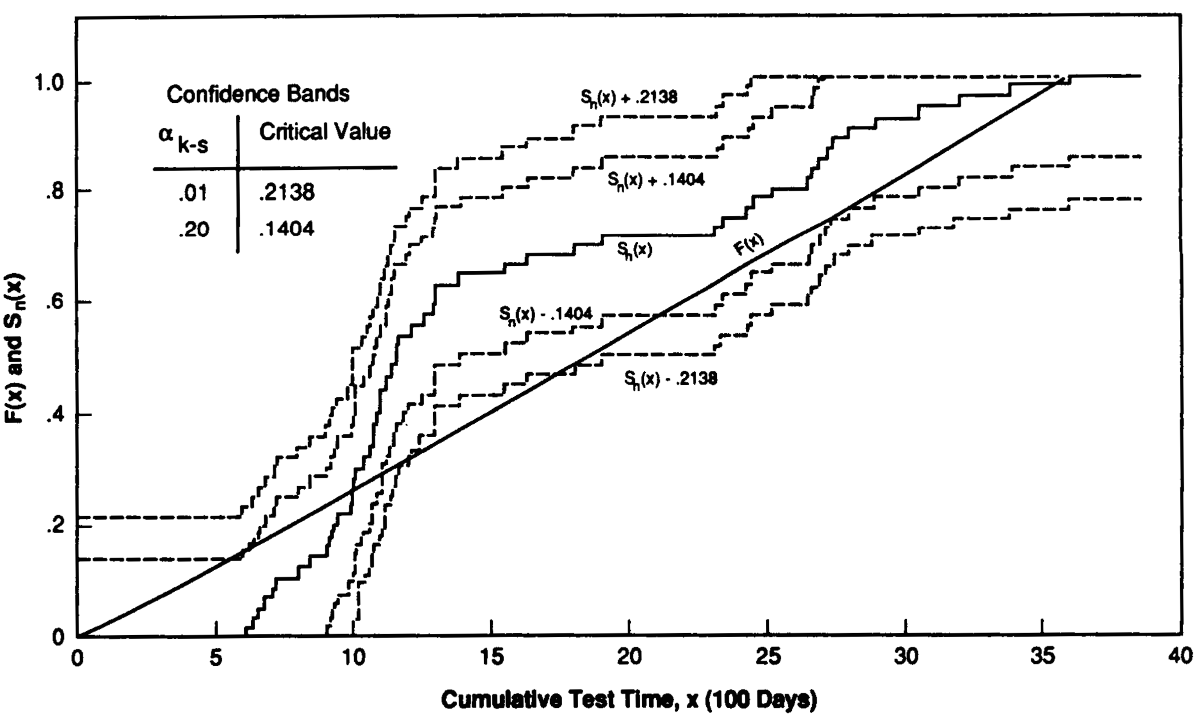

Figure 16. Diagram for Kolmogorov-Smirnov goodness-of-fit test with confidence bands for Weibull process for case D ($N = 56$).

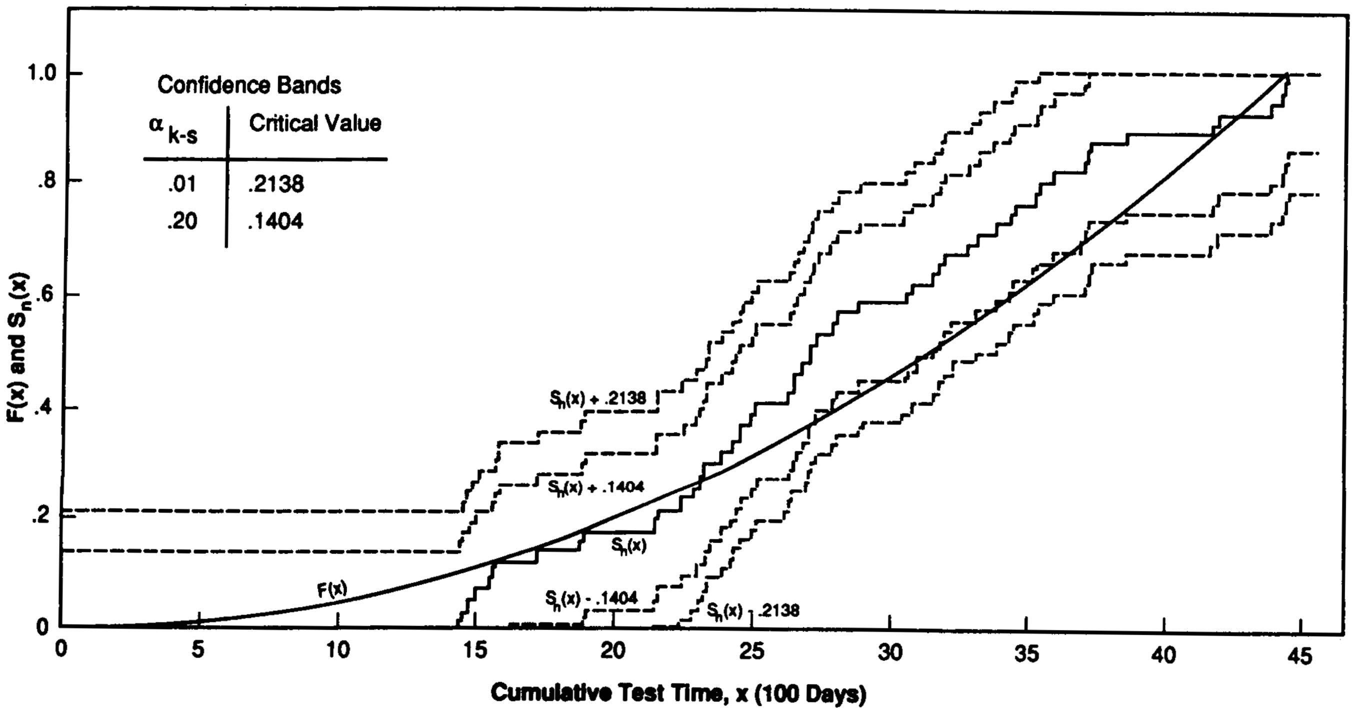

Figure 17. Diagram for Kolmogorov-Smirnov goodness-of-fit test with confidence bands for Weibull process for case E ($N=56$).

Table 14. Cramér-von Mises goodness-of-fit test for TTT and FTT.

Time-Terminated Testing

Case	M=N	$\hat{\beta}$	$\bar{\beta}$	C_m^2	Null hypothesis: Model is appropriate to represent the failure data
A	24	1.7964	1.7215	0.092066	α = 0.001-0.30: Not rejected
B	27	1.1312	1.0893	0.3283	α = 0.03-0.30: Rejected
					α = 0.001-0.01: Not rejected
C	38	1.0448	1.0174	0.6981	α = 0.001-0.30: Rejected
D	56	1.0140	0.9959	1.0942	α = 0.001-0.30: Rejected
E	56	1.9560	1.9211	0.5770	α = 0.001-0.30: Rejected

Failure-Terminated Testing

Case	M=N-1	$\hat{\beta}$	$\bar{\beta}$	C_m^2	Null hypothesis: Model is appropriate to represent the failure data
A	23	1.9831	1.8178	0.071770	α = 0.001-0.30: Not rejected
B	26	1.1674	1.0809	0.3646	α = 0.01-0.30: Rejected
					α = 0.001-0.005: Not rejected
C	37	1.0756	1.0190	0.7353	α = 0.001-0.30: Rejected
D	55	1.0430	1.0057	1.1158	α = 0.001-0.30: Rejected
E	55	2.0216	1.9494	0.5734	α = 0.001-0.30: Rejected

Notes:
(1) See tables 3A and 3B for critical values of Cramer-von Mises test statistic.
(2) α = level of significance.

Table 15. Kolmogorov-Smirnov goodness-of-fit test for FTT.

Failure-Terminated Testing

Case	N	$\lvert F(x)-S(x) \rvert$	Null hypothesis: Model is appropriate to represent the data
A	24	0.1344	α = 0.001-0.30: Not rejected
B	27	0.2267	α = 0.15-0.30: Rejected
			α = 0.001-0.10: Not rejected
C	38	0.3003	α = 0.005-0.30: Rejected
			α = 0.001: Not rejected
D	56	0.2796	α = 0.001-0.30: Rejected
E	56	0.1720	α = 0.10-0.30: Rejected
			α = 0.001-0.05: Not rejected

Notes:
(1) See tables 4A through 4D for critical values of Kolmogorov-Smirniv test statistic.
(2) $\lvert F(x)-S(x) \rvert$ = maximum absolute difference between observed and continuous cumulative distributions.

Table 16. Chi-square goodness-of-fit test for FTT.

Failure-Terminated Testing

Case	$\hat{\beta}$	$\hat{\lambda}$	ν	C_m^2	Null hypothesis: Model is appropriate to treat the data
A	1.3902	3.5190×10^{-4}	2	1.9752	α = 0.001-0.30: Not rejected
B	0.9630	1.0152×10^{-2}	1	4.7178	α = 0.05-0.30: Rejected
					α = 0.001-0.025: Not rejected
C	0.6164	0.2442	1	1.8796	α = 0.20-0.30: Rejected
					α = 0.001-0.15: Not rejected
D	0.5669	0.5396	3	5.3112	α = 0.20-0.30: Rejected
					α = 0.001-0.15: Not rejected
E	1.3380	5.1694×10^{-4}	4	17.2955	α = 0.005-0.30: Rejected
					α = 0.001: Not rejected

Notes:
(1) See tables 5A through 5D for critical values of chi-square test statistic.
(2) ν = number of degrees of freedom.

The chi-square goodness-of-fit test statistic is used to test whether the discrepancies between the observed and expected frequencies are attributed to contingency. The engine failure data for case C are grouped into frequency classes and compared to the expected number of failures based on the continuous cumulative distribution. After some iterations for grouping for 38 failures accumulated, the results are summarized in table 16 for the failure-terminated testing case.

For case C, the estimate of the shape parameter is calculated from equation (19) as $\hat{\beta} = 0.6164$. The scale parameter estimate from equation (16) is 0.2442. The goodness-of-fit statistic, equation (28), is computed to be 1.8796. The critical value for a χ^2 statistic with 1 degree of freedom at the 0.10 level of significance is obtained from table 5A to be 2.70554. Since the statistic is less than the critical value, the null hypothesis is not rejected so that applicability of the model for Weibull process is accepted.

CONCLUSIONS

In the total SSME test history, there are engine tests and flight failures, of which the numbers are reduced by means of screening for categorization into the five different groups (tables 6 through 10) for use in the reliability growth modeling analysis. Accordingly, the dates for first and last engine failure, as deducted from the epoch of May 19, 1975, are summarized below.

Case	Date of First Failure	FTT Date of Last Failure	TTT Predetermined Date
A	675	2,657	2,800
B	261	3,600	3,700
C	641	3,600	3,700
D	603	3,600	3,700
E	1,452	4,426	4,500

An important conclusion from this analysis is that using three goodness-of-fit methods of Cramér-von Mises, Kolmogorov-Smirnov, and chi-square to calculate the results of reliability growth modeling does result in adequate representation by the nonhomogeneous Poisson process with Weibull intensity function, known as the Weibull process, for only cases A and B. But cases C, D, and E do not have significant representation. The overall result is that all cases (A through E) incur the penalties of reliability deterioration, according to the statistical procedures, as evident in figures 18 through 22 which show declining MTBF's.

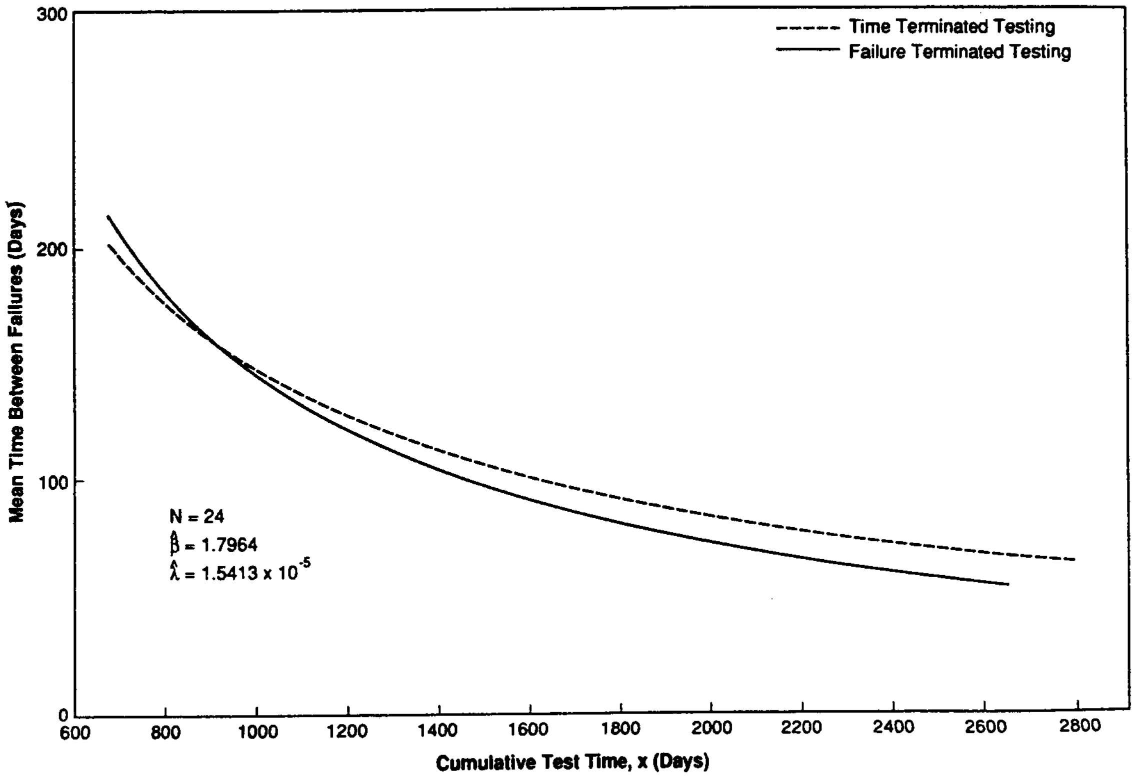

Figure 18. Weibull process: MTBF versus cumulative test time for case A ($N=24$).

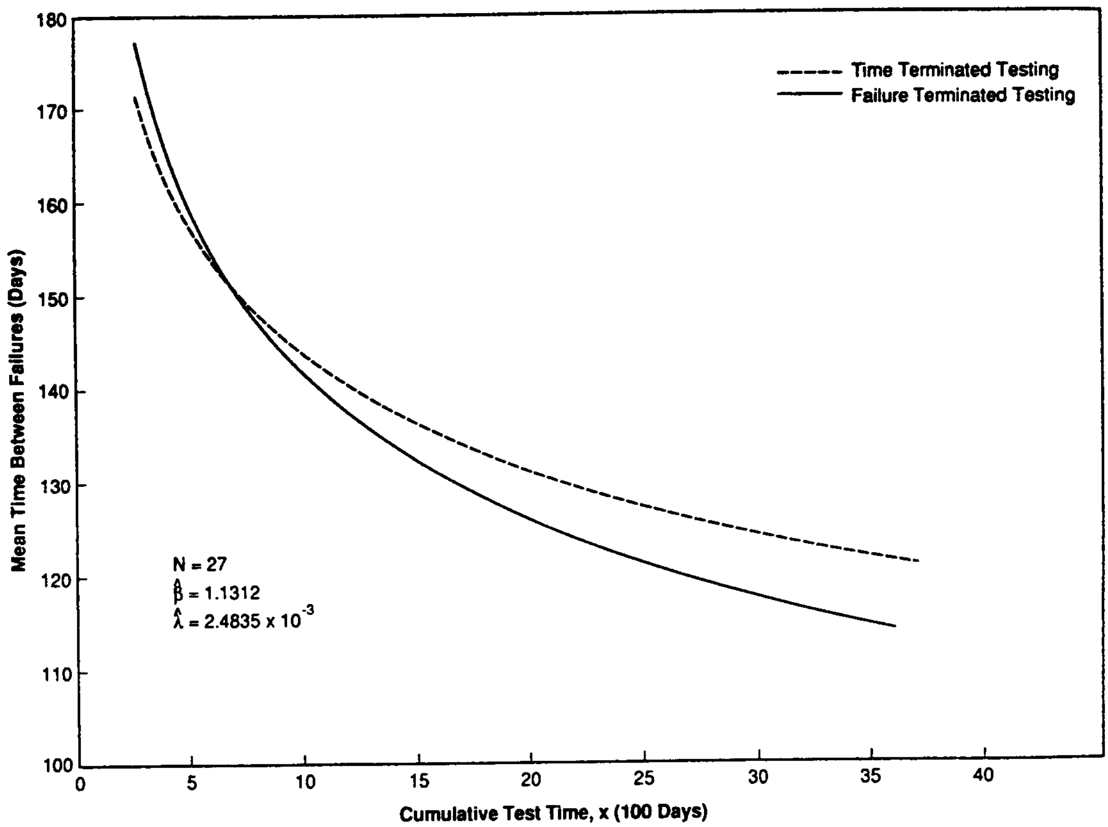

Figure 19. Weibull process: MTBF versus cumulative test time for case B ($N=27$).

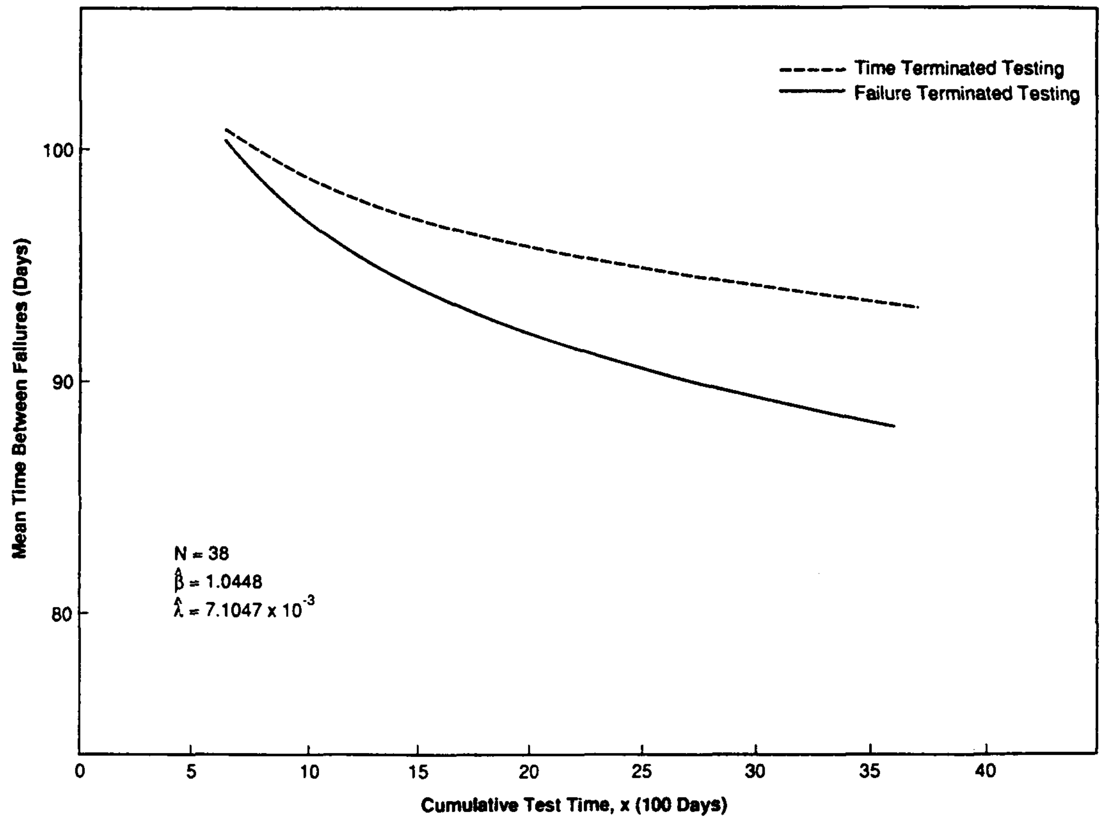

Figure 20. Weibull process: MTBF versus cumulative test time for case C ($N=38$).

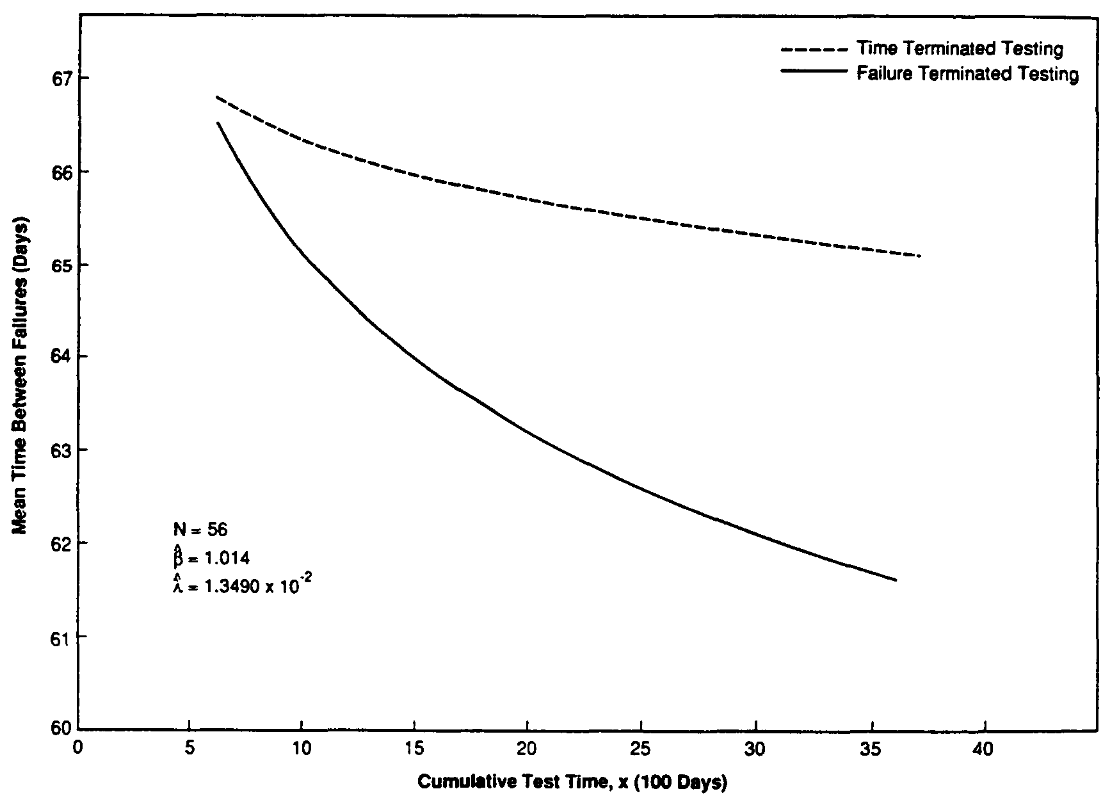

Figure 21. Weibull process: MTBF versus cumulative test time for case D ($N=56$).

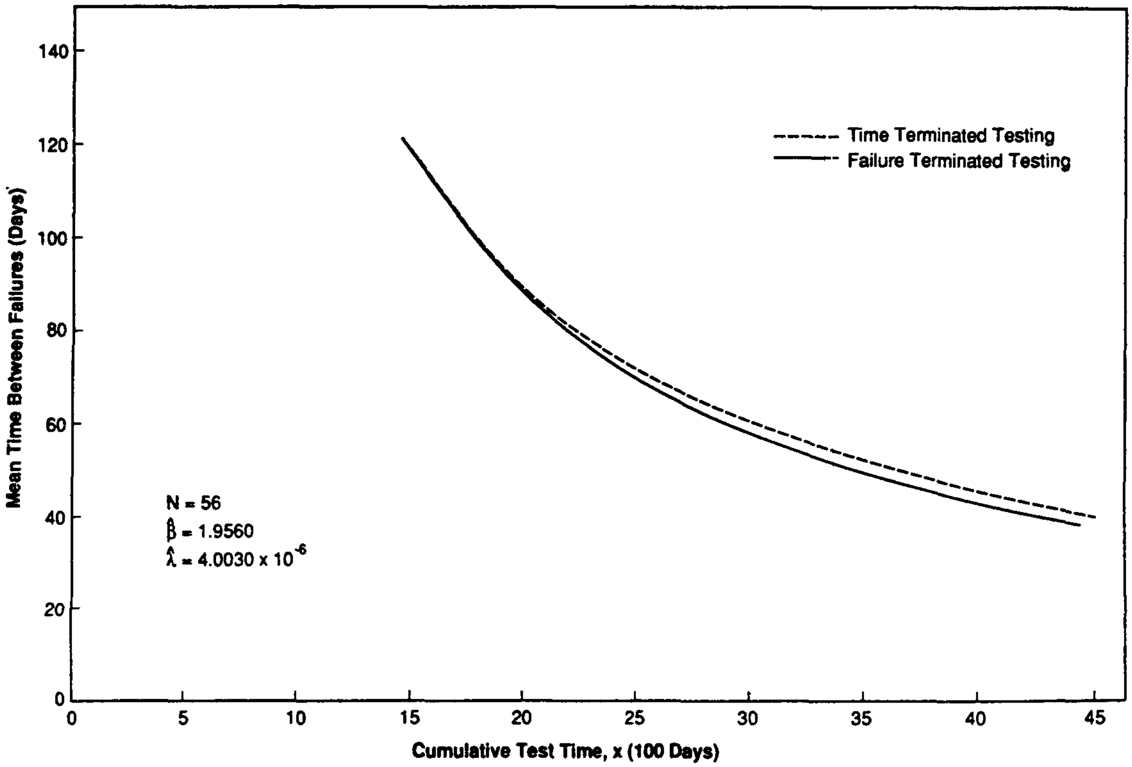

Figure 22. Weibull process: MTBF versus cumulative test time for case E ($N = 56$).

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APPROVAL

RELIABILITY GROWTH MODELING ANALYSIS OF THE SPACE SHUTTLE MAIN ENGINES BASED UPON THE WEIBULL PROCESS

By J.T. Wheeler

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

JAMES C. BLAIR
Director, Structures and Dynamics Laboratory

☆U.S. GOVERNMENT PRINTING OFFICE 1990-731-061/20206

NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM-103511		

4. Title and Subtitle	5. Report Date
Reliability Growth Modeling Analysis of the Space Shuttle Main Engines Based Upon the Weibull Process	August 1990
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
J.T. Wheeler	
	10. Work Unit No.
9. Performing Organization Name and Address	
George C. Marshall Space Flight Center Marshall Space Flight Center, Alabama 35812	11. Contract or Grant No.
	13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address	Technical Memorandum
National Aeronautics and Space Administration Washington, D.C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Prepared by Structures and Dynamics Laboratory, Science and Engineering Directorate.

16. Abstract

The Weibull process, identified as the nonhomogeneous Poisson process with the Weibull intensity function, is used to model the reliability growth assessment of the space shuttle main engine test and flight failure data. Additional tables of percentage-point probabilities for several different values of the confidence coefficient have been generated for setting $(1-\alpha)100$-percent two-sided confidence interval estimates on the mean time between failures. The tabled data pertain to two cases: (1) time-terminated testing and (2) failure-terminated testing. The critical values of the three test statistics, namely Cramér-von Mises, Kolmogorov-Smirnov, and chi-square, have been calculated and tabled for use in the goodness-of-fit tests for the engine reliability data. Numerical results are presented for five different groupings of the engine data that reflect the actual responses to the failures.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
reliability, reliability growth, nonhomogeneous Poisson process, Weibull intensity function, Duane postulate, stochastic process	Unclassified – Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	67	NTIS

NASA FORM 1626 OCT 86

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