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FREQUENCY ANALYSIS VIA THE METHOD OF MOMENT FUNCTIONALS¹

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Abstract

Several variants are presented of a linear-in-parameters least squares formulation for determining the transfer function of a stable linear system at specified frequencies given a finite set of Fourier series coefficients calculated from transient nonstationary input-output data. The basis of the technique is Shinbrot's classical method of moment functionals using complex Fourier based modulating functions to convert a differential equation model on a finite time interval into an algebraic equation which depends linearly on frequency-related parameters.

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1. INTRODUCTION

Methods for determining the transfer function of a stable linear system from input-output data include correlation and spectral analyses, as well as the direct sinusoidal measurements. Each of these "nonparametric" identification techniques require either a statistical stationarity assumption on the data, or a periodic steady state condition to be established, before initiating calculations of the transfer function at pertinent frequencies. Excellent summaries of these methods, as well as the analysis of noise effects and finite data lengths, can be found in Astrom [1], Ljung [2], Soderstrom and Stoica [3], and Unbehauen and Rao [4]. Notwithstanding noise considerations, long data lengths may be required in order to achieve good accuracy due to the stationarity or steady state assumption. By contrast, a method is proposed here that utilizes the frequency content in short data lengths in order to set up a least squares estimation of the transfer function at selected frequencies. Since short data lengths are used there is no assumption of steady state operation or stationarity of the data, though there must be present sufficient energy content in the data at the specified frequencies in order to avoid nondegeneracy in the least square estimate. The basis of the technique is the classical Shintz [5] method of moment functionals, also known as the modulating function technique, using *complex* Fourier based modulating functions. A forerunner of this approach can be found in Pearson and Lee [6] which utilized real valued Fourier based modulating functions, i.e., commensurable sinusoids, for the parameter estimation of linear differential systems. Therein also can be found a discussion of the background of this method with a listing of references. However, this paper appears to represent the first use of modulating functions in the context of the "nonparametric" system identification problem. Several variants of a deterministic least squares estimation of frequency-related parameters that underlie the transfer

function will be developed below.

2. Least Squares Formulations

Consider a stable linear system with input $u(t)$ and output $y(t)$ modeled on a finite time interval by the differential operator equation:

$$A(p)y(t)=B(p)u(t)+e(t) \quad (1)$$

where $(A(p), B(p))$ are polynomials in the differential operator $p=d/dt$ of degree less than or equal to an *a priori* integer n , and $e(t)$ represents the effect of modeling errors. The problem considered here is to estimate the transfer function $G(j\omega)=B(j\omega)/A(j\omega)$ at a finite set of frequencies $\{k\omega_0, k=1, 2, \dots, M\}$, where ω_0 is a user selected "resolving" frequency and M a chosen integer, given the input-output data $[u(t), y(t)]$ over a finite set of time intervals $\{[t_i, t_i+T], i=1, \dots, N\}$.¹ These time intervals are each chosen of length $T=2\pi/\omega_0$ and need not necessarily be disjoint. However, a certain degree of independence in the data collected over the different $[0, T]$ time intervals is necessary in order to avoid degeneracies in the least squares estimate to be discussed below. Understandably these degeneracies are more likely to be avoided in normal operating records if the intervals are disjoint. In addition to the upper bound on the system order n , the DC value of the transfer function is assumed given or can be measured from the step response, i.e., $G(0)=B(0)/A(0)$ is presumed known *a priori*. If this is not the case, then the estimated transfer function can be scaled by the parameter $G(0)$.

The Shinbrot method of moment functionals is a technique for converting a differential equation on a finite time interval into an algebraic equation by the use of "modulating func-

¹ If the system bandwidth ω_B is known, then choosing (M, ω_0) such that $M\omega_0 \approx \omega_B$ will cover the bandwidth at the knots $k\omega_0, k=1, 2, \dots, M$.

tions". As introduced by Shinbrot [5], $\phi(t)$ is a modulating function of order n relative to a finite time interval $0 \leq t \leq T$ if it is sufficiently smooth and satisfies the end point conditions:

$$\phi^{(i)}(0) = \phi^{(i)}(T) = 0, \quad i = 0, 1 \dots n-1 \quad (2)$$

where $\phi^{(i)}(t) = p^i \phi(t)$. Clearly, modulating functions can be constructed in many different ways.² Here a specific set of complex valued Fourier based modulating functions is defined in a way that is conducive to solving the problem at hand, viz., let

$$\phi_m(t) = e^{jm\omega_0 t} (1 - e^{j\omega_0 T})^n, \quad 0 \leq t \leq T = 2\pi/\omega_0 \quad (3a)$$

$$m = 0, 1 \dots M$$

define a set of modulating functions of order n with respect to the time interval $[0, T]$. Equivalently by the binomial expansion, each such function is representable by

$$\phi_m(t) = \sum_{k=0}^n b_k e^{j(m+k)\omega_0 t} \quad (3b)$$

where b_k is defined in relation to the binomial coefficient by

$$b_k = (-1)^k \binom{n}{k} \quad (4)$$

The first representation (3a) makes evident the fact that each $\phi_m(t)$ is indeed a modulating function of order n , i.e., (2) is satisfied,³ while the second representation (3b) implies that calculating linear functionals defined by each $\phi_m(t)$ on a set of functions specified over $[0, T]$ will entail calculating the Fourier series coefficients of these functions at the frequencies $k\omega_0$, $k = m, m+1 \dots m+n$, $m = 0, 1 \dots M$. In turn, these coefficients can be calculated efficiently by DFT/FFT methods which provides an important motivating factor for this analysis. This will

² See discussion in Pearson and Lee [6].

³ Notice that any modulating function of a fixed order is automatically a modulating function of any lower order relative to the same time interval. This property facilitates the formulation for any system of order less than or equal to the upper bound n .

be discussed further below. The important property of the functions defined in (3) is contained in the following ⁴

Modulation Property. Let $P(p)$ be a differential operator of order at most n , i.e., a polynomial in $p = d/dt$ of degree $\leq n$, and $z(t)$ any sufficiently smooth function defined on $[0, T]$. Then the modulation of $P(p)z(t)$ with $\phi_m(t)$ over $[0, T]$ satisfies

$$\int_0^T \phi_m(t) P(p) z(t) dt = \sum_{k=m}^{m+n} b_{k-m} P(-jk\omega_0) Z_{-k} \quad (5)$$

where Z_k is the k^{th} harmonic Fourier series coefficient of $z(t)$, i.e.,

$$Z_k = \int_0^T z(t) e^{-jk\omega_0 t} dt. \quad (6)$$

Note that owing to the end point constraints (2) satisfied by each $\phi_m(t)$ function, none of the boundary point derivatives $z^{(i)}(0)$ or $z^{(i)}(T)$ appear in (5). This is crucial to the ensuing analysis and, in fact, represents a primary reason for employing the modulating function technique.

2.1. Formulation 1

A direct application of the above property to the problem posed involves rewriting the differential operator model (1) in the equation error form followed by the modulation with $\phi_m(t)$; thus,⁵

⁴ Proof of this property is given in the Appendix in order to proceed directly with the development.

⁵ The process of going from the model (1) to equation (7) can be viewed as a projection from a space of functions on $[0, T]$ down into a finite dimensional space spanned by the modulating functions.

$$\int_0^T \phi_m(t) \left[A(p)y(t) - B(p)u(t) \right] dt = \int_0^T \phi_m(t) e(t) dt. \quad (7)$$

In view of (5) the preceding equation is equivalent to

$$\sum_{k=m}^{m+n} b_{k-m} \left[A(-jk\omega_0)Y_{-k} - B(-jk\omega_0)U_{-k} \right] = \sum_{k=m}^{m+n} b_{k-m} E_{-k}. \quad (8)$$

Define the real and imaginary parts of the polynomials $(A(jk\omega_0), B(jk\omega_0))$ as follows:

$$A(jk\omega_0) = \alpha_k + j\beta_k, \quad B(jk\omega_0) = \gamma_k + j\delta_k \quad (9)$$

and collect these together to form the 4×1 "parameter" vector:

$$\theta_k = \begin{bmatrix} \alpha_k \\ \beta_k \\ \gamma_k \\ \delta_k \end{bmatrix}. \quad (10)$$

Also, define as follows the 2×4 data matrix $\psi_k(i)$ in terms of the real and imaginary parts of the k^{th} harmonic Fourier series coefficients of the input-output data corresponding to the time interval $[t_i, t_i + T]$, $i = 1, 2, \dots, N$:

$$\psi_k(i) = \begin{bmatrix} Y_k^c(i) & Y_k^s(i) & -U_k^c(i) & -U_k^s(i) \\ -Y_k^s(i) & Y_k^c(i) & U_k^s(i) & -U_k^c(i) \end{bmatrix}. \quad (11)$$

The notation for the entries in (11) is explained by

$$Y_k^c(i) = \int_0^T y(t+t_i) \cos k\omega_0 t \, dt, \quad Y_k^s(i) = \int_0^T y(t+t_i) \sin k\omega_0 t \, dt \quad (12)$$

and similarly for $(U_k^c(i), U_k^s(i))$. Then the real and imaginary parts of the equation error (8) can be collected into the following real valued 2×1 vector equation which serves as the starting point for a least squares estimation:

$$\sum_{k=m}^{m+n} b_{k-m} \psi_k(i) \theta_k = \epsilon_m(i) \quad (13)$$

$$m = 0, 1, \dots, M, \quad i = 1, 2, \dots, N.$$

The equation error vector in (13) is related to the Fourier coefficients of the original equation error by

$$\epsilon_m(i) = \sum_{k=m}^{m+n} b_{k-m} \begin{bmatrix} E_k^c(i) \\ E_k^s(i) \end{bmatrix}.$$

Values of the transfer function $G(jk\omega_0) = B(jk\omega_0)/A(jk\omega_0)$ are seen from (9) and (10) to be related to the parameter vector θ_k by the real and imaginary part relations:

$$\text{Re}G(jk\omega_0) = \frac{\alpha_k \gamma_k - \beta_k \delta_k}{\alpha_k^2 + \beta_k^2}, \quad \text{Im}G(jk\omega_0) = \frac{\alpha_k \delta_k - \beta_k \gamma_k}{\alpha_k^2 + \beta_k^2} \quad (14)$$

or equivalently by the magnitude-phase relations:

$$|G(jk\omega_0)|^2 = \frac{\gamma_k^2 + \delta_k^2}{\alpha_k^2 + \beta_k^2}, \quad \angle G(jk\omega_k) = \tan^{-1} \frac{\delta_k}{\gamma_k} - \tan^{-1} \frac{\beta_k}{\alpha_k}. \quad (15)$$

Starting from a presumed knowledge of the DC value $G(0)$, which implies that $\theta_0 = [A(0), 0, B(0), 0]'$ is known, equation (13) can be rewritten in the standard regression equation format to estimate the parameters θ_k , $k=1, 2, \dots, M+n$ given the data over a sufficient number of $[t_i, t_i+T]$ intervals, $i=1, 2, \dots, N$. A consideration of this equation reveals the following:

- 1) The frequency range covered by the parameters in (13) is $(M+n)\omega_0$. Hence, if it is desired that the transfer function estimate cover a frequency range about 25% greater than the system bandwidth ω_B at a resolution ω_0 , a choice in (M, ω_0) such that

$$(M+n)\omega_0 \approx 1.25\omega_B \quad (16)$$

reflects this objective.

- 2) Counting unknowns in (10) and (13), the total number is $4(M+n)$. Since each equation in (13) is of dimension 2, counting equations suggests that the total number N of $[t_i, t_i+T]$ intervals should satisfy: $2(M+1)N \geq 4(M+n)$, i.e., $N \geq 2(M+n)/(M+1)$.

However, the equations in (13) are partially decoupled with respect to the index m . Therefore, it seems best to estimate $(\theta_1, \dots, \theta_n)$ at the first stage, which corresponds to $m=0$. This means that there are $4n$ unknowns for the first stage requiring that N satisfy $N \geq 2n$. Thereafter, the number of unknowns is just 4 for each stage corresponding to $m=1, 2, \dots, M$, which implies $N \geq 2$ assuming that the preceding estimates are used in each succeeding stage. This kind of "bootstrapping" of the least squares estimation facilitates keeping the number of unknowns to a modest level at each stage.

3) The two row vectors comprising $\psi_k(i)$ in (11) are seen to be mutually orthogonal for each k and i suggesting a maximal degree of independence for these equations in utilizing the information content in the data. This is a direct result of the Fourier nature of the underlying formulation.

Discussion: The above development shows that it is possible to formulate a linear-in-the-parameters least squares estimation problem for parameters (10) that underlie (via (14) or (15)) the transfer function $G(jk\omega_0)$ at each k^{th} harmonic frequency. The input-output data can be time-limited and transient, but must have sufficient energy content at the specified frequencies to avoid degeneracies in the least squares solution. Apart from being highly non-linear, the relations (14) and (15) involve the difference between parameter related quantities, e.g., $\alpha_k \gamma_k - \beta_k \delta_k$, whose values may be large for large k . This aspect of the formulation portends a potential source of error magnification which is alleviated by the formulation of the next section.

2.2. Formulation 2

Given that $[u(t), y(t)]$ satisfies the model (1) on a $[0, T]$ time interval, it follows that $[u(t), y(t)]$ also satisfies the model

$$A(-p)A(p)y(t) = A(-p)B(p)u(t) + A(-p)e(t). \quad (17)$$

Choosing a set of modulating functions of order $2n$ to accommodate the upper bound on the highest degree differential operator in (17) and modulating this equation with the m^{th} member of this set, the following projected equation error results which is analogous to (8):

$$\sum_{k=m}^{m+2n} \bar{b}_{k-m} \left[A(jk\omega_0)A(-jk\omega_0)Y_{-k} - A(jk\omega_0)B(-jk\omega_0)U_{-k} \right] = \sum_{k=m}^{m+2n} \bar{b}_{k-m} A(jk\omega_0)E_{-k} \quad (18)$$

where \bar{b}_k is defined by, cf. (4):

$$\bar{b}_k = (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}.$$

Noting that $A(jk\omega_0)A(-jk\omega_0)$ is real while $A(jk\omega_0)B(-jk\omega_0)$ is complex, define real quantities (a_k, α_k, β_k) by the relations:

$$a_k = A(jk\omega_0)A(-jk\omega_0), \quad \alpha_k + j\beta_k = A(jk\omega_0)B(-jk\omega_0) \quad (19)$$

and collect these into the 3×1 parameter vector $\bar{\theta}_k$ defined by

$$\bar{\theta}_k = \begin{bmatrix} a_k \\ \alpha_k \\ \beta_k \end{bmatrix}. \quad (20)$$

Also, define the 2×3 data matrix $\bar{\Psi}_k(i)$ by

$$\bar{\Psi}_k(i) = \begin{bmatrix} Y_k^c(i) & -U_k^c(i) & U_k^s(i) \\ Y_k^s(i) & -U_k^s(i) & -U_k^c(i) \end{bmatrix}. \quad (21)$$

where the notation for the entries in (21) is the same as defined in (12). Then the real and imaginary parts of the projected equation error (18) can be represented by the following real 2×1 vector equation:

$$\sum_{k=m}^{m+2n} \bar{b}_{k-m} \bar{\psi}_k(i) \bar{\theta}_k = \bar{\epsilon}_m(i) \quad (22)$$

$$m=0,1 \cdots M, \quad i=1,2 \cdots N.$$

Equation (22) can be rewritten into the standard regression equation format for setting up the least squares estimate of the parameters $(\bar{\theta}_1, \cdots \bar{\theta}_{M+2n})$ based on the data over $[t_i, t_i+T]$, $i=1,2 \cdots N$. Again, presumed knowledge of the DC value implies that $\bar{\theta}_0=A(0)[A(0),B(0),0]'$ is known or, if not, the resulting transfer function estimate can be scaled by the parameter $G(0)$. Here the estimates of the transfer function are related to the parameters by the real and imaginary part equations (as found from (19) and (20)):

$$\text{Re}G(jk\omega_0)=\frac{\alpha_k}{a_k}, \quad \text{Im}G(jk\omega_0)=-\frac{\beta_k}{a_k} \quad (23)$$

or equivalently by the magnitude-phase relations:

$$|G(jk\omega_0)|^2 = \frac{\alpha_k^2 + \beta_k^2}{a_k^2}, \quad \angle G(jk\omega_0) = -\tan^{-1} \frac{\beta_k}{\alpha_k}. \quad (24)$$

Consideration of the least squares formulation in this case leads to the following:

- 1) The frequency range covered by the parameters in (22) is $(M+2n)\omega_0$; hence, the guideline (analogous to (16)) for choosing the pair (M, ω_0) in this case is

$$(M+2n)\omega_0 \approx 1.25\omega_B. \quad (25)$$

- 2) Counting unknowns in (20) and (22), the total number is $3(M+2n)$ which would imply that the total number N of $[t_i, t_i+T]$ time intervals should satisfy: $2N(M+1) \geq 3(M+2n)$. However, the partially decoupled nature of the equation (22) with respect to the m index suggests bootstrapping the solution from the first stage. Thus, for the initial stage ($m=0$), N needs to satisfy $N \geq 3n$. In the succeeding stages, N needs to satisfy $N \geq 2$ (since there are 3 unknowns and $2N$ equations at each such

stage); this assumes that the preceding estimates are used at each succeeding stage ($m=1,2 \dots M$).

Discussion: Comparing (23) and (24) with (14) and (15) reveals that the second formulation avoids the potential error magnification problem of differencing large estimated quantities in calculating the transfer function at high frequencies. However, the second formulation requires estimating $6n$ unknowns at the first stage, i.e., the $m=0$ stage, verses $4n$ unknowns for the first stage of the first formulation.

2.3. Formulation 2-Dual

The dual to the formulation of the preceding section is to observe that a given pair $[u(t), y(t)]$ satisfying the model (1) on a $[0, T]$ time interval also implies that it satisfies (cf. (17))

$$B(-p)A(p)y(t) = B(-p)B(p)u(t) + B(-p)e(t). \quad (26)$$

Again choosing a set of modulating functions of order $2n$, a development similar to that of the previous section leads to the real 2×1 vector equation

$$\sum_{k=m}^{m+2n} \bar{b}_{k-m} \tilde{\psi}_k(i) \tilde{\theta}_k = \tilde{\epsilon}_m(i) \quad (27)$$

$$m=0, 1 \dots M, \quad i=1, 2 \dots N$$

where the data matrix $\tilde{\psi}_k(i)$ is defined by

$$\tilde{\psi}_k(i) = \begin{bmatrix} U_k^c(i) & -Y_k^c(i) & Y_k^s(i) \\ U_k^s(i) & -Y_k^s(i) & -Y_k^c(i) \end{bmatrix} \quad (28)$$

and the real 3×1 parameter vector $\tilde{\theta}_k$ is defined by

$$\tilde{\theta}_k = \begin{bmatrix} b_k \\ \alpha_k \\ -\beta_k \end{bmatrix} \quad (29)$$

with the entries in (29) defined by

$$b_k = B(jk\omega_0)B(-jk\omega_0), \quad \alpha_k + j\beta_k = A(jk\omega_0)B(-jk\omega_0). \quad (30)$$

Relations between the transfer function and the parameters in this case are found to be

$$\operatorname{Re} G(jk\omega_0) = \frac{\alpha_k b_k}{\alpha_k^2 + \beta_k^2}, \quad \operatorname{Im} G(jk\omega_0) = \frac{-\beta_k b_k}{\alpha_k^2 + \beta_k^2} \quad (31)$$

for the real and imaginary parts, or for the magnitude-phase:

$$|G(jk\omega_0)|^2 = \frac{b_k^2}{\alpha_k^2 + \beta_k^2}, \quad \angle G(jk\omega_0) = -\tan^{-1} \frac{\beta_k}{\alpha_k}. \quad (32)$$

Comparing (21) and (28) verifies the duality of the two formulations by virtue of the interchange of input and output. Note that each formulation has the same total number of unknowns - in general. However, the dual formulation has the potential advantage of reducing the total number of unknowns in the event of *a priori* information on a lower degree numerator polynomial than denominator polynomial in the transfer function. For example, an "all pole" model means that $b_k = (B(0))^2$ is known for all k , i.e., a total of $2(M+n)$ unknowns verses $3(M+2n)$ for the previous case. The formulation leading to (27) can easily be modified to reflect this consideration.

3. Conclusions

Three formulations of a linear-in-the parameters least squares estimation have been presented for determining the transfer function of a linear system at specified frequencies given transient nonstationary input-output data. While some comparisons have been noted in the discussions following each formulation, a clear indication of the pitfalls and advantages of

each will have to await a thorough simulation study including the effects of noise. Such a study is currently underway.

4. References

- [1] Astrom, K. J., *Lectures on System Identification, Chapt. 3: Frequency Response Analysis*. Report 7504, Dept. of Automatic Control, Lund Inst. of Tech., Sweden, Feb. 1975.
- [2] Ljung, L., *System Identification: Theory for the User*. Englewood Cliffs: Prentice Hall, 1987.
- [3] Soderstrom, T. and P. Stoica, *System Identification*. New York: Prentice Hall, 1989.
- [4] Unbehauen, H. and G. P. Rao, *Identification of Continuous Systems*. Amsterdam: North-Holland, 1987.
- [5] Shinbrot, M., "On the analysis of linear and nonlinear systems." *Trans. ASME*, vol. 79, pp. 547-552, 1957.
- [6] Pearson, A. E. and F. C. Lee, "Parameter identification of linear differential systems via Fourier based modulating functions," *Control-Theory and Adv. Tech.*, vol. 1, no. 4, pp. 239-266, 1985.

5. Appendix

To verify the modulation property (5) it is sufficient to consider the differential operator $P(p)=p^i$ for a fixed $i \leq n$. The result for a general n^{th} degree polynomial $P(p)$ then follows by superposition. Thus, for sufficiently smooth $z(t)$ on $[0,T]$ and $\phi_m(t)$ defined in (3), the left side of (5) in this case is

$$\int_0^T \phi_m(t) p^i z(t) dt = \int_0^T z(t) p^i \phi_m(t) dt \quad (33)$$

where integration-by-parts has been used i times taking into account the boundary conditions (2) possessed by each $\phi_m(t)$ function. Substituting the representation (3b) into the right side of (33) and changing the index of summation verifies (5) for $P(p)=p^i$ as purported.