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TRANSFORMATIONAL PART-COUNT IN LAYERED OCTAHEDRAL-TETRAHEDRAL TRUSS CONFIGURATIONS
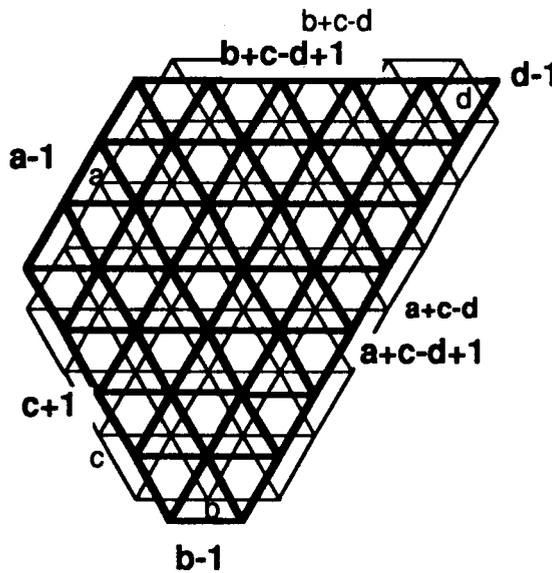
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CONTENTS

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1. Introduction

Many of NASA's projects have suggested the use of *triangulated*, modular space structures. Of these, the "tetrahedral truss" [1], and its orthogonal equivalent suggested by Mikulas [2] are candidate geometries for projects ranging from the Space Station and the Mars Vehicle to various solar reflectors, aerobrakes, the space crane, antennas, and so on. Since the "tetrahedral truss" is composed of close-packed *octa*hedra and *tetra*hedra, the term *octet truss* was suggested earlier by Fuller for the same configuration [3]. Besides the geometry, the number of elements or component parts, termed *part count*, is an important factor in the design, manufacture and assembly of modular space structures.

The part-count in triangulated space structures is the topic for this paper. Relations for part count for two types of double-layered tetrahedral trusses have been reported by Kenner, et. al., using a ring method [4]. General relations for determining the number of various component parts in a variety of profiles from layered tetrahedral-octahedral configuration are presented here using an alternative method. Expressions for part counts for a larger variety of space structure configurations of varying topology and dimension should be useful

and necessary when *space architecture* begins to experiment with alternative geometries.

2. The Euler Relations

The part count represents the number of nodes, struts, panels and cells, though the cells are used less frequently. These correspond to the number of topological elements : the vertices V, the edges E, the faces F and the cells C. The relation between the number of topological elements of a *3-dimensional* space structure, are constrained by the well-known Euler relation :

$$V - E + F - C = 1 \qquad \dots\dots\dots\dots (1)$$

Its various special cases are also well-known. For example, in *2-dimensional* structures there are no cells, making C=0; the relation becomes

$$V - E + F = 1 \qquad \dots\dots\dots\dots (2)$$

and in the case of convex *polyhedra* there is only one cell, making C=1; here the relation becomes

$$V - E + F = 2 \qquad \dots\dots\dots\dots (3)$$

In case of *n-dimensional* space structures, the relation is generalized by the Euler-Schlafli equation [5] which relates the number of all elements of varying dimensions within an n-dimensional space structure. In their projected states in 2 or 3 dimensions, n-dimensional space structures expand the repertoire of architectural space structures, both for terrestrial and space architecture. Such structures offer a promising direction in our efforts to discover new ways of structuring and defining useful and habitable space [6, Sec.9]. Note that the tetrahedral truss configuration is a 3-dimensional projection of a 6-dimensional space structure [6, Sec.8.2], with a special condition that some vertices overlap and some 3-cells are "flat".

3. The Tetrahedral-Octahedral Space-filling

The tetrahedral or the octet truss is a doubled layered slice from the space-filling of tetrahedra and octahedra [3]. This space-filling has four plane directions, and the double layer used in trusses is parallel to any one of these planes. The "unit cell", the minimum unit for translation, is a rhombohedron composed of two tetrahedra on opposite faces of one octahedron. It follows that in the infinite 3-dimensional space-filling, the proportion of octahedra to tetrahedra equals 1 to 2. The space-filling itself can be obtained by a recursive subdivision of the rhombohedron into smaller self-similar rhombohedra. At each stage of subdivision, the relation between the part counts of different elements is constrained by relation (1).

4. Layered Sections

Parallel sections through the space-filling, taken perpendicular to the 3-fold axis of symmetry, produce equilateral triangular grids in every layer. These single-layered grids are shifted in three different positions with respect to one another. The double layered configurations are also of three types. The first slice has a tetrahedron in the center, the second has the octahedron in the center and the third has an upside down tetrahedron in its center.

A large variety of triangulated profiles, i.e., finite portions from the infinite triangular grid, can be derived from each single layer and juxtaposed with another to produce a double-layered truss configuration. These profiles can be grouped by their symmetry or number of sides. Single layer profiles are first described and are followed by the derivation of the next layer.

4.1 Single Layer

All single-layer convex profiles can be derived from a generalized irregular hexagon. The expressions for its part-count are presented. As the profile transforms to another under special conditions, the general expressions also transform correspondingly into its special cases.

This irregular hexagon (Fig.1) is completely asymmetric and is the most complex of all convex profiles obtained from a triangular grid. It uses 4 independent variables to specify it completely. As the variables are reduced to 3, 2 or 1, the profiles become simpler and more symmetric. In the process of derivation of the profiles and their part count, the author proceeded in the reverse from simple to the most complex. The part counts for the regular triangle (Fig.7) and hexagon (Fig.6) were first derived using a single variable a. A second variable b was added to derive hexagons with a 3-fold symmetry (Fig.5). A third variable c was added to derive hexagons with a 2-fold symmetry (Fig.3). A fourth variable d was added to derive the irregular hexagon in Fig.1 which has a 1-fold symmetry. Other profiles were obtained as degenerate cases at each stage by assigning special values to the variables.

However, the number of variables in a profile is independent of the number of sides of a polygon which is, in some ways, more useful as a practical classification scheme. In this paper the profiles are thus organized in terms of the number of sides. All 6-sided profiles are described first, followed by 5-sided and 4-sided profiles. Among the 6-sided profiles, the most complex is described first. Alternative approaches include a constructive method using two primitive polygons, the triangle and a parallelogram, as the building blocks for all profiles. A more integrated approach is possible using conceptual organizing schemes and is briefly mentioned in the concluding Remarks.

4.11 Irregular Hexagon

Figure 1 shows an irregular hexagon. Its edge divisions can be described in terms of four independent variables a, b, c and d. Three of its sides are a, b and c, and the three corresponding opposite sides are a+c-d, b+c-d, and d. In the example shown, a=4, b=2, c=3 and d=1.
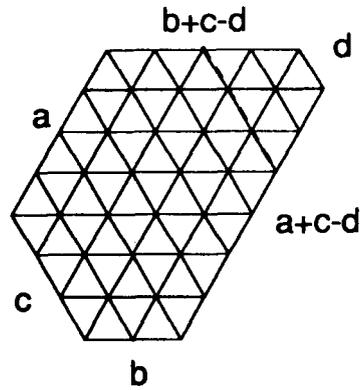
Figure 1

Its part counts are given by the following relations :

$$E = a + b + c + 3(ab + bc + ac) + (c-d)(3c+3d+1)/2 \quad \text{...............(4)}$$
$$V = a + b + c + ab + bc + ac + (c-d)(c+d+1)/2 + 1 \quad \text{...............(5)}$$
$$F = 2(ab + bc + ac) + (c-d)(c+d) \quad \text{...............(6)}$$
$$E_b = V_b = 2(a + b + c) + (c-d) \quad \text{...............(7)}$$

where E_b and V_b are the edges and vertices on the boundary.

The various special cases described below are obtained when the different variables are assigned specific values.

4.111 Hexagon with Bilateral Symmetry

In a special case when b=d, a bilaterally symmetric hexagon is obtained (Fig.2).

b = d
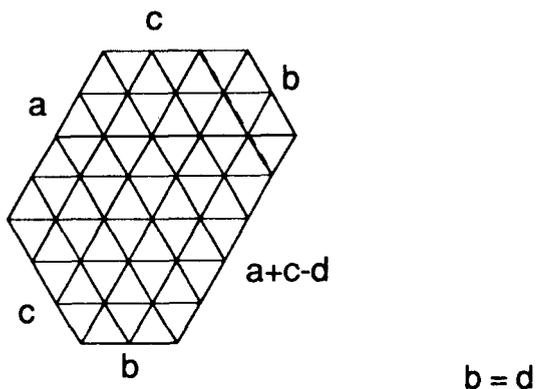
Figure 2

4.112 Hexagons with 2-fold Symmetry (c=d)

When c=d, a hexagon with a 2-fold rotational symmetry is obtained (Fig.3).

c=d
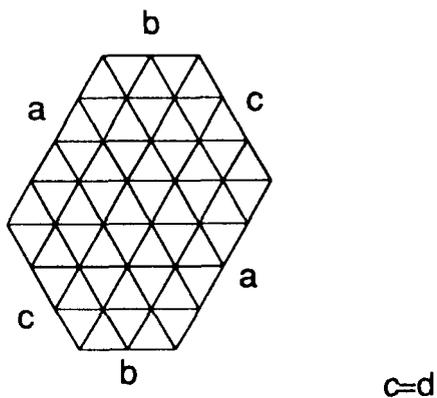
Figure 3

This is an interesting case since it degenerates into several useful profiles. The relations (4-7) transform to :

$$E = a + b + c + 3(ab + bc + ac) \quad \text{..............(8)}$$
$$V = a + b + c + ab + bc + ac + 1 \quad \text{..............(9)}$$
$$F = 2(ab + bc + ac) \quad \text{..............(10)}$$
$$Eb = Vb = 2(a + b + c) \quad \text{..............(11)}$$

When a=c=d, a hexagon with a 2-fold mirror symmetry is obtained (Fig.4). Note that this hexagon can also be produced in an alternative way by truncating the acute angles of a rhombus.

a=c=d
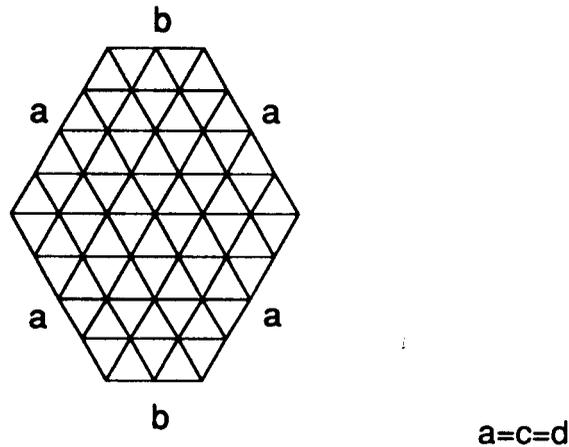
Figure 4

4.12   Profiles with 3- and 6-fold symmetry   (a=b=d)

A hexagon with 3-fold symmetry has the alternating sides equal (Fig.5) and is obtained by the special condition a=b=d in the relations (4-7).

a=b=d
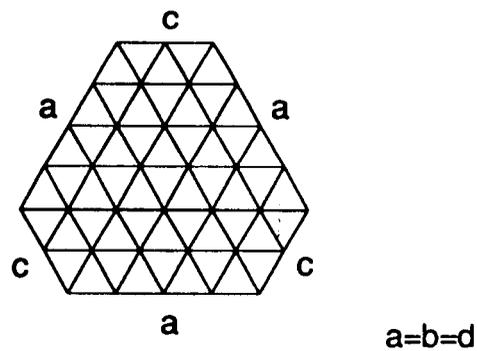
Figure 5

Hexagons of this type correspond to any layer obtained by slicing a subdivided octahedron away from its center. With this constraint, the relations (4-6) transform to :

$$E = \frac{3}{2}(a^2 + c^2) + \frac{3}{2}(a + c) + 6ac \qquad \dots\dots\dots (12)$$

$$V = \frac{1}{2}(a^2 + c^2) + \frac{3}{2}(a + c) + 2ac + 1 \qquad \dots\dots\dots (13)$$

$$F = a^2 + c^2 + 4ac \qquad \dots\dots\dots (14)$$

When a=c-1, the relations (12-14) transform further to the relations for the "three-sector" truss described in [4] since the profiles now lend themselves to a ring-count.

4.121   Hexagons with a 6-fold symmetry

When a=b=c=d, the hexagon of Figure 4 (or any earlier Figure) transforms to a regular hexagon which has a 6-fold symmetry (Fig. 6). This profile corresponds to the "six-sector" profile described in [4]. This layer is obtained when a slice is taken through the center of a subdivided octahedron.
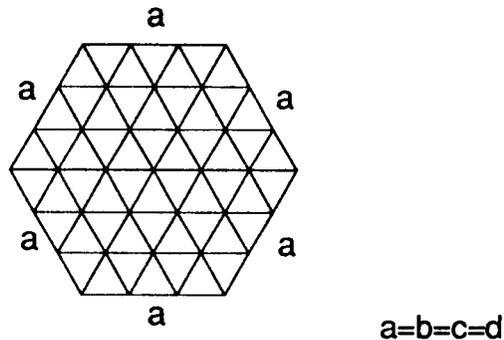
Figure 6

With these constraints, the special relations (12-14) further transform to the following and match the results reported in [4] :

$$E = 9a^2 + 3a \qquad \dots\dots\dots(15)$$

$$V = 3a^2 + 3a + 1 \qquad \dots\dots\dots(16)$$

$$F = 6a^2 \qquad \dots\dots\dots(17)$$

4.122 Equilateral Triangles

The minimum polygon obtained from the irregular hexagon is an equilateral triangle when a=b=d and c=0 (Fig. 7). It is the degenerate case of Figure 5 when c=0.
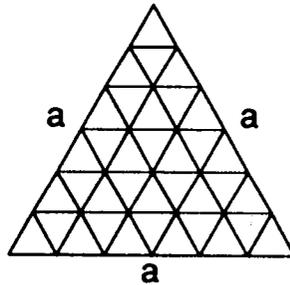
a=b=d

c=0

Figure 7

The special relations (12-14) transform to :

$$E = \frac{3}{2}a^2 + \frac{3}{2}a \qquad \text{.......... (18)}$$

$$V = \frac{1}{2}a^2 + \frac{3}{2}a + 1 \qquad \text{.......... (19)}$$

$$F = a^2 \qquad \text{.......... (20)}$$

4.13 Pentagonal Profiles

When d=0, the irregular hexagon of Fig. 1 transforms to an irregular pentagon (Fig. 8)
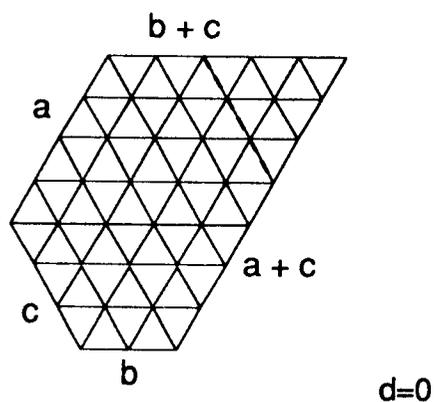
Figure 8

The irregular pentagon transforms to a pentagon with bilateral symmetry when a=b (Fig. 9).
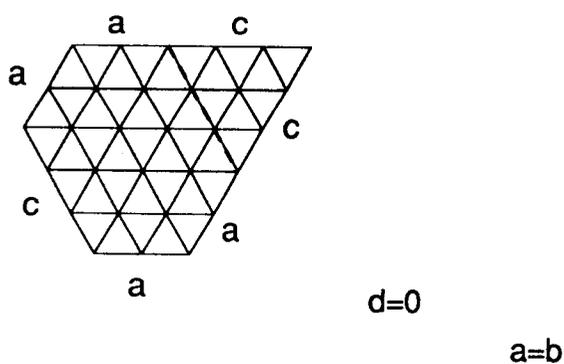
Figure 9

An interesting bilaterally symmetric pentagon is obtained when a=b=c and d=0 (Fig. 10). It is composed of a regular hexagon and a contiguous equilateral triangle and is a plane-filling pentagon.

a=b=c
d=0
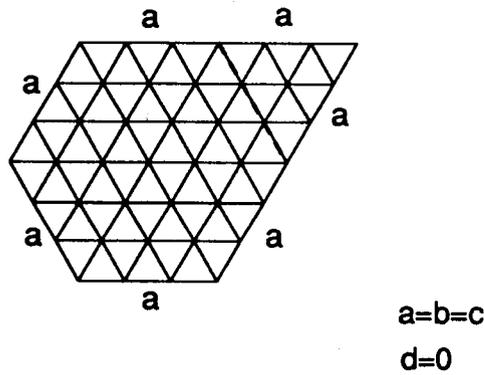
Figure 10

4.14   4-sided Polygons

The general hexagon of Figure 1 transforms to a parallelogram when c=d=0 (Fig.11), and further transforms to a rhombus when a=b (Fig. 12). The parallelogram is an interesting profile since all the profiles described here can be obtained from it by cutting off (truncating) its acute angles in varying degrees. Special cases are parallelogram "strips" when a=1.

c=d=0

Figure 11

a=b
c=d=0
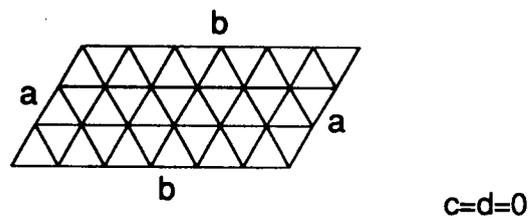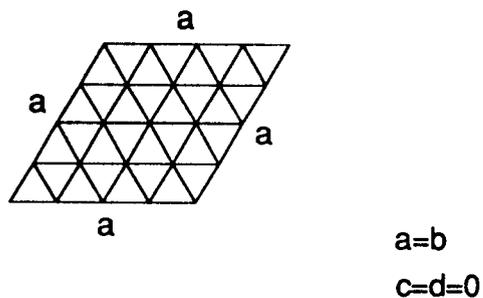
Figure 12

For the parallelogram, the relations (4-7) transform to :

$$E = a + b + 3ab \qquad \qquad ....(21)$$
$$V = a + b + ab + 1 \qquad \qquad ....(22)$$
$$F = 2ab \qquad \qquad ....(23)$$

The parallelogram of Figure 11 transforms to a trapezoid when a=d and c=0 (Fig. 13). It is composed of a parallelogram and a contiguous equilateral triangle. Special cases are trapezoidal "strips" when b=1. Note that the trapezoid can be derived from other profiles under different special conditions.
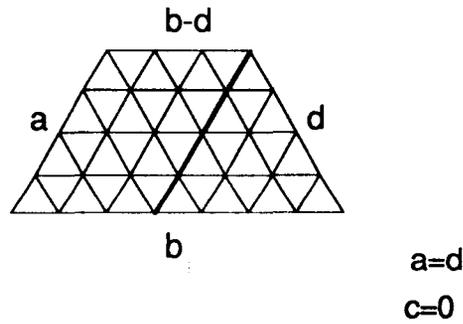
Figure 13

4.2 The Second Layer

Once a single layer profile is established, the next layer can be determined by simple rules. In the example shown here, the single layer already established is treated as the "bottom" layer. In Figure 14, the first (bottom) layer is shown in light lines and the second (top) layer in heavy lines. The bottom layer shown here is the irregular hexagon of Figure 1.
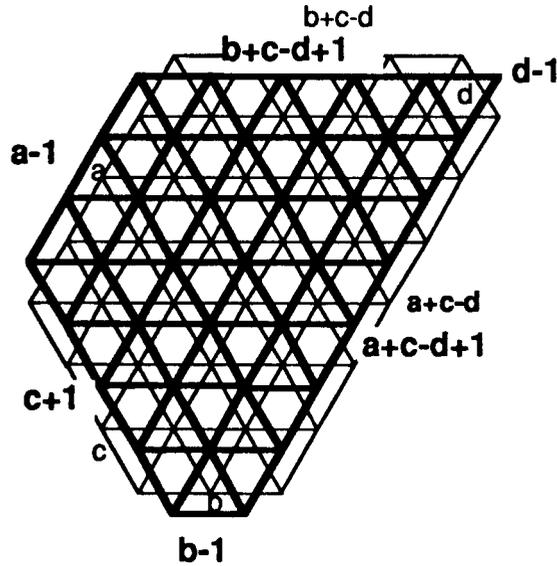
Figure 14

The successive edges on the boundary of the bottom layer are a, c, b, a+c-d, d, and b+c-d taken in a counter-clockwise sequence (these are indicated in smaller letters). The corresponding outer edges of the top layer are obtained by an alternation of -1 and +1 with respect to the outer edges of the bottom layer. That is, the six edges transform as follows: a becomes a-1, c becomes c+1, b becomes b-1, a+c-d becomes a+c-d+1, d becomes d-1, and b+c-d becomes b+c-d+1. The outer edges of the top layer can be described in terms of the new values a', b', c' and d' which correspond to a, b, c and d of the bottom layer. In the example shown, the new values change as follows :

$$a'=a-1 \qquad \text{.................. (24)}$$
$$b'=b-1 \qquad \text{.................. (25)}$$
$$c'=c+1 \qquad \text{.................. (26)}$$
$$d'=d-1 \qquad \text{.................. (27)}$$

The part count for the top layer can now be obtained from relations (4-7) by substituting a' for a, b' for b, c' for c and d' for d.

When the bottom layer changes to other profiles, the values of a', b', c' and d' vary according to the special conditions of the bottom layer. The following special conditions for the top layer are derived from the juxtaposition in Figure 14. Relations (24-27) hold for the top layer of the hexagon with 2-fold rotational symmetry shown in Figure 3. For the other hexagons, special conditions for the top layer are introduced, e.g., in the top layer of Figure 2, b'=d', and in the top of Figure 4, a'=d'. For the top layers of profiles with 3-fold or 6-fold symmetry (Figs. 5-7), a'=b'=d'. For the pentagonal profiles (Figs. 8-10), d'=0; in addition, for the pentagons of Figures 9 and 10, a'=b'. For the parallelogram and the rhombus of Figures 11 and 12, d'=0 in the top layer; for the rhombus a'=b' in addition. For the trapezoid of Figure 13, a'=d'.

In some cases, as in the triangular profile of Figure 7 and the 4-sided polygonal profiles of Figures 11-13, the top layer can be slightly modified by fixing c'=0. In the case of the triangle, this produces a larger triangle on the top layer with the special condition a'=b'=d'=a+1. In the case of the parallelogram and the rhombus of Figures 12 and 13, the top layer is identical to the bottom layer. In the case of the trapezoid of Figure 13, this restraint produces a larger trapezoid with the conditions a'=d'=a and b'=b+1.

Note that this procedure can be applied to the next (third) layer, and can be continued to any number of successive layers by modifying the values of a, b, c and d for each new layer.

4.3   The Core (Intermediate Layer) :

The number of edges (struts) in the inter-connecting "core" layer can be obtained from the number of vertices (nodes) in one layer, say, the bottom layer. Referring to Figure 14, if the nodes in the bottom layer are V_1, the number of struts E_c in the core layer are given by

$$E_c = 3. V_1 - (a + b + 3c - 2d - 3) \qquad \text{............... (28)}$$

This relation holds for all configurations derived from Figure 14. In each case the special conditions for the bottom layer described in the previous sections determine the constraints on a, b, c and d.

5. Remarks

The part-count relations for the *regular* tetrahedral-octahedral layered structures have been described; the term *regular* is based on the use of regular polygons (triangles in this case) which have equal lengths and equal angles. These relations hold for *any* truss configuration which is topologically isomorphic to the regular tetrahedral-octahedral truss. One example is Mikulas' "orthogonal tetrahedral truss" [2]. There are several other configurations from the 26-connected cubic node system which bear a one-to-one correspondence with the regular structure. These are derived by taking combinations of 6 vectors from the 13 vectors of the cubic node. In general, *any* 6 vector directions would suffice. Of these, interesting cases are the 'golden' tetrahedral-octahedral trusses which use two lengths in golden ratio, 1 and 1.61803......, [7]. In these cases, the node is based on icosahedral symmetry which permits a larger number of 6-vector combinations, and hence provides a greater geometric variety of layered octahedral-tetrahedral configurations.

It would be useful to extend the part-count relations to a broader class of space structures. These include a variety of polyhedra and zonohedra (a class of polyhedra with parallel faces), space-fillings, infinite polyhedra, toroidal structures and various classes of non-periodic space structures.

Meta-structural techniques, used by the author in previous works [8,9] would be useful in unifying the relations described. Meta-structures are structures-underlying-structures and provide a basis for a unified conceptual approach for a wide variety of morphological problems. Restricting to the geometry of space structures and to the triangulated profiles dealt with in this paper, *all* the profiles described can be mapped in a 4-dimensional space, where each direction of this space maps one of the four independent variables a, b, c and d. The 4-dimensional representation permits systematic inter-transformations of any profile to any other within this space by changing the variables in unit

increments. As before, each profile is a special case of the generalized hexagon of Figure 1. Such a unified organization of a space structure has implications at many levels ranging from interactive conceptual design exploration to automated manufacture and assembly of a variety of configurations.

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15. Supplementary Notes

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16. Abstract

The number of component part (nodes, struts and panels) termed part count, is an
important factor in the design, manufacture and assembly of modular space structures.
This paper presents part count expressions for a variety of profiles derived from
the layered octahedral-tetrahedral truss configuration. Referred to as the
"tetrahedral truss" in the NASA projects, this specific geometry has been used in
several missions. The general expressions presented here transforms to others as
one profile changes to another. Such transformational part count relations
provide a measure of flexibility and generality, and may be useful when dealing
with a wider range of geometric configurations.

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