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# THE EFFECTS OF GEAR REDUCTION ON ROBOT DYNAMICS

J. Chen
Department of Mechanical Engineering
University of Maryland
College Park, MD 20742

## Abstract

The effect of the joint drive system with gear reduction for a generic two-link system is studied. It is done by comparing the kinetic energy of such a system with that of a direct drive two-link system. The only difference are two terms involving the inertia of the motor rotor and gear ratio. Modifications of the equations of motion from a direct drive system are then developed and generalized to various cases encountered in robot manipulators.

## Introduction

Formulating the equations of motion for a robot is an important part of robot analysis that will provide necessary information for the design of control laws and mechanical components. Before the process of formulation can begin, idealization of the robot system into a model amenable to analysis has to be performed. Assumptions such as rigid bodies, perfect revolute joints, complete isolation between electrical phenomena and mechanical motions and idealized torque transmission in the gear trains are commonly made [1-5]. By removing one or several of these assumptions, one can come up with models with different levels of fidelity. The price to pay is the increased complexity in the resulted equations and possible numerical difficulties. But sometimes, the price has to be paid in order to obtain equations that correspond better with the important characteristics of the actual robot dynamics. In this paper, the effect of some idealizations of joint drive systems in the commonly used model will be investigated.

Although many robot joints are driven by motors through the use of gears with reasonably high (on the order of hundred) reduction ratio, the commonly used model does not include any detail of the drive system. Strictly speaking, the model of the generally used multi-body system is only directly related to a direct drive robot. For this model to be applied to a robot with torque transmission and amplification, certain rules are usually implied. It is generally believed that the torque at the joints are equal to the product of the corresponding motor torques and reduction ratios. It is also known to some that the rotational inertias of the motor rotors when amplified by the corresponding reduction ratios squared should be included in the link inertia to resist motion. These implications can be found in textbooks on automatic control, such as [6], but they are rarely spelled out in robot literature and their validity is not established.

In this paper, a simple model will be used to study the effect of gear reduction on system dynamics. Specifically, the necessary modifications to the equations of motion for the commonly used simply jointed robot model will be presented. An example will also be used to demonstrate the effects of gear reduction to the equations of motion.

## System Model

In the present study, it is assumed that the motor rotor is the only massive element in a joint drive system that will contribute to the modifications of the equations of motion from that of a multi-body direct drive system. To study this problem, one might be tempted to just study a particular system by including the rotors in the model and deriving equations for the system. Simulation can then be performed to hopefully reveal the effects caused by the inclusion of rotors in the model. This investigation, however, will be only specific for the particular system simulated and will not shed too much light for a system with very different geometry and mass distribution. The methodology adopted here is to understand the general effects of the rotors and thus to enumerate the suitable modifications of the equations due to them. In place of a specific model, a generic system model should be used and the resulted equations studied to identify and generalize the effects of rotors.

The generic model chosen is shown in Fig. 1. Body C is the carrier of the motor whose rotor together with the connecting shaft and attached gear forms a rigid body R. The driven link D has an attached gear which meshes with the gear in R. This is an idealized system of two links with simple drive system. Since it is assumed that the rotor is the only massive element in the drive system, this model is sufficiently comprehensive for the study. Instead of letting C be a link jointed to the base, it is allowed a general motion relative to the base. This is an intentional choice so that C can be any link of a multi-link system. The system can therefore be considered as a subsystem of an overall system. It can be seen that the linkage in Fig. 1 involves a closed-loop topology and formulations for the commonly studied open-loop multi-link system cannot be applied here. In particular, Newton-Euler formulation is not very convenient for this system and is especially difficult to generalize. Lagrange's or Kane's formulation is more suitable for the present study because generalized active and inertia forces can be considered as being contributed from individual elements of the system. The overall system can be thought of as having n generalized coordinates, $q_1, \ldots, q_n$, and q, the joint angle between C and D, can be one of them.

## Formulation of Dynamic Equations

Vector-dyadic formalism will be used in all the following formulations. Here, a vector and a dyadic are defined as abstract entities which are invariant with respect to unit vector bases [7-9]. They can be expressed in terms of any unit vector basis or bases but some operations among them can be reduced without being expressed in any basis. Vectors and dyadics as used in the following are therefore different from column matrices and square matrices, which are just congregates of numbers. However, when the vectors and the dyadics are represented in the same vector basis, their operations can be facilitated by matrix operations.

The first step in Lagrange's formulation is to derive the kinetic energy of the system. For the subsystem in Fig. 1, the contribution to the kinetic energy is

$$K = \frac{1}{2} \sum_{C+R+D} m({}^{N}v^{P})^{2} \tag{1}$$

where N is an inertia reference frame, m and ${}^{N}v^{P}$ are, respectively, the mass and the velocity in N of a generic particle P in the system, and $\sum_{C+R+D}$ denotes summation over all particles in bodies C, R and D. In the notation for a velocity, an acceleration, an angular velocity or an angular acceleration, a left superscript is used to denote the reference frame the quantity is referred to. In the sequel, when the left superscript is omitted in any of these notations, reference frame N is implied. Applying the theorem called one point moving on a rigid body [9], one can express the velocity of a generic particle P of R or D by

$$v^{P} = v^{\bar{C}} + {}^{C}v^{P} \tag{2}$$

where ${}^{C}v^{P}$ is the velocity of P in C, and $v^{\bar{C}}$ is the velocity of $\bar{C}$, which is a point of C that coincides with P at the instant under consideration. Substitution of equation (2) in equation (1) yields

$$K = \frac{1}{2} \{ \sum_{E} m(v^{P})^{2} + \sum_{R+D} m[({}^{C}v^{P})^{2} + 2\, v^{\bar{C}} \cdot {}^{C}v^{P}]\} \tag{3}$$

where E is a fictitious rigid body that moves exactly like C but has exactly the same mass distribution as that of C, R and D altogether at the instant under consideration.

With the application of a kinematic theorem for two points fixed on a rigid body [9], one can express the velocity of a generic particle P of E as

$$v^{P} = v^{Q} + \omega^{C} \times r^{QP} \tag{4}$$

where $\omega^{C}$ is the angular velocity of C in N, Q is an arbitrary reference point on C and $r^{QP}$ is the position vector from Q to P. The use of equation (4) brings the first term in the right hand side of equation (3) to the form of

$$\sum_{E} m(v^{P})^{2} = m_{E}(v^{Q})^{2} + \omega^{c} \cdot I^{E/Q} \cdot \omega^{c} + 2m_{E}\, v^{Q} \cdot (\omega^{c} \times r^{QE^{*}}) \tag{5}$$

where

$$I^{E/Q} = \sum_{E} m\, [(r^{QP})^{2}\, U - r^{QP}\, r^{QP}] \tag{6}$$

and, $m_E$ and $E^{*}$ are the mass and the mass center of E, respectively, while $r^{QE^{*}}$ is the position vector from Q to $E^{*}$, and U is a unit dyadic. It should be noticed that $E^*$ and $I^{E/Q}$ are not fixed in C. As for the second term in equation (3), the following equations can be similarly derived by suitably choosing a reference point. Firstly,

$$\sum_R m({}^C v^P)^2 = m_R({}^C v^{R^*})^2 + {}^C\omega^R \cdot I^{R/R^*} \cdot {}^C\omega^R \tag{7}$$

where

$$I^{R/R^*} = \sum_R m[(r^{R^*P})^2\, U - r^{R^*P}\, r^{R^*P}] \tag{8}$$

and, $m_R$ and $R^*$ are the mass and the mass center of R, while ${}^C v^{R^*}$ and ${}^C\omega^R$ are the velocity of $R^*$ in C and the angular velocity of R in C, respectively, and $r^{R^*P}$ is the position vector from $R^*$ to P. Next,

$$\sum_D m({}^C v^P)^2 = {}^C\omega^D \cdot I^{D/Q'} \cdot {}^C\omega^D \tag{9}$$

where

$$I^{D/Q'} = \sum_D m[(r^{Q'P})^2\, U - r^{Q'P}\, r^{Q'P}] \tag{10}$$

and, $r^{QP}$ and ${}^C\omega^D$ are the position vector from Q', a point on the joint axis, to P and the angular velocity of D in C, respectively. Then, making use of equation (4) for $v^P$, one can write

$$\sum_R m\, v^{\bar{C}} \cdot {}^C v^P = v^Q \cdot m_R\, {}^C v^{R^*} + \omega^C \cdot {}^C H^{R/Q} \tag{11}$$

where

$${}^C H^{R/Q} = \sum_R m\, r^{QP} \times {}^C v^P \tag{12}$$

and ${}^C H^{R/Q}$ is the angular momentum of R about Q in C. Similarly,

$$\sum_D m\, v^{\bar{C}} \cdot {}^C v^P = v^Q \cdot m_D\, {}^C v^{D^*} + \omega^C \cdot {}^C H^{D/Q} \tag{13}$$

where

$${}^C H^{D/Q} = \sum_D m\, r^{QP} \times {}^C v^P \tag{14}$$

Since $R^*$ is fixed in C which implies

$${}^C v^{R^*} = 0 \tag{15}$$

and

$$ {}^{C}\omega^{R} = \mu \, \dot{q} \, a \tag{16} $$

where $\mu$ is the gear ratio, it follows that

$$ {}^{C}H^{R/Q} = \mu \, J \, \dot{q} \, a \tag{17} $$

and

$$ {}^{C}\omega^{R} \cdot I^{R/R^*} \cdot {}^{C}\omega^{R} = \mu^2 \, J \, \dot{q}^2 \tag{18} $$

where J is the axial moment of inertia of R. Also since

$$ {}^{C}\omega^{D} = \dot{q} \, c \tag{19} $$

equation (9) can be rewritten as

$$ \sum_{D} m({}^{C}v^{P})^2 = \dot{q}^2 \, (c \cdot I^{D/Q'} \cdot c) \tag{20} $$

Substitution of equations (4)-(18) in equation (3) yields

$$ K = \frac{1}{2} \{m_E(v^Q)^2 + \omega^c \cdot I^{E/Q} \cdot \omega^c + \dot{q}^2(\mu^2 \, J + c \cdot I^{D/Q'} \cdot c)\} $$
$$ + \omega^c \cdot (r^{QE^*} \times m_E v^Q + \mu \, J \, \dot{q} \, a + {}^{C}H^{D/Q}) + m_D v^Q \cdot {}^{C}v^{D*} \tag{21} $$

It can be seen in equation (21) that the kinetic energy of the system involves only two terms, $\frac{1}{2} \mu^2 J\dot{q}^2$ and $\mu J\dot{q} \, \omega^c \cdot a$ that depend exclusively on the motor rotor R. Other contributions to K due to R are lumped in those terms involving the fictitious body E.

Consider now another system S' consisting of only two links C' and D' jointed together as shown in Fig. 2. The kinetic energy of S' will be the same as that in equation (21) less the two terms mentioned above if C, D and E are replaced by C', D' and E', respectively, and if E' is a fictitious body having the mass distribution of C' and D' at the instant under consideration. With perfect rotation of axisymmetric rotor R, the inertia dyadic of C and R together for Q is fixed C. Therefore a real rigid body C' can have the same mass distribution as C and R at all times and have the same motion as C. If this choice is made, and in addition, D' is chosen to have the mass distribution and the motion of D, then E' has the same mass distribution as that of E at all times. The kinetic energies K' and K , respectively, of S' and the original system S consisting of C, D and R can thus be related by

$$K = K' + \frac{1}{2}\mu^2 J \dot{q}^2 + \mu J \dot{q} \omega^C \cdot a \tag{22}$$

It is worth noticing that the common wisdom of simply adding $\mu^2 J$ to the "inertia of the driven link" to compensate for the drive system dynamics is only true when either $\omega^C$, the angular velocity of the carrier of the drive system, or $\omega^C \cdot a$ is zero. For many robots there are some drive systems that do not satisfy either condition.

The differences between generalized inertia force contributions due to S and S' can be worked out based on equation (22). Since generalized inertia force $F_r^*$ is related to kinetic energy K by

$$F_r^* = -\left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r}\right) \quad r = 1, \ldots, n \tag{23}$$

it follows that

$$(F_r^*)_S = (F_r^*)_{S'} + G_r^* \qquad r = 1, \ldots, n \tag{24}$$

where

$$G_r^* = \begin{cases} -\mu J(\mu\ddot{q} + \alpha^C \cdot a) & (q_r = q) \\ -\mu J\left\{\ddot{q}\,\dfrac{\partial(\omega^C \cdot a)}{\partial \dot{q}_r} + \dot{q}\left[\dfrac{d}{dt}\dfrac{\partial(\omega^C \cdot a)}{\partial \dot{q}_r} - \dfrac{\partial(\omega^C \cdot a)}{\partial q_r}\right]\right\} & (q_r \neq q) \end{cases} \tag{25}$$

The partial differentiations in equation (25) are most advantageously performed with C being the reference frame because a is fixed in C. Furthermore, if C is the ith link in an n-link articulated robot as shown in Fig. 3 and D is the subsequent link, then it can be shown that

$$\frac{\partial(\omega^C \cdot a)}{\partial \dot{q}_r} = \begin{cases} z_r \cdot a & (r \leq i) \\ 0 & (r > i) \end{cases} \tag{26}$$

$$\frac{\partial(\omega^C \cdot a)}{\partial q_r} = \frac{{}^C\partial\omega^C}{\partial q_r} \cdot a = \begin{cases} (\omega^{B_r} \times z_r) \cdot a & (r \leq i) \\ 0 & (r > i) \end{cases} \tag{27}$$

and

$$\frac{{}^C d}{dt} (z_r \cdot \underline{a}) = ({}^C\omega^{B_r} \times z_r) \cdot a = ({}^{B_i}\omega^{B_r} \times z_r) \cdot a \qquad (28)$$

With equations (26)-(28) in equation (25), one can rewrite $G_r^*$ as

$$G_r^* = \begin{cases} -\mu J (\mu \ddot{q}_{i+1} + {}^N\alpha^{B_i} \cdot a) & (r = i + 1) \\ -\mu J[(z_r \cdot a) \ddot{q}_{i+1} - \dot{q}_{i+1} ({}^N\omega^{B_i} \times z_r) \cdot a] & (r \leq i) \\ 0 & (r > i + 1) \end{cases} \qquad (29)$$

If the generalized inertia forces of system S' have been worked out separately, then that of S can be derived using equations (24) and (29).

As to the generalized active forces, consider that R is acted upon through electromagnetic or viscous damping interaction by C and the net result of this interaction is a couple of torque Ta. The laws of dynamics dictate that a couple of torque -Ta is also acted on C by R. The contribution of this pair of couples to the generalized active forces $F_r$ are simply

$$F_r = \begin{cases} \mu T & (r = i + 1) \\ 0 & (r \neq i + 1) \end{cases} \qquad (30)$$

No other interaction forces between the bodies in S contribute to the generalized active forces. For system S', it is easily seen that if a couple of torque μTc is assumed to act on D' by C', then the contributions of the interaction forces to the generalized active forces are the same as that in equation (30).

Generalization

In some situations, the joint drive system of a particular joint is mounted on the outward link rather than the inward link of the joint, such as that of the second joint of Unimation PUMA robots. The equations derived for S can still be applied with D being the inward link and C being the outward link. Consider that C is still the ith link, and D is the (i-1)th link. The difference terms $G_r^*$ in equation (24) become

$$G_r^* = \begin{cases} -\mu J [\mu q_i + {}^N\alpha^{B_i} \cdot a + \ddot{q}_i (z_i \cdot a) - \dot{q}_i({}^N\omega^{B_i} \times z_i) \cdot a] & (r = i) \\ -\mu J [(z_r \cdot a) \ddot{q}_i - \dot{q}_i({}^N\omega^{B_i} \times z_r) \cdot a] & (r < i) \\ 0 & (r > i) \end{cases} \qquad (31)$$

where unit vector a should be in the direction that is associated in the right hand sense with the rotation of R when the joint experiences a positive rotation. The above generalization is true because all the derivations for the generic system in Fig. 1 do not depend on whether C or D is the preceding link, with the exception of equations (16) and (19). When D is the link that precedes C, unit vectors **a** and **c** can be properly changed to maintain the validity of these equations. With this in mind, equation (22) remain valid for the new case. The difference from the original case is that $\omega^C$ in equation (22) is a function of q for the present case, and it therefore gives rise to the differences between the expressions in equations (29) and (31).

It is also observed that the above development applies to any axisymmetric body designated R that is carried on C and performs fixed axis rotation in C. Any gear in a gear train connecting a motor to the link it drives can be the rotor R and its contribution to the change of the generalized inertia forces can be identified. It should be noticed that there will be a different gear ratio and a different unit vector **a** for each gear.

Furthermore, one can see that the only role D plays in equations (29) or (31) is related to the definition of q which is used in equation (16). If the generalized coordinates can be properly introduced for the system and the angular velocity of R in C can be expressed as a function of these generalized coordinates, then the role of D can be eliminated. The above results can thus be extended to the drive system for linear joint. They can also be extended to complicated gear systems that make up many robot wrist mechanisms such as that discussed in [10]. For such a system with three degrees of freedom,

$$^{C}\omega^{R} = (\mu_1\dot{q}_1 + \mu_2\dot{q}_2 + \mu_3\dot{q}_3)\ \mathbf{a} \tag{32}$$

where $q_1$, $q_2$ and $q_3$ are the generalized coordinates associated with the system and $\mu_1$, $\mu_2$ and $\mu_3$ are the corresponding ratios for R, should be used instead of equation (16). With this, equation (22) is replaced by

$$K^S = K^{S'} + \frac{1}{2} J(\mu_1\dot{q}_1 + \mu_2\dot{q}_2 + \mu_3\dot{q}_3)^2 + J(\mu_1\dot{q}_1 + \mu_2\dot{q}_2 + \mu_3\dot{q}_3)\ \omega^C \cdot \mathbf{a} \tag{33}$$

where S is the system consisting of C and R while S' consists of C' only.

For a complete n degree of freedom motor-driven robot with speed reductions involved, the above procedure can be applied in the following manner:

1. Model the system as made up of n rigid bodies connected by the appropriate linear or revolute joints with each of these bodies having the mass distribution of the actual link and any rotational elements that it carries. Formulate the equations of motion for such a model.

2. For each of the rotational elements, figure out its additional contributions to the kinetic energy and in turn the contributions to the generalized inertia forces as developed above. Add these contributions to the equations of motion.

3. Use equation (30) to work out the generalized active forces.

## Significance of Reduction Effects

Dynamic equations for PUMA 560 robot has been explicitly derived with parameter values measured or estimated in [11]. Based on the parameter values listed, the effects of the drive systems can be estimated. For PUMA 560 robots, the 2nd and 3rd joint drive systems are mounted on the 2nd link while the 4th to 6th drive systems are mounted on the 3rd link. The motors are mounted in such a way that their axes of rotation are always perpendicular to the 2nd and the 3rd joint axes. If only the inertia matrix, which is the congregate of the coefficients of $\ddot{q}_i$, i= 1,..,6, and the motor rotor's contributions are considered, the coefficients to have additional terms include the diagonal elements as well as those with indices (1,i), i=2,..., 6, and the off-diagonal elements with indices (4,5), (4,6) and (5,6) due to the coupling of the drive systems of the wrist joints. It is understood that the inertia matrix is a symmetric matrix, and only the elements in the upper triangular part of the matrix are addressed.

Listed in [11] are inertia contributions of joint drive systems to the diagonal elements of the inertia matrix. Here, these are assumed to be mainly contributed by the motor rotors in the form of $\mu^2 J$ pursuant to equations (29) and (30). With this assumption, the constant coefficients of the dominant terms, involving sine and cosine functions of joint angles, of the matrix elements effected can be compared with those additional contributions proportional to $\mu J$ as computed based on equations (29) and (30). Table 1 shows the list.

Table 1 Effects of Drive System on Inertia Matrix

| Element | 1,1 | 2,2 | 3,3 | 4,4 | 5,5 | 6,6 | 1,2 | 1,3 | 1,4 | 1,5 | 1,6 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Dom. Coef. as in [2] | 2.57 | 6.79 | 1.16 | .20 | .18 | .19 | .69 | .13 | 1.64 E-3 | 1.25 E-3 | 4. E-5 |
| Coef. of add. term | 1.14 | 4.71 | .83 | .2 | .18 | .19 | .04 | .015 | 2.60 E-3 | 2.5 E-3 | 2.5 E-3 |

It can be seen from Table 1 that $\mu^2 J$ terms are dominant in the diagonal elements of the inertia matrix. They are included in the equations in [11] while the additional contributions to the off-diagonal elements that are proportional to $\mu J$ are not included. Due to the fact that their contributions remain constant when the robot posture is changed, the percentage variations of the diagonal elements is smaller than what it would be if the robot is a direct drive one. This makes fixed gain control more likely to succeed. Although some of the latter ones are dominant, they are still very small compared to the (1,1), or even the (4,4), (5,5) or (6,6) element. As to the elements (4,5), (4,6) and (5,6), since the coupling relationship between the drive systems are not discussed in [11], the additional contributions cannot be estimated. It is reasonable to predict that they will be more significant than those for other off-diagonal elements judging from equation (33).

Contributions due to the motor rotors to those terms second order in $\dot{q}_i$, i=1,.., 6, can also be estimated. Again, some of them may be dominant in the coefficient of a particular term, but their effect to the complete equations may not be significant unless $\dot{q}_i$, i=1,.., 6, assume significantly high values.

## Conclusions

Although it has been known to joint drive system designers that the inertia properties of the motor rotor and other elements connected to it are important factors in determining the system dynamic response, it has not been elaborated in so many articles on robot dynamics. The contributions in the form of $\mu^2 J$ in the inertia matrix can dominate some of the matrix elements. Other contributions proportional to $\mu J$ are less significant, but they may not be negligible in all cases. With more and very different robots to be developed in the future, it is important to know what need to be included in the dynamic model for it to have sufficient fidelity. The methodology set forth in this paper provides a means to gain the necessary information for a sound judgment.

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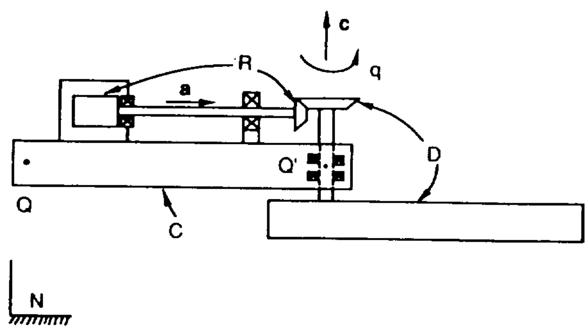


Fig. 1 Two Link System With Gear Reduction

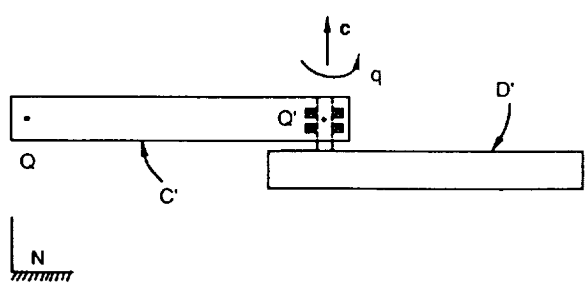


Fig. 2 Two Link Direct Drive System

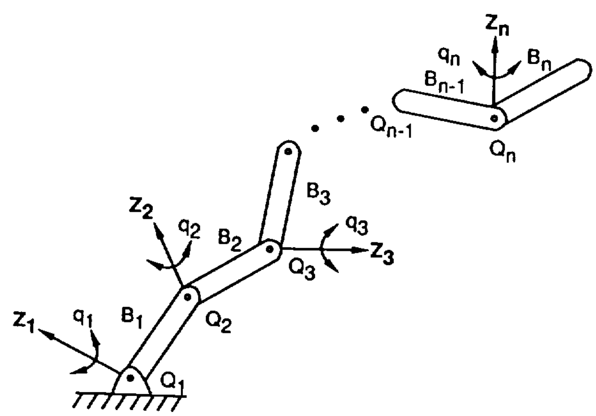


Fig. 3 Multi-Link System