RECURSIVE NEWTON-EULER FORMULATION OF MANIPULATOR DYNAMICS

M. G. Nasser

Lockheed Engineering & Sciences Company
P. O. Box 58561
Houston, Texas 77258

1. INTRODUCTION

This paper presents a new recursive Newton-Euler procedure for the formulation and solution of manipulator dynamical equations. The procedure includes rotational and translational joints and a topological tree. This model was verified analytically using a planar two-link manipulator. Also, the model was tested numerically against the Walker-Orin (ref. 1) model using the Shuttle Remote Manipulator System data. The hinge accelerations obtained from both models were identical. The computational requirements of the model vary linearly with the number of joints. The computational efficiency of this method exceeds that of Walker-Orin methods.

This procedure may be viewed as a considerable generalization of Armstrong's method (ref. 2). A six-by-six formulation is adopted which enhances both the computational efficiency and simplicity of the model.

In section 2.1, we begin with assuming an open chain, rotational joints, and prescribed base motion. In section 2.2, the procedure is extended to translational joints. Section 2.3 extends the formulation to a topological tree. Section 3 includes the algorithm summary and computational efficiency. The appendix contains descriptions of coordinate frames and notations and a summary of the standard kinematic relations used in the algorithm.

2. DYNAMICS FORMULATION

Let's begin with a quick look at the procedure. The first step is to set up the equations of motion for a generic link i (rotational) in the $i-1$ frame in a 6×6 form; namely, $S_i U_i = F_i^*$. U_i is a 6×1 vector consisting of the reaction loads from link $i-1$ on link i and $\ddot{\theta}_i$, the hinge acceleration of link i. S_i is a coefficient matrix, and F_i^* consists of the mass and inertia of link i (inertial parameters) acting on the inertial motion of the $i-1$ frame, nonlinear terms, body forces and torques, control torques, and reaction loads between link i and link $i+1$.

The procedure consists essentially of two phases, the inbound and the outbound. In the inbound phase, one begins at the free end, $i = N$. Since there is no outbound link, the reaction loads from link N on link $N+1$ are zero. Therefore, F^*_N is given by $F^*_N = A_{N,N-1}\, q_{N-1,N-1} + B_{N,N-1}$ where $A_{N,N-1}$ involves only link N inertial parameters.

Now $U^R_{N-1,N}$ [equation (2.1.7.1)] may be solved for in terms of $S_N^{-1}, A_{N,N-1}, q_{N-1,N-1}$, and $B_{N,N-1}$ but not $\ddot{\theta}_N$. Now we are ready to proceed to link $N-1$ and substitute $U^R_{N-1,N}$. However, $U^R_{N-1,N}$ must be transformed to the $N-2$ frame first. This transformation results in decomposing $(U^R_{N-1,N})_{N-2}$ into three terms: the first involving $\ddot{\theta}_{N-1}$; the second, $q_{N-2,N-2}$; and the third, a collection of nonlinear and forcing terms. This decomposition enables one to group these terms with their counterparts from link $N-1$. The resulting equation of motion is

$$L_{N-1} U_{N-1} = F_{N-1}$$

Note that in this equation of motion for link $N-1$, $\ddot{\theta}_N$ does not appear, only $\ddot{\theta}_{N-1}, \ddot{\theta}_{N-2}$, etc. Repeating the procedure by solving for $U^R_{i-1,i}$ for $i = N-2,\ N-3,\ \ldots$, we finally obtain the equation containing the hinge acceleration of the base link only.

For the outbound pass, beginning at the base link, link 2, we compute $\ddot{\theta}_2, (\dot{\mathbf{v}}_2)_2$, and $(\dot{\boldsymbol{\omega}}_2)_2$, then proceed to link 3 to compute $\ddot{\theta}_3, (\dot{\mathbf{v}}_3)_3$, and $(\dot{\boldsymbol{\omega}}_3)_3$, and so on to obtain all hinge accelerations.

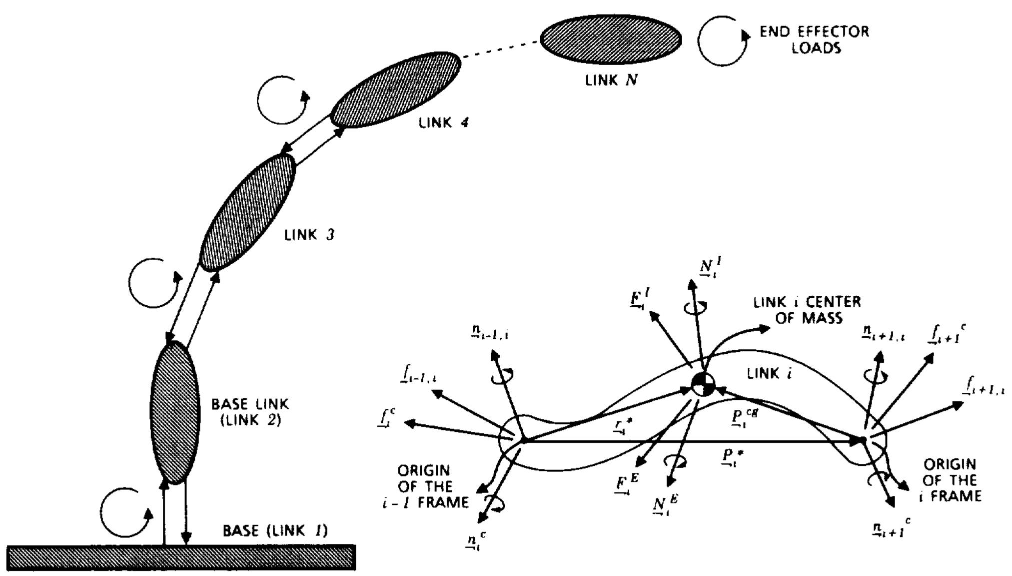
Now we proceed with a detailed description of the model.

2.1 MANIPULATOR WITH ROTATIONAL JOINTS

2.1.1 INBOUND PASS

The translational equation of motion for the center of mass of link i in the $i-1$ frame is (see figure 2-1 and the appendix)

$$\sum \mathbf{F}_i = \left(\frac{d\mathbf{p}_i}{dt}\right)_{i-1} = m_i\left(\dot{\mathbf{v}}_{i-1} + \boldsymbol{\omega}_i \times \left(\boldsymbol{\omega}_i \times \mathbf{r}_i^*\right) + \left(\dot{\boldsymbol{\omega}}_{i-1} + \boldsymbol{\omega}_{i-1} \times \mathbf{z}_{i-1}\dot{\theta}_i + \mathbf{z}_{i-1}\ddot{\theta}_i\right) \times \mathbf{r}_i^*\right) \quad (2.1.1)$$



OPEN KINEMATIC CHAIN

LINK i FREE BODY DIAGRAM

Definitions:

- $(\mathbf{F}_i^I)_{i-1}$ = the inertia forces developed in link i in the $i-1$ frame
- $(\mathbf{N}_i^I)_{i-1}$ = the inertia torques developed in link i in the $i-1$ frame
- $(\mathbf{f}_i^c)_{i-1}$ = the control forces applied at the proximal joint of link i in the $i-1$ frame
- $(\mathbf{f}_{i+1}^c)_{i-1}$ = the control forces applied at the distal joint of link i in the $i-1$ frame
- $(\mathbf{f}_{i+1,i})_{i-1}$ = the reaction force exerted on link i by link $i+1$ expressed in the $i-1$ frame
- $(\mathbf{n}_{i+1,i})_{i-1}$ = the reaction moment exerted on link i by link $i+1$ expressed in the $i-1$ frame
- $(\mathbf{f}_{i-1,i})_{i-1}$ = the reaction force exerted on link i by link $i-1$ expressed in the $i-1$ frame
- $(\mathbf{n}_{i-1,i})_{i-1}$ = the reaction moment exerted on link i by link $i-1$ expressed in the $i-1$ frame
- $(\mathbf{P}_i^*)_{i-1}$ = the position vector of the i frame relative to the $i-1$ frame and expressed in the $i-1$ frame
- $(\mathbf{n}_{i+1}^c)_{i-1}$ = the control torques applied at the distal joint of link i in the $i-1$ frame
- $(\mathbf{n}_i^c)_{i-1}$ = the control torques applied at the proximal joint of link i in the $i-1$ frame
- $(\mathbf{F}_i^E)_{i-1}$ = the external forces applied at the center of mass of link i in the $i-1$ frame
- $(\mathbf{N}_i^E)_{i-1}$ = the external torques applied at the center of mass of link i in the $i-1$ frame

Figure 2-1.

ORIGINAL PAGE IS
OF POOR QUALITY

$\Sigma \underset{\to}{F}_i$ is the total force exerted on the center of mass of link i in the $i-1$ frame. $\underset{\to}{p}_i$ is the linear momentum of the center of mass of link i in the $i-1$ frame.

$$\sum \underset{\to}{F}_i = \underset{\to}{f}_{i+1,i} + \underset{\to}{f}_{i-1,i} + \underset{\to}{F}_i^E - \underset{\to}{f}_{i+1}^c + \underset{\to}{f}_i^c \tag{2.1.2}$$

Substituting equation (2.1.2) into equation (2.1.1) yields the following translational equation of motion for any link i in the $i-1$ frame:

$$\underset{\to}{f}_{i-1,i} + m_i\left(\underset{\to}{r}_i^* \times \underset{\to}{z}_{i-1}\ddot{\theta}_i\right) = \underset{\to}{\alpha}_i + \underset{\to}{\beta}_i + \underset{\to}{f}_{i,i+1} - \underset{\to}{f}_i^c - \underset{\to}{f}_{i+1}^c + \underset{\to}{F}_i^E \tag{2.1.3}$$

$$\underset{\to}{\alpha}_i = m_i\left(\underset{\to}{\dot{v}}_{i-1} + \underset{\to}{\dot{\omega}}_{i-1} \times \underset{\to}{r}_i^*\right) \tag{2.1.3.1}$$

$$\underset{\to}{\beta}_i = m_i\left(\underset{\to}{\omega}_i \times \left(\underset{\to}{\omega}_i \times \underset{\to}{r}_i^*\right) + \left(\underset{\to}{\omega}_{i-1} \times \underset{\to}{z}_{i-1}\dot{\theta}_i\right) \times \underset{\to}{r}_i^*\right) \tag{2.1.3.2}$$

The rotational equations of motion for link i in the $i-1$ frame (torque balance about the proximal joint of link i) are

$$\left(\sum \underset{\to}{N}_i\right)_{i-1} = \frac{d}{dt}\left(\underset{\to}{r}_i^* \times \underset{\to}{p}_i + I_i \underset{\to}{\omega}_i\right)_{i-1} + \left(\underset{\to}{v}_{i-1} \times \underset{\to}{p}_i\right)_{i-1} \tag{2.1.4}$$

or

$$\left(\sum \underset{\to}{N}_i\right)_{i-1} = \underset{\to}{r}_i^* \times m_i \underset{\to}{\dot{v}}_i^{cg} + \left(I_i\right)_{i-1} \underset{\to}{\dot{\omega}}_i + \underset{\to}{\omega}_i \times \left(I_i\right)_{i-1} \underset{\to}{\omega}_i \tag{2.1.5}$$

$$\left(I_i\right)_{i-1} = R_{i-1,i} I_i R_{i,i-1} = J_i \tag{2.1.5.1}$$

$$\left(\sum \underset{\to}{N}_i\right)_{i-1} = \underset{\to}{n}_{i+1,i} + \underset{\to}{n}_{i-1,i} + \underset{\to}{N}_i^t + \underset{\to}{P}_i^* \times \underset{\to}{f}_{i+1,i} \tag{2.1.5.2}$$

$$\underset{\to}{N}_i^t = \underset{\to}{N}_i^E + \underset{\to}{r}_i^* \times \underset{\to}{F}_i^E - \underset{\to}{P}_i^* \times \underset{\to}{f}_{i+1}^c - \underset{\to}{n}_{i+1}^c + \underset{\to}{n}_i^c \tag{2.1.5.3}$$

The rotational equation of motion for arbitrary link i is:

$$\begin{aligned}\underset{\to}{n}_{i-1,i} + m_i \underset{\to}{r}_i^* \times \left(\underset{\to}{r}_i^* \times \underset{\to}{z}_{i-1}\ddot{\theta}_i\right) - J_i\left(\underset{\to}{z}_{i-1}\ddot{\theta}_i\right) &= \underset{\to}{\alpha}_i^* + \underset{\to}{\beta}_i^* + \underset{\to}{r}_i^* \times \underset{\to}{\beta}_i - \underset{\to}{N}_i^t + \underset{\to}{r}_i^* \times \underset{\to}{\alpha}_i \\ &\quad + \underset{\to}{n}_{i,i+1} - \underset{\to}{P}_i^* \times \underset{\to}{f}_{i+1,i}\end{aligned} \tag{2.1.6}$$

$$\underset{\to}{\alpha}_i^* = J_i \underset{\to}{\dot{\omega}}_{i-1} \tag{2.1.6.1}$$

$$\underset{\to}{\beta}_i^* = J_i\left[\underset{\to}{\omega}_{i-1} \times \underset{\to}{z}_{i-1}\dot{\theta}_i\right] + \underset{\to}{\omega}_i \times J_i \underset{\to}{\omega}_i \tag{2.1.6.2}$$

Equations (2.1.3) and (2.1.6) may be combined and written in the following matrix form:

$$S_i U_i = F_i^* \tag{2.1.7}$$

$$U_i = \left[U_{i-1,i}^R \ \ \ddot{\theta}_i\right] = \left[\underset{\to}{f}_{i-1,i}(1) \ \ \underset{\to}{f}_{i-1,i}(2) \ \ \underset{\to}{f}_{i-1,i}(3) \ \ \underset{\to}{n}_{i-1,i}(1) \ \ \underset{\to}{n}_{i-1,i}(2) \ \ \ddot{\theta}_i\right]^T \tag{2.1.7.1}$$

There is no reaction torque in the drive direction.

$$S_i = I - Z_i Z_i^T - A_{i,i-1} Z_i Z_i^T - Z_i Z_i^T J_i^a \tag{2.1.7.2}$$

where I is a 6×6 identity matrix, Z_i is its last column, and J_i^a is the actuator inertia associated with hinge i.

$$F_i^* = A_{i,i-1} q_{i-1,i-1} + B_{i,i-1} + \begin{bmatrix} U_{i,i+1}^R \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \underset{\to}{P}_i^* \times \underset{\to}{f}_{i,i+1} \end{bmatrix} \tag{2.1.7.3}$$

$$A_{i,i-1} \triangleq \begin{bmatrix} m_i[I] & -m_i\left[\tilde{r}_i^*\right] \\ m_i\left[\tilde{r}_i^*\right] & J_i - m_i\left[\tilde{r}_i^*\right]^2 \end{bmatrix} \tag{2.1.7.4}$$

$$q_{i-1,i-1} \triangleq \begin{bmatrix} \left(\dot{v}_{i-1}\right)_{i-1} \\ \left(\dot{\omega}_{i-1}\right)_{i-1} \end{bmatrix} \tag{2.1.7.5}$$

$$B_{i,i-1} \triangleq \begin{bmatrix} \beta_i - f_i^c + f_{i+1}^c - F_i^E \\ \beta_i^* + r_i^* \times \beta_i - N_i^t \end{bmatrix} \tag{2.1.7.6}$$

$[I]$ is a 3×3 identity matrix, and $[\tilde{r}_i^*]$ is a skew symmetric matrix associated with r_i^*.

Since the formal structure of equation (2.1.7) has been defined, consider link N (the link at the free end) and make use of the following boundary conditions:

$$\begin{bmatrix} f_{N,N+1} \\ n_{N,N+1} \end{bmatrix} = \phi = 0 \tag{2.1.8}$$

$$L_N \triangleq S_N \tag{2.1.9}$$

Therefore, equation (2.1.7) applied to link N is

$$F_N = A_{N,N-1} q_{N-1} + B_{N,N-1} \tag{2.1.10}$$

$$G_i \triangleq L_i^{-1} \quad , \quad \forall\, i = 1, 2, \ldots, N \tag{2.1.11}$$

$$U_N = G_N F_N \tag{2.1.12}$$

Although the expression for $\ddot{\theta}_N$ was obtained in equation (2.1.12), $\ddot{\theta}_N$ cannot be computed until $\dot{v}_{N-1}$ and $\dot{\omega}_{N-1}$ are. Therefore, proceed to link $N-1$ and set up equation (2.1.7) for $i = N-1$.

When transforming $(U^R_{i-1,i})$ into the $i-2$ frame, the following recursive relation is used:

$$\left(q_{i,i}\right)_{i-1} = P_i^T\left(q_{i-1,i-1} + \sigma_{i,i-1} + \sigma^*_{i,i-1}\right) \quad , \quad \forall\, i = 1, 2, \ldots, N \tag{2.1.13}$$

$$\sigma_{i,i-1} = \ddot{\theta}_i Z_i \tag{2.1.13.1}$$

$$\sigma^*_{i,i-1} = \begin{bmatrix} \omega_i \times \left(\omega_i \times P_i^*\right) \\ -\omega_{i-1} \times z_{i-1} \dot{\theta}_i \end{bmatrix} \tag{2.1.13.2}$$

$$P_i^T = \begin{bmatrix} I & -\left[\tilde{P}_i^*\right] \\ \phi & I \end{bmatrix} \tag{2.1.13.3}$$

I is a 3×3 identity matrix, and $[\tilde{P}_i^*]$ is a skew symmetric matrix associated with P_i^*.

$$\begin{bmatrix} U^R_{N-1,N} \\ 0 \end{bmatrix}_{N-2} = \left(A_{N,N-1}\right)_{N-2} P^T_{N-1}\left(q_{N-2} + \sigma_{N-1,N-2} + \sigma^*_{N-1,N-2}\right)$$

$$F^*_{N-1} = A_{N-1,N-2} q_{N-2,N-2} + B_{N-1,N-2} + \begin{bmatrix} U^R_{N-1,N} \\ 0 \end{bmatrix}_{N-2} + \begin{bmatrix} \phi \\ \underset{\rightharpoondown}{P}^*_{N-1} \times \underset{\rightharpoondown}{f}_{N-1,N} \end{bmatrix}_{N-2}$$

$$\begin{bmatrix} U^R_{N-1,N} \\ 0 \end{bmatrix}_{N-2} = \left[R^*_{N-2,N-1} \gamma G_N A_{N,N-1} R^{*T}_{N-2,N-1} \right] \left(q_{N-1,N-1} \right)_{N-2} + R^*_{N-2,N-1} \gamma G_N B_{N,N-1}$$

$$A_{N,N-2} = R^*_{N-2,N-1} \gamma G_N A_{N,N-1} R^{*T}_{N-2,N-1} \tag{2.1.14}$$

$$A^*_{N,N-2} = P_{N-1} A_{N,N-2} P^T_{N-1} \tag{2.1.15}$$

$$B_{N,N-2} = R^*_{N-2,N-1} \gamma G_N B_{N,N-1} \tag{2.1.16}$$

$$B^*_{N,N-2} = P_{N-1} B_{N,N-2} + A^*_{N,N-2} \sigma^*_{N-1,N-2} \tag{2.1.17}$$

The superscript T denotes the transpose operator.

$$R^*_{i-1,i} = \begin{bmatrix} R_{i-1,i} & \phi \\ \phi & R_{i-1,i} \end{bmatrix} \tag{2.1.18}$$

Obviously, upon substituting for $U^R{}_{N-1,N}$ into equation (2.1.18) for $i = N - 1$, we get

$$F^*_{N-1} = \left(A_{N-1,N-2} + A^*_{N,N-2} \right) q_{N-2,N-2} + B_{N-1,N-2} + B^*_{N,N-2} + A^*_{N,N-2} \sigma_{N-1,N-2}$$

Since $A^*{}_{N,N-2}\, \sigma_{N-1,N-2}$ is a function of $\ddot{\theta}_{N-1}$ only, it can be moved to the left-hand side to combine with its counterpart from link $N - 1$.

Thus, in general, the equation of motion for any link i takes the following form:

$$L_i U_i = F_i \tag{2.1.19}$$

$$L_i = S_i - A^*_{i+1,i-1} Z_i Z_i^T \tag{2.1.19.1}$$

$$F_i = A^*_{i,i-1} q_{i-1,i-1} + B^*_{i,i-1} \tag{2.1.19.2}$$

$$A^*_{i,i-1} = A_{i,i-1} + A^*_{i+1,i-1} \tag{2.1.19.3}$$

$$A^*_{i+1,i-1} = P_i R^*_{i-1,i} \gamma_{i+1} G_{i+1} A^*_{i+1,i} R^{*T}_{i-1,i} P^T_i \tag{2.1.19.4}$$

$$B^*_{i,i-1} = B_{i,i-1} + B^*_{i+1,i-1} \tag{2.1.19.5}$$

$$B^*_{i+1,i-1} = P_i B_{i+1,i-1} + A^*_{i+1,i-1} \sigma^*_{i,i-1} \tag{2.1.19.6}$$

$$B_{i+1,i-1} = R^*_{i-1,i} \gamma_{i+1} G_{i+1} B^*_{i+1,i} \tag{2.1.19.7}$$

2.1.2 OUTBOUND PASS

Assume a prescribed base motion. In this case, $\dot{\underset{\rightharpoondown}{v}}_1$, $\dot{\underset{\rightharpoondown}{\omega}}_1$, and $(\underset{\rightharpoondown}{v}_1, \underset{\rightharpoondown}{\omega}_1)$ are given. First compute F_2 and then solve for $\ddot{\theta}_2$ from the following equation.

$$\ddot{\theta}_2 = Z_2^T G_2 F_2 \tag{2.1.20}$$

Once $\ddot{\theta}_2$ is obtained, $(\dot{\underset{\sim}{v}}_2)_2$ and $(\dot{\underset{\sim}{\omega}}_2)_2$ can be computed. This completes the outbound computational cycle for the base link. Next we can move on to link 3 and repeat the same sequence – namely, compute F_3, $\ddot{\theta}_3$, $(\dot{\underset{\sim}{v}}_3)_3$, $(\dot{\underset{\sim}{\omega}}_3)_3$, $\ddot{\theta}_4$, etc., until all hinge accelerations are determined. Then we proceed to the integration phase.

2.2 MANIPULATOR WITH TRANSLATIONAL JOINTS

Some manipulators contain a mixture of translational and rotational joints. The procedure developed in the previous section for rotational joints is still applicable with slight modifications of the expressions involved (using the kinematics for translational link). These expressions include U_i, Z_i, $\sigma_{i,i-1}$, $\sigma_{i,i-1}^*$, $\underset{\sim}{\beta}_i$, and $\underset{\sim}{\beta}_i^*$. If we denote these variables by a prime to distinguish them from their rotational counterparts, we get

$$\underset{\sim}{\beta}_i' = m_i\left(\underset{\sim}{\omega}_{i-1} \times \left(\underset{\sim}{\omega}_{i-1} \times \underset{\sim}{v}_i^*\right) + 2\,\underset{\sim}{\omega}_{i-1} \times \underset{\sim}{z}_{i-1}\dot{\theta}_i\right) \tag{2.2.1}$$

$$\underset{\sim}{\beta}_i^{*'} = \underset{\sim}{\omega}_{i-1} \times J_i\,\underset{\sim}{\omega}_{i-1} \tag{2.2.2}$$

$$U_i' = \left[\underset{\sim}{f}_{i-1,i}(1) \quad \underset{\sim}{f}_{i-1,i}(2) \quad \ddot{\theta}_i \quad \underset{\sim}{n}_{i-1,i}(1) \quad \underset{\sim}{n}_{i-1,i}(2) \quad \underset{\sim}{n}_{i-1,i}(3)\right] \tag{2.2.3}$$

$$B_{i,i-1}' = \begin{bmatrix} \underset{\sim}{\beta}_i' - \underset{\sim}{f}_i^c - \underset{\sim}{f}_{i+1}^c + \underset{\sim}{F}_i^E \\ \underset{\sim}{\beta}_i^{*'} + \underset{\sim}{r}_i^* \times \underset{\sim}{\beta}_i' - \underset{\sim}{N}_i^t \end{bmatrix} \tag{2.2.4}$$

$$\sigma_{i,i-1}' = \begin{bmatrix} \underset{\sim}{z}_{i-1}\ddot{\theta}_i \\ \phi \end{bmatrix} \tag{2.2.5}$$

$$\sigma_{i,i-1}^{'*} = \begin{bmatrix} \underset{\sim}{\omega}_{i-1} \times \left(\underset{\sim}{\omega}_{i-1} \times \underset{\sim}{P}_i^*\right) + 2\,\underset{\sim}{\omega}_{i-1} \times \underset{\sim}{z}_{i-1}\dot{\theta}_i \\ \phi \end{bmatrix} \tag{2.2.6}$$

$$Z_i' = [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T \tag{2.2.7}$$

The remaining variables are defined as in the rotational joints case.

Therefore, the equations of motion for any link i may be written in the following form:

$$L_i U_i = F_i$$

where the formulas obtained in the rotational link case still hold. Note that the only distinction between rotational and translational joints is through the use of either $\underset{\sim}{\beta}_i$, $\underset{\sim}{\beta}_i^*$, U_i, $\sigma_{i,i-1}$, $\sigma^*_{i,i-1}$, and Z_i for rotational links or $\underset{\sim}{\beta}_i'$, $\underset{\sim}{\beta}_i^{*'}$, U_i', $\sigma'_{i,i-1}$, $\sigma^{*'}_{i,i-1}$, and Z_i' for translational links.

2.3 TOPOLOGICAL TREE

The case of a manipulator with tree topology does not alter the formulation in a fundamental manner. In fact, only the root links must be treated differently.

Consider the system shown in figure 2-2. For any branch b_i, we can proceed as in the open chain case until the root link is reached. Denote the root link by K; hence,

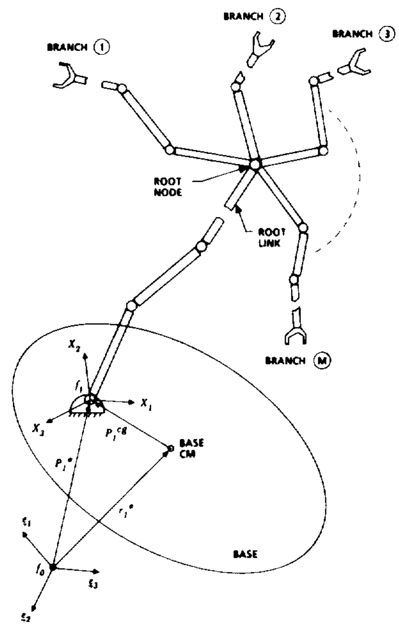


Figure 2-2.

$$\left(U^{R}_{K,K+1}\right)_{K} = \left(U^{R^{1}}_{K,K+1}\right)_{K} + \left(U^{R^{2}}_{K,K+1}\right)_{K} + \ldots + \left(U^{R^{m}}_{K,K+1}\right)_{K} \tag{2.3.1}$$

Recall that in the open chain case $(U_{i,i+1}{}^{R})_i$ was transformed to the $i-1$ frame and expanded in terms of $A^*{}_{i+1,i-1}$, $q_{i-1,i-1}$, and $B^*{}_{i+1,i-1}$.

Therefore, we get

$$\left(U^{R}_{K,K+1}\right)_{K-1} = \left(U^{R^{1}}_{K,K+1}\right)_{K-1} + \left(U^{R^{2}}_{K,K+1}\right)_{K-1} + \ldots + \left(U^{R^{m}}_{K,K+1}\right)_{K-1} \tag{2.3.2}$$

or

$$A^{*}_{K,K-1} = A_{K,K-1} + \sum_{j=1}^{m} A^{*j}_{K+1,K-1} \tag{2.3.3}$$

$$B^{*}_{K,K-1} = B_{K,K-1} + \sum_{j=1}^{m} B^{*j}_{K+1,K-1} \tag{2.3.4}$$

For any j, the definition of $A^{*j}{}_{K+1,K-1}$ and $B^{*j}{}_{K+1,K-1}$ is the same as that of the open chain.

3. ALGORITHM SUMMARY AND COMPUTATIONAL EFFICIENCY

3.1 OPEN KINEMATIC CHAIN

Start at the free end, $i = N$.

3.1.1 INBOUND PASS

Repeat the following sequence for $i = N, N-1, \ldots$:

1. Compute $A^*{}_{i+1,i-1}$ and $B^*{}_{i+1,i-1}$ (may be skipped for link N).
2. Compute $A^*{}_{i,i-1}$ and $B^*{}_{i,i-1}$.

3. Compute L_i and G_i.
4. $i = i - 1$ and repeat until $i = 2$.

3.1.2 OUTBOUND PASS

Prescribed base motion: $\dot{\underset{\sim}{v}}_1, \dot{\underset{\sim}{\omega}}_1, \underset{\sim}{\omega}_1$, and $\underset{\sim}{v}_1$ are given. Repeat the following steps for $i = 2, 3, \ldots N$.

1. Compute either F_i or $F_i{}'(i = 1)$.
2. Compute $\ddot{\theta}_i$.
3. Compute $(\dot{\underset{\sim}{\omega}}_i)_i$ and $(\dot{\underset{\sim}{v}}_i)_i$.
4. $i = i + 1$ and repeat steps 1 through 3.

3.2 TOPOLOGICAL TREE

3.2.1 INBOUND PASS

Apply the open kinematic chain procedure to all branches until the base node is reached in this case.

1. Compute $A^{*j}{}_{K+1,K-1}$ and $B^{*j}{}_{K+1,K-1}$ or $A^{*j'}{}_{K+1,K-1}$ and $B^{*j'}{}_{K+1,K-1}$ for all $j = 1, 2, \ldots, m$ where m is the number of branches at the base node.
2. Compute $A^{*}{}_{K,K-1}$ and $B^{*}{}_{K,K-1}$.
3. Repeat steps 2, 3, and 4 as in the open chain unless another is reached; in such case, repeat steps 1 and 2.

3.2.2 OUTBOUND PASS

No change.

3.3 COMPUTATIONAL EFFICIENCY

The number of multiplies is equal to $258N - 119$, and the number of adds is equal to $191N - 83$, where N is the number of links.

4. CONCLUSIONS

A general procedure for the formulation and solution of the equations of motion for a rigid manipulator has been presented. This procedure includes a solution for the tree topology. The extension to a closed kinematic chain follows naturally. However, the presentation of this extension is pending formal implementation and verification.

5. ACKNOWLEDGMENTS

I wish to thank Mr. Ken Hopping for the model implementation and for his suggestions and recommendations. I wish to express my deep appreciation to Mr. Carl Adams for his support throughout the development and verification process – in particular, in taking on the tedious task of verifying the two-link planar manipulator case analytically.

APPENDIX
LINK COORDINATE FRAME AND NOTATION

We adopt a dynamic reference frame. This frame is used here with the Denavit and Hartenberg convention (ref. 3). The joints are points of articulation between links and are numbered such that joint i connects link $i - 1$ and link i. Consequently, joints i and $i + 1$ are the proximal and distal joints, respectively, of link i. Each link i is assigned a Cartesian coordinate frame, (x_i, y_i, z_i), which is fixed on the link and therefore moves with it. (See figure A-1.)

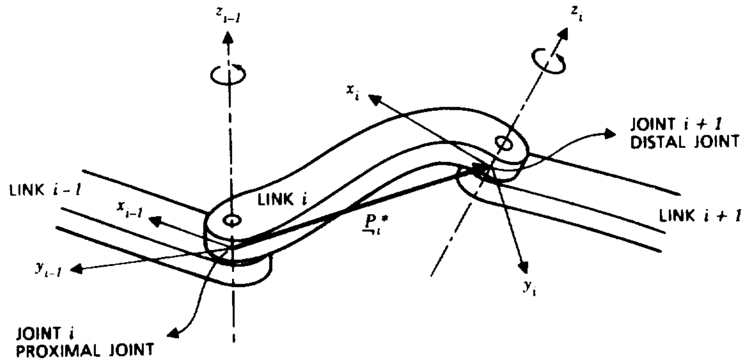


Figure A-1

The $\underset{\sim}{z}_i$ axis is the axis of the rotation/translation of the distal joint of link i. The $\underset{\sim}{x}_i$ axis is directed along the common normal from $\underset{\sim}{z}_{i-1}$ to $\underset{\sim}{z}_i$. The $\underset{\sim}{y}_i$ axis equals $\underset{\sim}{z}_i \times \underset{\sim}{x}_i$ to complete the right-handed system.

In order to associate a particular vector with the coordinate frame, an indexed parenthesis notation is introduced as follows.

$(\underset{\sim}{\theta}_i)_{i-1}$ = the link i relative displacement with respect to and expressed in the $i-1$ frame
$(\underset{\sim}{P}^*_i)_{i-1}$ = the position vector of the i frame relative to and expressed in the $i-1$ frame

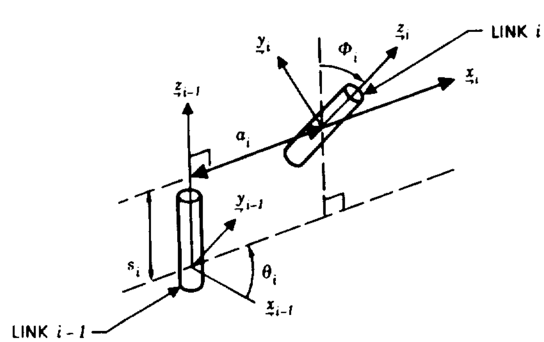
To relate two neighboring coordinate frames, a transformation <u>from</u> the $i-1$ frame <u>to</u> the i frame is defined as successive rotations of θ_i about the $\underset{\sim}{z}_{i-1}$ axis followed by ϕ_i about the x_i axis. (See figure A-2.) This is denoted as

$$R_{i,i-1} = Rot_{x_i}\left(\phi_i\right) Rot_{z_{i-1}}\left(\theta_i\right)$$

$$= \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\cos\phi_i \sin\theta_i & \cos\phi_i \cos\theta_i & \sin\phi_i \\ \sin\phi_i \sin\theta_i & -\sin\phi_i \cos\theta_i & \cos\phi_i \end{bmatrix} \tag{A.1.1}$$

$$R^{-1}_{i,i-1} = R^T_{i,i-1} = R_{i-1,i} \tag{A.1.2}$$

$$\left(\underset{\sim}{P}^*_i\right)_{i-1} = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ s_i \end{bmatrix} \tag{A.1.3}$$



Note: When the z_{i-1} and z_i axes are aligned, it implies that $\theta_i = 0$.

Figure A-2

The following is a set of standard kinematic relations (see figure A-3) for the motion of a rigid body relative to a moving reference frame.

$$\left(\underset{\rightarrow}{\omega}_s\right)_{i-1} = \begin{cases} \underset{\rightarrow}{z}_{i-1}\dot{\theta}_i \\ 0 \end{cases} \quad and \quad \left(\underset{\rightarrow}{\dot{\omega}}_s\right)_{i-1} = \begin{cases} \underset{\rightarrow}{z}_{i-1}\ddot{\theta}_i & if\,link\ i\ is\ rotational \\ 0 & if\,link\ i\ is\ translational \end{cases} \tag{A.2.1}$$

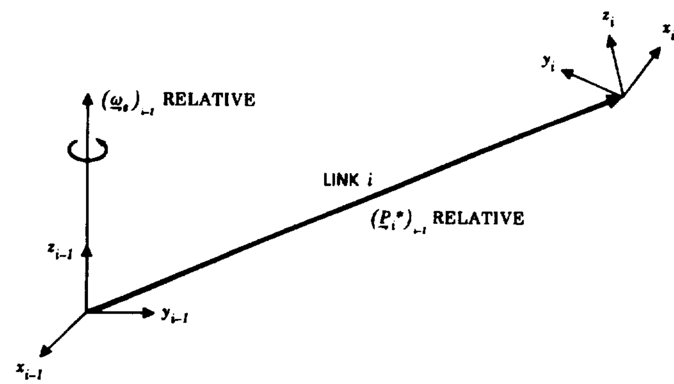


Figure A-3

$$\left(\underset{\rightarrow}{\theta}_i\right)_{i-1} = \begin{bmatrix} 0 & 0 & \theta_i \end{bmatrix}^T_{i-1} \tag{A.2.2}$$

$$\left(\underset{\rightarrow}{\dot{\theta}}_i\right)_{i-1} = \left(\underset{\rightarrow}{\omega}_s\right)_{i-1} = \begin{bmatrix} 0 & 0 & \dot{\theta}_i \end{bmatrix}^T_{i-1} \tag{A.2.3}$$

$$\left(\underset{\rightarrow}{\ddot{\theta}}_i\right)_{i-1} = \left(\underset{\rightarrow}{\dot{\omega}}_s\right)_{i-1} = \begin{bmatrix} 0 & 0 & \ddot{\theta}_i \end{bmatrix}^T_{i-1} \tag{A.2.4}$$

$$\left(\underset{\rightarrow}{\omega}_i\right)_{i-1} = \left(\underset{\rightarrow}{\omega}_{i-1}\right)_{i-1} + \left(\underset{\rightarrow}{\omega}_s\right)_{i-1} \tag{A.2.5}$$

REFERENCES

1. Orin, D. E.; and Walker, M. W.: Efficient Dynamic Computer Simulation of Robot Mechanisms. J. Dynamic Systems, Measurement, Control 104, 1982, pp. 205-211.

2. Armstrong, W. W.: Recursive Solution to the Equations of Motion of an n-Link Manipulator. Proc. 5th World Congress on Theory of Machines and Mechanisms, Montreal, July 1979, pp. 1343-1346.

3. Denavit, J.; and Hartenberg, R.S.: A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices. ASME Journal of Applied Mechanics, June 1955, pp. 215-221.