

PIEZOELECTRICALLY FORCED VIBRATIONS OF ELECTRODED DOUBLY ROTATED
QUARTZ PLATES BY STATE SPACE METHOD

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PROCEDURE OUTLINE

The purpose of this investigation is to develop an analytical method to study the vibration characteristics of piezoelectrically forced quartz plates. The procedure is schematically shown in Figure 1, and can be summarized as follows. The three dimensional governing equations of piezoelectricity, the constitutive equations and the strain-displacement relationships are used in deriving the final equations. For this purpose, a state vector consisting of stresses and displacements are chosen and the above equations are manipulated to obtain the projection of the derivative of the state vector with respect to the thickness coordinate on to the state vector itself. The solution to the state vector at any plane is then easily obtained in a closed form in terms of the state vector quantities at a reference plane. To simplify the analysis, simple thickness mode and plane strain approximations are used.

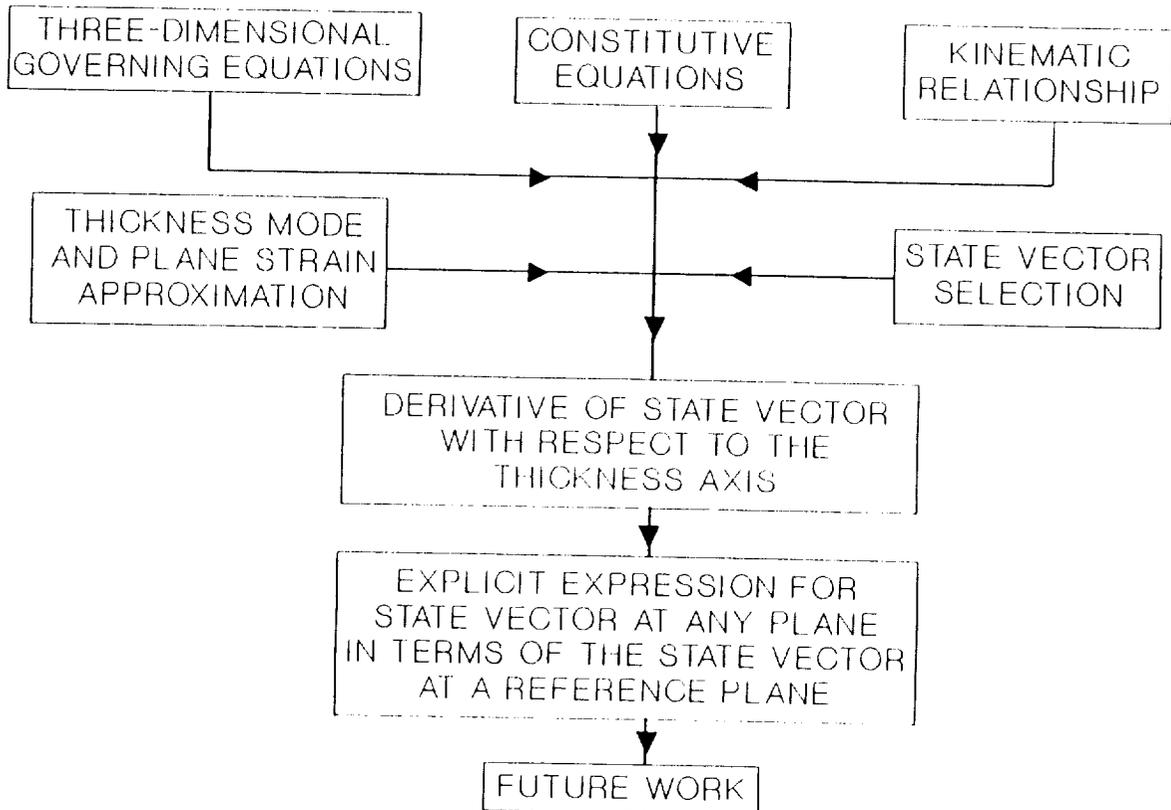


Figure 1

The governing equations of piezoelectricity consisting of the equations of motion and the charge equations of electrostatics are given by Equations (1) and (2). The quantities σ_{ij} , u_i and D_i are the components of stress, mechanical and electrical displacements. The constitutive equations are presented in Equations (3) and (4), where C_{ijkl} is the elastic stiffness, and ϵ_{kl} , e_{ijk} , E_i and S_{ij} are respectively the components of mechanical strain, piezoelectric strain constants, electric field and dielectric permittivity. The relationship between mechanical strain and displacement, and the relationship between electric field and electric potential are given in Equations (5) and (6) respectively.

EQUATIONS OF MOTION

$$\sigma_{ij,j} = \rho u_{i,tt} \quad (1)$$

CHARGE EQUATION OF ELECTROSTATICS

$$D_{i,i} = 0 \quad (2)$$

CONSTITUTIVE EQUATIONS

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} - e_{kij} E_k \quad (3)$$

$$D_i = e_{ijk} \epsilon_{jk} + S_{ij} E_j \quad (4)$$

$$\epsilon_{ij} = 0.5 (u_{j,i} + u_{i,j}) \quad (5)$$

$$E_i = -\phi_{,i} \quad (6)$$

Figure 2

The plane x_1 - x_3 is taken to be the plane of the plate, and the x_2 -direction is considered as the thickness coordinate. The simple thickness mode approximation, in which the various quantities are just functions of the thickness coordinate, is used in the analysis. Also, the system is considered to be under plane strain conditions. Invoking the above assumptions, and using the contracted notation given by Equation (7), the surviving system of equations are presented in Equations (8) through (10). Differentiating the last of the equations (10), using the third of Equation (8) and integrating the resulting equation twice, the expression for ϕ is obtained (Equation 11), where A and B are constants of integration. A constant field does not produce any electric field, hence the constant B in Equation (11) is neglected. Substituting Equation (11) in Equation (10), the expressions for the non zero stress components are obtained, and are given in Equations 12 and 13.

CONTRACTED NOTATION

$$11, 22, 33, 23 \text{ OR } 32, 31 \text{ OR } 13, 12 \text{ OR } 21 \quad (7)$$

$$\downarrow$$

$$1, 2, 3, 4, 5, 6$$

INVOKING SIMPLE THICKNESS MODE AND PLANE STRAIN ASSUMPTIONS

$$\left. \begin{aligned} \sigma_{6,x_2} &= \rho u_{1,tt} \\ \sigma_{2,x_2} &= \rho u_{2,tt} \\ D_{2,x_2} &= 0 \end{aligned} \right\} \quad (8)$$

$$\left. \begin{aligned} \epsilon_2 &= u_{2,x_2} \\ \epsilon_6 &= 1/2 u_{1,x_2} \\ E_2 &= -\phi_{,x_2} \end{aligned} \right\} \quad (9)$$

$$\left. \begin{aligned} \sigma_2 &= 1/2 C_{26} u_{1,x_2} + C_{22} u_{2,x_2} + e_{22} \phi_{,x_2} \\ \sigma_6 &= C_{66} u_{1,x_2} + C_{62} u_{2,x_2} + e_{26} \phi_{,x_2} \\ D_2 &= 1/2 e_{26} u_{1,x_2} + e_{22} u_{2,x_2} - S_{22} \phi_{,x_2} \end{aligned} \right\} \quad (10)$$

$$\phi = (1/2 e_{26} u_1 + e_{22} u_2) / S_{22} + Ax_2 + B \quad (11)$$

$$\sigma_2 = a_{26} u_{1,x_2} + a_{22} u_{2,x_2} + e_{22} A \quad (12)$$

$$\sigma_6 = a_{66} u_{1,x_2} + a_{62} u_{2,x_2} + e_{26} A \quad (13)$$

Figure 3

A state vector $\{V\}$ defined by Equation (14) is chosen. The derivatives of the state vector with respect to x_2 is obtained from Equations (8) through (13) and the resulting expressions are given in Equations (17) through (20). The elements of the matrices B_1 , B_2 and B_3 are made up of the material constants and derivatives with respect to x_2 and time (t).

STATE VECTOR

$$\{V\} = [\{V_1\}^T \{V_2\}^T]^T \quad (14)$$

$$\{V_1\} = [u_1 \sigma_2]^T \quad ; \quad \{V_2\} = [\sigma_6 u_2]^T$$

$$\{V_1\}_{,x_2} = [B_1] \{V_2\} - A \{b_1\} \quad (15)$$

$$\{V_2\}_{,x_2} = [B_2] \{V_1\} - A \{b_2\} \quad (16)$$

$$\{V\}_{,x_2} = [B_3] \{V\} - A \{b_3\} \quad (17)$$

$$B_1 = \begin{bmatrix} 1/a_{66} & -(a_{62}/a_{66}) \partial/\partial x_2 \\ 0 & \rho \partial^2/\partial t^2 \end{bmatrix} \quad b_1 = \begin{bmatrix} e_{26}/a_{66} \\ 0 \end{bmatrix} \quad (18)$$

$$B_2 = \begin{bmatrix} \rho \partial^2/\partial t^2 & 0 \\ -(a_{26}/a_{22}) \partial/\partial x_2 & 1/a_{22} \end{bmatrix} \quad b_2 = \begin{bmatrix} 0 \\ e_{22}/a_{22} \end{bmatrix} \quad (19)$$

$$\begin{aligned} a_{62} &= C_{62} + e_{26}e_{22}/S_{22} & ; & \quad a_{66} = 1/2 \{C_{66} + e_{26}^2/S_{22}\} \\ a_{26} &= 1/2 \{C_{26} + e_{22}e_{26}/S_{22}\} & ; & \quad a_{22} = C_{22} + e_{22}^2/S_{22} \end{aligned} \quad (20)$$

Figure 4

A solution to the differential equation given in Equation (11) can be easily obtained and is given in Equation (21), where $\{V_0\}$ is the state vector evaluated at $x_2=0$. The analyst has the flexibility of choosing any plane as the appropriate reference plane. The exponential term in Equation (21) can be expressed in an infinite series, and the powers of the matrix B_3 can conveniently be grouped as shown in Equation (23).

$$V = e^{B_3 x_2} \{V_0\} + B_3^{-1} A b_3 \quad (21)$$

$$\{V_0\} = \{V\}_{x_2=0}$$

$$e^{B_3 x_2} \{V_0\} = \left[I + B_3 x_2 + \frac{(B_3 x_2)^2}{2!} + \frac{(B_3 x_2)^3}{3!} + \dots \right] \{V_0\} \quad (22)$$

$$B_3^2 = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \quad B_3^3 = \begin{bmatrix} 0 & B_1 Q \\ B_2 P & 0 \end{bmatrix} \quad (23)$$

$$B_3^4 = \begin{bmatrix} P^2 & 0 \\ 0 & Q^2 \end{bmatrix} \quad B_3^5 = \begin{bmatrix} 0 & B_1 Q^2 \\ B_2 P^2 & 0 \end{bmatrix}$$

$$[P] = [B_1] [B_2]$$

$$[Q] = [B_2] [B_1]$$

Figure 5

Using the expressions given in Equation (23), the infinite series expansion for the exponential term can be conveniently grouped as shown in Equation (24). The elements present in Equation (24) can be recognized as a convergent series. The resultant expression is given by Equation (25). Substituting this expression in Equation (21), the final equation for the state vector at any reference plane in terms of the state vector at a reference plane is obtained (Equation 26).

$$\begin{aligned}
 e^{B3x2} \{V0\} = & \left[I + \begin{bmatrix} 0 & B1 \\ B2 & 0 \end{bmatrix} x2 + \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x2^2/2! + \right. \\
 & \left. \begin{bmatrix} 0 & B1Q \\ B2P & 0 \end{bmatrix} x2^3/3! + \begin{bmatrix} P^2 & 0 \\ 0 & Q^2 \end{bmatrix} x2^4/4! + \right. \\
 & \left. \begin{bmatrix} 0 & B1Q^2 \\ B2P^2 & 0 \end{bmatrix} x2^5/5! + \dots \right] \{V0\} \quad (24)
 \end{aligned}$$

$$= \begin{bmatrix} \cosh (x2/P) & B1 \sqrt{Q} \sinh (x2/Q) \\ B2/P \sinh (x2/P) & \cosh (x2/Q) \end{bmatrix} \{V0\} \quad (25)$$

$$= [R] \{V0\}$$

$$\{V\} = [R] \{V0\} + A [B3]^{-1} \{b3\} \quad (26)$$

Figure 6

Symbols and Abbreviations

A, B	Constants of integration
a_{ij}	Constant coefficients
C_{ijkl}	Elastic stiffness
D_i	Components of electric displacement
e_{kij}	Components of piezoelectric strain constant
E_k	Components of electric field
S_{ij}	Components of dielectric permittivity
u_i	Components of mechanical displacement
ϵ_{kl}	Components of strain
ϕ	Electric potential
ρ	Mass density
σ_{ij}	Stress components