Time Dependent Viscous Incompressible Navier-Stokes Equations

By

John W. Goodrich
Computational Fluid Dynamics Branch
Internal Fluid Mechanics Division
NASA Lewis Research Center
Cleveland, Ohio 44135

TIME DEPENDENT INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) - \frac{1}{Re} \triangle \mathbf{u} = -\nabla p + \mathbf{F}, \quad \text{for } \mathbf{x} \text{ in } \Omega, \text{ and } t > 0,$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{for } \mathbf{x} \text{ in } \Omega, \text{ and } t > 0.$$

* Ω is a bounded open region in $\mathbf{R}^2$, with boundary $\partial\Omega$.
* Initial and boundary conditions must be supplied.
* $\mathbf{F}$ is the volume force per unit mass, assumed to be 0.

$\Longrightarrow$ The continuity equation is not given in a time evolution form.

$\Longrightarrow$ The pressure gradient couples the continuity equation to the momentum equations.

STREAMFUNCTION EQUATIONS FOR UNSTEADY INCOMPRESSIBLE FLOW

$$\frac{\partial \triangle \psi}{\partial t} = \frac{1}{Re}\triangle^2 \psi + \frac{\partial \psi}{\partial x}\triangle\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y}\triangle\frac{\partial \psi}{\partial x}, \quad \text{for } \mathbf{x} \text{ in } \Omega, \text{ and } t > 0;$$

with

$$u(\mathbf{x},t) = \frac{\partial \psi}{\partial y}, \text{ and } v(\mathbf{x},t) = -\frac{\partial \psi}{\partial x}, \quad \text{for } \mathbf{x} \text{ in } \Omega, \text{ and } t \geq 0.$$

* For Ω in $\mathbf{R}^2$.

* Initial and boundary conditions must be supplied.

$\Longrightarrow$ Vorticity and pressure do not enter into the streamfunction formulation.

$\Longrightarrow$ The velocity solution is always divergence free, and so incompressible.

THE STREAMFUNCTION ALGORITHM FOR UNSTEADY INCOMPRESSIBLE FLOW

$$
\begin{aligned}
&\mathrm{La}(\tilde{\mathbf{z}}^{n+1}) - \frac{\Delta t}{2Re}\mathrm{Bi}(\tilde{\mathbf{z}}^{n+1}) \\
&= \mathrm{La}(\tilde{\mathbf{z}}^{n}) + \frac{\Delta t}{2Re}\mathrm{Bi}(\tilde{\mathbf{z}}^{n}) - \frac{3\Delta t}{2}\left[\delta_x\left(\delta_y(\tilde{\mathbf{z}}^{n})\mathrm{La}(\tilde{\mathbf{z}}^{n})\right) - \delta_y\left(\delta_x(\tilde{\mathbf{z}}^{n})\mathrm{La}(\tilde{\mathbf{z}}^{n})\right)\right] \\
&\quad + \frac{\Delta t}{2}\left[\delta_x\left(\delta_y(\tilde{\mathbf{z}}^{n-1})\mathrm{La}(\tilde{\mathbf{z}}^{n-1})\right) - \delta_y\left(\delta_x(\tilde{\mathbf{z}}^{n-1})\mathrm{La}(\tilde{\mathbf{z}}^{n-1})\right)\right],
\end{aligned}
$$

with

$$u^n_{i,j} = \frac{1}{2\Delta y}(z^n_{i,j+1} - z^n_{i,j-1}), \quad \text{and} \quad v^n_{i,j} = -\frac{1}{2\Delta x}(z^n_{i+1,j} - z^n_{i-1,j}).$$

* La and Bi are central difference approximations to the Laplace and Biharmonic operators.
* δ_x and δ_y are conventional centered difference operators.

$\Longrightarrow$ In $\mathbf{R}^2$ there is one unknown $\{z^n_{i,j}\}$ per grid cell instead of three.

$\Longrightarrow$ The velocity components and streamfunction are all defined at each grid point.

$\Longrightarrow$ The discrete solution is exactly incompressible, $\delta_x(u^m_{i,j}) + \delta_y(v^m_{i,j}) = 0$.

$\Longrightarrow$ Stability limit is Courant number < 1.

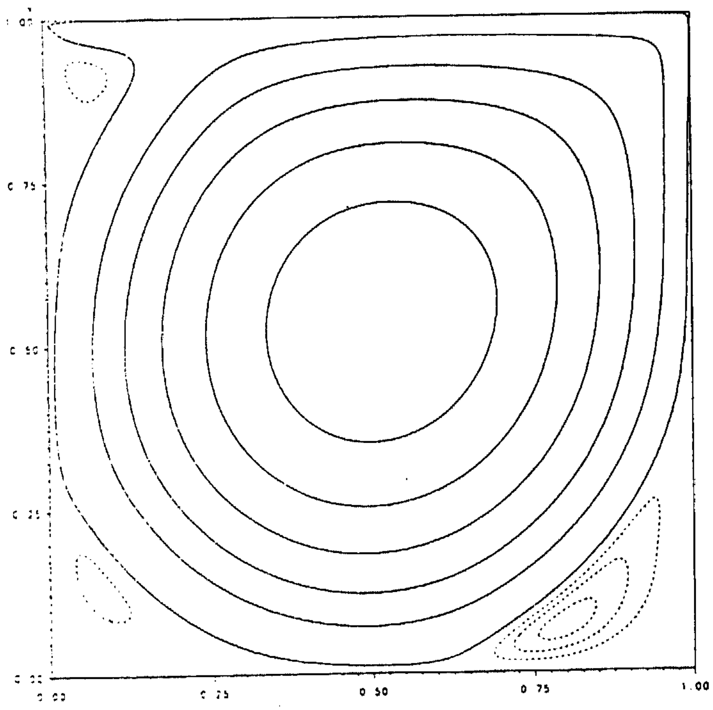
A MULTIGRID SOLVER FOR THE LINEAR IMPLICIT EQUATIONS

$$\mathrm{La}(\tilde{\mathbf{z}}^{n+1}) - \frac{\Delta t}{2Re}\mathrm{Bi}(\tilde{\mathbf{z}}^{n+1}) = Source\ \ Term(\tilde{\mathbf{z}}^{n}, \tilde{\mathbf{z}}^{n-1})$$

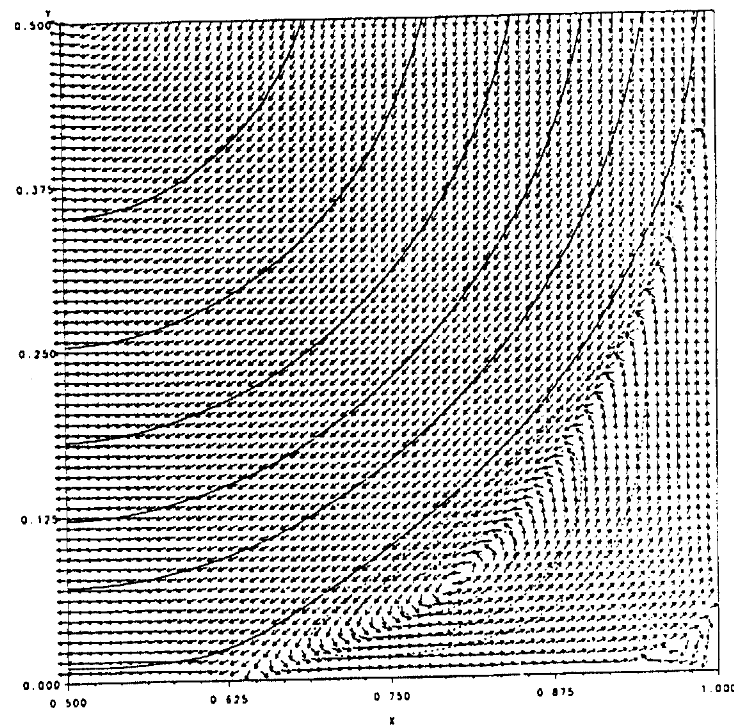
$\Longrightarrow$ Use a multigrid solver for the implicit equations at each time step.

* The Biharmonic operator is factored as two Laplacians.
* On a 256 by 256 fine grid, 7 grid levels are used in 6.8 MBytes storage.
* Point Gauss-Seidel smoothing, linear restriction and prolongation.
* A V-cycle with 3 iterations per grid level while coarsening, none while refining.
* 10 to 15 iteration cycles reduce residuals to less than 5.0×10^{-11}.

STREAM FUNCTION CONTOURS
Re=5k, 128*128 grid, t=491.80625

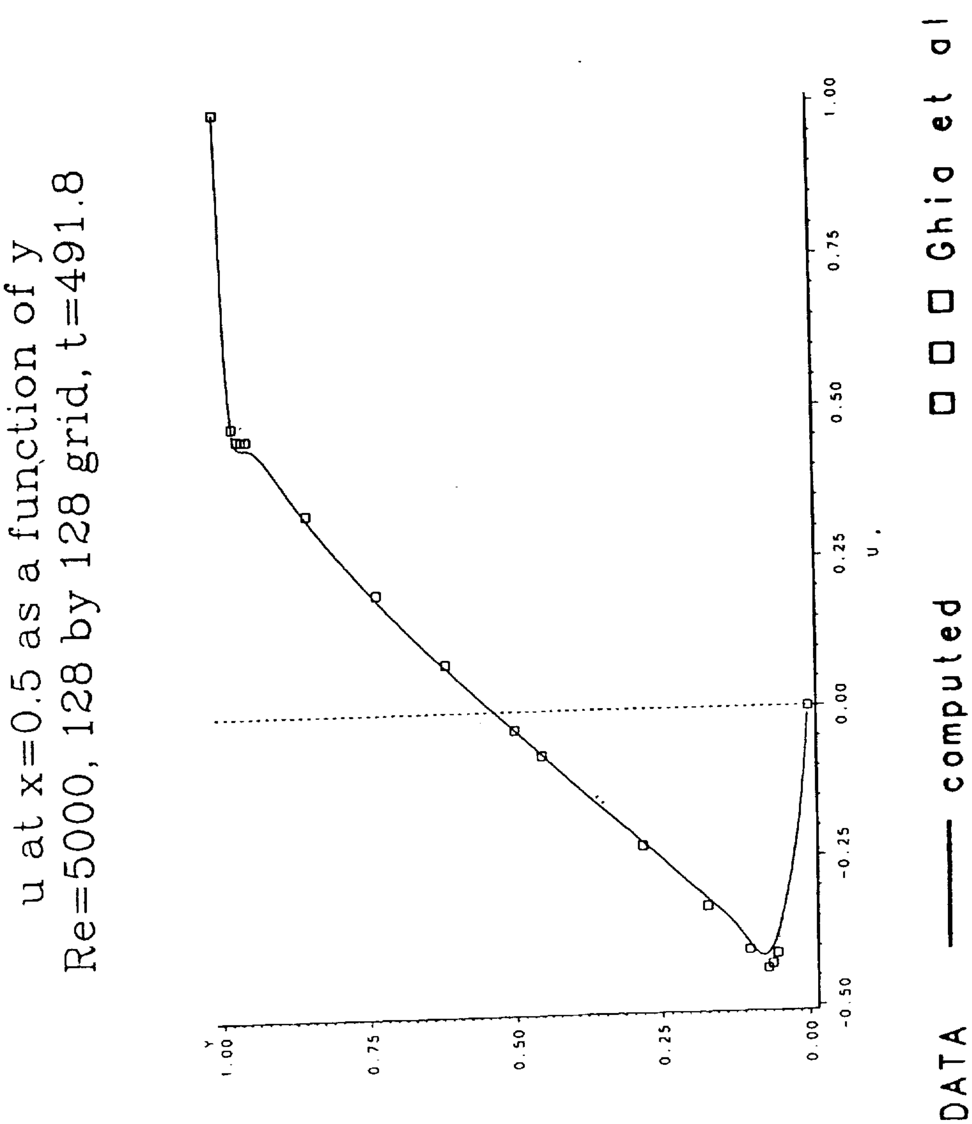


STREAM FUNCTION CONTOURS – NORMALIZED VECTOR PLOTS
Re=5k, 128*128 grid, t=491.80625
0.5<=x<=1.0 and 0.0<=y<=0.5

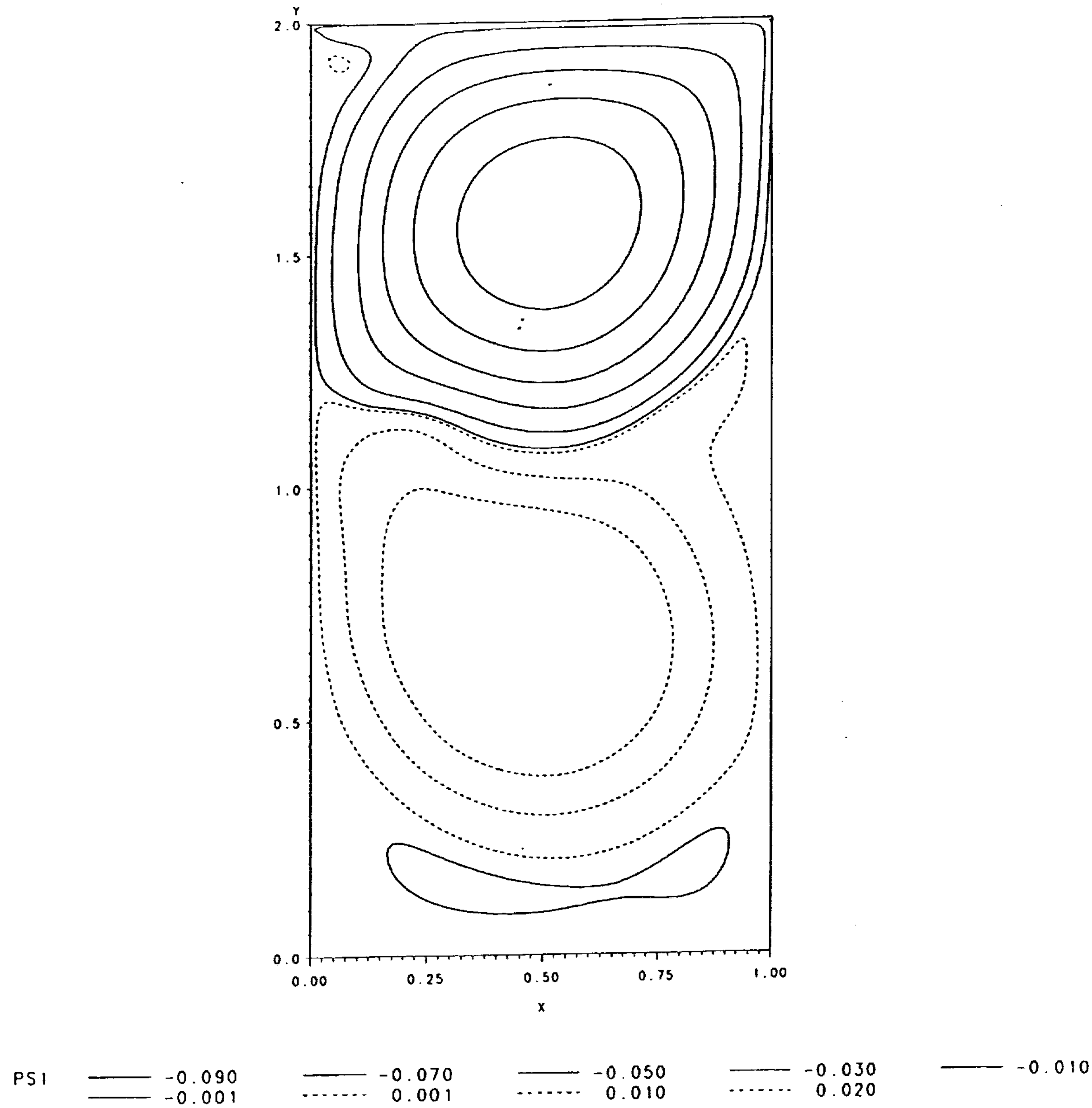


PSI
-0.100 / -0.001
-0.080 / 0.001
-0.060 / 0.002
-0.040 / 0.003
-0.020

ORIGINAL PAGE IS
OF POOR QUALITY

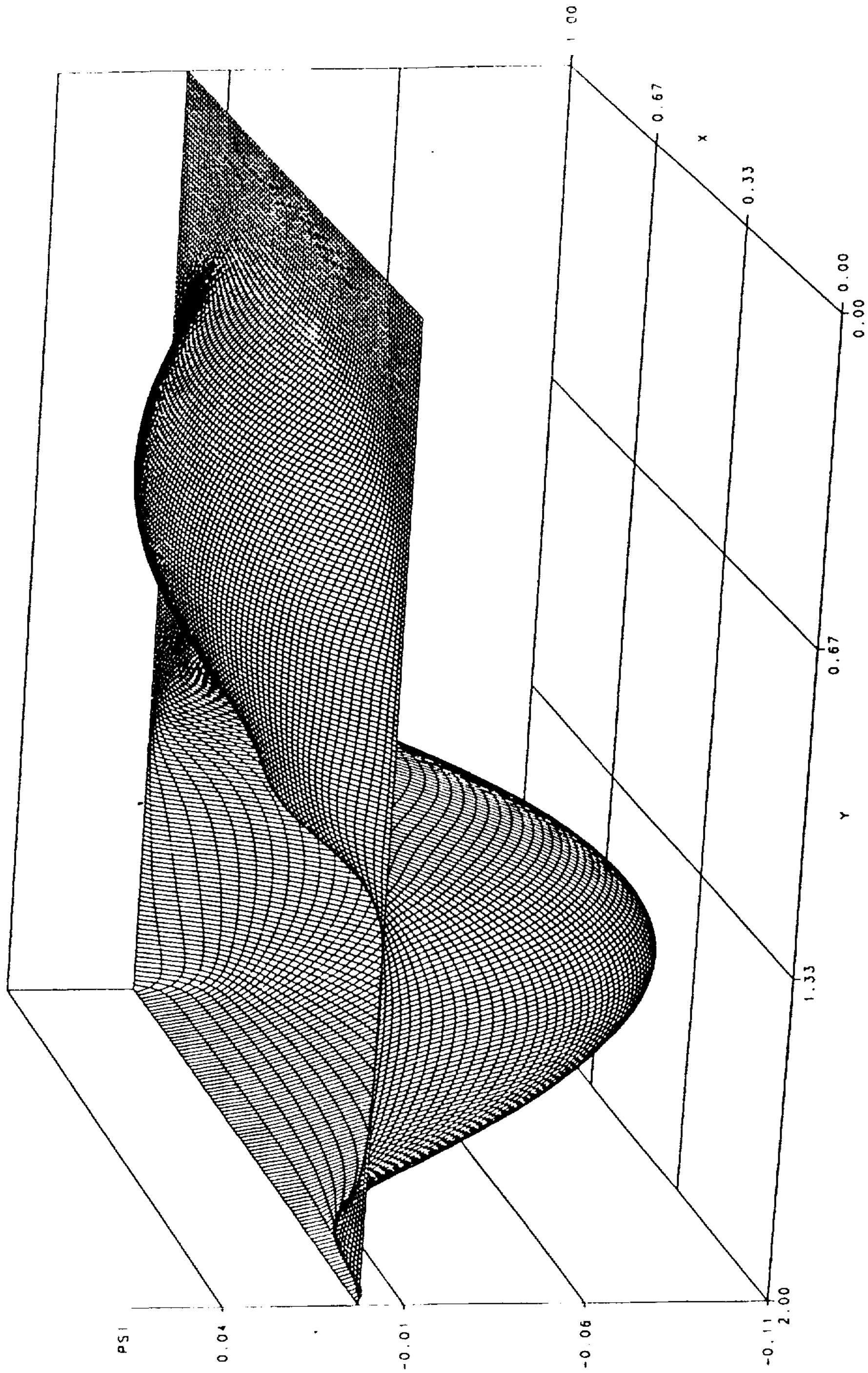

u at x=0.5 as a function of y
Re=5000, 128 by 128 grid, t=491.8
Y
1.00
0.75
0.50
0.25
0.00
-0.50
-0.25
0.00
0.25
0.50
0.75
1.00
U,
DATA
computed
Ghia et al

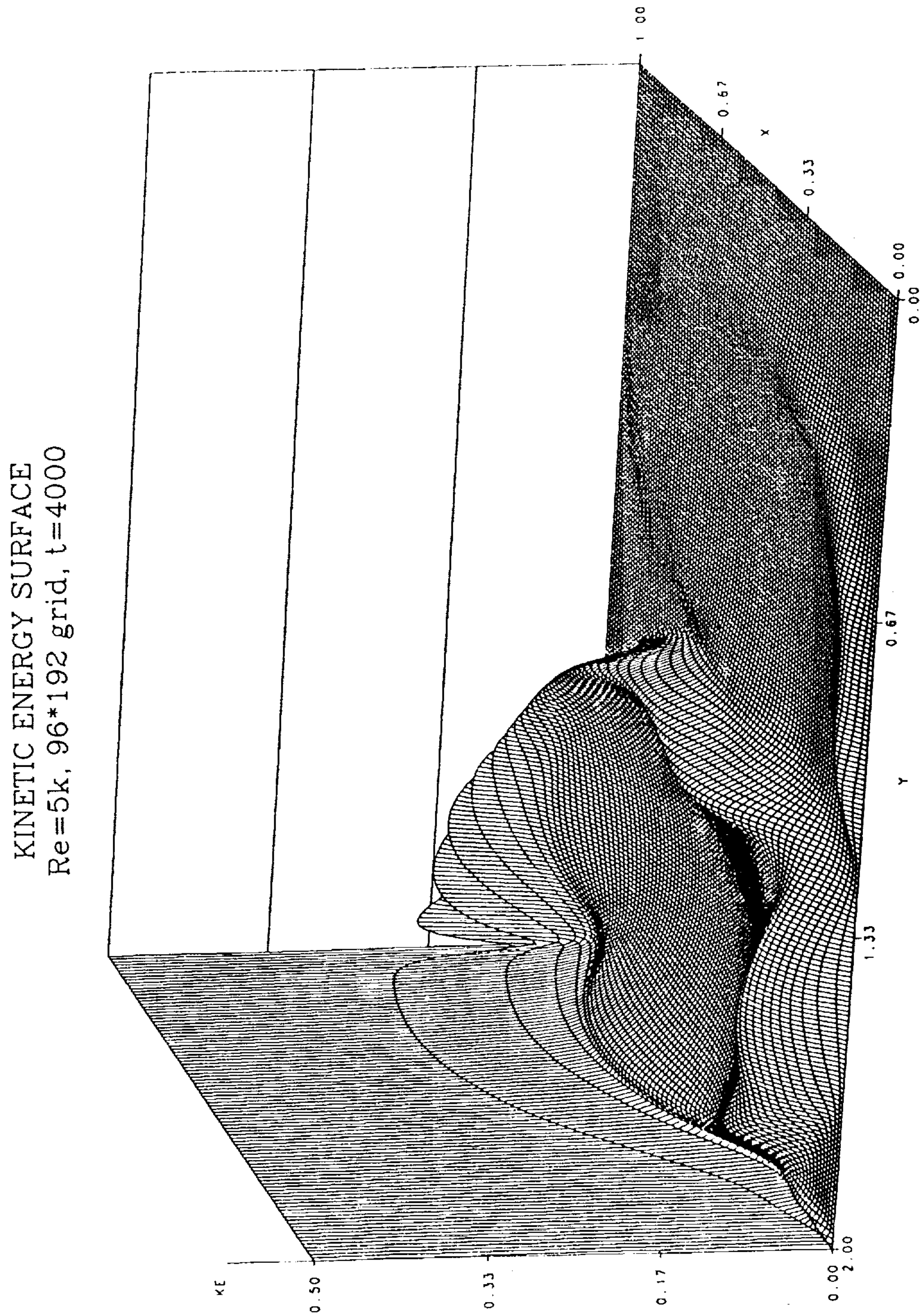
STREAM FUNCTION CONTOURS
Re=5k, 96*192 grid, t=4000



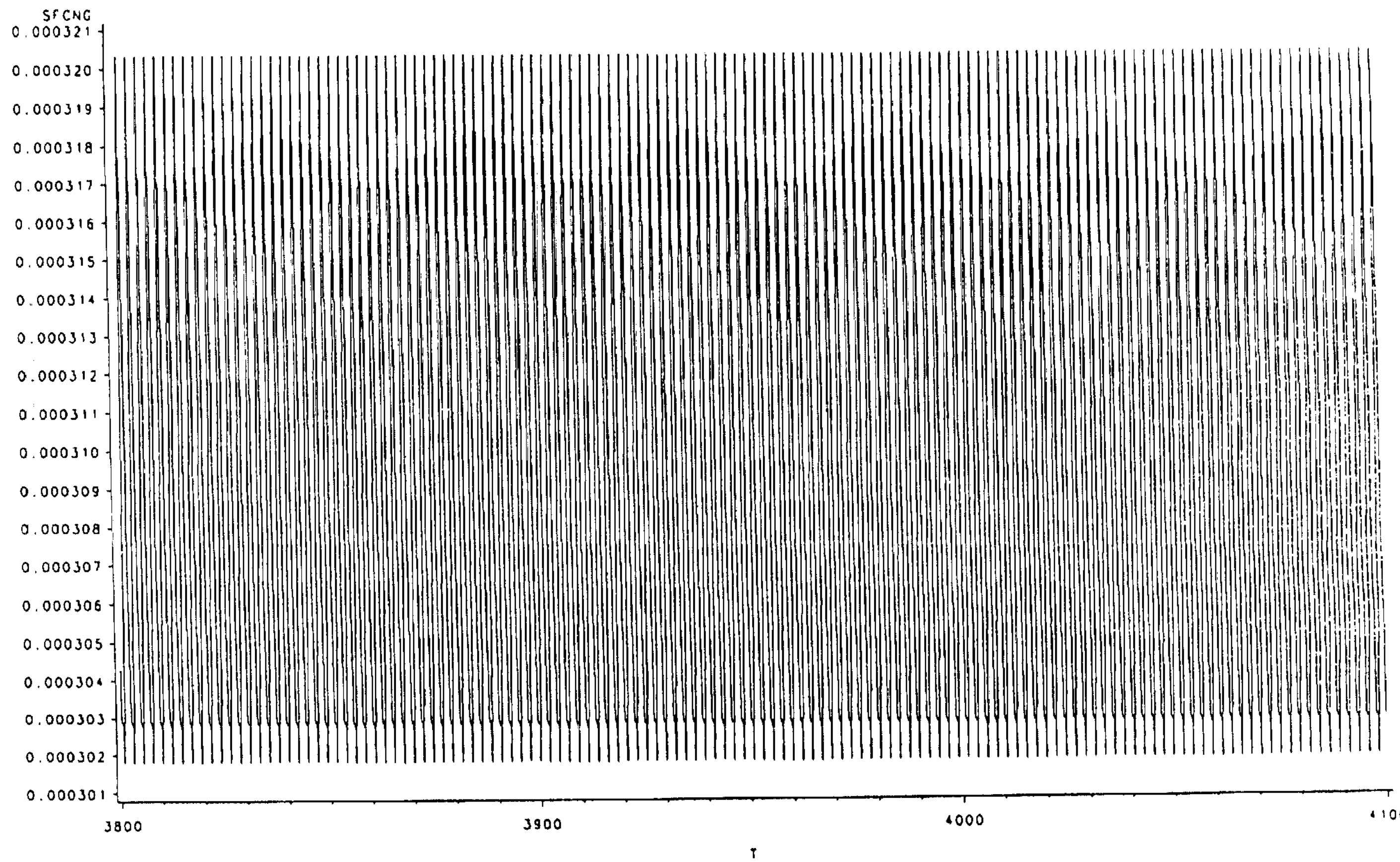
ORIGINAL PAGE IS
OF POOR QUALITY

STREAM FUNCTION SURFACE
Re=5k, 96*192 grid, t=4000

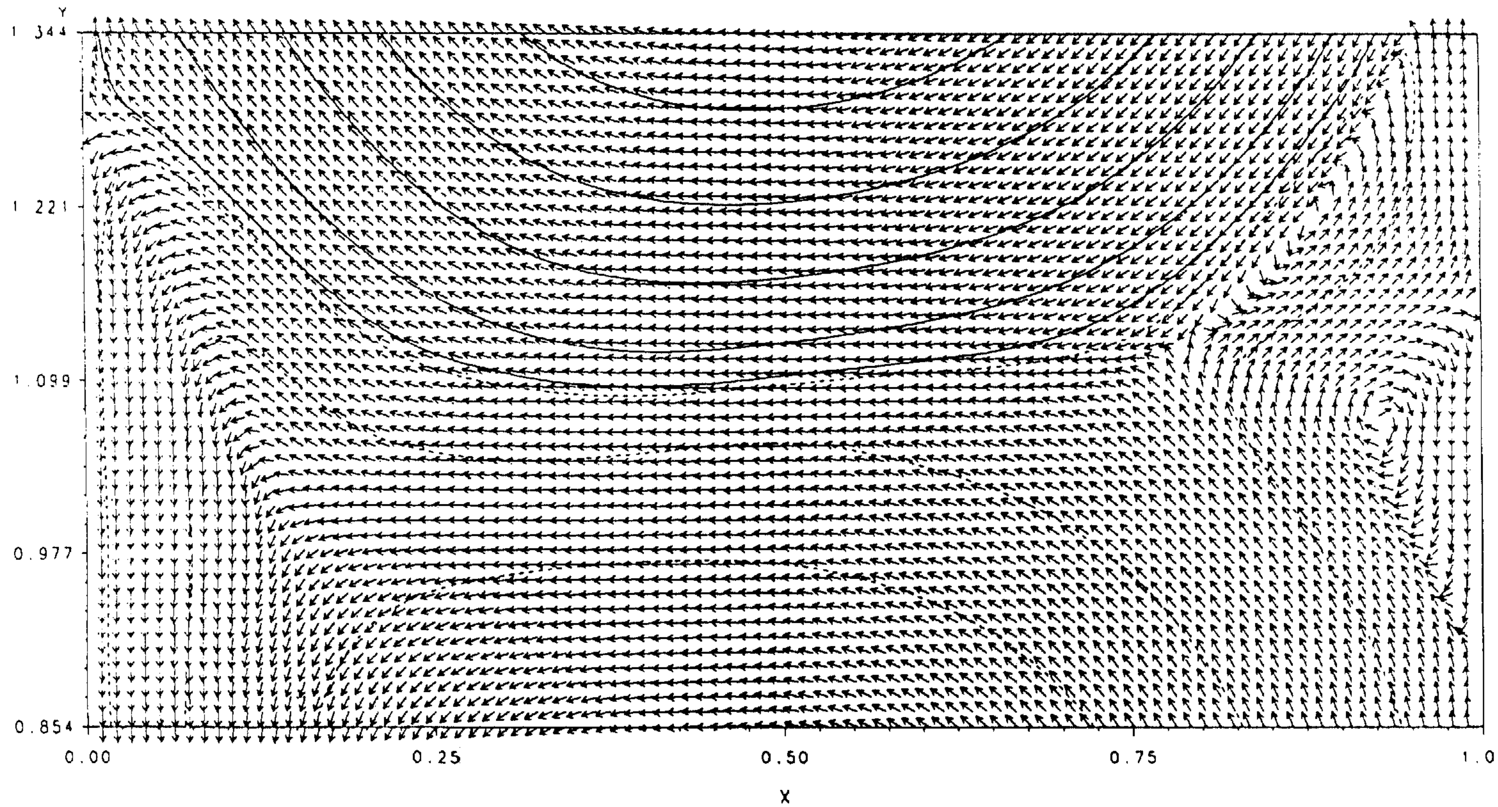



KINETIC ENERGY SURFACE
Re=5k, 96*192 grid, t=4000
KE
0.50
0.33
0.17
0.00
2.00
1.33
Y
0.67
0.00
0.33
X
0.67
1.00

STREAMFUNCTION CHANGE PER TIME STEP
Relative L1 norm for the change
Re=5k, 96*192 grid, 3800<=t<=4100



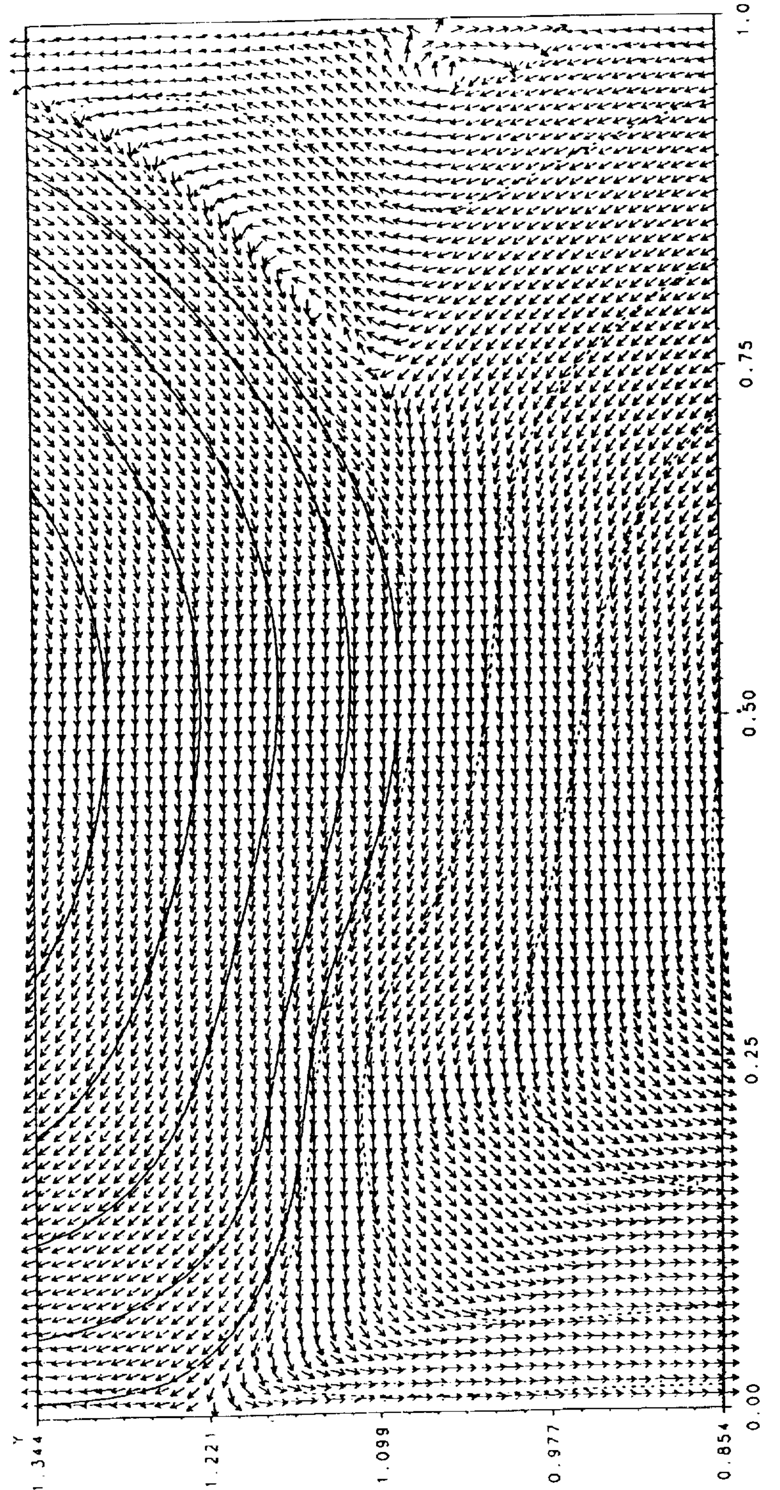
STREAM FUNCTION CONTOURS – NORMALIZED VECTOR PLOTS
Re=5k, 96*192 grid, t=4100.25
0.0<=x<=1.0 and 0.85<=y<=1.35



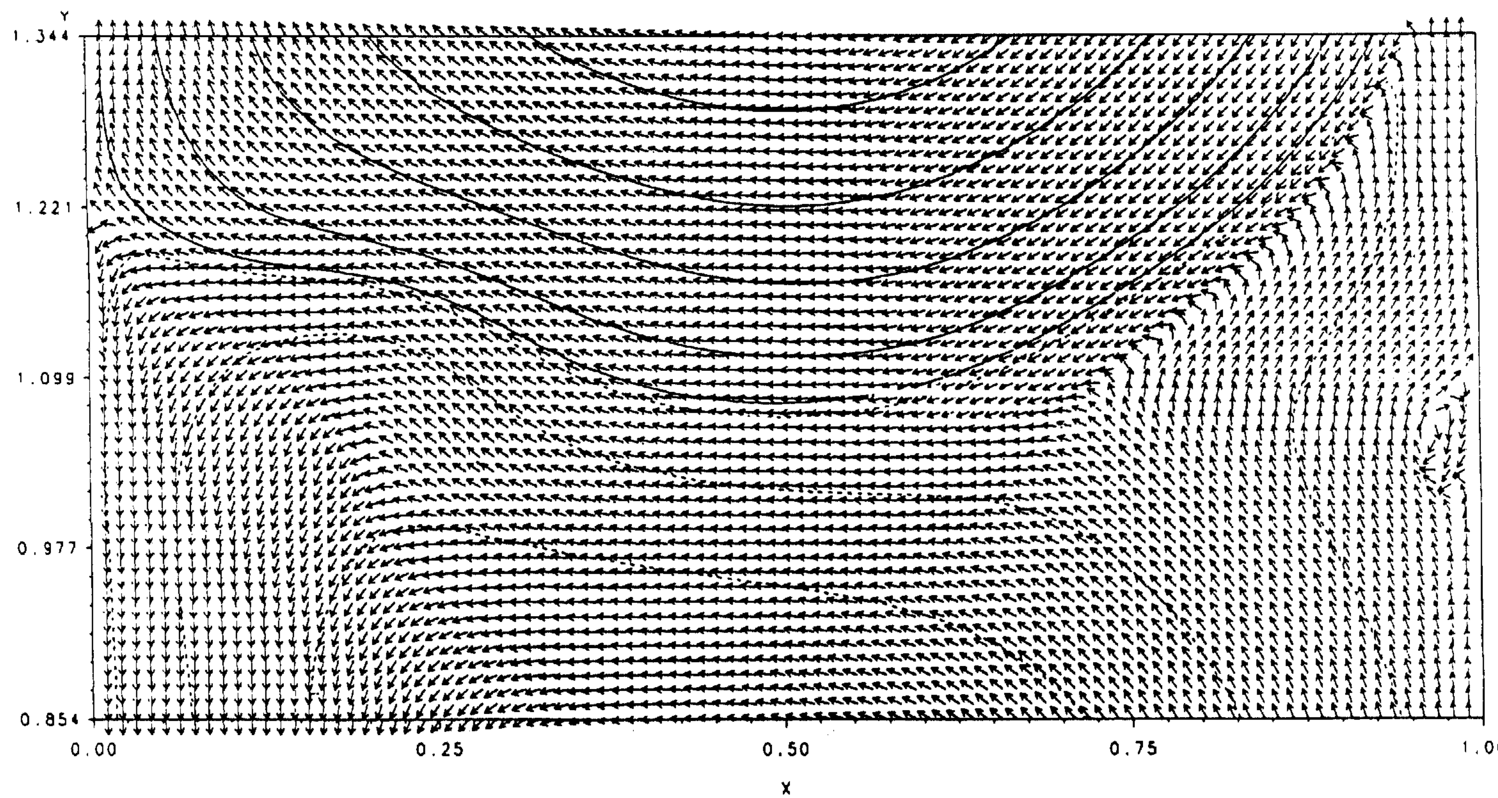
STREAM FUNCTION CONTOURS - NORMALIZED VECTOR PLOTS

Re=5k, 96*192 grid, t=4101.25

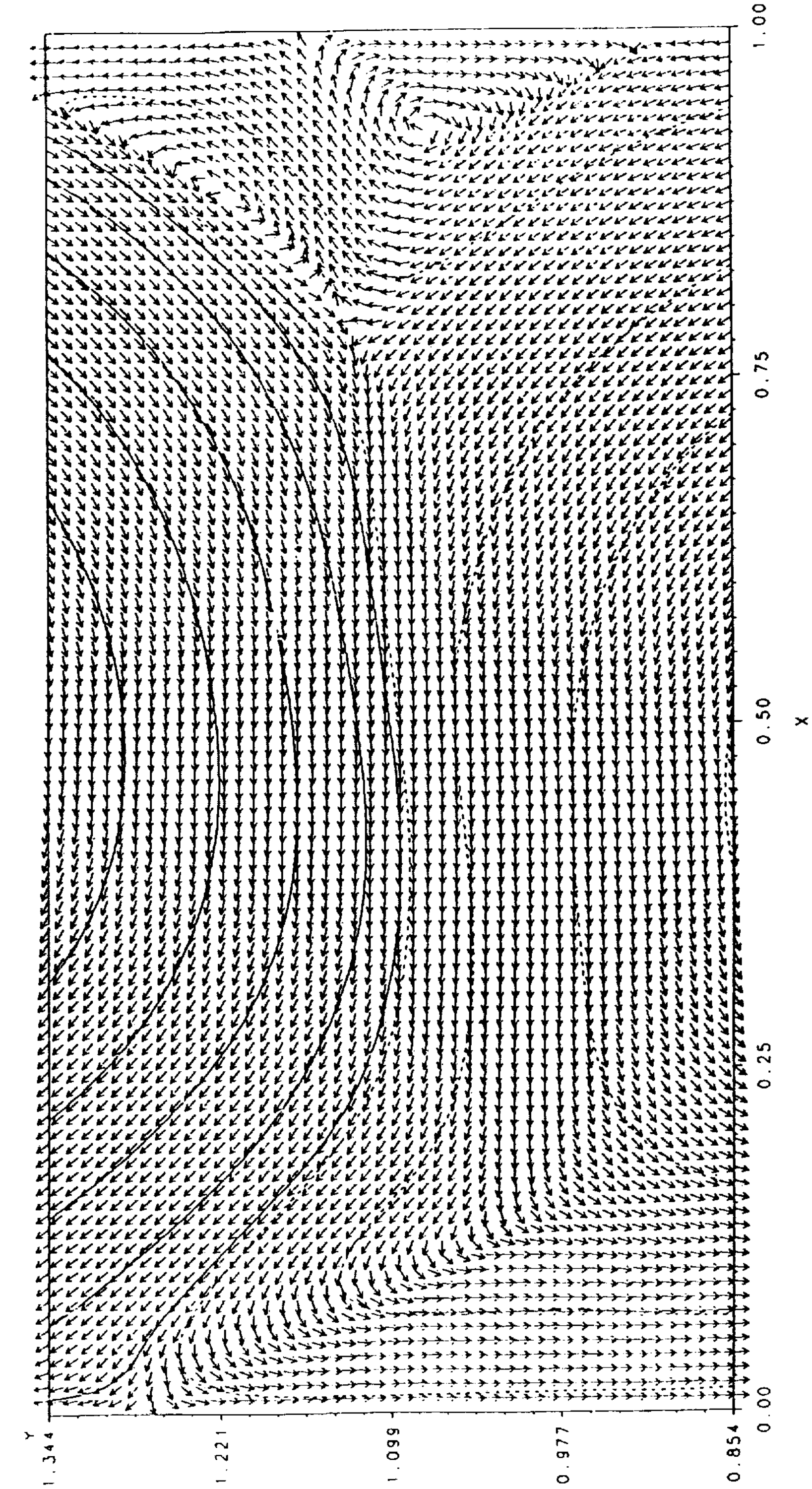
0.0<=x<=1.0 and 0.85<=y<=1.35



STREAM FUNCTION CONTOURS – NORMALIZED VECTOR PLOTS
Re=5k, 96*192 grid, t=4101.50
0.0<=x<=1.0 and 0.85<=y<=1.35



STREAM FUNCTION CONTOURS – NORMALIZED VECTOR PLOTS
Re=5k, 96*192 grid, t=4102.50
0.0<=x<=1.0 and 0.85<=y<=1.35



SUMMARY: A NEW ALGORITHM

$\Longrightarrow$ has one unknown per grid cell in two space dimensions;

$\Longrightarrow$ requires storage that increases linearly with the number of grid points;

$\Longrightarrow$ CPU time per time step increases linearly with the number of grid points;

$\Longrightarrow$ is second order accurate in both time and space;

$\Longrightarrow$ stability limit is Courant number < 1;

$\Longrightarrow$ is robust with respect to Reynolds number.

SUMMARY: A NEW PERIODIC FLOW SOLUTION

$\Longrightarrow$ is exactly periodic;

$\Longrightarrow$ does not use a time dependent forcing term;

$\Longrightarrow$ has no periodic or artifical throughflow boundary conditions;

$\Longrightarrow$ is probably driven by the wall jet descending from the lid;

$\Longrightarrow$ is evidence of a Hopf bifurcation;

$\Longrightarrow$ may lead to period doubling bifurcations and a chaotic flow.