Lower Bound on Reliability for Weibull Distribution
When Shape Parameter is not Estimated Accurately

Zhaofeng Huang; Rocketdyne Div., Rockwell Int'l; Canoga Park
Albert A. Porter; Rocketdyne Div., Rockwell Int'l; Canoga Park

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SUMMARY AND CONCLUSION

This paper investigates the mathematical relationships between the shape parameter β and estimates of reliability and a life limit lower bound for the two parameter Weibull distribution. It shows that under rather general conditions, both the reliability lower bound and the allowable life limit lower bound (often called a tolerance limit) have unique global minimums over a range of β. Hence lower bound solutions can be obtained without assuming or estimating β. The existence and uniqueness of these lower bounds are proven in the Appendix. Some real data examples are given to show how these lower bounds can be easily established and to demonstrate their practicality. The method developed in the paper has proven to be extremely useful when using the Weibull distribution in analysis of no-failure or few-failures data. The results are applicable not only in the aerospace industry but anywhere that system reliabilities are high.

INTRODUCTION

The two parameter Weibull distribution is widely used for reliability and life limit estimations. However, there is often insufficient information, e.g., failure data, to accurately estimate the shape parameter β of the distribution. Traditionally, either a value of β is assumed and then used to establish the scale parameter α and estimate a reliability lower bound, or a nonparametric method is invoked to estimate a reliability lower bound. In the former situation the assumption of β is often controversial and difficult to justify. In the latter case, the reliability bound estimated is usually too conservative. This paper presents an alternate method which attempts to overcome the drawbacks in the traditional methods.

NOTATION AND ASSUMPTION

Notation:

Cdf	Cumulative distribution function
β	Weibull distribution shape parameter (slope value)
α	Weibull distribution scale parameter (characteristic value)
$\hat{p}$	Estimate of any parameter, p
n	Total number of test units
r	Number of failed units
t_i	Operating time (seconds, cycles)
t_m	$\max(t_1, t_2, \ldots, t_n)$
z_i	t_i/t_m, normalized time
T	Specific operating time or life limit lower bound
R	Reliability for a given life limit
T_c	Conditional life limit lower bound
R_c	Conditional reliability

MTBF	Mean time between failure
τ	Confidence level
Ln	Natural logarithmn

Assumption:

For the hardware units, the distribution of operating time to failure is the 2-parameter Weibull.

MATHEMATICAL RESULTS

The 2-parameter Weibull Cdf is:

$$F(t) = 1 - EXP(-(\frac{t}{\alpha})^{\beta}), \quad t>0 \tag{1}$$

Where t is a variable generally given in terms of time or cycles, β is the shape or slope parameter of the distribution and α is the scale or characteristic life parameter. Assume n units are tested. Let $t_1,t_2,...,t_r$ be failure data and $t_{r+1},t_{r+2},...,t_n$ be censored data (either failure censoring or time censoring).

A lower bound on α results from the fact that if t is distributed as Weibull (β,α) then t^{β} has an exponential distribution with MTBF parameter α^{β}. Thus given a value of β, a 100τ percent confidence lower bound on α (Refs. 1 - 4) is given by:

$$\hat{\alpha} = (\frac{\Sigma t_i^{\beta}}{c(\tau)})^{1/\beta}, \tag{2}$$

where $c(\tau)$ is a positive constant which depends on the confidence level τ, the number of failures r and the censoring scheme. For example, for type II censoring, $c(\tau) = .5*X^2{}_{\tau,2r}$; for type I censoring with replacement, $c(\tau) = .5*X^2{}_{\tau,2r+2}$ (Ref. 4), where $X^2{}_{\tau,k}$ is the 100τ percentile of the Chi-square distribution with degrees of freedom k.

With high reliability systems a value of β is frequently hard to come by. Either there are no failures in which case β cannot be estimated or there are few failures in which case the estimate of β is subject to considerable uncertainty, and no assumption about β based on similar failure modes or similar hardware is available. Fortunately, it turns out that under certain conditions, which are not so restrictive, reliability for a given life or life limit for a given reliability experiences a minimum value as a function of β. In other words there is a "worst case β", say β_0, such that reliability or life limit attains a strict minimum when $\beta=\beta_0$. The equations and conditions are shown below.

<u>Reliability lower bound:</u>

When the unit is tested to time T, it is easily seen that the 100τ percent confidence lower bound on reliability is

$$\hat{R} \equiv \hat{R}(\beta) = \mathrm{EXP}\left(- \frac{T^\beta}{\hat{\alpha}^\beta}\right) = \mathrm{EXP}\left(- c(\tau)\frac{T^\beta}{\Sigma t_i{}^\beta} \right) \qquad (3)$$

<u>Theorem 1:</u> Under the condition: $(t_1 t_2 \ldots t_n)^{1/n} < T < \max(t_1, t_2, \ldots, t_n)$, there exists a unique $\beta_0 > 0$ such that

$$\hat{R}(\beta) > \hat{R}(\beta_0), \text{ for all } \beta > 0 \text{ and } \beta \neq \beta_0;$$

and $\hat{R}(\beta)$ is a monotone increasing function of β when $\beta \geq \beta_0$,
and $\hat{R}(\beta)$ is a monotone decreasing function of β when $\beta \leq \beta_0$.

For all theorems presented here, when the conditions are not met, the function in question is either monotonically increasing or monotonically decreasing over the range of β, in either case a worst case β does not exist.

<u>Allowable life limit lower bound:</u>

Assuming the reliability is a given constant R, we solve for T from (3), then a 100τ percent confidence lower bound on allowable life limit (tolerance limit) is

$$\hat{T} \equiv \hat{T}(\beta) = \left(- \mathrm{Ln}(R) \frac{\Sigma t_i^{\beta}}{c(\tau)} \right)^{1/\beta} \quad (4)$$

<u>Theorem 2:</u> Under the condition: $k < c(\tau)/(-\mathrm{Ln}(R)) < n$, where k = number of t_i's with values = $\max(t_1, t_2, \ldots, t_n)$,

there exists a unique $\beta_0 > 0$ such that:

$$\hat{T}(\beta) > \hat{T}(\beta_0), \text{ for all } \beta > 0 \text{ and } \beta \neq \beta_0;$$

and $\hat{T}(\beta)$ is a monotone increasing function of β when $\beta \geq \beta_0$, and $\hat{T}(\beta)$ is a monotone decreasing function of β when $\beta \leq \beta_0$.

Rocket engine reliability is more commonly thought of in terms of mission reliability rather than life time reliability. Thus there is interest in the reliability of the next t_0 second mission given that the unit has already accumulated $T-t_0$ seconds. The equations and conditions for the conditional reliability and life limit are given below.

<u>Conditional reliability lower bound:</u>

We define the conditional reliability R_c as the probability that the unit will not fail during the next t_0 seconds, given that the unit has survived the first $T-t_0$ seconds. Then R_c is computed by

$$R_c = \mathrm{EXP}\left(- \frac{T^{\beta} - (T-t_0)^{\beta}}{\alpha^{\beta}} \right). \quad (5)$$

Substituting equation (2) into equation (5), we get the 100τ percent confidence lower bound on R_c:

$$\hat{R}_c \equiv \hat{R}_c(\beta) = \mathrm{EXP}\left(- c(\tau) \frac{T^{\beta} - (T-t_0)^{\beta}}{\Sigma t_i^{\beta}} \right) \quad (6)$$

<u>Theorem 3:</u> Under the conditions: $0 < t_0 < T < \max(t_1, t_2, \ldots, t_n)$ and $\Sigma t_i (1 + \mathrm{Ln}(T/t_i)) < 0$ there exists a $\beta_0 > 1$ such that

$$\hat{R}_C(\beta) \geq EXP(-c(\tau)\frac{t_0\beta_0T^{\beta_0-1}}{\Sigma t_i^{\beta_0}}), \text{ for all } \beta>1. \quad (7)$$

Conditional operating life limit lower bound:

We frequently need to assess a lower bound for a conditional operating life limit T_C which is defined by (5) for a given conditional reliability R_C. While T_C can not be explicitly solved for from (5), it can be treated as an implicit function of other variables.

Theorem 4. Under the conditions:

$$R_C < EXP(- c(\tau)\frac{t_0}{\Sigma t_i}) \text{ and } \Sigma t_i * Ln(\frac{t_i}{t_0}) > 0, \quad (8)$$

there exists a $\beta_0 > 1$ such that

$$\hat{T}_C(\beta) \geq (-Ln(R_C)\frac{\Sigma t_i^{\beta_0}}{c(\tau)\beta_0 t_0})^{1/(\beta_0-1)}, \text{ for all } \beta>1. \quad (9)$$

Note that Theorems 3 and 4 have been restricted to $\beta>1$ (wear out type failures). Similar work for $\beta\leq1$ (infant mortality type failures and constant failure rate type failures) will be the subject of a future paper.

The proofs of the Theorems 1 to 4 are given in the Appendix and a computer program has been written to solve for the β_0's and the minimums.

ILLUSTRATIVE EXAMPLES

Example 1. Of the 59 units of a particular component that were tested only one failed. The operating times (seconds) of the 59 units are: 14176 (failure), 28587, 20359, 18248, 17256, 16769, 13890, 13357 13118, 13118, 13061, 12893, 12640, 11878, 11858, 10378, 8172, 7918, 7379, 7326, 6700, 6442, 6263, 6168, 5809, 5383, 5305,4788, 4788, 4784, 4748, 4701, 4453, 4258, 3696, 3145, 3138, 3059, 2867, 2837, 2777, 2438, 2300, 2178, 2027, 1807, 1739, 1567, 1384, 1357, 1303, 1161, 1062, 1062, 1042, 903, 819, 299, 299.

We wish to use the Weibull distribution to assess the reliability lower bound at the 50% confidence level for a new unit operated for 10,000 seconds.

Since we have just one failure, we are not able to obtain a good estimate of β. Using Theorem 1 with:

$\tau = .50$; $c(\tau) = X^2{}_{\tau,4} = 3.357$ and $T = 10{,}000$ seconds,

we have $\hat{R}(\beta) = EXP(-.5*3.357\frac{10000^\beta}{\Sigma t_i{}^\beta})$

From Theorem 1, we know there exists a unique $\beta_0>0$ (Figure 1) such such that

$$\hat{R}(\beta) \geq \hat{R}(\beta_0) \text{ for all } \beta_0>0.$$

The computer program computes: $\beta_0 = 1.083$ and $\hat{R}(\beta_0) = .958$. Therefore we may conclude with 50% confidence that the reliability of a unit operated for 10,000 seconds is <u>at least</u> .958.

Example 2. Sixteen units of a particular component experienced the following cycles with no failure: 35, 31, 26, 21, 40, 30, 38, 27, 23, 38, 18, 19, 12, 14, 13, 9. Given these data, what is the minimum number of tests which can be run without failure at .95 reliability and 50% confidence level?

We have: $R=.95$, $\tau=.50$ and $c(\tau)=.5*X^2{}_{\tau,2}=1.386$. Using Theorem 2 (see Figure 2), we get:

$$\beta_0 = 1.529 \text{ and } \hat{T}(\beta_0) = 28.6 \text{ starts}$$

So we conclude that at the 50% confidence level and 0.95 reliability, the component could be tested for <u>at least</u> 28 cycles without failure.

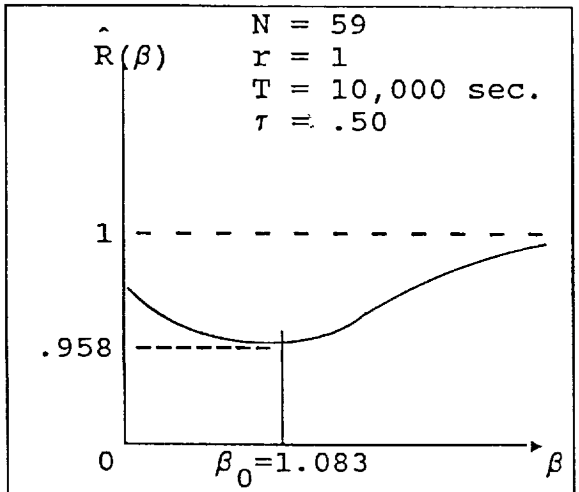

Fig.1 Example of Theorem 1

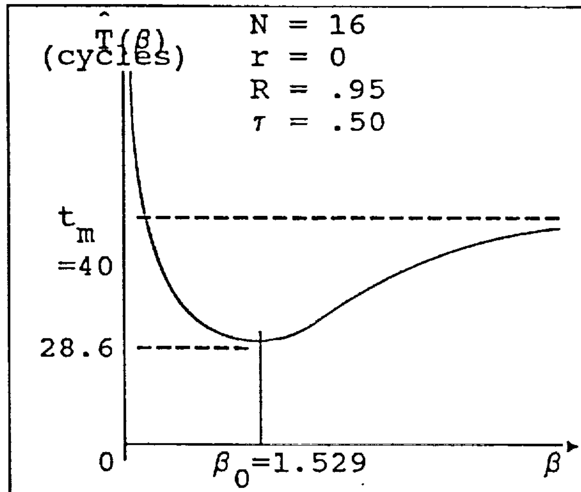

Fig.2 Example of Theorem 2

Example 3. Using the data in Example 1 compute the 90% confidence lower bound on reliability for 520 seconds of operation given that the unit had already operated for 5000 seconds without failing.

We have T=5,000+520=5520 seconds, t_0=520 seconds, τ=.90 and $c(\tau)=.5*X^2{}_{\tau,4}=7.779$. Using Theorem 3 (see Figure 3), we get:

β_0 = 1.321 and equation (7) gives

$$\hat{R}_C(\beta) \geq \mathrm{EXP}\left(-.5*7.779\frac{520*1.321*5520^{1.321-1}}{\Sigma t_i^{\,1.321}}\right) = .9945, \text{ for all } \beta>1.$$

Thus we conclude, with 90% confidence, that the conditional reliability of a unit operating for 520 seconds given that it had already operated for 5000 seconds is at least .9945.

Example 4. Again using Example 1 data compute the life limit such that a 520 second operation reliability of at least .99 is guaranteed at the 95% confidence level.

We have t_0=520 seconds, τ=.95, $c(\tau)=.5*X^2{}_{\tau,4}=9.488$ and R_C=.99. Using Theorem 4 (see Figure 4), we get:

β_0 = 2.188 and equation (9) gives

$$\hat{T}_C(\beta) \geq \left(-\mathrm{Ln}(.99)\frac{\Sigma t_i^{\,2.188}}{.5*9.488*2.188*520}\right)^{1/(2.188-1)}$$

=9,486 seconds, for all $\beta>1$.

So we see, in order to meet 0.99 reliability lower bound at 95% confidence level, the conditional life limit is <u>at least</u> 9,486 seconds.

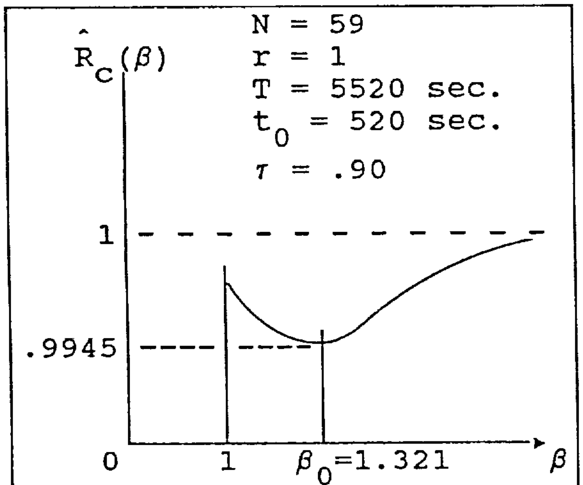

Fig.3 Example of Theorem 3

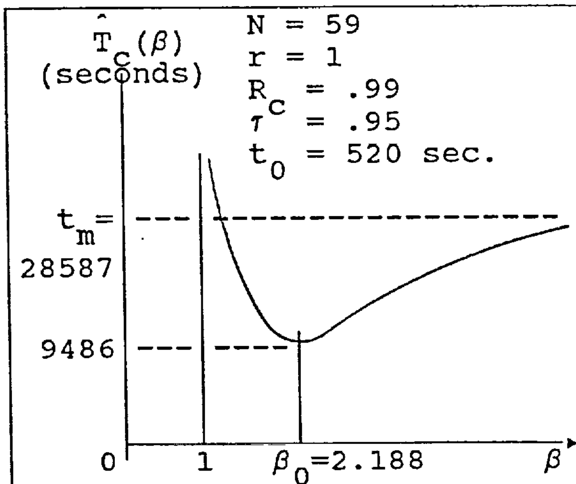

Fig.4 Example of Theorem 4

APPENDIX

1. Proof of Theorem 1:

From (3), we get: $\hat{R}(\beta) = EXP(\ -c(\tau)\frac{T^\beta}{\Sigma t_i^\beta})$

It is obvious that finding a minimum for $\hat{R}(\beta)$ is equivalent to finding a minimum for $g(\beta) \equiv \Sigma(t_i/T)^\beta$.

The first derivative and the second derivative of $g(\beta)$ are

$$g'(\beta) = \Sigma\ Ln(\frac{t_i}{T})*(\frac{t_i}{T})^\beta \tag{1.1}$$

and

$$g''(\beta) = \Sigma(Ln\frac{t_i}{T})^2*(\frac{t_i}{T})^\beta \tag{1.2}$$

From (1.2), and the given condition:

$$(t_1t_2...t_n)^{1/n} < T < \max(t_1,t_2,...,t_n), \quad (1.3)$$

we see $g''(\beta)>0$. So $g'(\beta)$ is a strictly monotone increasing function of β. Also because of (1.3), we have

$$\lim_{\beta \to 0} g'(\beta) = \Sigma \mathrm{Ln}(\frac{t_i}{T}) = \mathrm{Ln}(\frac{t_1}{T}\frac{t_2}{T}\cdots\frac{t_n}{T}) < 0$$

and $\lim_{\beta \to +\infty} g'(\beta) = +\infty > 0$. Therefore there exists a unique $\beta_0 > 0$ such that:

$g'(\beta_0)=0$ and $g(\beta)$ attains a minimum at β_0;

and $g(\beta)$ is a monotone increasing function of β when $\beta \geq \beta_0$,

and $g(\beta)$ is a monotone decreasing function of β when $\beta \leq \beta_0$.

Transferring the result from $g(\beta)$ back to $\hat{R}(\beta)$, we conclude the Theorem 1.

2. Proof of Theorem 2:

From (4) we get $\hat{T}(\beta) = (-\mathrm{Ln}(R)\frac{\Sigma t_i{}^{\beta}}{c(\tau)})^{1/\beta}$

Using the notations $t_m = \max(t_1, t_2, \ldots, t_n)$, $z_i = t_i/t_m$, $i=1,2,\ldots n$; and $C = -\mathrm{Ln}(R)/c(\tau)$. We have

$$\frac{\hat{T}(\beta)}{t_m} = (C\Sigma z_i{}^{\beta})^{1/\beta} = \mathrm{EXP}(\frac{\mathrm{Ln}(C)+\mathrm{Ln}(\Sigma z_i{}^{\beta})}{\beta}) \qquad (2.1)$$

Define $h(\beta) \equiv (\frac{\mathrm{Ln}(C)+\mathrm{Ln}(\Sigma z_i{}^{\beta})}{\beta})$. We have $\frac{\hat{T}(\beta)}{t_m} = \mathrm{EXP}(h(\beta))$.

The first derivative of $h(\beta)$ is

$$h'(\beta) = (-\mathrm{Ln}C + \beta\frac{\Sigma \mathrm{Ln}(z_i) z_i{}^{\beta}}{\Sigma z_i{}^{\beta}} - \mathrm{Ln}(\Sigma z_i{}^{\beta})) / \beta^2$$

Defining $h_1(\beta) = -\mathrm{Ln}C + \beta\frac{\Sigma \mathrm{Ln}(z_i) z_i{}^{\beta}}{\Sigma z_i{}^{\beta}} - \mathrm{Ln}(\Sigma z_i{}^{\beta})$, (2.2)

we have $h'(\beta) = h_1(\beta)/\beta^2$. The first derivative of $h_1(\beta)$ is

$$h_1'(\beta) = \frac{\Sigma \mathrm{Ln}(z_i) z_i{}^{\beta}}{\Sigma z_i{}^{\beta}} + \beta\frac{(\Sigma(\mathrm{Ln}z_i)^2 z_i{}^{\beta})(\Sigma z_i{}^{\beta}) - (\mathrm{Ln}(z_i) z_i{}^{\beta})^2}{(\Sigma z_i{}^{\beta})^2}$$
$$- \frac{\Sigma \mathrm{Ln}(z_i) z_i{}^{\beta}}{\Sigma z_i{}^{\beta}}$$
$$= \beta\frac{(\Sigma(\mathrm{Ln}z_i)^2 z_i{}^{\beta})(\Sigma z_i{}^{\beta}) - (\Sigma \mathrm{Ln}(z_i) z_i{}^{\beta})^2}{(\Sigma z_i{}^{\beta})^2} \qquad (2.3)$$

Using the Cauchy Inequality (Ref.6, Page 11), we have $h_1'(\beta) \geq 0$.

But by the given condition $k < c(\tau)/(-\mathrm{Ln}(R)) < n$, where k=number of

t_i's with values=t_m, the equality does not hold. So $h_1'(\beta)>0$. Hence $h_1(\beta)$ is a strictly monotone increasing function of β.

Using the condition $k<c(\tau)/(-Ln(R))<n$ again, we have

$$\lim_{\beta \to 0} h_1(\beta) = -LnC + 0 - Ln(n) = -Ln(Cn) < 0$$

and $$\lim_{\beta \to +\infty} h_1(\beta) = -LnC + 0 - Ln(k) = -Ln(Ck) > 0.$$

Therefore, there exists a unique $\beta_0>0$ such that $h_1(\beta_0)=0$,

and $h_1(\beta)<0$ when $\beta\le\beta_0$,

and $h_1(\beta)>0$ when $\beta\ge\beta_0$.

Noticing $h_1(\beta)=h'(\beta)*\beta^2$, we can conclude there exists a unique $\beta_0>0$ such that $h'(\beta_0)=0$ and $h(\beta)$ attains a minimum at β_0; and $h(\beta)$ is a monotone increasing function of β when $\beta\ge\beta_0$, and $h(\beta)$ is a monotone decreasing function of β when $\beta\le\beta_0$.

Transferring the result from $h(\beta)$ back to $\hat{T}(\beta)$, we conclude the Theorem 2.

3. Proof of Theorem 3:

From (6), we have

$$\hat{R}_C(\beta) = EXP(-c(\tau)\frac{T^\beta-(T-t_0)^\beta}{\Sigma t_i^\beta})$$

Using the notations defined in the Notation Section, we have

$$\frac{-Ln(\hat{R}_C(\beta))}{c(\tau)} = \frac{(T/t_m)^\beta-((T-t_0)/t_m)^\beta}{\Sigma z_i^\beta} \tag{3.1}$$

Using the first order Taylor expansion, R.H.S. (right hand side) of (3.1) $= \dfrac{\beta(t_0/t_m)((T-\theta t_0)/t_m)^{\beta-1}}{\Sigma z_i^\beta}$, where $0\le\theta\le1$

Considering $\beta>1$, we get $$\frac{-Ln(\hat{R}_C(\beta))}{c(\tau)} \le \frac{\beta(t_0/t_m)(T/t_m)^{\beta-1}}{\Sigma z_i^\beta} \equiv z(\beta) \tag{3.2}$$

Define $z_1(\beta) \equiv Ln(z(\beta)t_m/t_0) = Ln\beta+(\beta-1)Ln(T/t_m)-Ln(\Sigma z_i^\beta)$.

The first derivative and the second derivative of $z_1(\beta)$ are

$$z_1'(\beta) = \frac{1}{\beta} + Ln(\frac{T}{t_m}) - \frac{\Sigma z_i^{\beta} Lnz_i}{\Sigma z_i^{\beta}} \tag{3.3}$$

$$z_1''(\beta) = -\frac{1}{\beta^2} - \frac{(\Sigma z_i^{\beta}(Lnz_i)^2)(\Sigma z_i^{\beta}) - (\Sigma z_i^{\beta} Lnz_i)^2}{(\Sigma z_i^{\beta})^2} \tag{3.4}$$

By the Cauchy Inequality, the second term of R.H.S. of (3.4) ≤ 0. So $z_1''(\beta)<0$. Therefore $z_1'(\beta)$ is a strictly monotone decreasing function of β. Because of the given conditions, we also have

$$\lim_{\beta \to 1} z_1'(\beta) = 1 + Ln(\frac{T}{t_m}) - \frac{\Sigma z_i Lnz_i}{\Sigma z_i} > 0$$

and

$$\lim_{\beta \to +\infty} z_1'(\beta) = 0 + Ln(T/t_m) - 0 < 0.$$

So there exists a unique $\beta_0>0$ such that $z_1'(\beta)=0$ and $z_1(\beta)$ attains a maximum at β_0. But noticing

$$z(\beta) = (t_0/t_m) EXP(z_1(\beta)),$$

we could extend the result to $z(\beta)$. So we have

$$\frac{-Ln(\hat{R}_c(\beta))}{c(\tau)} \leq \frac{\beta t_0 T^{\beta-1}}{\Sigma t_i^{\beta}} \equiv z(\beta) \leq z(\beta_0) \equiv \frac{\beta_0 t_0 T^{\beta_0 - 1}}{\Sigma t_i^{\beta_0}}, \quad \text{for all } \beta>1.$$

That is, $\hat{R}_c(\beta) \geq EXP(-c(\tau)\frac{t_0 \beta_0 T^{\beta_0-1}}{\Sigma t_i^{\beta_0}})$, for all $\beta>1$.

4. Proof of Theorem 4:

From (6), we have

$$\frac{-LnR_c}{c(\tau)} = \frac{T_c^{\beta} - (T_c - t_0)^{\beta}}{\Sigma t_i^{\beta}} \equiv \frac{(T_c/t_m)^{\beta} - ((T_c - t_0)/t_m)^{\beta}}{\Sigma z_i^{\beta}} \tag{4.1}$$

First, we have to show, for any R_c, $c(\tau)$, t_0, β, t_i's and under the given conditions (8), a unique $\hat{T}_c \equiv \hat{T}_c(\beta)$ can be solved for from (4.1).

Define $f(T_c) \equiv T_c^{\beta} - (T_c - t_0)^{\beta} - \Sigma t_i^{\beta}(\frac{-LnR_c}{c(\tau)})$.

Taking the derivative with respect to T_c, we have

$$f'(T_c)=\beta T_c^{\beta-1}-\beta(T_c-t_0)^{\beta-1}>0.$$

We also have $\lim_{T_c\to+\infty} f(T_c) = +\infty >0$ and

$$\lim_{T_c\to t_0} f(T_c) = t_0^{\beta} - \Sigma t_i^{\beta}\left(\frac{-LnR_c}{c(\tau)}\right) < 0, \text{ because of (8).}$$

Therefore, a unique solution $\hat{T}_c\equiv\hat{T}_c(\beta)$ exists for the equation (4.1).

Using a Taylor expansion, we have

$$\frac{-LnR_c}{c(\tau)} = \frac{\beta(t_0/t_m)((T_c-\theta t_0)/t_m)^{\beta-1}}{\Sigma z_i^{\beta}}, \text{ where } 0\le\theta\le1 \tag{4.2}$$

Solving for $\hat{T}_c(\beta)$ from (4.2), we have

$$\frac{\hat{T}_c(\beta)}{t_m} = \left(\frac{-LnR_c}{c(\tau)}\frac{\Sigma z_i^{\beta}}{\beta}\frac{t_m}{t_0}\right)^{1/(\beta-1)} + \theta\frac{t_0}{t_m}$$

$$\ge \left(\frac{-LnR_c}{c(\tau)}\frac{\Sigma z_i^{\beta}}{\beta}\frac{t_m}{t_0}\right)^{1/(\beta-1)} \equiv w(\beta)$$

Let $W \equiv t_m(-LnR_c)/c(\tau)/t_0$, (constant, independent of β) and

$$w_1(\beta) \equiv Lnw(\beta) = \frac{LnW+Ln(\Sigma z_i^{\beta})-Ln\beta}{\beta-1}.$$

The first derivative of $w_1(\beta)$ is

$$w_1'(\beta) = \frac{-LnW}{(\beta-1)^2} + \frac{\frac{\Sigma z_i^{\beta}Lnz_i}{\Sigma z_i^{\beta}}(\beta-1)-Ln(\Sigma z_i^{\beta})}{(\beta-1)^2} - \frac{(\beta-1)/\beta-Ln\beta}{(\beta-1)^2}$$

$$= \frac{-1+1/\beta-LnW+Ln\beta+\frac{\Sigma z_i^{\beta}Lnz_i}{\Sigma z_i^{\beta}}(\beta-1)-Ln(\Sigma z_i^{\beta})}{(\beta-1)^2} \tag{4.3}$$

Let the numerator of R.H.S of (4.3) = $w_2(\beta)$. Then

$$w_2'(\beta) = \frac{\beta-1}{\beta^2} + \frac{\Sigma z_i^{\beta}Lnz_i}{\Sigma z_i^{\beta}} + (\beta-1)\frac{(\Sigma z_i^{\beta}(Lnz_i)^2)(\Sigma z_i^{\beta})-(\Sigma z_i^{\beta}Lnz_i)^2}{(\Sigma z_i^{\beta})^2} - \frac{\Sigma z_i^{\beta}Lnz_i}{\Sigma z_i^{\beta}}$$

$$= \frac{\beta-1}{\beta^2} + (\beta-1)\frac{(\Sigma z_i^{\beta}(Lnz_i)^2)(\Sigma z_i^{\beta})-(\Sigma z_i^{\beta}Lnz_i)^2}{(\Sigma z_i^{\beta})^2} \tag{4.4}$$

Using the Cauchy Inequality and noticing $\beta-1>0$, we have $w_2'(\beta)>0$.

So $w_2(\beta)$ is a strictly monotone increasing function of β.

Using the given condition: $R_c < EXP\left(-c(\tau)\frac{t_0}{\Sigma t_i}\right)$, we have

$$\lim_{\beta \to 1} w_2(\beta) = -\mathrm{Ln}W - \mathrm{Ln}(\Sigma z_i) < 0$$

and $\lim_{\beta \to +\infty} w_2(\beta) = +\infty > 0$.

So there exists a unique $\beta_0>1$ such that $w_2(\beta_0) = 0$ and

$$w_2(\beta) < 0 \text{ when } \beta < \beta_0;\ w_2(\beta) > 0 \text{ when } \beta > \beta_0.$$

But $w_1'(\beta)=\frac{w_2(\beta)}{(\beta-1)^2}$, so the same conclusions hold for $w_1'(\beta)$ for $\beta>1$.

Therefore, $w_1(\beta)$ attains a minimum at β_0. Hence $w(\beta)\equiv\mathrm{EXP}(w_1(\beta))$ attains a minimum at β_0. That is,

$$\frac{\hat{T}_c(\beta)}{t_m} \geq w(\beta) \geq w(\beta 0)=\left(\frac{-\mathrm{Ln}R_c}{c(\tau)}\,\frac{\Sigma z_i{}^{\beta_0}}{\beta_0}\,\frac{t_m}{t_0}\right)^{1/(\beta_0-1)}.$$

Therefore we have: $\hat{T}_c(\beta) \geq \left(-\mathrm{Ln}(R_c)\frac{\Sigma t_i{}^{\beta_0}}{c(\tau)\beta_0 t_0} \right)^{1/(\beta_0-1)}$, for all $\beta>1$.

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BIOGRAPHIES

Zhaofeng Huang
Internal Mail Code AA48
Rocketdyne Div.
Rockwell International
6633 Canoga Ave.
Canoga Park, CA. 91303 USA

Mr. Huang is a reliability engineer in the Space Shuttle Main Engine program at the Rocketdyne Division of Rockwell International. He is currently involved in the Space Shuttle Main Engine anomaly reporting, reliability prediction and estimation, statistical and mathematical methodology development. He received a B.S. in computational mathematics in 1982 from Shanghai University of Science and Technology, an M.A. in mathematics in 1986 from Temple University and an M.S. in statistics in 1987 from Iowa State University.

Albert A. Porter

Internal Mail Code AA48

Rocketdyne Division

Rockwell International

6633 Canoga Ave.

Canoga Park, CA. 91303 USA

Mr. Porter is a reliability engineer at the Rocketdyne Division of Rockwell International where he has been a probability/statistical consultant to engineering, quality and manufacturing for 26 years. He has supported all the company major rocket engine contracts from Apollo to Space Shuttle Main Engine as well as nuclear and laser programs. He obtained a B.A. in mathematics in 1963 and an M.A. in statistics in 1964, both from the University of Missouri.