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**Estimability of Geodetic Parameters from
Space VLBI Observables**

by

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PREFACE

This project is under the supervision of Professor Ivan I. Mueller, Department of Geodetic Science and Surveying, The Ohio State University. The Science Advisor is Dr. David E. Smith, Code 920, and the Technical Officer is Dr. Herbert Frey, Code 921, both at NASA Goddard Space Flight Center, Greenbelt, Maryland 20771. The work is carried out under NASA Grant No. NSG 5265, OSU Research Foundation Project No. 711055.

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ABSTRACT

In the 1990's dedicated radio telescopes will be launched into Earth orbit and will be integrated in the *ground-based VLBI* networks. A straightforward extension from present ground-based VLBI to space is called *space VLBI*, which uses radio-antennas in space. In the simplest version of the space VLBI technique, one station in orbit observes in conjunction with a second station on the ground. However, in practice we have a number of networks of ground antennas observing the common celestial radio sources simultaneously with a conventional VLBI technique. Moreover, joint observations of two or more space VLBI satellites will supposedly be performed in the future. Therefore, in our research work a combined use of simultaneous space and ground-based VLBI observations are considered from the geodetic and geodynamic point of view.

This investigation studies the feasibility of space VLBI observables for geodesy and geodynamics. A brief review of space VLBI systems from the point of view of potential geodetic application is given. A selected notational convention is used to jointly treat the VLBI observables of different type of baselines within a combined ground/space VLBI network. The basic equations of the space VLBI observables appropriate for covariance analysis are derived and included. The corresponding equations for the ground-to-ground baseline VLBI observables are also given for a comparison. The simplified expression of the mathematical models for both space VLBI observables (time delay and delay rate) include the ground station coordinates, the satellite orbital elements, the earth rotation parameters, the radio source coordinates, and clock parameters. The observation equations with these parameters have been examined in order to determine which of them are separable or nonseparable. Singularity problems arising from coordinate system definition and critical configuration are studied. Linear dependencies between partials are analytically derived.

The mathematical models for ground-space baseline VLBI observables have been tested with simulation data in the frame of some numerical experiments. Singularity due to datum defect is confirmed.

Recommendations are given for future research work.

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LIST OF ABBREVIATIONS

AT.....	The Australia Telescope
CCRS.....	Conventional Celestial Reference System
CIS.....	Conventional Inertial System
CTS	Conventional Terrestrial System
DSN.....	NASA/JPL Deep Space Network
ESA	European Space Agency
EVN.....	European VLBI Network
GPS.....	Global Positioning System
GSFC	Goddard Space Flight Center
IERS	International Earth Rotation Service
IKI	Space Research Institute of the Academy of Science in USSR
ISAS	The Institute for Space and Astronautical Science in Japan
IVS.....	International VLBI Satellite
JPL.....	Jet Propulsion Laboratory
NASA.....	National Aeronautics and Space Administration
OVLBI.....	Orbiting VLBI
PRARE	Precise Range and Range-Rate Equipment
QUASAT.....	QUAsar SATellite
RADIOASTRON.....	Radio Astronomical Satellite
TDRSE	The current TDRSS satellite which is situated at 41°W in the geostationary orbit
TDRSS	Transfer and Data Relay Satellite System
VLBA	Very Long Baseline Array
VLBI.....	Very Long Baseline Interferometry
ΔVLBI.....	Differential VLBI
VSOP.....	VLBI Space Observatory Program

1. INTRODUCTION

Two dedicated space VLBI projects are currently in preparation to launch one or more VLBI radio telescopes in orbit. The first space VLBI mission that will be implemented in the near future is RADIOASTRON in the Soviet Union. It is already an approved and funded mission. The satellite will be launched in 1993. Its trajectory is planned to be 3,000 km in perigee and 69,000 km in apogee. The satellite will carry a 10 m antenna. The second project is a Japanese orbiting VLBI mission called VSOP. The expected launch date of the satellite is 1995. Its trajectory is planned to be 1,000 km in perigee and 20,000 km in apogee. Both projects are now in progress. Most probably, a combined use of RADIOASTRON and VSOP observations will be performed in the future. There is a Western European mission with NASA participation in Phase A Study at ESA called QUASAT. Although this project was not approved by ESA at the selection round of October 1988, a second generation space VLBI mission called IVS has been proposed to ESA at the end of 1989.

The feasibility of and potential for using a dedicated VLBI observatory in space has already been demonstrated successfully by the NASA TDRSS satellite with a nearly five meter antenna. In 1987 during the XIX General Assembly of IUGG in Vancouver, IAG Special the Study Group 2.109—Application of space VLBI in the Field of Astronomy and Geodesy—was established to study the usefulness of space VLBI for astrometry and geodesy.

It is expected that space VLBI will be a reality in the current decade. Orbiting radio telescopes will be used to make interferometric observations of extragalactic radio sources in conjunction with the major ground-based VLBI arrays in Europe, USA, Australia, Japan and the USSR. It is planned to determine the orbits of these radio telescopes in space with high accuracy by the missions themselves and possibly using additional tracking systems (e.g., GPS, PRARE). The main goals of all space VLBI projects are to carry out astrophysical investigations. The current missions are devoted to improve imaging quality and angular resolution of compact galactic and extragalactic radio sources. At the same time, the radioastronomers are going to plan the scientific goals of space VLBI missions to be as wide-ranging as possible.

The anticipated use of a space VLBI system poses several important questions with respect to its usefulness in geodesy and geodynamics.

The potential applications of the space VLBI system in these areas are related to the connection and unification of reference systems and frames inherent in the technique. Since the currently available space geodetic techniques are effectively used for these applications by station collocation in the frame of the International Earth Rotation Service (IERS) [Boucher et al., 1988] and [Mueller, 1988], one should investigate how the space VLBI system could complement these techniques. The capabilities of the new space techniques to provide valuable information on the systematic differences between the frames of the various Conventional Inertial Systems (CIS) and Conventional Terrestrial Systems (CTS) is now widely investigated [Kovalevsky et al., 1989] because of the high importance of the connection and unification of all reference frames in geodesy and geodynamics. In principle, a space VLBI system offers an opportunity to connect the reference frames of the CTS and two types of CIS: 1) Radio Source-CIS, and 2) Dynamic (that is Satellite Orbit)-CIS inherent in a space VLBI system. The establishment of the relationship between them would be of scientific interest. Note that the currently adopted Conventional Celestial Reference System (CCRS) is based on radio source positions.

Another important problem area that needs investigation is that related to the usefulness of space VLBI data for gravity field determination. The present gravity field determinations using artificial satellites are based mainly on range and range-rate observations which contain no direct directional information [Marsh et al., 1988] and [Rapp, 1989]. Therefore, they are only indirectly linked to the inertial reference frame. One should investigate how the space VLBI observation containing directional information could be applied to improve the gravity field determination. The space VLBI missions offer and provide new types of satellite observables (VLBI time delay, delay rate, and differential VLBI tracking data) with high accuracy for these potential applications as well.

Earlier investigations on the potential applications of space VLBI for astrometry, geodesy and geodynamics include Adam (1989), Adam and Mueller (1989), Bartel (1989), Fejes et al., (1986, 1987 and 1989a,b), Kawaguchi (1989), Koyama (1989), and Takahashi (1989).

On the basis of the background mentioned above, it was considered appropriate and worthwhile to pursue the present investigation. The aim of which is to briefly review the

space VLBI systems in progress emphasizing the geodetic aspects, to derive mathematical models for both space VLBI observables (time delay and delay rate) suitable for least squares covariance analysis of the parameters of geodetic interest, and to explore estimability problems inherent in the space VLBI technique. An additional important aim is to numerically check the mathematical models by test computations.

Consequently, the whole work is divided into five chapters, each of which treats aspects of the geodetic applicability of space VLBI. Chapter 2 gives a brief review of space VLBI missions in progress. Chapter 3 derives the mathematical models for space VLBI observables and studies the estimability of geodetic parameters involved in the models. Chapter 4 describes the theoretical basis of the datum problem of a space VLBI network within singular Gauss-Markov model and shows the results of test computations.

2. BRIEF REVIEW OF SPACE VLBI SYSTEMS

2.1 Introduction

The application of the VLBI technique to the investigation of the nuclei of galaxies and quasars has allowed the mapping of their angular radio brightness distributions and the studying of the kinematics of their cores. Substantial improvements in the resolving power of VLBI might enable the proper motion of a galaxy or a quasar to be measured. Microarcsecond angular resolution will allow community of radio astronomers to study several phenomena, see, e.g., [Bartel et al., 1986 and 1988], [Kardashev and Slysh, 1988]. Annual parallax, proper motion and position determinations (quasar cores, nearby galaxy cores, galactic centers, pulsars and radio stars) at microarcsecond (μas) level are also very important for geodesy and geodynamics.

There are two ways of increasing the angular resolution: (1) use of short radio wavelengths (mm-VLBI), and (2) use of radiotelescopes in space (space VLBI). The first encounters problems caused by atmospheric phase fluctuations and technical difficulties. The second is more straightforward and has no limitations in principle to the baseline length, see [Burke, 1983], [Kardashev and Slysh, 1988], [Preston, 1983], [Sagdeev, 1984] and [Schilizzi et al., 1984].

A very straightforward extension from present *ground-based VLBI* is called *space VLBI*, which uses radio antennas in space. Moreover, radioastronomers need multiple elements in space (introducing *space-space VLBI*) in order to go to higher angular resolution and/or higher dynamic range mapping. One of such conceptions is suggested by Hirabayashi (1989b) for two orbiting VSOP observatory with modestly near earth orbits.

2.2 Basic Concept of Space VLBI

The usual geodetic VLBI system shown in Fig. 1a [Counselman and Shapiro, 1978] consists of an array of at least two antennas that observe the same radio source simultaneously. A direct electrical connection is not maintained between the antennas, thus allowing them to be separated by thousands of kilometers. At each antenna station, the radio interferometry signal (RF) received from the observed radio source is converted to a

lower, “intermediate” frequency (IF) by mixing with a local oscillator (LO) signal. At each site, the IF signal is tape recorded with a reference time base derived from the same frequency standard. Tapes recorded simultaneously at the antenna sites are later collected

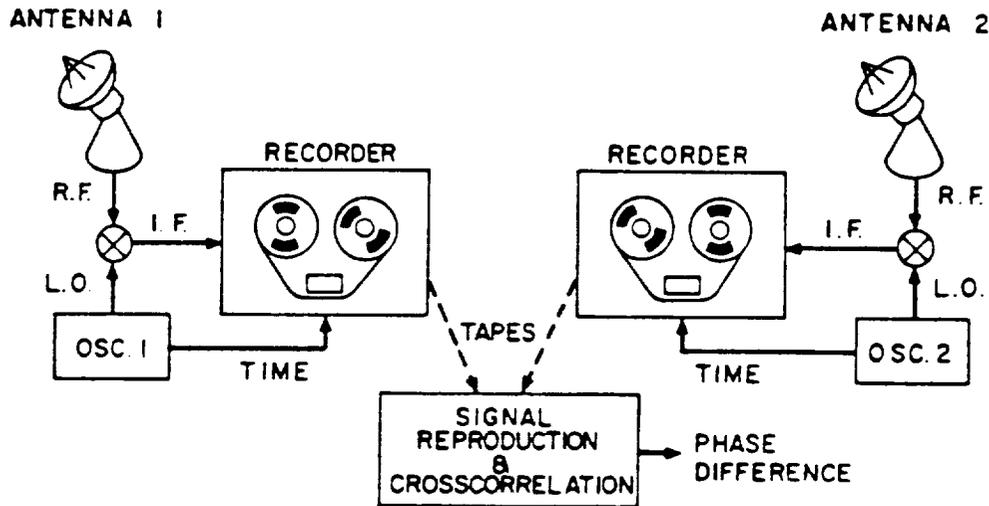


Fig. 1a Ground-based geodetic VLBI system

and played back together at a central processing station where the reproduced signals are cross-correlated to obtain the basic VLBI observables.

The space VLBI system does not differ conceptually from the case of a ground-based array of radio telescopes. A schematic diagram of the space/ground VLBI system is represented in Fig. 1b [Schilizzi et al., 1984] and [Schilizzi, 1988a]. The space-borne antenna will observe the same radio sources in conjunction with networks of antennas, and relay the received signals via a digital or analogue link directly to telemetry stations on the ground. A phase/frequency reference for the antenna in space will be based on hydrogen maser oscillators on the ground and relayed directly to the satellite from the telemetry stations in turn (phase transfer). The stability required for this phase transfer is very high (about 1×10^{-14}). After transmission to the ground, the IF data will be recorded on VLBI magnetic tapes on the ground in exactly the same way as for the ground based elements of the array. The tapes will then be brought together with tapes from the ground VLBI arrays at a central processing station for cross-correlation and image processing. After cross-correlation and calibration, the obtained basic VLBI data will be used for scientific investigations.

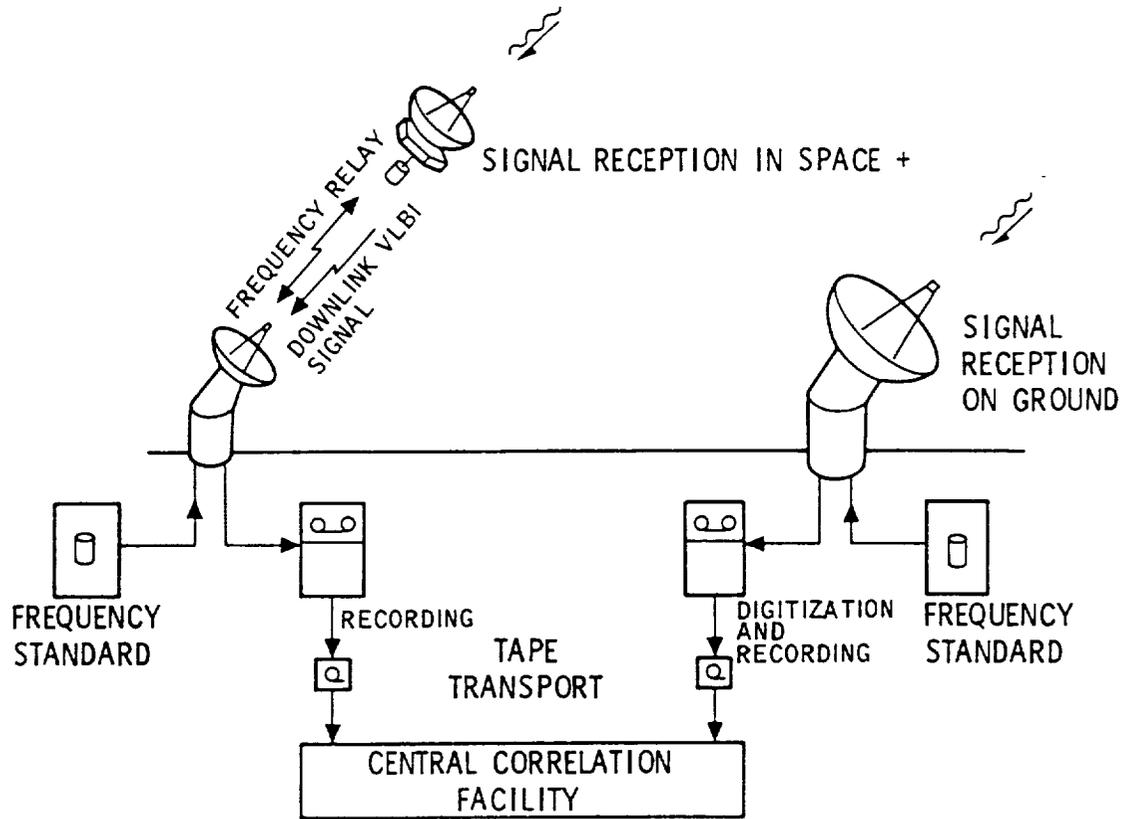


Fig. 1b A schematic diagram of a Space/ground VLBI system

All communication with the space VLBI antenna will be through one or more telemetry/control (T/C) stations in the network. The two way or multiple way coherent link(s) by telemetry stations provide range, range-rate, and phase data which can be used for orbit determination.

2.3 TDRSS-OVLBI Demonstration Experiment

The feasibility of and potential for using a dedicated VLBI observatory in space has already been demonstrated (see [Levy et al., 1986, 1987 and 1989], [Linfield et al., 1988 and 1989], [Hirabayashi, 1988], and [Nishimura and Hirabayashi, 1988]). A communications satellite in the Tracking and Data Relay Satellite System (TDRSS) of NASA was successfully used for the first space VLBI test experiment in 1986-87. The idea of using existing TDRSS for demonstrating space VLBI has been proposed by scientists at the Jet Propulsion Laboratory (JPL). The TDRS has two identical high-gain antennas with a 4.9 m diameter for S- and K-bands (see Fig. 2). The TDRSE which is situated at 41° W in the geostationary orbit was used in combination with ground 64 m

diameter antennas in Tidbinbilla, Australia, and in Usuda, Japan. It was the most suitable existing satellite for orbiting VLBI (OVLBI) because of the design of its local oscillator chain and its sensitivity. The test experiment called *TDRSS-OVLBI experiment* was done with the observing frequency of 13 cm (2.3 GHz). The received signal from radio sources at TDRSS is down-linked to the ground, frequency converted, and recorded on magnetic tapes. The reference signal in turn is uplinked to the satellite to phase lock the receivers on board to maintain coherence of the interferometer. Fig. 3 shows the concept of the TDRSS-OVLBI demonstration experiment.

The TDRSS-OVLBI experiment was performed in two phases (July-August, 1986 and January, 1987). For the observation, radio sources in the Southern Hemisphere ranged in declination from 0° to -30° were selected due to the limits of the TDRSE satellite. In the July-August 1986 session, fringes were successfully obtained among three stations (TDRSE, Usuda and Tidbinbilla) for three quasars. The maximum projected baseline obtained was 17,800 km (about 1.4 times Earth's diameter) by the TDRSS-Usuda baseline. In January, 1987, additional observations were made using the same ground antennas. Of the 25 observed sources, 23 were detected on orbiter-ground baselines, with baseline lengths as large as 2.16 Earth diameters. Also, the radio source structures were estimated from the January, 1987, session. Note that in 1988 an experiment at 2.3 GHz (13 cm) and 15 GHz (2 cm) frequencies was also successful.

Tracking information for TDRSE is obtained through triangulation with ground transponders located at White Sands Ground Terminal (WSGT) in New Mexico (32° N latitude, 107° W longitude) and Ascension Island (8° S latitude, 15° W longitude) (see [Levy et al., 1989]). The forward link uses a pseudo-noise code. The return link uses a different but synchronized pseudo-noise code modulation. By comparing epochs of the pseudo-noise code (all measurements are referenced to the WSGT cesium standard), the range is obtained. The Doppler frequency is also extracted and counted to obtain the range rate. The ranging and Doppler tracking data were processed by K.B. Blaney at Goddard Space Flight Center (GSFC) for orbit determination by a differential correction technique (ibid.). The calculation procedure is a least-squares fitting of weighted range and Doppler residuals to a corrected orbit. A data span (and solution interval) of 30-34 hr was used. The solution yielded three position components, their associated velocity components, and their time derivatives at 60 s intervals.

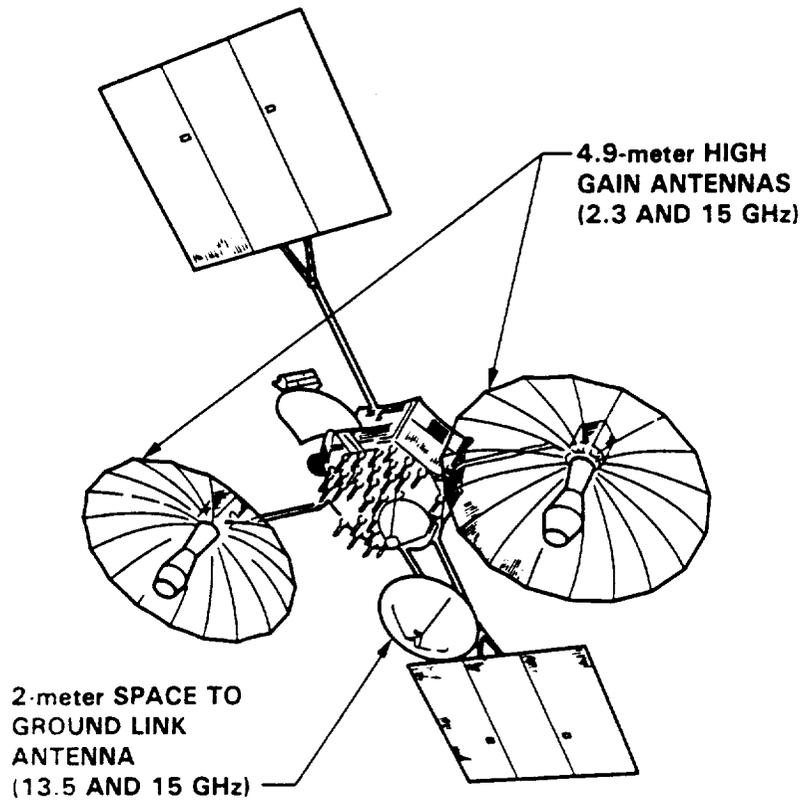


Fig. 2 Geometry of the TDRSE spacecraft. The antennas used in the OVLBI experiment are labeled [Levy et al., 1989].

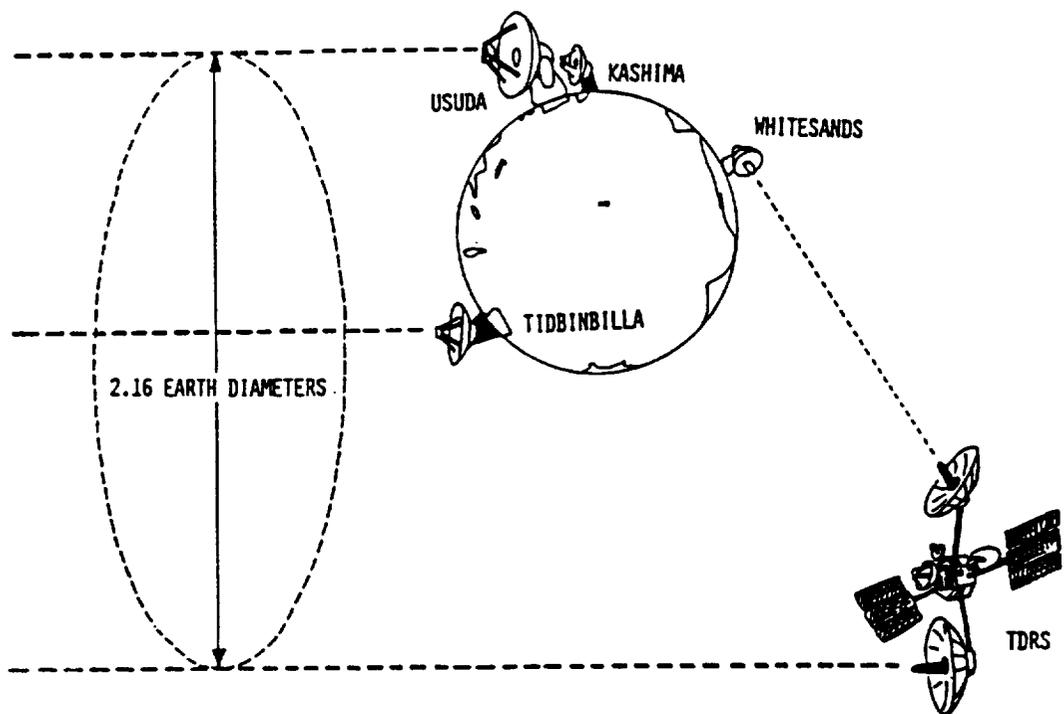


Fig. 3 The concept of TDRSS-OVLBI demonstration experiment [Hirabayashi, 1988].

As a summary, the test of the technical concept and the probe of the scientific potential for space VLBI were successful. The TDRSS-OVLBI experiment shows that an accurate knowledge of the spacecraft orbit is essential in order to achieve the coherence of the measured data with high sensitivity and calibration accuracy.

2.4 Future Space VLBI Missions

Projects to place one or more telescopes in space for VLBI purposes have been proposed or planned by several space agencies:

- QUASAT (Quasar Satellite) by ESA and NASA, see, e.g., [Schilizzi et al., 1984] and [Schilizzi, 1988a]
- RADIOASTRON (Radio Astronomical Satellite) by The Space Research Institute (IKI) of the Academy of Sciences in the USSR, see, e.g., [Sagdeev, 1984], [Kardashev and Slysh, 1988] and [Kardashev, 1989]
- VSOP (VLBI Space Observatory Program) by The Institute for Space and Astronautical Science (ISAS) in Japan, see, e.g., [Hirabayashi, 1984 and 1988] and [Nishimura and Hirabayashi, 1988].
- IVS (International VLBI Satellite), a proposal to ESA to follow RADIOASTRON and VSOP [Schilizzi et al., 1989].

The orbital parameters of the planned space VLBI satellites are given in Table 1.

2.4.1 The RADIOASTRON Mission. The first space VLBI mission that will be implemented in the near future is RADIOASTRON. It is an approved and funded mission in the Soviet Union. The design study is in progress with significant contributions from abroad. The RADIOASTRON satellite will carry a 10 m antenna, and its trajectory is planned to be 3,000 km in perigee and 69,000 km in apogee. The launching is expected around the end of 1993.

Fig. 3 is an overall schematic diagram of the Radioastron system. The satellite radio telescope (Fig. 4) will be a prime focus system with a deployable parabolic reflector 10 m in diameter. The reflector has a fixed inner part of 3 m diameter and 24 unfoldable panels.

Table 1. Comparison of the Orbital Parameters of QUASAT, RADIOASTRON and VSOP Satellites.

	QUASAT	RADIOASTRON	VSOP
h_a (km)	36,000 & 22,000	69,000	20,000
h_p (km)	5,000 & 5,000	3,000	1,000
T (hrs)	12.2 & 7.75	24	6.06
i ($^\circ$)	30	65 fixed	46.4 or 31.0
ω ($^\circ$)	variable	285 + 315 to be chosen	variable
ϵ		0.82	0.379
Ω		to be chosen	variable

h_a = apogee height, h_p = perigee height, T = period, i = inclination

ω = argument of the periapsis, ϵ = eccentricity, Ω = longitude of the ascending node

The reflector is made of reinforced carbon fiber and has an rms surface accuracy of 0.5 mm. Four dual circular polarization receivers will be provided on the satellite. The spacecraft operation time will be from two to five years.

The down-link to the ground station is at X-band (8.2 GHz) with a two-channel transmitter and high-gain parabolic antenna. The ground-based station of the RADIOASTRON mission in Suffa, USSR consists of one transmitting (command and S-band local oscillator up-link) antenna and two receiving antennas (S-band local oscillator down-link and X-band high-data rate down-link). The 70 m antenna will be used as a radio telescope to receive signals from a radio source. Both X-band data and 70 m radio telescope outputs will be recorded on two tape recorders in the VLBA format.

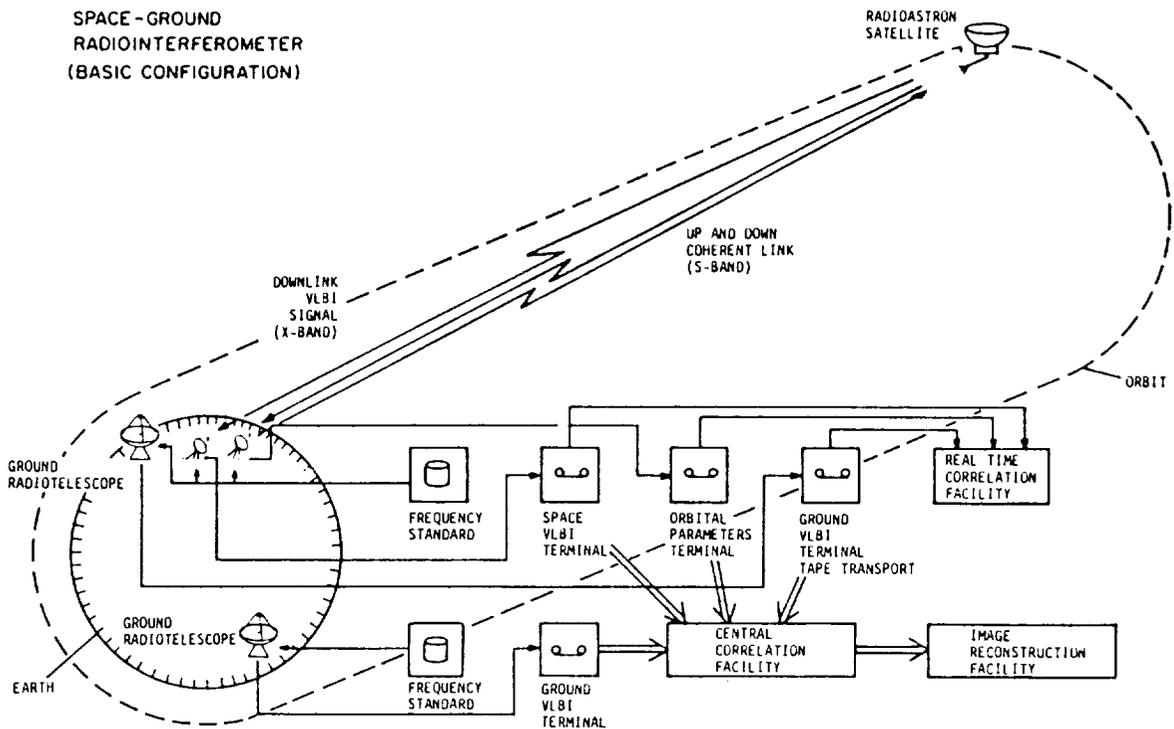


Fig. 4 The basic configuration of the space-ground radio interferometer in the RADIOASTRON mission [Kardashev and Slysh, 1988].

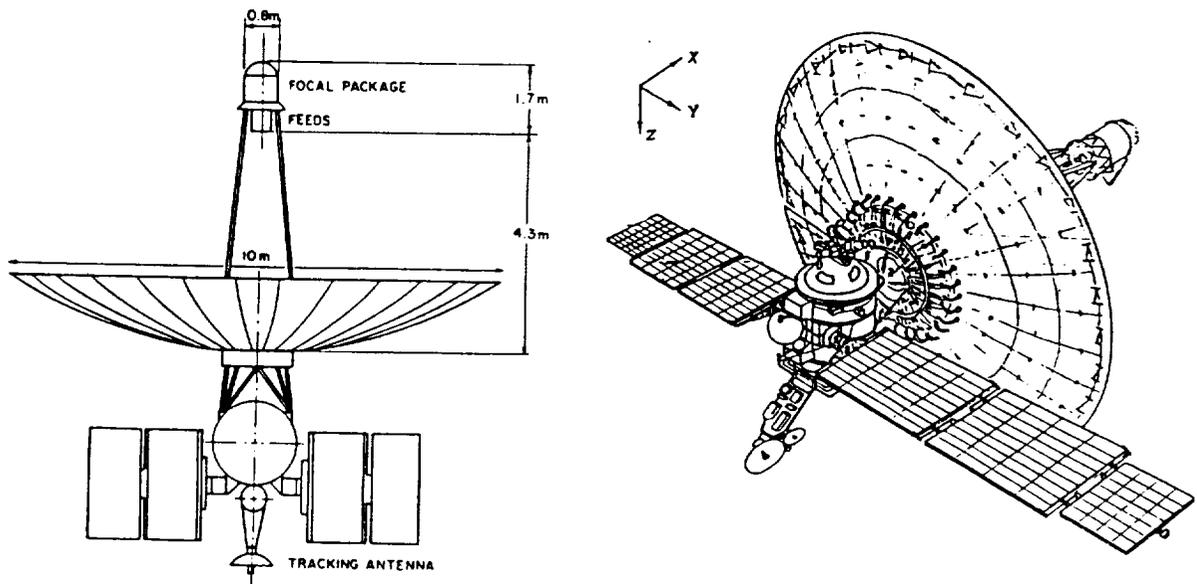


Fig. 5 The RADIOASTRON satellite [Kardashev, 1989].

2.4.2 The VSOP Mission. The Japanese VSOP satellite will carry a 10-m antenna, and its trajectory is 1,000 km in perigee and 20,000 km in apogee. This orbit provides good imaging capability of VSOP with the combination of the telescope arrays on the Earth. Fig. 6 is a rough sketch of the VSOP satellite. The surface accuracy of the 10m antenna will be better than 0.5mm RMS to be operable up to 22 GHz (wavelength of 1.3 cm) reception. The pointing accuracy of the high gain antenna is planned to be 0.01° and thus a high precision star sensor will be employed. The local oscillators on-board the VSOP satellite must be phase-locked to the frequency standard on the ground. VSOP will have cooled amplifiers, giving more than one order of magnitude better sensitivity than TDRS.

The position and the velocity of the satellite will be tracked much more precisely than other satellites by a sophisticated tracking scheme. VSOP is now in progress and the expected launch date is 1995. The space VLBI program VSOP itself has been informally approved by the Japanese government very recently [Hirabayashi, personal communication, May 30, 1989]. More details about VSOP mission can be found in [Hirabayashi, 1984; 1988; 1989 a, b, c] and [Nishimura and Hirabayashi, 1988].

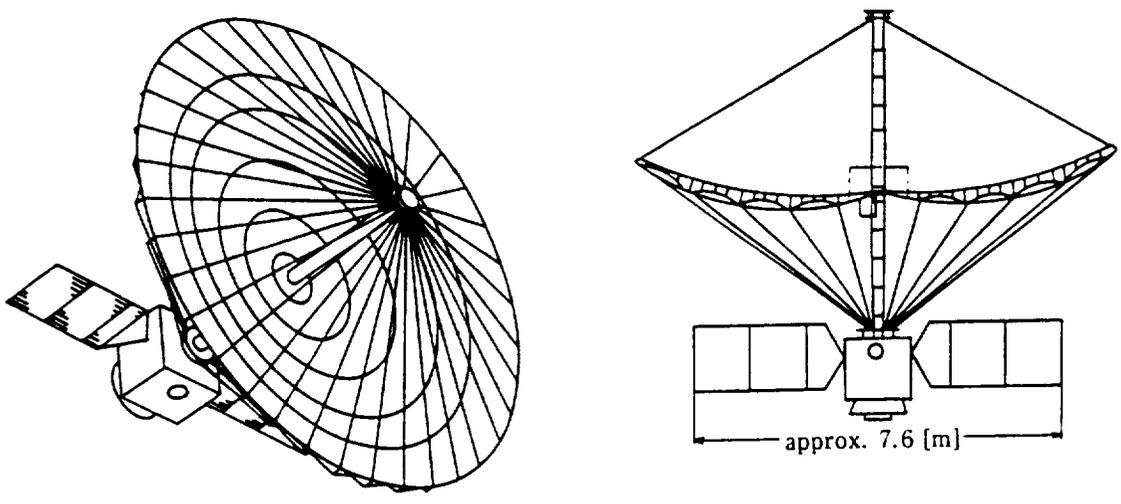


Fig. 6 The VSOP satellite [Nishimura and Hirabayashi, 1988].

2.4.3 The QUASAT Mission. The proposed QUASAT mission is a free-flying spacecraft carrying a 15 m radio antenna (Fig. 7). Two orbital situations have been considered for the mission. One of the orbital situations is featured by the first operational elliptic orbit with perigee of about 5,000 km and apogee of about 36,000 km. It was planned that the spacecraft's apogee be lowered later to 22,000 km in order to fulfill the various scientific requirements. The orbiting radio telescope was planned to make interferometric observations of radio sources in conjunction with the major ground-based VLBI arrays in Europe, USA, Australia and the USSR. The mission design lifetime was planned to be two years, but an operational lifetime of five years was expected. More details about QUASAT mission can be found in, e.g., [Proc. of Workshop on QUASAT: 1984], [Schilizzi et al., 1984], [Frisk et al., 1988] and [Schilizzi, 1988a].

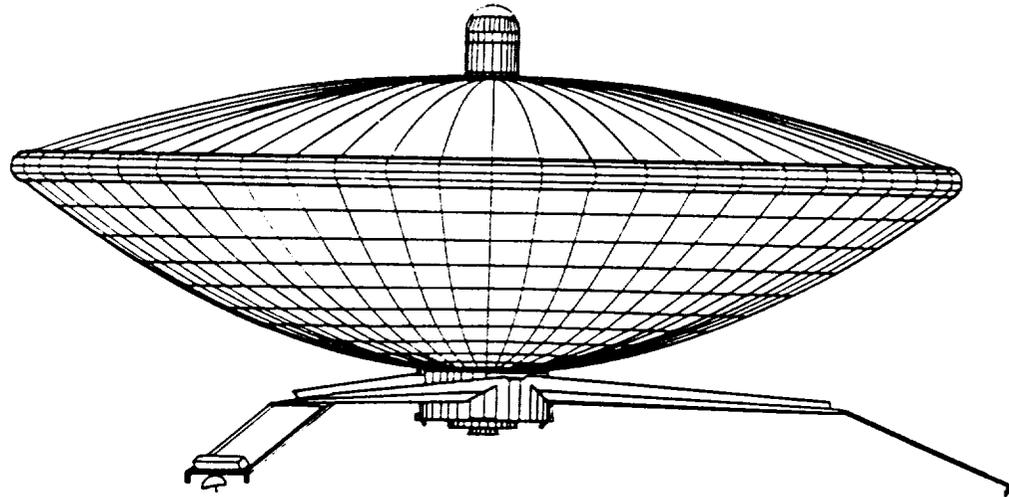


Fig. 7 The QUASAT satellite [Schilizzi, 1988a].

2.4.4 The IVS Mission. The IVS mission is conceived to be a major radio telescope in space funded by the principal space agencies. Orbiting the Earth, the 25 m diameter telescope will provide high quality images of radio sources to wavelengths spanning the radio band from meters to millimeters with resolutions as high as 25 micro arcseconds. It is proposed that IVS be placed initially in an orbit whose apogee height is in the range 20,000 km (VSOP-like) to 40,000 km (QUASAT-like), but whose inclination is higher than either VSOP or QUASAT. Possible orbital parameters are therefore: apogee

height 20,000-40,000 km, perigee height about 5,000 km and inclination about 60 degrees. Maximum baseline lengths of 30,000-50,000 km can be generated in this manner, three to five times longer than the maximum available on Earth. IVS is a second generation mission well matching the ground VLBI arrays in both sensitivity and wavelength coverage. The goals of the IVS mission will require the position of the satellite to be known to 30 cm or better at all times [Schilizzi et al., 1989].

2.4.5 Joint Space VLBI Observations. The space VLBI missions are expected to be operational in the next decade. However, QUASAT was not approved by ESA at the end of 1988. Therefore, the Soviet RADIOASTRON and the Japanese VSOP will be the first two dedicated space VLBI missions in the 1990's. Supposedly, QUASAT will again be considered by ESA to realize. These missions will have many improvements compared to the TDRSS which was used for the first space VLBI test experiment. The u-v coverage in the TDRSS-OVLBI experiment was not satisfactory because of the geostationary nature of the satellite and of the small number of ground radio telescopes which joined the experiment.

The RADIOASTRON and VSOP orbits are complementary in the u-v coverage they will provide. A simulation study by Murphy (1989) shows that joint space-VLBI observations can produce better results in imaging processing. Therefore, further simulation tests will be carried out to determine the optimal orbital elements of RADIOASTRON for possible simultaneous observation with VSOP. Their combined use is most probable and desirable. *It is expected that a combined use of RADIOASTRON and VSOP observations will be performed in order to have much improved images.*

Radioastronomers need multiple elements in space for higher angular resolution and/or higher dynamic range mapping. There are plans to have multiple elements in space within both missions introducing space-space VLBI. Hirabayashi (1989b) presented a plan for two orbiting VSOP observatory with modestly near earth orbits. VSOP mission with two space elements might be very appropriate particularly for geodetic and geodynamical purposes as well.

The coordination for international collaboration is a critical item for both projects—RADIOASTRON and VSOP. The design study of both missions is in progress with significant contributions from abroad. For space VLBI to become operational, coordination of observing, management of ground telescopes and tracking operations, data

transfer and processing is needed. Therefore, international workshops are organized. The first was to discuss QUASAT in June 18-22, 1984 in Austria [Proc. of Workshop in QUASAT 1984]. The International Radioastron Review Meetings have been held twice a year. The eighth meeting was held in Green Bank, West Virginia, USA [Memorandum of the 8th RADIOASTRON Meeting 1989]. The ninth one was in Tashkent, USSR in November, 1989. Domestic VSOP meetings have been held in Japan [Proc. of a Domestic Workshop on "The Research of Space VLBI" 1989]. An international symposium was organized also in Japan December 5-7, 1989 in order to discuss worldwide the VSOP mission. An other international meeting "Space VLBI—The Missions and the Science" organized by Committee on Space Research (COSPAR) in The Hague, Netherlands, July 2-3, 1990 reviewed all aspects of space VLBI, with special emphasis on the two approved missions RADIOASTRON and VSOP.

2.5 The Ground-Based VLBI Networks

The ground-based arrays are the other major element of the space VLBI missions. All missions assume an orbiting observatory to be coherently connected to the radiotelescopes distributed widely on the earth. They require a global network of ground antennas to be able to form subnets. Therefore, the involvement of all possible ground antennas is important not only for radioastronomical purposes but for application of them in the field of geodesy and geodynamics as well.

There are two main networks in operation—the European VLBI Network (EVN) and the U.S. Very Long Baseline Array (VLBA), both of which provide observing opportunities on regional as well as international scales, see [Romney, 1988] and [Schilizzi, 1988b]. Networks of both types, tracking/link and observing, respectively in the USSR and in Japan, are very important from the viewpoint of their space VLBI missions. They will constitute the main support to the orbiting elements. However, other networks in the Southern Hemisphere (e.g., the Australia Telescope (AT) and Australian VLBI antennas), in Asia (e.g., China, India) and in Canada are also likely to participate. Stations of these networks will be the co-observing stations for the space VLBI missions.

Note that VLBA construction is now in progress with expected completion of 1992. EVN and AT constructions are also in progress. A construction of dedicated VLBI network "QUASAR" of six radiotelescopes in the Soviet Union with possible extensions by other telescopes from abroad (China, India) has been started very recently [Finkelstein

and Yatzkiv, 1989]. These VLBI networks (and correlators) are expected to work jointly with space VLBI missions.

Tracking networks like the NASA/JPL Deep Space Network (DSN) will collaborate on precise tracking of both RADIOASTRON and VSOP satellites. The different networks of VLBI antennas are shown in Figs. 8 and 9, which reflect the strong international character of this research. The observing schedule, data processing, the managements of ground telescopes and tracking operations need to be discussed and organized worldwide.

2.6 Orbital Tracking of Space VLBI Satellites

The astrometric, geodetic and geodynamic potential of the space VLBI missions is strongly dependent on orbit determination accuracy of the satellites. Therefore, high accurate knowledge of the satellite orbit is very important for the space VLBI missions. The following tracking techniques are recommended for precise orbit determination of space VLBI satellites, see [Fejes, 1989] memorandum of the 8th RADIOASTRON Review Meeting 1989 [Tang, 1984].

The first category which requires no systems modifications or additional equipment:

1. Two-way or multiple-way coherent link(s) by telemetry stations (range, range-rate, phase). A good telemetry coverage is essential.
2. VLBI delay and delay-rate as tracking data.
3. Differential VLBI (Δ VLBI) using extended (worldwide) network of ground radio telescopes during selected observing programs.

Results of a computational simulation study by Konopliv (1989) show that it might be possible to reach meter-level position accuracy in orbit determination of RADIOASTRON satellite with range and Δ VLBI measurements by NASA/JPL DSN stations. Borza et al., (1990) show by simulations that the dominant part of the orbital errors of space VLBI satellites originates from errors in the solar reflectivity models. All other effects can be modeled so accurately that they only yield small or negligible errors. According to the test computations carried out by Fejes et al., (1989b), the space VLBI time delay and delay-rate observables can contribute significantly to orbit accuracy improvement for the missions. It is in a good agreement with the statement of Tang (1984).

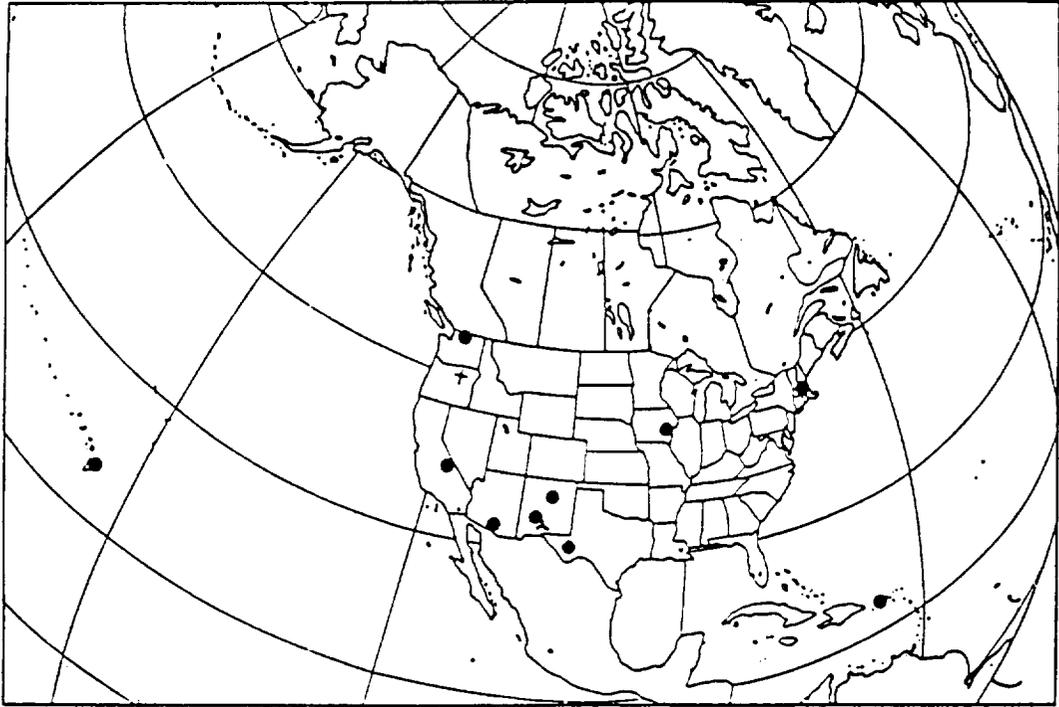


Fig. 8 The Very Long Baseline Array—VLBA.

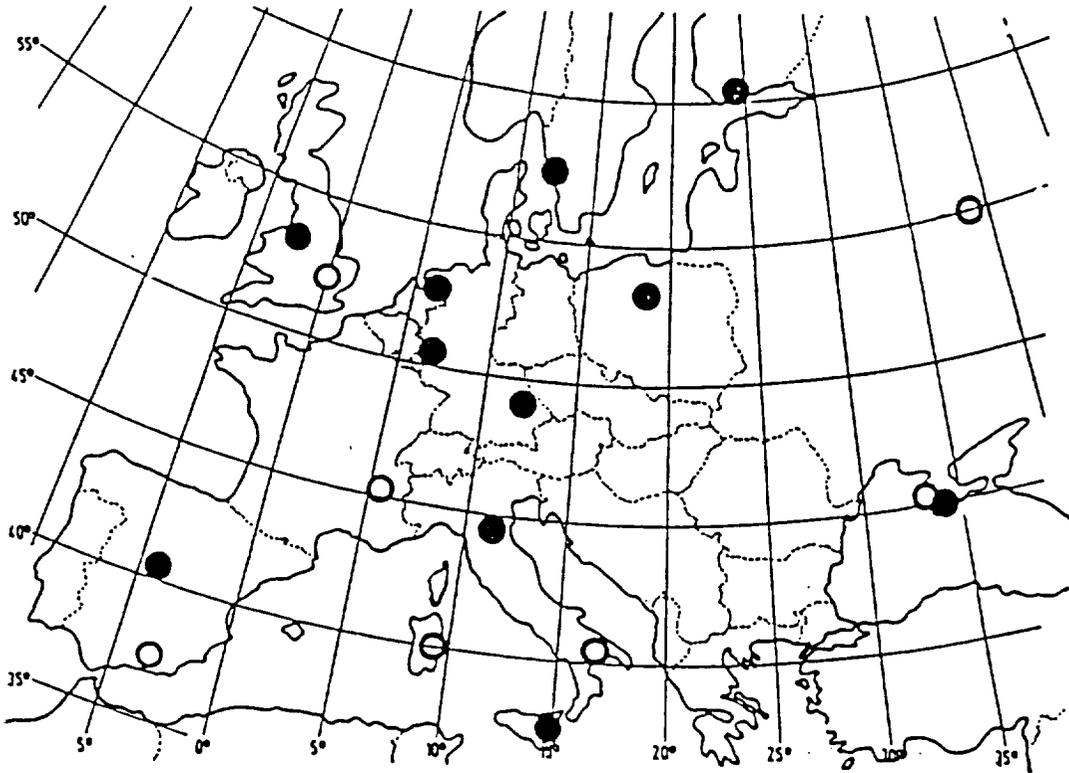


Fig. 9 Stations of the European VLBI Network (EVN). Station currently in operation (full circles) or expected to be operational by 1995 (open circles).

The second category which requires system modification and additional ground and space equipment:

1. Microwave techniques: (a) One-way satellite-based systems (GPS, GLONASS); and (b) two-way satellite-based systems (PRARE)
2. Laser ranging
3. Micro-accelerometer on board

Note that the inclusion of PRARE System for Radioastron is considered. It is planned that the VSOP satellite will carry a Global Positioning System (GPS) receiver on board to provide tracking data for orbit determination. A set of corner-cube reflectors on board of a space VLBI satellite (e.g., VSOP) would increase the potential of space VLBI in geodesy and geodynamics.

3. MATHEMATICAL MODELS FOR SPACE VLBI OBSERVABLES

3.1 Introduction

Since there are no conceptual differences between ground-based and space-based VLBI, existing formalism of the ground-based geodetic VLBI carries over easily to the space-based case. In ground-based VLBI the necessary relative motion of the stations is provided by the Earth's rotation. In space-based VLBI the relative motion of the antennas is provided by both Earth's rotation and orbital motion of the space VLBI satellite around the Earth.

In the simplest version of space VLBI, one station in orbit observes in conjunction with a second station on the ground. However, in practice we have a number of networks of ground antennas observing the common celestial radio sources simultaneously with a conventional VLBI technique. Moreover, joint observations of two or perhaps more space VLBI satellites will supposedly be performed in the future. Therefore, a combined use of simultaneous space and ground VLBI observations should be considered.

3.2 Basic Observables and Their Models

We have selected notational conventions which allow us to treat the VLBI observables of the different types of baselines within a combined ground/space VLBI network uniquely. A space VLBI network is formed by ground VLBI stations P_i, P_j and orbiting VLBI satellites' positions S^I and S^J . The lower index with small letters (i, j) refers to ground points while the upper index with capital letters (I, J) to satellite points at high altitude. Therefore, in our conventions we can distinguish a *ground-to-ground baseline* between two ground VLBI stations P_i and P_j by B_{ij} , a *ground-to-space baseline* between a ground VLBI station P_i and an orbiting VLBI telescope S^J by B_i^J , and a *space-to-space baseline* between two orbiting VLBI telescopes S^I and S^J by B^{IJ} . We use these notations to distinguish the corresponding VLBI observables, i.e., time delay and delay rate as well.

The basic geometry for a typical baseline in the *ground-based VLBI* and space-based VLBI or simply *space VLBI* is shown in Figs. 10-12. Generally, a certain segment of a wavefront from the natural radio source will arrive at one site before it arrives at another (Figs. 10-12). This time delay is the basic observable of VLBI. Delay observations are

Fig. 10 Geometry of a ground-to-ground baseline

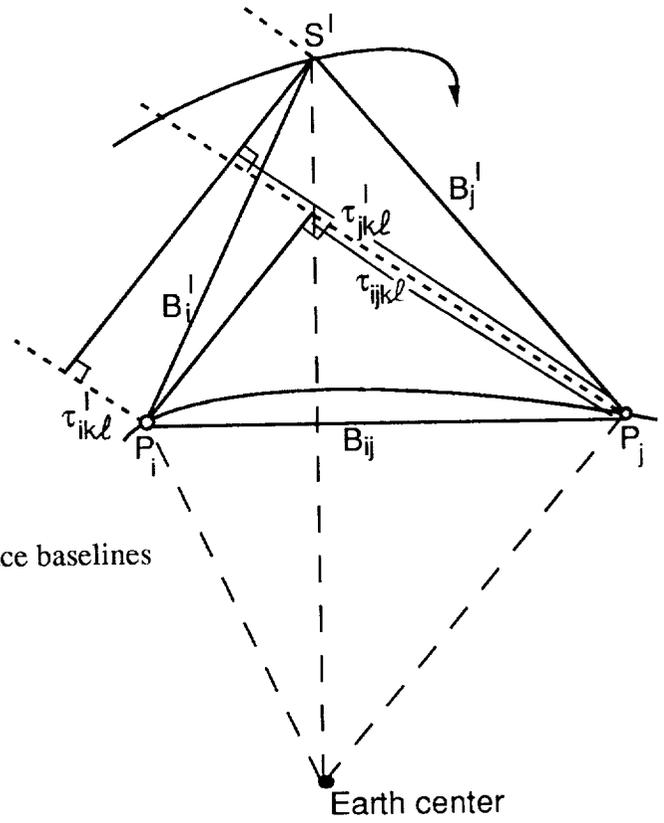
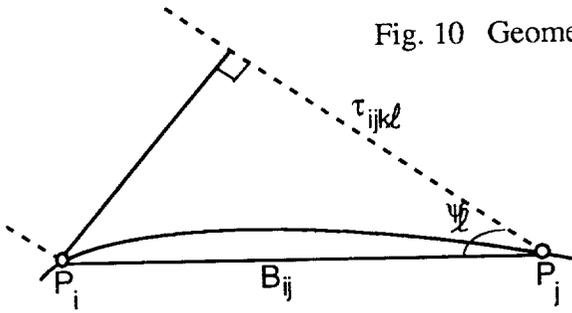


Fig. 11 Geometry of the ground-to-space baselines

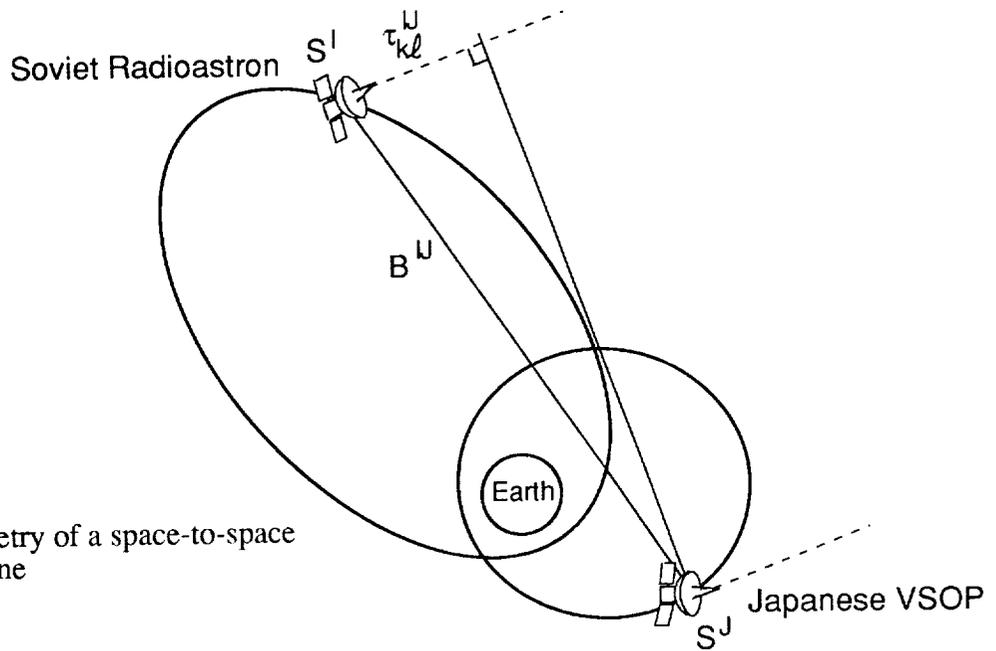


Fig. 12 Geometry of a space-to-space baseline

composed of three components. The first component is due to the geometry of the station locations and source location, as illustrated in Figs.10-12. This component contains all of the information about the geodetic and astrometric parameters that are of interest. The second component is due to instrumental effects, principally clock errors. The third component is due to effects of the propagation medium (e.g., the atmosphere and the ionosphere). This latter component is not considered here.

In the *ground-based VLBI network*, two ground radio telescopes P_i and P_j form a *ground-to-ground baseline* B_{ij} (Fig. 10). The signal from the ℓ^{th} radio source arrives at the k^{th} epoch of observation to the ground VLBI stations with time delay $\tau_{ijk\ell}$ which can be written as

$$\tau_{ijk\ell} = \frac{1}{c} d_{ijk\ell} = \frac{1}{c} B_{ij}(t_k) \cos \psi_{\ell} , \quad (1)$$

or in vector notation, the dot denoting the inner product,

$$\tau_{ijk\ell} = -\frac{1}{c} B_{ij}(t_k) \cdot \mathbf{e}_{\ell} \quad (2)$$

where \mathbf{e}_{ℓ} is the unit vector in the direction of the ℓ^{th} radio source, B_{ij} is the vector separation of the receiving stations P_i and P_j , and c is the speed of light. The minus sign is introduced to follow convention. The changing geometrical configuration gives rise to the delay rates $\dot{\tau}_{ijk\ell}$

$$\dot{\tau}_{ijk\ell} = -\frac{1}{c} \frac{dB_{ij}(t_k)}{dt} \cdot \mathbf{e}_{\ell} \quad (3)$$

assuming that $\dot{\mathbf{e}}_{\ell} = 0$.

The third VLBI observable is phase delay $\phi_{ijk\ell}$ which is related to time delay (2) by

$$\phi_{ijk\ell} = \omega \tau_{ijk\ell} , \quad (4)$$

where ω is the angular frequency ($2\pi f$) of the received radio signal. The phase delay is the difference in phase of the signals received at each site.

Note that time (group) delay $\tau_{ijk\ell}$ is the derivative of (fringe) phase $\phi_{ijk\ell}$ with respect to angular frequency. The time delay contains the same information as phase for

astrometric and geodetic purposes but is inherently less accurate than phase. On the other hand, it can be estimated unambiguously. The delay observable yields a full baseline solution and, therefore, plays the most important role in geodetic VLBI. The delay rate is not often used in ground-based VLBI. However, the role of the rate observable will be more important in space VLBI.

In the following the speed of light will be set to unity so that the time delay and delay rate will be expressed in units of distance and its time variation, respectively. Assume that they have been corrected for discrepancies between modeled and true distance (or distance variation), e.g., atmospheric errors, relativistic effects and aberration. The time delay is now

$$d_{ijk\ell} = c \tau_{ijk\ell} = -B_{ij}(t_k) \cdot e_{\ell} , \quad (5)$$

and the time delay rate is

$$\dot{d}_{ijk\ell} = c \dot{\tau}_{ijk\ell} = -\frac{dB_{ij}(t_k)}{dt} \cdot e_{\ell} . \quad (6)$$

Adding a two term polynomial, whose coefficients ΔC_{0ij} and ΔC_{1ij} correspond to a relative offset and rate, respectively, between the two clocks at the ends of the $P_i P_j^{\text{th}}$ baseline, the geometric time delay (expressed in units of length) can be modelled as

$$d_{ijk\ell} = -B_{ij}(t_k) \cdot e_{\ell} + c [\Delta C_{0ij} + \Delta C_{1ij}(t_k - t_0)] , \quad (7)$$

where t_0 is the initial epoch of observation. The time delay rate is then

$$\dot{d}_{ijk\ell} = -\frac{dB_{ij}(t_k)}{dt} \cdot e_{\ell} + c \Delta C_{1ij} . \quad (8)$$

The clock parameters ΔC_{0ij} , ΔC_{1ij} are nuisance parameters, included here to make the model more realistic.

In *space VLBI network*, an Earth-orbiting VLBI telescope S^I with two ground VLBI stations P_i and P_j at a moment t_k forms two *ground-to-space baselines* $B_i^I(t_k)$, $B_j^I(t_k)$ in addition to the ground-to-ground baseline $B_{ij}(t_k)$ (Fig. 11). The corresponding time delays $\tau_{ik\ell}^I$, $\tau_{jk\ell}^I$ and delay rates $\dot{\tau}_{ik\ell}^I$, $\dot{\tau}_{jk\ell}^I$, including the clock parameters, can be

expressed by similar observation equations which are for the ground-to-space baseline $B_j^I(t_k)$ as follows:

$$d_{jk\ell}^I = c \tau_{jk\ell}^I = -B_j^I(t_k) \cdot e_\ell + c [\Delta C_{0rj}^I + \Delta C_{1rj}^I(t_k - t_0)] \quad (9)$$

and

$$\dot{d}_{jk\ell}^I = c \dot{\tau}_{jk\ell}^I = -\frac{dB_j^I(t_k)}{dt} \cdot e_\ell + c \Delta C_{1rj}^I \quad (10)$$

where the delay rate ($\dot{\tau}_{jk\ell}^I$) plays a more important role than at the ground-based VLBI network due to the larger relative motions between a ground and an orbiting antenna.

The lower index r of the clock parameters in Eqs. (9) and (10) refers to the reference clock at the telemetry/control (T/C) ground station. There will be no clock on-board of the orbiting radio telescope (see Chapter 2.2).

Let us suppose that there will be two orbiting space VLBI satellites in operation in the future (e.g., RADIOASTRON and VSOP, IVS may be later in addition, or two VSOP satellites as a system in themselves) having simultaneous observations for the same extragalactic radio source. Two orbiting VLBI satellites, S^I and S^J , at a moment t_k form a *space-to-space baseline* $B^{IJ}(t_k)$ (Fig. 12). Then, including the clock parameters, the corresponding time delay and delay rate are modelled

$$d_{k\ell}^{IJ} = c \tau_{k\ell}^{IJ} = -B^{IJ}(t_k) \cdot e_\ell + c[\Delta C_{0rq}^{IJ} + \Delta C_{1rq}^{IJ}(t_k - t_0)] \quad (11)$$

$$\dot{d}_{k\ell}^{IJ} = c \dot{\tau}_{k\ell}^{IJ} = -\frac{dB^{IJ}(t_k)}{dt} \cdot e_\ell + c \Delta C_{1rq}^{IJ}, \quad (12)$$

where the lower indexes r and q refer to the clocks at telemetry/control ground stations r and g , respectively for the satellites S^I and S^J .

By examining the geometric part of the time delay observable models we recognize that the time delays will change if the length of the baseline changes, or if the orientation of the baseline changes with respect to the plane of the wavefront, or if both change simultaneously. In the case of a *ground-to-ground baseline*, the time delay changes due to the variation of baseline orientation with respect to the plane of the wavefront. In the *ground-based VLBI*, the length of a baseline is supposed to be unchanged for a short

period (the geodynamical effects, e.g., tides, can be accounted for). It is evident that time delay (and delay rate) observations can be used to recover baseline length and baseline orientation with respect to the radio source direction or recover the source orientation with respect to the baseline direction. The first two applications are of geodetic interest while the last one is of astrometric.

In the space VLBI the time delay changes because of the continuous changes of baseline length and of baseline orientation with respect to the plane of the wavefront. Both changes are due to the continuous relative motion between both ends of the baseline. For the case of a *ground-to-space baseline*, the relative motion between both ends of the baseline is a result of the Earth's rotation and satellite motion. Changes in both length and orientation of a *space-to space baseline*, result from the relative motion of satellites with respect to each other. Since the baselines in space VLBI do not repeat themselves, their length and orientation therefore can not be recovered from the usual VLBI technique itself. Some of the parameters (e.g., satellite's coordinates and/or source coordinates) should be determined by some other means.

3.3 Datum Content of Space VLBI Observables

Following an evaluation by Wells et al., (1987) for GPS datum definition, in this chapter we describe datum content of the space VLBI time delay observables. For this purpose, consider a space VLBI network of points divided into satellite points with an orbiting radio telescope S^I and a ground-based VLBI station P_J . A single time delay observation (ignoring clock parameters, and expressing the time delay in units of distance) is given by

$$d_{jk\ell}^I = -B_{jk}^I \cdot e_\ell = -[R_k X_j - X_k^I]^T e_\ell,$$

in matrix form, where X_J contains the Earth-fixed coordinates of the station P_J , X_k^I contains the coordinates of satellite S^I in a true-of-date geocentric inertial coordinate frame at epoch t_k , R_k is a transformation matrix (transpose of the matrix S), see Eqs. (7-11) in [Moritz and Mueller, 1987; p. 417], the dot denotes the scalar product and the T is the transpose sign.

Translating the datum by ΔX we have

$$\begin{aligned}\overline{R_k X_j} &= R_k X_j + \Delta X \\ \overline{X_k^I} &= X_k^I + \Delta X\end{aligned}$$

$$\begin{aligned}
\bar{d}_{jk\ell}^I &= -\left[\overline{R_k X_j} - \bar{X}_k^I\right]^T \mathbf{e}_\ell \\
&= -\left[(R_k X_j + \Delta X) - (X_k^I + \Delta X)\right]^T \mathbf{e}_\ell \\
&= -\left[R_k X_j - X_k^I\right]^T \mathbf{e}_\ell \\
&= d_{jk\ell}^I
\end{aligned} \tag{13a}$$

that is, time delays are invariant with respect to translation.

Rotating the datum, we have

$$\begin{aligned}
\widetilde{R_k X_j} &= R_o(R_k X_j) \\
\widetilde{X}_k^I &= R_o X_k^I \\
\widetilde{\mathbf{e}}_\ell &= R_o \mathbf{e}_\ell
\end{aligned}$$

where R_o represents a proper orthogonal transformation ($R_o^T R_o = I$). Then

$$\begin{aligned}
\widetilde{d}_{jk\ell}^I &= -\left[\widetilde{R_k X_j} - \widetilde{X}_k^I\right]^T \widetilde{\mathbf{e}}_\ell \\
&= -\left[R_o(R_k X_j) - R_o X_k^I\right]^T R_o \mathbf{e}_\ell \\
&= -\left[R_k X_j - X_k^I\right]^T R_o^T R_o \mathbf{e}_\ell \\
&= -R_o^T R_o \left[R_k X_j - X_k^I\right]^T \mathbf{e}_\ell \\
&= -\left[R_k X_j - X_k^I\right]^T \mathbf{e}_\ell \\
&= d_{jk\ell}^I
\end{aligned} \tag{14}$$

that is, time delays are invariant with respect to rotation.

Changing the datum scale by s , we have

$$\begin{aligned}
\widehat{R_k X_j} &= s R_k X_j \\
\widehat{X}_k^I &= s X_k^I \\
\widehat{\mathbf{e}}_\ell &= s \mathbf{e}_\ell
\end{aligned}$$

Then

$$\begin{aligned}
\widehat{d}_{jk\ell}^I &= -\left[\widehat{R}_k X_j - \widehat{X}_k^I\right]^T \widehat{e}_\ell \\
&= -\left[sR_k X_j - sX_k^I\right]^T s e_\ell \\
&= -s\left[R_k X_j - X_k^I\right]^T s e_\ell \\
&= -s^2\left[R_k X_j - X_k^I\right]^T e_\ell \\
&= s^2 d_{jk\ell}^I
\end{aligned} \tag{15}$$

that is, time delays are not invariant with respect to scale change.

For the sake of completeness and a comparison, the corresponding equations for the ground-based VLBI time delay observable can be derived in order to show its translational and rotational invariance. In the following only the translational invariance will be deduced pointing at some difference in the sequel. Consider a network of ground-based VLBI stations P_i and P_j . A single time delay observation (ignoring again clock parameters and expressing the time delay in units of distance) is given here in matrix form

$$d_{ijk\ell} = -(X_j - X_i)^T R_k e_\ell ,$$

where X_i and X_j is the i th and j th station within a ground-based geodetic VLBI network, R_k is a transformation matrix, see Eq. (19).

Translating the datum by ΔX we have

$$\begin{aligned}
\overline{X}_j &= X_j + \Delta X \\
\overline{X}_i &= X_i + \Delta X .
\end{aligned}$$

Then

$$\begin{aligned}
\overline{d}_{ijk\ell} &= -(\overline{X}_j - \overline{X}_i)^T R_k e_\ell \\
&= -[(X_j + \Delta X) - (X_i + \Delta X)]^T R_k e_\ell \\
&= -(X_j - X_i)^T R_k e_\ell \\
&= d_{ijk\ell}
\end{aligned} \tag{13b}$$

that is, ground-based VLBI time delays are invariant with respect to translation. A comparison of Eqs. (13a) and (13b) is discussed in section 3.5.1.

Therefore, time delay observations cannot provide datum origin and orientation. Consequently, the basic formulations are independent of the reference frame used. Since the geometric observables are the dot product of two vectors ($B(t)$ and e_ℓ , $dB(t)/dt$, and e_ℓ , respectively) it is evident that they do not depend on the location and orientation of the coordinate frame in which the components of these vectors refer.

Eqs. (7-12) can be applied for covariance analysis either in terrestrial frame or in the true celestial frame. In the former case, e_ℓ is transformed from the true celestial to the terrestrial frame, and in the latter case the baseline vector (B_{ij} , B_i^I or B^{IJ}) is transformed from the terrestrial to the true celestial frame. On the other hand, the observables are affected by the scale of the coordinate frame which is implied by the adopted speed of light. Therefore, from observations of both time delay and delay rate it is impossible to recover the origin and absolute orientation of the coordinate frame.

3.4 Mathematical Model for Ground-to-Ground VLBI Observations

The basic theory of ground-based geodetic VLBI is very well documented by, e.g., Whitney (1974), Robertson (1975), Counselmann (1976), Counselmann and Shapiro (1978), Ma (1978), Shapiro (1978), Bock (1980), Brouwer (1985), Harvey (1985), Campbell (1987), Moritz and Mueller (1987) and Sovers and Fanselow (1987). Therefore, only those aspects of the theory which are essential to the research in this report, are briefly summarized.

Since a combined use of ground-based and space VLBI observables will be used in the future, a detailed comparison of the corresponding mathematical models is desirable. Only the essentially necessary equations for ground-to-ground baseline observables are given here in order to make a comparison between the mathematical models for different type of baseline observations.

Therefore, some of the important mathematical equations will be derived in rigorous manner.

3.4.1 Time Delay Model. In the ground-based geodetic VLBI, the mathematical model for the time delay observable can basically be written as the inner product of the baseline vector rotated from an Earth-fixed system into the true-of-date inertial system and the quasar unit vector in the true-of-date inertial system plus polynomial terms of clock

parameters. For baseline P_iP_j observing source ℓ at epoch k the path difference (time delay times the speed of light) which the incoming signal must travel after its reception at station P_i till its arrival at station P_j (see Fig. 10) can be expressed in a simplified manner suitable for least squares covariance analysis as follows (the T is the transpose sign),

$$d_{ijk\ell} = - \begin{bmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{bmatrix}^T R_2(-\xi) R_1(-\eta) R_3(\theta_k) \begin{bmatrix} \cos\delta_\ell & \cos\alpha_\ell \\ \cos\delta_\ell & \sin\alpha_\ell \\ \sin\delta_\ell & \sin\alpha_\ell \end{bmatrix} + c [\Delta C_{0ij} + \Delta C_{1ij} (t_k - t_0)], \quad (16)$$

where

- $\Delta X_{ij}, \Delta Y_{ij}, \Delta Z_{ij}$ are the coordinate differences of the baseline P_iP_j in an Earth-fixed system;
- α_ℓ, δ_ℓ are the true right ascension and declination of the ℓ th quasar;
- ξ, η are the components of polar motion that relate the true celestial pole to the terrestrial pole;
- θ_k is the Greenwich Apparent Sidereal Time (GAST) at epoch t_k , that is
- $$\begin{aligned} \theta_k &= \theta_0 + W_d \text{UT1}_k \\ &= \theta_0 + W_d [\text{TAI} - (\text{TAI} - \text{UTC}) - (\text{UTC} - \text{UT1})]_k + \text{Eq.E} \end{aligned} \quad (17)$$

- θ_0 GAST at initial epoch t_0
- TAI International Atomic Time
- UTC Coordinated Universal Time
- UT1 observed Universal Time corrected for polar motion
- W_d conversion factor from Universal to Sidereal Time
- Eq.E Equation of Equinox

$\Delta C_{0ij}, \Delta C_{1ij}$ are the clock offset and drift between the ground stations P_i and P_j ,

$R_i(\phi)$ is the rotation matrix for a right handed rotation ϕ about the axis i , see [Moritz and Mueller, 1987; p. 417]:

$$R_2(-\xi) = \begin{bmatrix} \cos \xi & 0 & \sin \xi \\ 0 & 1 & 0 \\ -\sin \xi & 0 & \cos \xi \end{bmatrix} \quad (18a)$$

$$R_1(-\eta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \eta & -\sin \eta \\ 0 & \sin \eta & \cos \eta \end{bmatrix} \quad (18b)$$

$$R_3(\theta_k) = \begin{bmatrix} \cos \theta_k & \sin \theta_k & 0 \\ -\sin \theta_k & \cos \theta_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (18c)$$

that is

$$R = R_2(-\xi) R_1(-\eta) R_3(\theta_k) = \begin{bmatrix} \cos \xi \cos \theta_k - \sin \xi \sin \eta \sin \theta_k & \cos \xi \sin \theta_k + \sin \xi \sin \eta \cos \theta_k & \sin \xi \cos \eta \\ -\cos \eta \sin \theta_k & \cos \eta \cos \theta_k & -\sin \eta \\ -\sin \xi \cos \theta_k - \cos \xi \sin \eta \sin \theta_k & -\sin \xi \sin \theta_k + \cos \xi \sin \eta \cos \theta_k & \cos \xi \cos \eta \end{bmatrix} \quad (19)$$

Expression of Eq. (16) may be rewritten as

$$\begin{aligned} d_{ijk\ell} = & -\Delta X_{ij} [\cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) \\ & + \sin \xi \cos \eta \cos \delta_\ell] + \\ & + \Delta Y_{ij} [\cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \eta \sin \delta_\ell] + \\ & + \Delta Z_{ij} [\sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) \\ & - \cos \xi \cos \eta \sin \delta_\ell] + \\ & + c [\Delta C_{0ij} + \Delta C_{1ij} (t_k - t_0)] \end{aligned} \quad (20)$$

Differentiation of Eq. (20) yield the usual form of the adjustment model (error equations),

$$d(d_{ijkl}) = \sum_p A_p dP_p, \quad (21)$$

where the index p stands for the unknowns

$$\{\Delta X_{ij}, \Delta Y_{ij}, \Delta Z_{ij}, \alpha_\ell, \delta_\ell, \xi, \eta, \kappa, \Delta C_{0ij}, \Delta C_{1ij}\} \quad (22)$$

The parameter κ is used to model the departure of the earth's sidereal rotation from uniformity, thus, e.g.,

$$\kappa = \text{UT1} - \text{UTC}. \quad (23)$$

The partial derivatives A_p are as follows:

$$A_{\Delta X_{ij}} = - [\cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \sin \xi \cos \eta \cos \delta_\ell] \quad (24)$$

$$A_{\Delta Y_{ij}} = \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \eta \sin \delta_\ell \quad (25)$$

$$A_{\Delta Z_{ij}} = \sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \cos \eta \sin \delta_\ell \quad (26)$$

$$A_\xi = \Delta X_{ij} [\sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \cos \eta \sin \delta_\ell] + \Delta Z_{ij} [\cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \cos \eta \sin \delta_\ell] \quad (27)$$

$$A_\eta = \Delta X_{ij} [\sin \xi \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \sin \delta_\ell] - \Delta Y_{ij} [\sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \eta \sin \delta_\ell] + \Delta Z_{ij} [\cos \xi \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \sin \delta_\ell] \quad (28)$$

$$A_\kappa = W_d \{ \Delta X_{ij} [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] + \Delta Y_{ij} \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \Delta Z_{ij} [\sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \} \quad (29)$$

$$A_{\alpha_\ell} = - \{ \Delta X_{ij} [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] + \Delta Y_{ij} \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \Delta Z_{ij} [\sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \} \quad (30)$$

$$A_{\delta_\ell} = \sin \delta_\ell \{ \Delta X_{ij} [\cos \xi \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos (\theta_k - \alpha_\ell)] - \Delta Y_{ij} \cos \eta \sin (\theta_k - \alpha_\ell) - \Delta Z_{ij} [\sin \xi \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \sin (\theta_k - \alpha_\ell)] \} - \cos \delta_\ell [\Delta X_{ij} \sin \xi \cos \eta - \Delta Y_{ij} \sin \eta + \Delta Z_{ij} \cos \xi \cos \eta] \quad (31)$$

$$A_{\Delta C_{0ij}} = c \quad (32)$$

$$A_{\Delta C_{1ij}} = c (t_k - t_0) \quad (33)$$

The set of unknown parameters in Eq. (22) can usually be determined in a standard least squares adjustment from the information present in the observables. It is not the purpose of this report to describe the adjustment procedure in detail. However, the estimability aspect of the adjustment problem will be emphasized. In the present context the term simply refers to parameters which can be estimated through the adjustment process. The detection of what is estimable and what is not is through the design matrix A of the partial derivatives. Therefore, it has particular significance in the following. If there exist linear relationships between the columns of the design matrix A , its column rank will be deficient (will not be full) and the normal matrix $A^T P A$ (P is the weight matrix of the observables) will consequently be singular implying that not all of the parameters are estimable and the establishment of a new set of the parameters is required.

The following linear relationships can be detected among the partial derivatives in Eqs. (24) - (30):

$$A_{\xi} = \Delta X_{ij} A_{\Delta Z_{ij}} - \Delta Z_{ij} A_{\Delta X_{ij}} \quad (34)$$

$$A_{\eta} = \cos \xi (\Delta Z_{ij} A_{\Delta Y_{ij}} - \Delta Y_{ij} A_{\Delta Z_{ij}}) + \sin \xi (\Delta X_{ij} A_{\Delta Y_{ij}} - \Delta Y_{ij} A_{\Delta X_{ij}}) \quad (35)$$

$$A_{\kappa} = W_d [\sin \eta (\Delta X_{ij} A_{\Delta Z_{ij}} - \Delta Z_{ij} A_{\Delta X_{ij}}) + \sin \xi \cos \eta (\Delta Y_{ij} A_{\Delta Z_{ij}} - \Delta Z_{ij} A_{\Delta Y_{ij}}) + \cos \xi \cos \eta (\Delta X_{ij} A_{\Delta Y_{ij}} - \Delta Y_{ij} A_{\Delta X_{ij}})] \quad (36)$$

$$A_{\alpha_1} = -A_{\kappa}/W_d \quad (37)$$

The equations (34) to (36) with

$$\begin{aligned} \sin \xi &\cong \sin \eta \cong 0 \\ \cos \xi &\cong \cos \eta \cong 1 \end{aligned} \quad (38)$$

yield in matrix form

$$\begin{bmatrix} A_{\xi} \\ A_{\eta} \\ A_{\kappa}/W_d \end{bmatrix} = \begin{bmatrix} -\Delta Z_{ij} & 0 & \Delta X_{ij} \\ 0 & \Delta Z_{ij} & -\Delta Y_{ij} \\ -\Delta Y_{ij} & \Delta X_{ij} & 0 \end{bmatrix} \begin{bmatrix} A_{\Delta X_{ij}} \\ A_{\Delta Y_{ij}} \\ A_{\Delta Z_{ij}} \end{bmatrix} \quad (39)$$

Since there exist linear dependencies among the partial derivatives, it is not possible to estimate all of the parameters of interest from ground-to-ground baseline VLBI time delay observations. Of the initial 10 parameters of interest only six may be estimated simultaneously.

It is obvious that the system of normal equations $N = A^T P A$ of the least squares adjustment will be singular. Two reasons have to be mentioned for this: 1) rank deficiencies due to coordinate system definition, and 2) rank deficiencies due to a critical configuration.

There are certain conventions to overcome rank deficiencies of a ground-based geodetic VLBI network due to coordinate system definitions, see Bock (1980), Brouwer (1985), Dermanis and Mueller (1978), and Papo and Saleh (1988). Since in a Euclidean space, the coordinate system definition requires seven parameters; therefore, in practice the most common way to define the coordinate system is to constrain seven well-chosen “conventions” in the least squares adjustment. One possible choice for these seven parameters for ground-based geodetic VLBI is [Brouwer, 1985]:

- the X, Y and Z coordinate of one station to define the origin (three translations)
- the epoch ephemeris pole position to define the equatorial plane (two rotations)
- the right ascension of one source to define the orientation in the equatorial plane
(one rotation)
- the velocity of light in vacuo as a scale parameter

An other convention for the coordinate system definition may be the use of a minimum norm solution. This procedure yields a minimal trace for the variance-covariance matrix so that smoother looking variances appear (for details see Chapter 4.1). Here we mention that there exists a rectangular matrix G of m rows and r linear independent columns such that $AG^T = 0$. The result is that a nullspace L of $N = A^T P A$ is described. The basis of the nullspace is formed by the column vectors of the G matrix, the dimension of the nullspace is r . Such a matrix for the ground-based geodetic VLBI is given by, e.g., Brouwer (1985; Appendix E) and Dermanis and Mueller (1978).

In order to circumvent the estimability problem, the earth rotation parameters (ERP: ξ, η, κ) as one of the possibilities and right ascension (α_1) may be redefined by introducing

$$\Delta\xi_0 = \xi - \xi_0 \quad (40a)$$

$$\Delta\eta_0 = \eta - \eta_0 \quad (40b)$$

$$\Delta\kappa_0 = \kappa - \kappa_0 \quad (40c)$$

$$\Delta\alpha_{0\ell} = \alpha_\ell - \alpha_0, \quad (41)$$

where ξ_0 , η_0 , κ_0 are the adopted ERP providing the initial reference orientation of the network (CTS) with respect to the true equator and equinox, and α_0 is the fixed right ascension of one radio source (3C273B or β Persei) providing the reference origin of right ascension. Therefore, the corresponding estimable parameters are the ERP differences ($\Delta\xi_0$, $\Delta\eta_0$, $\Delta\kappa_0$) as changes in the network orientation relative to the initial orientation and the right ascension differences $\Delta\alpha_{0\ell}$ with respect to the fixed right ascension α_0 . However, the errors in the four adopted basic parameters of orientation bias the baseline components. Thus, from this point of view, the baseline components ΔX_{ij} , ΔY_{ij} , ΔZ_{ij} are nonestimable and are replaced by the corresponding set of estimable components $\Delta X'_{ij}$, $\Delta Y'_{ij}$, $\Delta Z'_{ij}$, contaminated by the above errors and defined by the following equations [Arnold, 1974], [Bock, 1980] and [Moritz and Mueller, 1987]

$$\begin{bmatrix} d\Delta X_{ij} \\ d\Delta Y_{ij} \\ d\Delta Z_{ij} \end{bmatrix} = \begin{bmatrix} d\Delta X'_{ij} \\ d\Delta Y'_{ij} \\ d\Delta Z'_{ij} \end{bmatrix} - \begin{bmatrix} -\Delta Z_{ij} & 0 & \Delta Y_{ij} \\ 0 & \Delta Z_{ij} & -\Delta X_{ij} \\ \Delta X_{ij} & -\Delta Y_{ij} & 0 \end{bmatrix} \begin{bmatrix} d\xi_0 \\ d\eta_0 \\ d\beta_0 \end{bmatrix}, \quad (42)$$

where $d\Delta X_{ij}$, $d\Delta Y_{ij}$, $d\Delta Z_{ij}$ are the differential changes of the baseline components and

$$d\beta_0 = d\alpha_0 - W_d d\kappa_0, \quad (43)$$

implying that the two differential rotations $d\alpha_0$ and $d\kappa_0$ are inseparable, $d\xi_0$, $d\eta_0$, and $d\beta_0$ are the small errors of the initial reference orientations. Note that the baseline length is unaffected by the errors in the reference orientation being invariant of coordinate system definition.

A list of estimable parameters recoverable from delay observations is:

$$\{\Delta X'_{ij}, \Delta Y'_{ij}, \Delta Z'_{ij}, \delta_\ell, \Delta\alpha_{0\ell}, \Delta\xi_0, \Delta\eta_0, \Delta\kappa_0, \Delta C_{0ij}, \Delta C_{1ij}\} \quad (44)$$

A number of limiting conditions should be considered in the geodetic analysis phase of VLBI observations, among others some critical configurations. These are the result of an insufficient measurement design that yield a (nearly) singular system of normal equations in the least squares adjustment. The following two equations are useful to describe critical configuration situations.

Rearranging Eq. (39) we arrive at the equation

$$\Delta Z_{ij} A_{\kappa} = W_d (\Delta Y_{ij} A_{\xi} + \Delta X_{ij} A_{\eta}). \quad (45)$$

Furthermore, under the assumptions in Eq. (38) ,

$$\begin{aligned} A_{\delta_{\ell}} = & \Delta Z_{ij} A_{\Delta Z_{ij}} \operatorname{ctg} \delta_{\ell} - \\ & - (\Delta X_{ij} A_{\Delta X_{ij}} + \Delta Y_{ij} A_{\Delta Y_{ij}}) \operatorname{tg} \delta_{\ell} \end{aligned} \quad (46)$$

can be derived which with Eq. (45) is appropriate for studying critical baseline configurations, cf., [Bock, 1980], [Dermanis, 1980] and [Moritz and Mueller, 1987]. A well-known example of critical configuration in ground-based geodetic VLBI that yields a singular case is the single baseline experiment with observed sources at only one declination, see Eq. (46).

Recognizing that

$$\cos \xi \cong \cos \eta \cong 1, \quad (47a)$$

$$\sin \xi \cong \xi, \quad (47b)$$

$$\sin \eta \cong \eta, \quad (47c)$$

and neglecting products $\xi\eta$, an examination of Eqs. (1) and (20) reveals that since

$$d_{ijk\ell} = K_1 \cos(\theta_k - \alpha_{\ell}) + K_2 \sin(\theta_k - \alpha_{\ell}) + K_3 + K_4(t_k - t_0) \quad (48)$$

where

$$K_1 = -(\Delta X_{ij} - \xi \Delta Z_{ij}) \cos \delta_{\ell} \quad (49a)$$

$$K_2 = (\Delta Y_{ij} + \eta \Delta Z_{ij}) \cos \delta_{\ell} \quad (49b)$$

$$K_3 = -(\Delta Z_{ij} + \xi \Delta X_{ij} - \eta \Delta Y_{ij}) \sin \delta_{\ell} + c \Delta C_{0ij} \quad (49c)$$

$$K_4 = c \Delta C_{1ij} \quad (49d)$$

or more simply

$$d_{ijk\ell} = K \sin (\omega_e t_k + \phi) + K_3 + K_4(t_k - t_0) \quad (50)$$

where

$$K = (K_1^2 + K_2^2)^{1/2} \quad (51a)$$

$$\phi = \alpha_\ell - \alpha_0 \quad (51b)$$

α_0 is the right ascension of the baseline at the initial epoch t_0 ,

a time delay observation of a single ground-to-ground baseline using a single source represents a sinusoid superimposed on a straight line. Eq. (48) and (50) may help us in obtaining a better understanding of the dependencies among the unknown parameters.

Counselmann and Shapiro (1978), Bock (1980), Dermanis and Grafarend (1981), Shapiro (1978), and Moritz and Mueller (1987) show which parameters are estimable from ground-based VLBI measurements. Of the geodetic parameters only baseline lengths and source declinations are estimable from time delay. All other parameters are variational in nature, that is they are determined relative to the initial orientation of the inertial and terrestrial reference frames. Only their variations (changes) are estimable. Therefore the initial values for right ascension and earth rotation parameters (polar motion components and UT1-UTC value) must be held fixed at a priori values with the use of external information (e.g., derived from an independent technique). Geocentric site coordinates are not estimable because the observations are primarily a function of the differences of these coordinates for each baseline. In addition these baseline components are contaminated by errors in the reference orientation, polar motion and UT1-UTC variations.

Note that from one baseline time delay observations, only two of the earth rotation parameters are estimable. Therefore, multi-baseline configurations are needed to estimate all three earth orientation parameters.

3.4.2 Time Delay Rate Model. Differentiating Eq. (20) with respect to time we get

$$\begin{aligned} \dot{d}_{ijk\ell} = & \omega_e \{ \Delta X_{ij} [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \\ & + \Delta Y_{ij} \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \\ & - \Delta Z_{ij} [\sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \} + \\ & + c \Delta C_{1ij} , \end{aligned} \quad (52)$$

where

$$\omega_e = \frac{d\theta_k}{dt} = |\overline{\omega}_e| \quad (53)$$

is the spin rate of the earth, $\overline{\omega}_e$ the instantaneous earth rotation vector.

One sees that time delay rate is effectively insensitive to the ΔZ_{ij} component of the baseline, consequently, only the equatorial projection of the baseline can be estimated. In addition, the delay rate is unaffected by clock offset variations, ΔC_{oij} .

Furthermore, since

$$\frac{d\overline{\mathbf{B}}}{dt} = \overline{\omega}_e \times \overline{\mathbf{B}} \quad (54)$$

is orthogonal to $\overline{\omega}_e$ and, thus, the origin of declination is undefined as well as the right ascension origin.

Taking the differential of $\dot{d}_{ijk\ell}$ with respect to the parameters

$$d(\dot{d}_{ijk\ell}) = \sum_p B_p dP_p, \quad (55)$$

where the index p stands for the unknowns

$$\{\Delta X_{ij}, \Delta Y_{ij}, \Delta Z_{ij}, \alpha_\ell, \delta_\ell, \xi, \eta, \kappa, \Delta C_{ij}\} \quad (56)$$

The B_p 's in Eq. (55) are the required partial derivatives of the time delay with respect to the parameters subscripted p as follows:

$$B_{\Delta X_{ij}} = \omega_e [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \quad (57)$$

$$B_{\Delta Y_{ij}} = \omega_e \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) \quad (58)$$

$$B_{\Delta Z_{ij}} = -\omega_e [\sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \quad (59)$$

$$B_\xi = \omega_e \{ \Delta X_{ij} [\cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell)] - \Delta Z_{ij} [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \} \quad (60)$$

$$B_\eta = \omega_e [\Delta X_{ij} \sin \xi \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \Delta Y_{ij} \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \Delta Z_{ij} \cos \xi \cos \eta \cos \delta_\ell (\theta_k - \alpha_\ell)] \quad (61)$$

$$B_\kappa = \omega_e W_d \{ \Delta X_{ij} [\cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell)] - \Delta Y_{ij} \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \Delta Z_{ij} [\sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell)] \} \quad (62)$$

$$B_{\alpha_\ell} = \omega_e \{ \Delta X_{ij} [\sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] + \Delta Y_{ij} \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \Delta Z_{ij} [\sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell)] \} \quad (63)$$

$$B_{\delta_\ell} = -\omega_e \sin \delta_\ell \{ \Delta X_{ij} [\cos \xi \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos (\theta_k - \alpha_\ell)] + \Delta Y_{ij} \cos \eta \cos (\theta_k - \alpha_\ell) - \Delta Z_{ij} [\sin \xi \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos (\theta_k - \alpha_\ell)] \} \quad (64)$$

$$B_{\Delta C_{1ij}} = c. \quad (65)$$

The following linear relationships are evident

$$B_\xi = \Delta X_{ij} B_{\Delta Z_{ij}} - \Delta Z_{ij} B_{\Delta X_{ij}} \quad (66)$$

$$B_\eta = \cos \xi (\Delta Z_{ij} B_{\Delta Y_{ij}} - \Delta Y_{ij} B_{\Delta Z_{ij}}) + \sin \xi (\Delta X_{ij} B_{\Delta Y_{ij}} - \Delta Y_{ij} B_{\Delta X_{ij}}) \quad (67)$$

$$B_\kappa = W_d \left[\sin \eta (\Delta X_{ij} B_{\Delta Z_{ij}} - \Delta Z_{ij} B_{\Delta X_{ij}}) + \sin \xi \cos \eta (\Delta Y_{ij} B_{\Delta Z_{ij}} - \Delta Z_{ij} B_{\Delta Y_{ij}}) + \cos \xi \cos \eta (\Delta X_{ij} B_{\Delta Y_{ij}} - \Delta Y_{ij} B_{\Delta X_{ij}}) \right] \quad (68)$$

$$B_{\alpha_\ell} = -B_\kappa / W_d \quad (69)$$

indicating that it is impossible to estimate all of the parameters of interest from ground-to-ground baseline VLBI delay rate observations. Eqs. (66) - (69) are formally very similar to Eqs. (34) - (37).

The Eqs. (66) - (68) with assumptions in Eq. (38) yield in matrix form

$$\begin{bmatrix} B_\xi \\ B_\eta \\ B_\kappa/W_d \end{bmatrix} = \begin{bmatrix} -\Delta Z_{ij} & 0 & \Delta X_{ij} \\ 0 & \Delta Z_{ij} & -\Delta Y_{ij} \\ -\Delta Y_{ij} & \Delta X_{ij} & 0 \end{bmatrix} \begin{bmatrix} B_{\Delta X_{ij}} \\ B_{\Delta Y_{ij}} \\ B_{\Delta Z_{ij}} \end{bmatrix} \quad (70)$$

The discussion of the parameters estimable from delay rate is identical to that of time delays except that in this case ΔZ_{ij} ($\Delta Z'_{ij}$) and ΔC_{oij} are deleted, and declination differences

$$\Delta \delta_{o\ell} = \delta_\ell - \delta_o \quad (71)$$

replace δ_ℓ . Thus, the list of the estimable parameters recoverable from delay rate observations is

$$\left\{ \Delta X'_{ij}, \Delta Y'_{ij}, \Delta \delta_{o\ell}, \Delta \alpha_{o\ell}, \Delta \xi_o, \Delta \eta_o, \Delta \kappa_o, \Delta C_{1ij} \right\}. \quad (72)$$

It can easily be shown that the equatorial baseline components are also contaminated by the errors of the initial orientation in the following sense

$$\begin{bmatrix} d\Delta X_{ij} \\ d\Delta Y_{ij} \end{bmatrix} = \begin{bmatrix} d\Delta X'_{ij} \\ d\Delta Y'_{ij} \end{bmatrix} - \begin{bmatrix} -\Delta Z_{ij} & 0 & \Delta Y_{ij} \\ 0 & \Delta Z_{ij} & -\Delta X_{ij} \end{bmatrix} \begin{bmatrix} d\xi_o \\ d\eta_o \\ d\beta_o \end{bmatrix} \quad (73)$$

where $d\beta_o$ equals with Eq. (43) implying again that the two differential rotations $d\alpha_o$ and $d\kappa_o$ are inseparable.

Similar derivation to the Eqs. (45) and (46) gives

$$\Delta Z_{ij} B_\kappa = W_d (\Delta Y_{ij} B_\xi + \Delta X_{ij} B_\eta) \quad (74)$$

$$B_{\delta_\ell} = -\text{tg}\delta_\ell (\Delta X_{ij} B_{\Delta X_{ij}} + \Delta Y_{ij} B_{\Delta Y_{ij}} + \Delta Z_{ij} B_{\Delta Z_{ij}}) \quad (75)$$

Eqs. (74) and (75) can be used to detect critical configurations and for sensitivity analysis as well.

An examination of Eq. (52) or differentiating the Eq. (50) with respect to time, we get

$$\dot{d}_{ijk\ell} = K \omega_e \cos (\omega_e t_k + \phi) + K_4, \quad (76)$$

which is useful for an analysis. The coefficients K and K₄ are the same expressed by Eqs. (51a) and (49d).

The delay rate model includes a reduced parameter set in comparison with the time delay model. The third component of the baseline is non-estimable and only declination differences may be estimated, see [Bock, 1980] and [Moritz and Mueller, 1987]. Therefore, it is not possible with delay rates alone to estimate all of the parameters of geodetic and astrometric interest.

3.5 Mathematical Models for Ground-to-Space VLBI Observables

We are interested in forming equations which relate to observables to the unknown parameters suitable for least squares covariance analysis. Our main purpose is not to derive explicit observation equations, but rather to develop qualitative expressions to demonstrate the relationship between the observables (time delay and delay rate) and the parameters and to explore estimability problems inherent in the space VLBI system.

3.5.1 Time Delay Model. From Eq. (9) we can get a simplified expression (similar to Eq. (16) [Pavlis, 1986]) suitable for the analysis as follows (the T is the transpose sign), see Fig. 13:

$$d_{jk\ell}^I = - \left\{ \begin{array}{l} \left[\begin{array}{c} X_j \\ Y_j \\ Z_j \end{array} \right]^T R_2(-\xi) R_1(-\eta) R_3(\theta_\kappa) - \left[\begin{array}{c} X_k^I \\ Y_k^I \\ Z_k^I \end{array} \right]^T \end{array} \right\} \begin{bmatrix} \cos \delta_\ell & \cos \alpha_\ell \\ \sin \delta_\ell & \sin \alpha_\ell \\ \sin \delta_\ell & \end{bmatrix} + \quad (77)$$

$$+ c \left[\Delta C_{0rj}^I + \Delta C_{1rj}^I (t_k - t_0) \right],$$

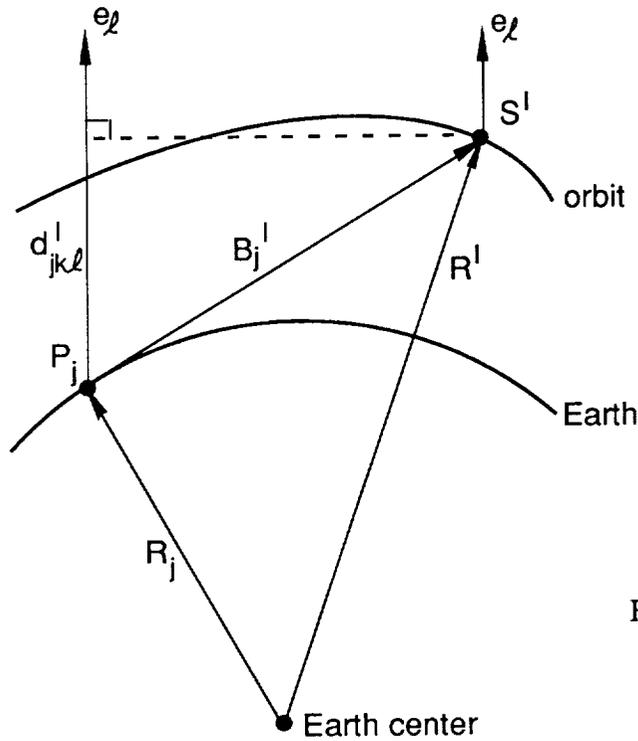


Fig. 13 Space VLBI time delay observable

where

X_j, Y_j, Z_j are the Earth-fixed coordinates of the station P_j ,

X_k^I, Y_k^I, Z_k^I are the coordinates of satellite S^I in a true-of-date geocentric inertial coordinate frame at the epoch t_k ,

α_ℓ, δ_ℓ is the right ascension and declination of the ℓ^{th} radio source in the same as above true-of-date coordinate frame,

ξ, η are the polar motion components that relate the instantaneous rotation axis of the Earth with the average terrestrial pole,

θ_k is the Greenwich Apparent Sidereal Time (GAST) at epoch t_k ,

c is the speed of light,

t_0 is the initial epoch of observation,

$\Delta C_{0rj}^I, \Delta C_{1rj}^I$ is the clock offset and drift between the reference clock at the telemetry/control station and the clock of station P_j , and

$R_i(\phi)$ is the rotation matrix for a right handed rotation ϕ about the axis i .

The expression of Eq. (77) may be rewritten as

$$\begin{aligned}
d_{jk\ell}^I = & -X_j [\cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \\
& + \sin \xi \cos \eta \sin \delta_\ell] + \\
& + Y_j [\cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \eta \sin \delta_\ell] + \\
& + Z_j [\sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \\
& - \cos \xi \cos \eta \sin \delta_\ell] + \\
& + X_k^I \cos \delta_\ell \cos \alpha_\ell + \\
& + Y_k^I \cos \delta_\ell \sin \alpha_\ell + \\
& + Z_k^I \sin \delta_\ell + \\
& + c \left[\Delta C_{0rj}^I + \Delta C_{1rj}^I (t_k - t_0) \right]
\end{aligned} \tag{78}$$

In the following, the Cartesian coordinates X_k^I, Y_k^I, Z_k^I of satellite S^I are transformed into Keplerian orbital elements consisting of the semimajor axis a , the eccentricity e specifying the elongation of the orbital conic section, the inclination i specifying the orientation of the satellite's orbital plane with respect to the equator of the Earth, the right ascension Ω of the ascending node, i.e., the angle measured eastward along the equator between the vernal equinox and the point where the satellite crosses the equator traveling in a northerly direction, the argument ω of perigee, i.e., angle between the ascending node and the perifocal point measured positive with increasing mean anomaly, and the mean anomaly M , i.e., the sum of the mean anomaly at epoch and the product of the mean motion and the elapsed time from epoch. A transformation of Kepler elements

$$\{a, e, i, \omega, \Omega, M\} \tag{79}$$

into geocentric orthogonal coordinates $\{X, Y, Z\}$ is given by the following formulas (dropping the lower and upper indexes) [El'yasberg, 1967; p. 63] and [Grafarend and Livieratos, 1978]:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = r \begin{bmatrix} \cos \Omega \cos u - \sin \Omega \sin u \cos i \\ \sin \Omega \cos u + \cos \Omega \sin u \cos i \\ \sin u \sin i \end{bmatrix} \tag{80}$$

where

$$r = a (1 - e \cos E) \quad (81a)$$

$$u = \omega + f \quad (81b)$$

$$\operatorname{tg} f/2 = \sqrt{(1+e)/(1-e)} \operatorname{tg} E/2 \quad (81c)$$

$$M = E - e \sin E \quad (81d)$$

$$\sin f = \sqrt{1-e^2} \sin E / (1 - e \cos E) \quad (81e)$$

$$\cos f = (\cos E - e) / (1 - e \cos E) \quad (81f)$$

E is the eccentric anomaly.

Equation (78) with Eq. (80) [and Eq. (77)] expresses the functional relationship of the time delay observation with the listed parameters. Of direct geodetic interest are the station coordinates X_j, Y_j, Z_j , the Keplerian orbital elements Eq. (79) of the orbiting radiotelescope S^I and the earth rotation parameters (ERP) ξ, η, κ . The source coordinates α_ℓ and δ_ℓ are of astrometric interest, while the clock offset ΔC_{0rj}^I and rates ΔC_{1rj}^I are nuisance parameters defined to make the mathematical model more realistic.

The GAST, θ_k as well as the Earth rotation parameters κ are reformulated identically to that of ground-to-ground baseline VLBI observables, see Chapter 3.4, i.e., Eqs. (17) and (23). Assume that for an observation session (one day or one revolution of Keplerian orbit) only one set of earth rotation parameters (ξ, η, κ) is determined. Thus, our parameter set will contain the following 16 parameters:

$$\{X_j, Y_j, Z_j, a, e, i, \omega, \Omega, M, \xi, \eta, \kappa, \alpha_\ell, \delta_\ell, \Delta C_{0rj}^I, \Delta C_{1rj}^I\} \quad (82)$$

In the following we examine the complete observation equation Eq. (78) with Eq. (80) for the parameters, the coordinates of the ground VLBI station P_j , the Keplerian orbital elements of the orbiting radiotelescope S^I , the ℓ th radio source positions, the earth rotation parameters, and the clock parameters to determine which of them are separable or nonseparable and, therefore, design experiments in which those parameters of primary interest will be estimable.

Since Eq. (78) is non-linear with respect to that parameter set, we proceed as usual to linearize Eq. (78). Taking the differential of $d_{jk\ell}^I$ with respect to the parameters,

$$d\left(d_{jk\ell}^I\right) = \sum_p C_p dP_p, \quad (83)$$

where the C_p 's are the required partial derivatives of the time delay with respect to the parameters of interest, indexed by p . They are as follows:

$$C_{X_j} = -[\cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell)] \\ + \sin \xi \cos \eta \sin \delta_\ell \quad (84)$$

$$C_{Y_j} = \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \eta \sin \delta_\ell \quad (85)$$

$$C_{Z_j} = \sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \\ - \cos \xi \cos \eta \sin \delta_\ell \quad (86)$$

$$C_\xi = X_j [\sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \\ - \cos \xi \cos \eta \sin \delta_\ell] + \\ + Z_j [\cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \\ + \sin \xi \cos \eta \sin \delta_\ell] \quad (87)$$

$$C_\eta = X_j [\sin \xi \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \sin \delta_\ell] - \\ - Y_j [\sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \eta \sin \delta_\ell] + \\ + Z_j [\cos \xi \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \sin \delta_\ell] \quad (88)$$

$$C_k = W_d \{ X_j [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell)] + \\ + Y_j \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \\ - Z_j [\sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \} \quad (89)$$

$$C_a = (1 - e \cos E) \{ \cos \delta_\ell [\cos u \cos (\Omega - \alpha_\ell) - \\ - \sin u \cos i \sin (\Omega - \alpha_\ell)] + \sin \delta_\ell \sin u \sin i \} \quad (90)$$

$$C_i = r [\sin u \sin i \sin (\Omega - \alpha_\ell) \cos \delta_\ell + \sin u \cos i \sin \delta_\ell] \quad (91)$$

$$C_\omega = -r \{ \cos \delta_\ell [\sin u \cos (\Omega - \alpha_\ell) + \cos u \cos i \sin (\Omega - \delta_\ell)] - \\ - \sin \delta_\ell \cos u \sin i \} \quad (92)$$

$$C_\Omega = -r \cos \delta_\ell [\cos u \sin (\Omega - \delta_\ell) + \sin u \cos i \cos (\Omega - \delta_\ell)] \quad (93)$$

$$C_e = \frac{a(e - \cos E)}{(1 - e \cos E)^2} C_a + \left[\frac{\sin f}{1 - e \cos E} + \frac{\sin f}{1 - e^2} \right] C_\omega \quad (94)$$

$$C_M = \frac{a e \sin E}{(1 - e \cos E)^2} C_a + \frac{\sin f}{\sin E (1 - e \cos E)} C_\omega \quad (95)$$

$$\begin{aligned}
C_{\alpha_\ell} = & - \{ X_j [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] + \\
& + Y_j \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \\
& - Z_j [\sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \} + \\
& + r \cos \delta_\ell [\cos u \sin (\Omega - \alpha_\ell) + \sin u \cos i \cos (\Omega - \alpha_\ell)]
\end{aligned} \tag{96}$$

$$\begin{aligned}
C_{\delta_\ell} = & \sin \delta_\ell \{ X_j [\cos \xi \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \sin (\theta_k - \alpha_\ell)] - \\
& - Y_j \cos \eta \sin (\theta_k - \alpha_\ell) - \\
& - Z_j [\sin \xi \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \sin (\theta_k - \alpha_\ell)] - \\
& - r [\cos u \cos (\Omega - \alpha_\ell) - \sin u \cos i \sin (\Omega - \alpha_\ell)] \} - \\
& - \cos \delta_\ell [X_j \sin \xi \cos \eta - Y_j \sin \eta + Z_j \cos \xi \cos \eta - r \sin u \sin i]
\end{aligned} \tag{97}$$

$$C_{\Delta C_{0_{t_j}^I}} = c \tag{98}$$

$$C_{\Delta C_{1_{t_j}^I}} = c (t_k - t_0) \tag{99}$$

In order to get the partial derivatives of time delay ($d_{jk\ell}^I = d$) with respect to Keplerian orbital elements, the following relationship:

$$\begin{aligned}
\begin{bmatrix} C_a = \frac{\partial d}{\partial a} \\ C_e = \frac{\partial d}{\partial e} \\ C_i = \frac{\partial d}{\partial i} \\ C_\omega = \frac{\partial d}{\partial \omega} \\ C_\Omega = \frac{\partial d}{\partial \Omega} \\ C_M = \frac{\partial d}{\partial M} \end{bmatrix} &= \begin{bmatrix} \frac{\partial X}{\partial a} & \frac{\partial Y}{\partial a} & \frac{\partial Z}{\partial a} \\ \frac{\partial X}{\partial e} & \frac{\partial Y}{\partial e} & \frac{\partial Z}{\partial e} \\ \frac{\partial X}{\partial i} & \frac{\partial Y}{\partial i} & \frac{\partial Z}{\partial i} \\ \frac{\partial X}{\partial \omega} & \frac{\partial Y}{\partial \omega} & \frac{\partial Z}{\partial \omega} \\ \frac{\partial X}{\partial \Omega} & \frac{\partial Y}{\partial \Omega} & \frac{\partial Z}{\partial \Omega} \\ \frac{\partial X}{\partial M} & \frac{\partial Y}{\partial M} & \frac{\partial Z}{\partial M} \end{bmatrix} \begin{bmatrix} \frac{\partial d}{\partial X^I} = C_{X^I} \\ \frac{\partial d}{\partial Y^I} = C_{Y^I} \\ \frac{\partial d}{\partial Z^I} = C_{Z^I} \end{bmatrix}
\end{aligned} \tag{100}$$

was applied, where the partial derivatives of time delay with respect to the orbiting radiotelescope's cartesian coordinates are as follows:

$$C_{X^I} = \cos \delta_\ell \cos \alpha_\ell \tag{101}$$

$$C_{Y^I} = \cos \delta_\ell \sin \alpha_\ell \tag{102}$$

$$C_{Z^I} = \sin \delta_\ell \tag{103}$$

The elements of the coefficient matrix in Eq. (100), called as a linear Kepler transformation matrix K by Grafarend and Livieratos (1978), are given in Appendix A. We call this matrix here as a linear Kepler transformation matrix K_c of position (or coordinate) type.

A useful check of the partial derivatives of the Cartesian inertial coordinates with respect to Keplerian orbital elements $\left(\frac{\partial X_i}{\partial K_j}\right)$ is their use with the corresponding partial derivatives of Keplerian elements with respect to the Cartesian parameters $\left(\frac{\partial K_j}{\partial X_k}\right)$ to confirm the relation

$$\sum_{j=1}^6 \frac{\partial X_i}{\partial K_j} \frac{\partial K_j}{\partial X_k} = \delta_{ik} , \quad (104)$$

for all $i, k \in \{1,2,3\}$

where δ_{ik} is the Kronecker-delta, X represents the Cartesian coordinates $\{X, Y, Z\}$ and K represents the Keplerian orbital elements $\{a, e, i, \omega, \Omega, M\}$. Both types of partial derivatives in Eq. (104) are collected in Appendix A. To compute the partial derivatives of Keplerian elements with respect to the Cartesian parameters, the partials given in Hill (1989) are used. Appendix A also includes a computer program with subroutines as well as a practical example to confirm the relation in Eq. (104).

The partial derivatives [Eqs. (84) - (99) and Eqs. (101) - (103)] of the time delay observable with respect to the corresponding parameters of the mathematical model Eq. (78) with Eq. (80) constitute the elements of the design matrix C in Eq. (83) that forms the normal matrix $N = C^T P C$ (P is the weight matrix of the observables). The inverse of that normal matrix yields the a priori covariance matrix of parameters. The magnitude of a particular partial derivative which reflects the sensitivity of an observation to a particular parameter determines its numerical contribution to the normal matrix. Most of the partial derivatives listed in Eq. (84) - (99) are diurnal sinusoids, some others are composed of diurnal sinusoids, functions of station coordinates, and Keplerian orbital elements (or varying satellite positions).

Examining the analytical expressions for the partial derivatives we can derive some interesting sensitivity relations between the time delay observable and the parameters to be solved for. Since the partial derivatives of delay with respect to the polar components of the ground station and satellite position and to the declination of the sources (when $Z_j \equiv Z_k^I$)

go to zero as the declination goes to zero, the sensitivity of delay to these parameters decreases as the declinations go to zero. The tendency of the sensitivity to the right ascension of the sources and the earth rotation parameters κ is just opposite, that is, from C_κ and C_{α_ℓ} it is evident that the time delay is sensitive to the declinations of sources near the equator. The orbiting radiotelescope's coordinates (Keplerian elements) effect only the partial derivatives of delay with respect to radio source position components, see Eqs. (96) - (97). From the analytical expression of these partial derivatives (C_{α_ℓ} and C_{δ_ℓ}) we can gather that their numerical values will in general increase as the coordinate separation between the station and the orbiting radiotelescope increases (advantage of space VLBI).

With three coordinates for the ground station, six Keplerian orbital elements for the orbiting radiotelescope, two parameters for the radio source position, three for the ERP and two for the clock, there are 16 parameters in total in the model Eq. (78). Therefore, the design matrix C given in Eq. (83) will have a minimum of 16 columns, if one ground station and one radio source is considered. In identifying linear dependencies between the parameters, we can separate C into five submatrices according to the five major groups of parameters, namely, the station coordinates, the orbital elements, earth rotation parameters, radio source positions and clock parameters. From Eqs. (84) - (99) we can state that there are no linear dependencies between columns of the design matrix within the same group. However, dependencies do exist between columns of the design submatrices among different groups with the exception of the ones belonging to the clock parameters. The columns of the station coordinates can be written as linear combinations of those of the earth rotation parameters. Therefore, we cannot separate the ground station parameters and the ones for the earth rotation in a simultaneous adjustment.

Furthermore, one can easily write the linear relationship between columns of the station coordinates, the right ascension of radio sources and the longitude of satellite orbit's ascending node (Ω). The linear dependence between the columns of the right ascension (α) and earth rotation parameter (κ) is also obvious. Consequently, the following linear relationships are evident among the partial derivatives:

$$C_\xi = X_j C_{Z_j} - Z_j C_{X_j} \quad (105)$$

$$C_\eta = \cos \xi (Z_j C_{Y_j} - Y_j C_{Z_j}) + \sin \xi (X_j C_{Y_j} - Y_j C_{X_j}) \quad (106)$$

$$\begin{aligned}
C_{\kappa} = W_d & \left[\sin \eta (X_j C_{Z_j} - Z_j C_{X_j}) \right. \\
& + \sin \xi \cos \eta (Y_j C_{Z_j} - Z_j C_{Y_j}) \\
& \left. + \cos \xi \cos \eta (X_j C_{Y_j} - Y_j C_{X_j}) \right]
\end{aligned} \tag{107}$$

$$C_{\alpha_{\ell}} = -C_{\kappa}/W_d - C_{\Omega} \tag{108}$$

Recognizing that $\sin \xi \cong \sin \eta \cong 0$ and $\cos \xi \cong \cos \eta \cong 1$, the linear relationships through Eqs. (105) and (107) are reduced to the following matrix form which is more clearly indicating the dependencies among the corresponding partial derivatives:

$$\begin{bmatrix} C_{\xi} \\ C_{\eta} \\ C_{\kappa}/W_d \end{bmatrix} = \begin{bmatrix} -Z_j & 0 & X_j \\ 0 & Z_j & -Y_j \\ -Y_j & X_j & 0 \end{bmatrix} \begin{bmatrix} C_{X_j} \\ C_{Y_j} \\ C_{Z_j} \end{bmatrix} \tag{109}$$

The linear dependencies derived analytically by Eqs. from (105) to (108) are formally very similar to ones existing among the partial derivatives of ground-based time delay observable (see Eqs. (34) - (37)). It is quite obvious from a comparison of Eqs. (109) and (39). The only difference is the use of station coordinates instead of baseline components. However, a comparison of the Eqs. (37) and (108) shows that in the latter equation a new term (C_{Ω}) appears due to the additional linear dependency of radio source's right ascension with the ascending node Ω of the satellite's Keplerian orbit in the space VLBI. A geometric interpretation of this reason is that all three parameters, i.e., right ascension α_{ℓ} , Greenwich Apparent Sidereal Time (θ_{κ}) (or earth rotation parameter κ) and ascending node Ω of the Keplerian orbit are "measured" in the same plane, that is, in the equator (see Fig. 14).

In the following, it is useful to present the geometric interpretations of the rank deficiencies expressed in Eqs. (109) and (108). Eq. (109) shows a linear dependence between various combinations of ground station coordinates X_j, Y_j, Z_j and the earth rotation parameters ξ, η, κ . These indicate a rank deficiency of three due to a lack of absolute orientation of the network of ground stations with respect to the true-of-date celestial frame which cannot be sensed by the observables. Eq. (108) is due to a lack of reference direction (origin or right ascension of the radio source and of the ascending node

of satellite orbit), for the true-of-date celestial frame, i.e., the observables are insensitive to the orientation of the true-of-date inertial frame in right ascension (see Fig. (14)).

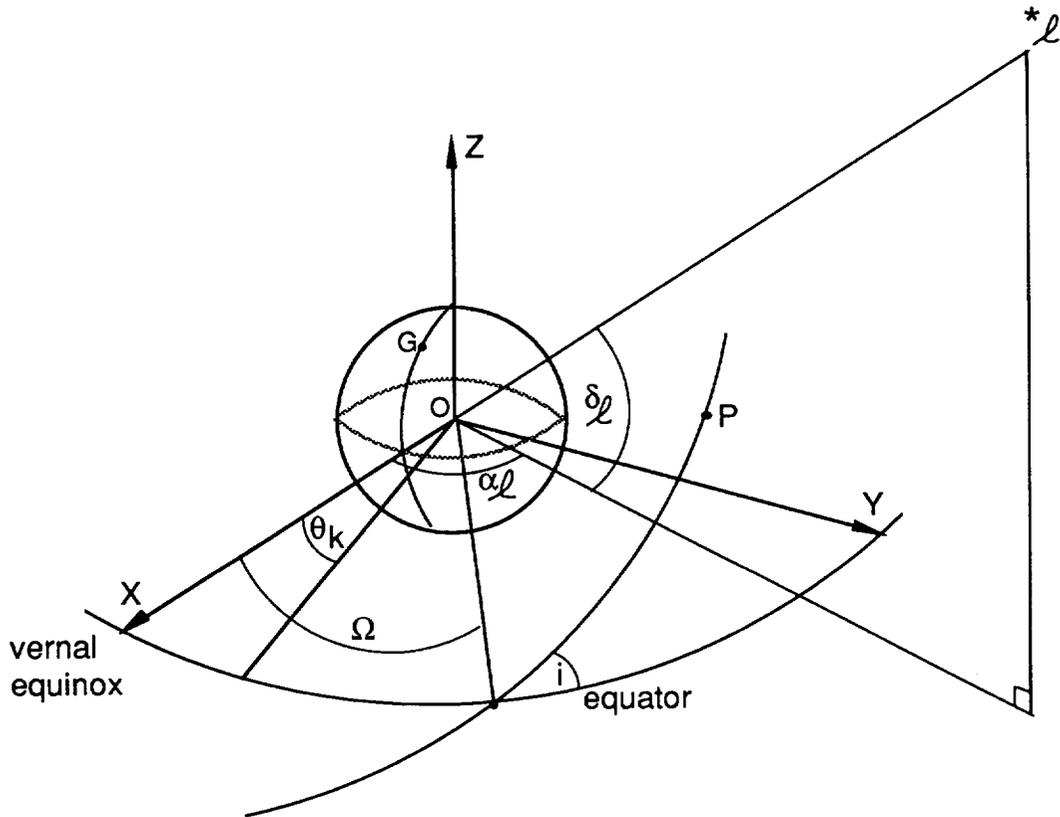


Fig. 14 A geometric configuration of the Greenwich Apparent Sidereal Time (GAST), the right ascension (α_{ℓ}) of a radio source and the ascending node (Ω) of the Keplerian orbit.

It has been shown in Chapter 3.3 that the time delay observables (or time delay times the speed of light as the path difference) are invariant under datum translations and rotations, but not invariant under datum scale change. The observable is insensitive in the location and orientation of the coordinate system where the coordinates of the end points of the baseline are given. Therefore, time delay observations cannot provide datum origin and orientation, but only scale. The missing origin and orientation components are the datum defect.

There are principally two reference frames associated with the mathematical model. The first is an earth-fixed frame, and the second is a true-of-date inertial frame. Since time delay can only provide scale, nine parameters define the two frames—three for the origin

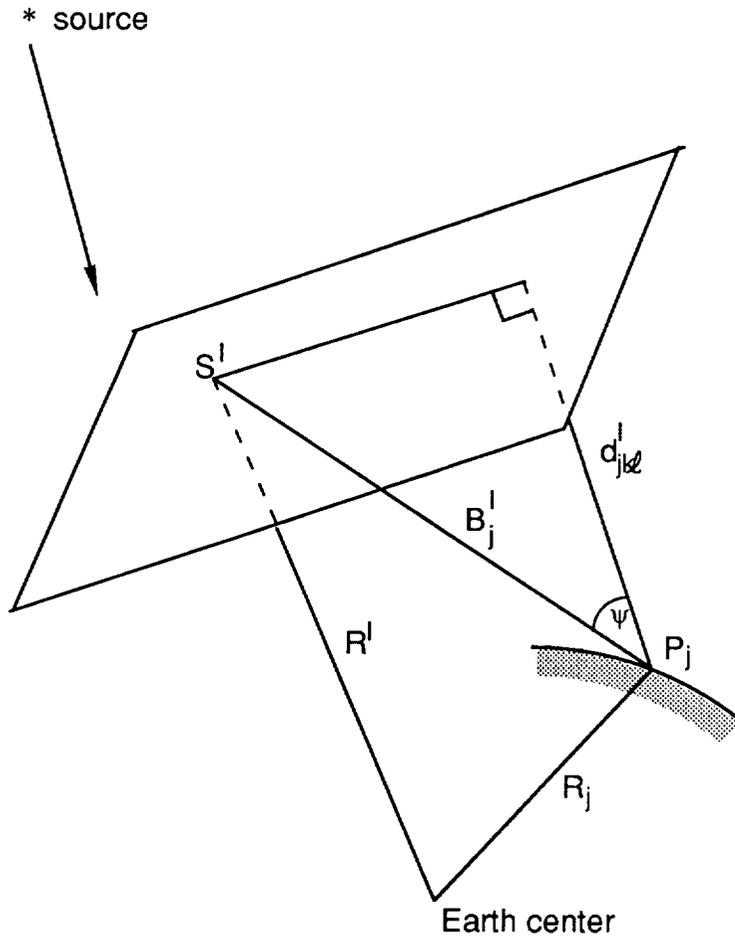


Fig. 15 Geometry of time delay observable

and three for the orientation of the earth-fixed frame, plus three for the orientation of the true-of-date inertial frame with respect to the earth-fixed frame. The design matrix C contains partial derivatives of time delay with respect to parameters which include station, satellite and radio source coordinates. Because such coordinates are not estimable quantities, the design matrix should have a column defect of at least nine, provided there are no configuration defects.

However, the time delay observable will remain unchanged if the wavefront rotates in such a way that it remains tangent to a cone with axis the baseline and half angle equal to ψ (Fig. 9 and 15), see [Counselman and Shapiro, 1978]. From that fact, two additional degrees of freedom exist (three rotations of a plane minus one constraint of tangency to the cone). Furthermore, in practice, instead of three coordinates we need only two directional

coordinates of source, that is, the right ascension α and declination δ to define the true-of-date inertial reference frame. Thus, seven rank deficiencies should appear in the mathematical model Eq. (77) of the observable.

We have shown that the three translational and the three rotational parameters cannot be found from the either ground-based and space VLBI time delay observations. There are certain conventions in ground-based VLBI to define the origin, that is, e.g., to fix the X,Y and Z coordinates of one station or to use a minimum norm solution. In the latter case, we will have an optimal datum definition in the classical sense with the help of the standard inner constraints for 3D networks [Brouwer, 1985; Appendix E].

Note that we were searching to analytically find linear constraints for “translational” datum defect for space VLBI observation equations. In the case of ground-based VLBI, it is easy to get a nullspace basis in such a way that $AG_t^T = 0$ for translational defect in the time delay observation equation [Brouwer, 1985] and [Dermanis and Mueller, 1978]. However, a comparison of Eqs. (13a) and (13b) shows that in the case of space VLBI, the derivation of such a G_t^T matrix ($CG_t^T = 0$) for definition of the origin in the minimum norm solution is impossible. Therefore, it implies that some fixed values should be applied to fix the origin of the coordinate system for space VLBI observables.

Contrary to ground-based VLBI network, in the space-VLBI the origin of the reference frames is fixed by the given Keplerian orbital elements at some reference epoch, T_0 . In fact, the orbital elements are not time invariant, i.e., their derivatives with respect to time, \dot{E}_i , will not be zero. The variation of a given element from some reference epoch, T_0 , to the epoch of utilization, T , can symbolically be described by the following equation

$$E_i = E_i^p + \int_{T_0}^T \dot{E}_i dT \quad (110)$$

where E_i^p is the element in question at the initial epoch T_0 , and E_i at T (Fig. 16), see [Moritz and Mueller, 1987; pp. 368-369]. The integral represents the perturbation of the element E_i .

A rank deficiency of a normal equation system may easily occur as a combined effect of coordinate system definition and critical configurations which are the result of an

insufficient measurement set-up. Therefore, the consideration of critical configurations is also very important.

From Eq. (109) one obtains

$$Z_j C_{\kappa} = W_d (Y_j C_{\xi} + X_j C_{\eta}) \quad (111)$$

Considering the assumptions in Eq. (38)

$$C_{\delta_{\ell}} = \text{ctg } \delta_{\ell} Z_j C_{Z_j} - \text{tg } \delta_{\ell} (X_j C_{X_j} + Y_j C_{Y_j} + a C_a) + r \sin u \sin i / \cos \delta_{\ell} \quad (112)$$

is derived. In searching of the rank deficiencies the following equation

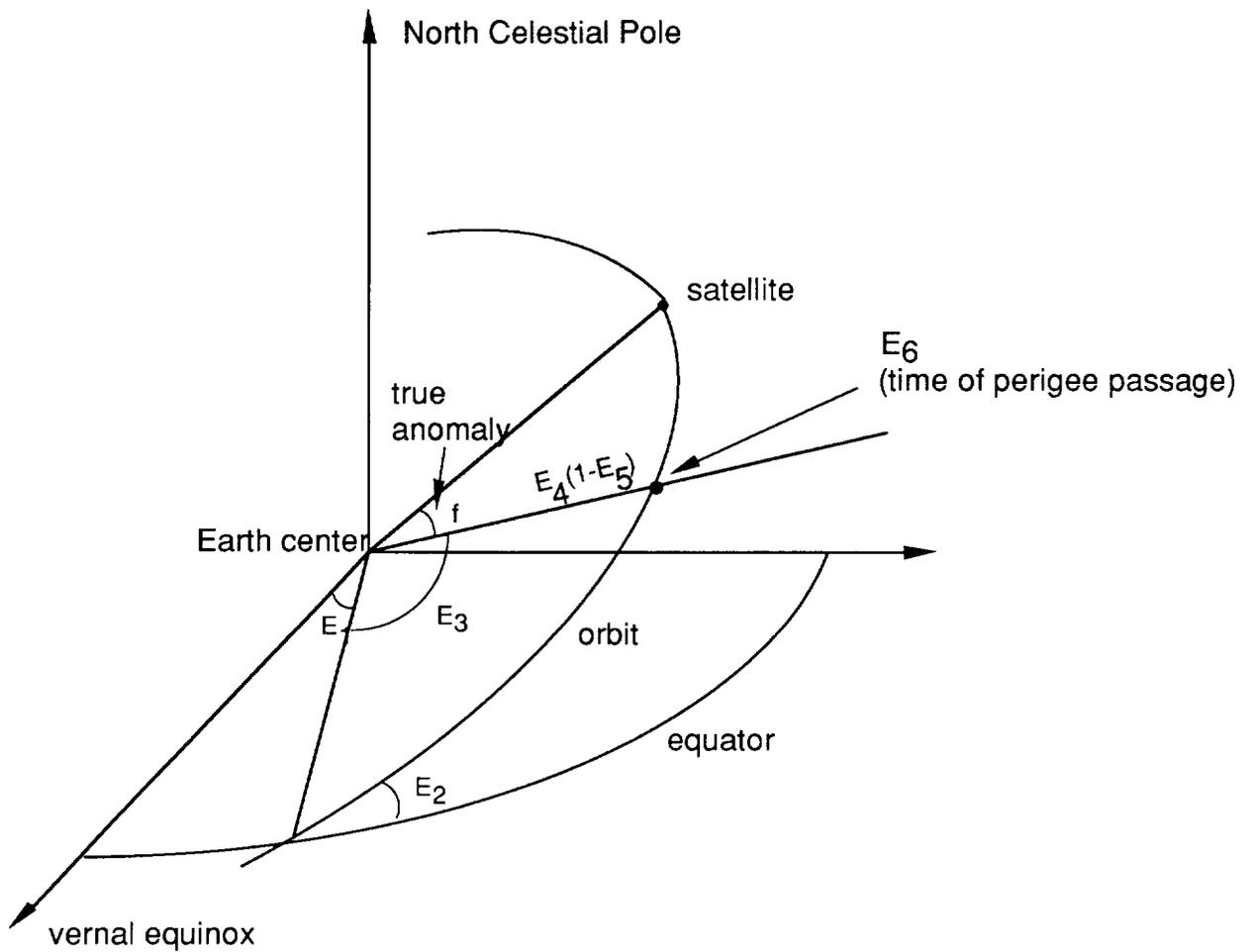


Fig. 16 Keplerian orbital elements: Right ascension of node, E_1 ; inclination, E_2 ; argument of perigee, E_3 ; semidiameter of the elliptic orbit, E_4 ; eccentricity of the ellipse, E_5 ; time of perigee passage, E_6

$$\cos i C_\omega = \text{ctg } u \sin i C_i + C_\Omega \quad (113)$$

has been obtained. Eqs. (111) - (113) can be used for studying the critical station and satellite configurations. They may be useful in observation design works as well. For instance, additional rank defect occurs by a special choice of Keplerian orbital elements. If $i = 0^\circ$, then from Equation (113) it follows that

$$C_\omega = C_\Omega \quad (114)$$

indicating an inseparable situation for the parameters ω and Ω . In the space VLBI network this special situation gives a rank defect of internal type [Graferend and Livieratos, 1978]

Because of the linear relationships Eqs. (108) and (109) among the coefficients C_p of the error Eq. (83), it is not possible to determine all of the 16 unknowns in Eq. (82). Therefore, a new set of estimable parameters must be introduced. Concerning the linear relationships expressed in Eqs. (105) - (107) [or see Eq. (109)] and in Eq. (108), the ERP differences $\Delta\xi_o$, $\Delta\eta_o$, $\Delta\kappa_o$ and right ascension difference $\Delta\alpha_{o\ell}$ may be introduced as the corresponding new estimable parameters like that of the ground-based VLBI time delay observable, see Eqs. (40) - (41). However, the errors in the four adopted basic parameters of the initial orientation (ξ_o , η_o , κ_o , $\alpha_{o\ell}$) bias the station coordinates. Thus, from this point of view, the station coordinates X_j^\cup , Y_j^\cup , Z_j^\cup are non estimable. They are replaced by the new set of estimable station coordinates X'_j , Y'_j , Z'_j contaminated by the errors of the initial orientation ($d\xi_o$, $d\eta_o$, $d\kappa_o$, $d\alpha_o$). It can be shown that the errors of the corresponding parameters can be expressed in the following manner:

$$\begin{bmatrix} dX_j \\ dY_j \\ dZ_j \end{bmatrix} = \begin{bmatrix} dX'_j \\ dY'_j \\ dZ'_j \end{bmatrix} - \begin{bmatrix} -Z_j & 0 & Y_j \\ 0 & Z_j & -X_j \\ X_j & -Y_j & 0 \end{bmatrix} \begin{bmatrix} d\xi_o \\ d\eta_o \\ d\beta_o \end{bmatrix}, \quad (115)$$

where dX_j , dY_j , dZ_j are the differential changes of the station coordinates and $d\beta_o$ is identical with Eq. (43), that is, the two differential relations $d\alpha_o$ and $d\kappa_o$ are inseparable.

Furthermore, the error of the initial reference orientation in right ascension makes its influence in the space coordinates of the orbiting radiotelescope. Consequently, the

satellite coordinates (only the equatorial ones, X_k^I and Y_k^I) will be contaminated by the error $d\alpha_o$ in the following manner:

$$\begin{bmatrix} dX_k^I \\ dY_k^I \end{bmatrix} = \begin{bmatrix} dX_k^I \\ dY_k^I \end{bmatrix} - \begin{bmatrix} 0 & d\alpha_o \\ -d\alpha_o & 0 \end{bmatrix} \begin{bmatrix} X_k^I \\ Y_k^I \end{bmatrix}, \quad (116)$$

where dX_k^I and dY_k^I are the differential changes of the orbiting radiotelescope's equatorial coordinates.

Note that the declination of the reference source should be nearly equatorial to provide a strong definition, similarly to that of the case of ground-based VLBI. This can be seen by an examination of C_{α_ℓ} , Eq. (96).

In order to show the main differences between the ground-based and space VLBI time delay observables, under the assumptions of Eq. (47) we rewrite Eq. (78) in a similar way to that of Eq. (48) as follows:

$$d_{jk\ell}^I = K_1 \cos(\theta_k - \alpha_\ell) + K_2 \sin(\theta_k - \alpha_\ell) + K_3(t_k) + K_4(t_k - t_0), \quad (117)$$

where

$$K_1 = -(X_j - \xi Z_j) \cos \delta_\ell \quad (118a)$$

$$K_2 = (Y_j + \eta Z_j) \cos \delta_\ell \quad (118b)$$

$$\begin{aligned} K_3(t_k) = & -(Z_j + X_j \xi - Y_j \eta) \sin \delta_\ell + C \Delta C_{o_{rj}}^I \\ & + X_k^I \cos \delta_\ell \cos \alpha_\ell \\ & + Y_k^I \cos \delta_\ell \cos \alpha_\ell \\ & + Z_k^I \sin \delta_\ell \end{aligned} \quad (118c)$$

$$K_4 = c \Delta C_{i_{rj}}^I \quad (118d)$$

or more simply

$$d_{jk\ell}^I = K \sin(\omega_e t_k + \phi) + K_3(t_k) + K_4(t_k - t_0) \quad (119)$$

where ϕ is the phase of the sinusoids relative to some initial epoch, ω_e is the rotation rate of the earth, and

$$K = (K_1^2 + K_2^2)^{1/2} \quad (120)$$

Eqs. (117) and (119) shows that a space VLBI time delay observation between a single ground station and the orbiting radio telescope using a single source represents a combined function of a sinusoid and a periodic function with a period of the orbital revolution.

3.5.2 Time Delay Rate Model

The geometric delay rate is the time derivative of the geometric delay. Including the clock parameters, the delay rate may be expressed by Eq. (10). Differentiating Eq. (78) with respect to time we get

$$\begin{aligned} d_{jkl}^I = \omega_e \{ & X_j [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] + \\ & + Y_j \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \\ & - Z_j [\sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \} \\ & + \dot{X}_k^I \cos \delta_\ell \cos \alpha_\ell + \\ & + \dot{Y}_k^I \cos \delta_\ell \sin \alpha_\ell + \\ & + \dot{Z}_k^I \sin \delta_\ell + \\ & + c \Delta C_{1rj}^I, \end{aligned} \quad (121)$$

where

ω_e is the spin rate of the earth expressed by Eq. (53) and

$\dot{X}_k^I, \dot{Y}_k^I, \dot{Z}_k^I$, are the velocity components of satellite S^I in a true-of-date geocentric inertial coordinate frame at the epoch t_k .

Instead of the Cartesian velocity components $\dot{X}_k^I, \dot{Y}_k^I, \dot{Z}_k^I$ of the orbiting radiotelescope S^I , their equivalents expressed in Keplerian orbital elements are used by the following formulas (dropping the lower and upper indexes) [El'yasberg, 1967; p. 63]:

$$\begin{aligned} \dot{X} = v_r (\cos \Omega \cos u - \sin \Omega \sin u \cos i) - \\ - v_u (\cos \Omega \sin u + \sin \Omega \cos u \cos i) \end{aligned} \quad (122a)$$

$$\begin{aligned} \dot{Y} = v_r (\sin \Omega \cos u + \cos \Omega \sin u \cos i) - \\ - v_u (\sin \Omega \sin u - \cos \Omega \cos u \cos i) \end{aligned} \quad (122b)$$

$$\dot{Z} = v_r \sin u \sin i + v_u \cos u \sin i , \quad (122c)$$

where

$$v_r = \sqrt{\mu/p} e \sin f \quad (123a)$$

$$v_u = \sqrt{\mu/p} (1 + e \cos f) \quad (123b)$$

$$p = a (1 - e^2) \quad (123c)$$

μ is the gravitational constant times the mass of the Earth .

Our parameter set contains the same unknown parameters indicating by Eq. (82) with the exception of clock offset $\Delta C_{o_{rj}}^I$ to which the time delay rate is unaffected,

$$\{X_j, Y_j, Z_j, a, e, i, \omega, \Omega, M, \xi, \eta, \kappa, \alpha_\ell, \delta_\ell, \Delta C_{I_{rj}}^I\} \quad (124)$$

Taking the differential of $\dot{d}_{jk\ell}^I$ with respect to the parameters listed in the previous chapter

$$d(\dot{d}_{jk\ell}^I) = \sum_p D_p dP_p , \quad (125)$$

where the D_p 's are the required partial derivatives of the time delay with respect to the parameters subscripted p as follows:

$$D_{X_j} = \omega_e [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \quad (126)$$

$$D_{Y_j} = \omega_e \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) \quad (127)$$

$$D_{Z_j} = -\omega_e [\sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \quad (128)$$

$$D_\xi = \omega_e \{X_j [\cos \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell)] - Z_j [\cos \xi \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)]\} \quad (129)$$

$$D_\eta = \omega_e [X_j \sin \xi \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - Y_j \sin \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + Z_j \cos \xi \cos \eta \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] \quad (130)$$

$$D_k = \omega_e W_d \{ X_j [\cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) - \sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \delta_\ell)] - \\ - Y_j \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \\ - Z_j [\sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \delta_\ell)] \} \quad (131)$$

$$D_a = \cos \delta_\ell \left\{ -\frac{v_r}{2a} [\cos u \cos (\Omega - \alpha_\ell) - \sin u \cos i \sin (\Omega - \alpha_\ell)] + \right. \\ \left. + \frac{v_u}{2a} [\sin u \cos (\Omega - \alpha_\ell) + \cos u \cos i \sin (\Omega - \alpha_\ell)] \right\} - \\ - \sin \delta_\ell \left(\frac{v_r}{2a} \sin u \sin i + \frac{v_u}{2a} \cos u \sin i \right) \quad (132)$$

$$D_i = \cos \delta_\ell [v_r \sin u \sin i \sin (\Omega - \alpha_\ell) + \\ + v_u \cos u \sin i \sin (\Omega - \alpha_\ell)] + \\ + \sin \delta_\ell (v_r \sin u \cos i + v_u \cos u \cos i) \quad (133)$$

$$D_\omega = -\cos \delta_\ell \{ v_r [\sin u \cos (\Omega - \alpha_\ell) + \cos u \cos i \sin (\Omega - \alpha_\ell)] + \\ + v_u [\cos u \cos (\Omega - \alpha_\ell) - \sin u \cos i \sin (\Omega - \alpha_\ell)] \} + \\ + \sin \delta_\ell (v_r \cos u \sin i - v_u \sin u \sin i) \quad (134)$$

$$D_\Omega = -\cos \delta_\ell \{ v_r [\cos u \sin (\Omega - \alpha_\ell) + \sin u \cos i \cos (\Omega - \alpha_\ell)] - \\ - v_u [\sin u \sin (\Omega - \alpha_\ell) - \cos u \cos i \cos (\Omega - \alpha_\ell)] \} \quad (135)$$

$$D_e = \cos \delta_\ell \left\{ -v_r \frac{1}{e(1-e \cos E)} [\cos u \cos (\Omega - \alpha_\ell) - \sin u \cos i \sin (\Omega - \alpha_\ell)] - \right. \\ \left. - v_u \frac{e + \cos f}{(1-e^2)(1+e \cos f)} [\sin u \cos (\Omega - \alpha_\ell) + \cos u \cos i \sin (\Omega - \alpha_\ell)] \right\} + \\ + \sin \delta_\ell \left\{ -v_r \frac{1}{e(1-e \cos E)} \sin u \sin i + \right. \\ \left. + v_u \frac{e + \cos f}{(1-e^2)(1+e \cos f)} \cos u \sin i \right\} \quad (136)$$

$$D_M = \frac{v_r \cos f - v_u \sin f}{\sin E (1 - e \cos E)} \{ \cos \delta_\ell [\cos u \cos (\Omega - \alpha_\ell) - \sin u \cos i \sin (\Omega - \alpha_\ell)] + \\ + \sin \delta_\ell \sin u \sin i \} . \quad (137)$$

$$\begin{aligned}
D_{\alpha_\ell} = & \omega_e \{ X_j [\sin \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) - \cos \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell)] + \\
& + Y_j \cos \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell) + \\
& + Z_j [\sin \xi \cos \delta_\ell \cos (\theta_k - \alpha_\ell) + \cos \xi \sin \eta \cos \delta_\ell \sin (\theta_k - \alpha_\ell)] \} + \\
& + \cos \delta_\ell \{ v_r [\cos u \sin (\Omega - \alpha_\ell) + \sin u \cos i \cos (\Omega - \alpha_\ell)] - \\
& - v_u [\sin u \sin (\Omega - \alpha_\ell) - \cos u \cos i \cos (\Omega - \alpha_\ell)] \}
\end{aligned} \tag{138}$$

$$\begin{aligned}
D_{\delta_\ell} = & -\omega_e \sin \delta_\ell \{ X_j [\cos \xi \sin (\theta_k - \alpha_\ell) + \sin \xi \sin \eta \cos (\theta_k - \alpha_\ell)] + \\
& + Y_j \cos \eta \cos (\theta_k - \alpha_\ell) - \\
& - Z_j [\sin \xi \sin (\theta_k - \alpha_\ell) - \cos \xi \sin \eta \cos (\theta_k - \alpha_\ell)] \} - \\
& - \sin \delta_\ell \{ v_r [\cos u \cos (\Omega - \alpha_\ell) - \sin u \cos i \sin (\Omega - \alpha_\ell)] - \\
& - v_u [\sin u \cos (\Omega - \alpha_\ell) + \cos u \cos i \sin (\Omega - \alpha_\ell)] \} + \\
& + v_r \sin u \sin i \cos \delta_\ell + v_u \cos u \sin i \cos \delta_\ell .
\end{aligned} \tag{139}$$

$$D_{\Delta C_{1rj}^I} = c \tag{140}$$

In order to get the partial derivatives of time delay rate ($\dot{d}_{jk\ell}^I = \dot{d}$) with respect to Keplerian orbital elements, the following chain-rule differentiation

$$\begin{aligned}
\left[\begin{array}{c} D_a = \frac{\partial \dot{d}}{\partial a} \\ D_e = \frac{\partial \dot{d}}{\partial e} \\ D_i = \frac{\partial \dot{d}}{\partial i} \\ D_\omega = \frac{\partial \dot{d}}{\partial \omega} \\ D_\Omega = \frac{\partial \dot{d}}{\partial \Omega} \\ D_M = \frac{\partial \dot{d}}{\partial M} \end{array} \right] &= \left[\begin{array}{ccc} \frac{\partial \dot{X}}{\partial a} & \frac{\partial \dot{Y}}{\partial a} & \frac{\partial \dot{Z}}{\partial a} \\ \frac{\partial \dot{X}}{\partial e} & \frac{\partial \dot{Y}}{\partial e} & \frac{\partial \dot{Z}}{\partial e} \\ \frac{\partial \dot{X}}{\partial i} & \frac{\partial \dot{Y}}{\partial i} & \frac{\partial \dot{Z}}{\partial i} \\ \frac{\partial \dot{X}}{\partial \omega} & \frac{\partial \dot{Y}}{\partial \omega} & \frac{\partial \dot{Z}}{\partial \omega} \\ \frac{\partial \dot{X}}{\partial \Omega} & \frac{\partial \dot{Y}}{\partial \Omega} & \frac{\partial \dot{Z}}{\partial \Omega} \\ \frac{\partial \dot{X}}{\partial M} & \frac{\partial \dot{Y}}{\partial M} & \frac{\partial \dot{Z}}{\partial M} \end{array} \right] \left[\begin{array}{c} \frac{\partial \dot{d}}{\partial \dot{X}^I} = D_{\dot{X}^I} \\ \frac{\partial \dot{d}}{\partial \dot{Y}^I} = D_{\dot{Y}^I} \\ \frac{\partial \dot{d}}{\partial \dot{Z}^I} = D_{\dot{Z}^I} \end{array} \right]
\end{aligned} \tag{141}$$

was applied, where the partial derivatives of time delay rate with respect to the orbiting radiotelescope's Cartesian velocity components are as follows:

$$D_{\dot{X}^I} = \cos \delta_\ell \cos \alpha_\ell \tag{142}$$

$$D_{\dot{Y}}I = \cos \delta_\ell \sin \alpha_\ell \quad (143)$$

$$D_{\dot{Z}}I = \sin \alpha_\ell \quad (144)$$

The elements of the coefficient matrix in Eq. (141) are derived and given in Appendix A. A useful check of the partial derivatives of the Cartesian velocity components with respect to Keplerian orbital elements $\partial\dot{X}_i/\partial K_j$ is their use with the corresponding partial derivatives of Keplerian elements with respect to the rectangular velocity components $\partial K_j/\partial\dot{X}_k$ to confirm the relation

$$\sum_{j=1}^6 \frac{\partial\dot{X}_i}{\partial K_j} \frac{\partial K_j}{\partial\dot{X}_k} = \delta_{ik},$$

for all $i, k \in \{1, 2, 3\}$ (145)

where δ_{ik} is the Kronecker-delta, \dot{X} represents the Cartesian velocity components $\{\dot{X}, \dot{Y}, \dot{Z}\}$ and K stands for the Keplerian orbital elements $\{a, e, i, \omega, \Omega, M\}$.

The magnitude of the terms containing ξ and η in Eq. (121) are small, indicating that the delay rate is effectively insensitive to the Z_j component of the ground station. More precisely, it is completely insensitive to the coordinate component of the ground station parallel to the Earth's angular velocity vector. This is a disadvantage for geodetic applications of space VLBI delay rate observables. In addition, the delay rate is unaffected by clock offset variations, ΔC_{orj}^I . From $D_{\dot{Z}_k}I$ it is evident that the delay rate is insensitive to the polar component of the orbiting radiotelescope's velocity as the declination goes to zero. Only the partial derivatives of delay rate with respect to radio source position components contain the velocity components of orbiting radiotelescope.

In the ground-based VLBI, the delay rate observable is insensitive to the declinations of sources near equator due to the fact that B_{δ_ℓ} in Eq. (64) contains only the polar component of the declination. However, in the space VLBI the delay rate observable is more sensitive to the declinations of sources near the equator, since the corresponding partial derivative $D_{\delta_\ell}I$ in Eq. (139) contains the equatorial component of the radio source's declination as well. Therefore, an advantage of space VLBI over ground-based VLBI is that the space VLBI may be used for accurate determination of the declination of those radio sources near equator by the use of such ground-to-space baselines that possess large polar components

and relatively large values for the polar component of the radiotelescope's velocity, compare Eqs. (64) and (139).

The following linear relationships can be detected among the partial derivatives D_p :

$$D_\xi = X_j D_{Z_j} - Z_j D_{X_j} \quad (146)$$

$$D_\eta = \cos \xi (Z_j D_{Y_j} - Y_j D_{Z_j}) + \sin \xi (X_j D_{Y_j} - Y_j D_{X_j}) \quad (147)$$

$$D_\kappa = W_d \left[\sin \eta (X_j D_{Z_j} - Z_j D_{X_j}) + \sin \xi \cos \eta (Y_j D_{Z_j} - Z_j D_{Y_j}) + \cos \xi \cos \eta (X_j D_{Y_j} - Y_j D_{X_j}) \right] \quad (148)$$

$$D_{\alpha_\ell} = -D_\kappa/W_d - D_\Omega \quad (149)$$

Eqs. (146) - (148) with assumptions in Eq. (38) yield in matrix form

$$\begin{bmatrix} D_\xi \\ D_\eta \\ D_\kappa/W_d \end{bmatrix} = \begin{bmatrix} -Z_j & 0 & X_j \\ 0 & Z_j & -Y_j \\ -Y_j & X_j & 0 \end{bmatrix} \begin{bmatrix} D_{X_j} \\ D_{Y_j} \\ D_{Z_j} \end{bmatrix} \quad (150)$$

The relations from (146) to (150) are formally similar to the ones derived for the time delay model, see Eqs. (105) - (109). There is a formal similarity even with the analytically derived equations of the linear dependencies among parameters of the ground-based VLBI time delay and delay rate observation models. Compare the Eqs. (34) - (39), Eqs. (66) - (70), Eqs. (105) - (109) and Eqs. (146) - (150).

From (150) we can write

$$Z_j D_\kappa = W_d (Y_j D_\xi + X_j D_\eta) \quad (151)$$

Similar to Eqs. (112) and (113) for time delay observable, we can derive

$$D_{\delta_\ell} = -\text{tg } \delta_\ell (X_j D_{X_j} + Y_j D_{Y_j} + Z_j D_{Z_j} - 2aD_a) + (v_r \sin u + v_u \cos u) \sin i / \cos \delta_\ell \quad (152)$$

and

$$\cos i D_\omega = D_\Omega + \sin i \frac{v_r \cos u - v_u \sin u}{v_r \sin u + v_u \cos u} D_i \quad (153)$$

Eqs. (151), (152) and (153) are useful to study the critical configurations.

The discussion of the parameters estimable from delay rate is identical to that of time delays except that in this case $Z_j (Z'_j)$ and ΔC_{oTj}^I are deleted. Initial reference orientation parameters ξ_o, η_o, κ_o and α_o may be introduced again in order to overcome the datum rank defect problem. However, their errors effect the equatorial coordinates of the ground station and the equatorial velocity components of the orbiting radiotelescope in the following manner:

$$\begin{bmatrix} dX_j \\ dY_j \end{bmatrix} = \begin{bmatrix} dX'_j \\ dY'_j \end{bmatrix} - \begin{bmatrix} -Z_j & 0 & Y_j \\ 0 & Z_j & -X_j \end{bmatrix} \begin{bmatrix} d\xi_o \\ d\eta_o \\ d\beta_o \end{bmatrix} \quad (154a)$$

where $d\beta_o$ is identical with Eq. (43), that is, the two differential rotations $d\alpha_o$ and $d\kappa_o$ are inseparable and

$$\begin{bmatrix} d\dot{X}_k^I \\ d\dot{Y}_k^I \end{bmatrix} = \begin{bmatrix} d\dot{X}'_k^I \\ d\dot{Y}'_k^I \end{bmatrix} - \begin{bmatrix} 0 & d\alpha_o \\ -d\alpha_o & 0 \end{bmatrix} \begin{bmatrix} \dot{X}_k^I \\ \dot{Y}_k^I \end{bmatrix} \quad (154b)$$

Differentiating Eq. (117) we get

$$\dot{d}_{jk\ell}^I = K \omega_e \cos(\omega_e t_k + \phi) + \dot{K}_3(t_k) + K_4, \quad (155)$$

where

$$\begin{aligned} \dot{K}_3(t_k) &= \dot{X}_k^I \cos \delta_\ell \cos \alpha_\ell \\ &+ \dot{Y}_k^I \cos \delta_\ell \sin \alpha_\ell \\ &+ \dot{Z}_k^I \sin \delta_\ell, \end{aligned} \quad (156)$$

showing that a space VLBI time delay rate observable represents a combined function of a sinusoid and a periodic function of satellite's velocity components with a period of the orbital revolution.

3.6 Mathematical Models for Space-to-Space VLBI Observations

3.6.1 Time Delay Model. Eq. (11) may be rewritten as

$$d_{k\ell}^{\text{II}} = \begin{bmatrix} X_k^{\text{I}} - X_k^{\text{J}} \\ Y_k^{\text{I}} - Y_k^{\text{J}} \\ Z_k^{\text{I}} - Z_k^{\text{J}} \end{bmatrix} \begin{bmatrix} \cos \delta_\ell \cos \alpha_\ell \\ \cos \delta_\ell \sin \alpha_\ell \\ \sin \delta_\ell \end{bmatrix} + c \left[\Delta C_{\text{orq}}^{\text{II}} + \Delta C_{\text{Irq}}^{\text{II}} (t_k - t_0) \right] \quad (157)$$

where $X_k^{\text{I}}, Y_k^{\text{I}}, Z_k^{\text{I}}$ and $X_k^{\text{J}}, Y_k^{\text{J}}, Z_k^{\text{J}}$ are the coordinates of the orbiting radiotelescope S^{I} and S^{J} , respectively, in the true-of-date inertial reference frame at the epoch t_k .

In the model Eq. (157) we have ten unknown parameters

$$\left\{ X_k^{\text{I}}, Y_k^{\text{I}}, Z_k^{\text{I}}, X_k^{\text{J}}, Y_k^{\text{J}}, Z_k^{\text{J}}, \alpha_\ell, \delta_\ell, \Delta C_{\text{orq}}^{\text{II}}, \Delta C_{\text{Irq}}^{\text{II}} \right\} \quad (158)$$

to be determined. Taking the differential of $d_{k\ell}^{\text{II}}$ with respect to the parameters,

$$d_{k\ell}^{\text{II}} = \sum_p E_p dP_p, \quad (159)$$

where the E_p 's are the required partial derivatives of the time delay with respect to the parameters of interest, indexed by p . They are as follows:

$$E_{X_k^{\text{I}}} = -\cos \delta_\ell \cos \alpha_\ell \quad (160)$$

$$E_{Y_k^{\text{I}}} = -\cos \delta_\ell \sin \alpha_\ell \quad (161)$$

$$E_{Z_k^{\text{I}}} = -\sin \delta_\ell \quad (162)$$

$$E_{X_k^{\text{J}}} = \cos \delta_\ell \cos \alpha_\ell \quad (163)$$

$$E_{Y_k^{\text{J}}} = \cos \delta_\ell \sin \alpha_\ell \quad (164)$$

$$E_{Z_k^{\text{J}}} = \sin \delta_\ell \quad (165)$$

$$E_{\alpha_\ell} = \cos\delta_\ell \left[(X_k^I - X_k^J) \sin\alpha_\ell - (Y_k^I - Y_k^J) \cos\alpha_\ell \right] \quad (166)$$

$$E_{\delta_\ell} = \sin\delta_\ell \left[(X_k^I - X_k^J) \cos\alpha_\ell + (Y_k^I - Y_k^J) \sin\alpha_\ell - (Z_k^I - Z_k^J) \cos\delta_\ell \right] \quad (167)$$

$$E_{\Delta C_{0rq}^{IJ}} = c \quad (168)$$

$$E_{\Delta C_{1rq}^{IJ}} = c (t_k - t_0) \quad (169)$$

The following relationships are immediately apparent from above set

$$E_{X_k^I} = -E_{X_k^J} \quad (170)$$

$$E_{Y_k^I} = -E_{Y_k^J} \quad (171)$$

$$E_{Z_k^I} = -E_{Z_k^J} \quad (172)$$

These parameters cannot be separated at all, and new combined unknowns must be introduced:

$$\Delta X_k^{IJ} = X_k^I - X_k^J \quad (173)$$

$$\Delta Y_k^{IJ} = Y_k^I - Y_k^J \quad (174)$$

$$\Delta Z_k^{IJ} = Z_k^I - Z_k^J, \quad (175)$$

with partial derivatives

$$E_{\Delta X_k^{IJ}} = E_{X_k^I} \quad (176)$$

$$E_{\Delta Y_k^{IJ}} = E_{Y_k^I} \quad (177)$$

$$E_{\Delta Z_k^{IJ}} = E_{Z_k^I} \quad (178)$$

The new parameters are coordinate differences of the orbiting radiotelescopes S^I and S^J at the epoch t_k .

If the Keplerian orbital elements are used instead of the satellite's Cartesian coordinates, then the partials in Eqs. (90) - (95) may be used for the formulation. In this case, it can be shown that the ascending nodes (Ω_1 and Ω_2) of these two satellites (S^I and S^J) and the right ascension (α_ℓ) of the radio source cannot be separated in a common adjustment, since they are appeared in the partials through $(\Omega_1 - \alpha_\ell)$ and $(\Omega_2 - \alpha_\ell)$. This fact indicates that a space-to-space VLBI time delay observable is insensitive in the location

of the true-of-date coordinate system where the coordinate difference components (coordinates of the endpoints) of a space-to-space baseline are given. Because of this linear interrelationship between these parameters, a new estimable parameter, namely $\Delta\alpha_{o\ell}$ given by Eq. (41) must be introduced as unknown similarly to that of the cases of ground-to-ground and ground-to-space baseline VLBI observables. However, it can be shown that the equatorial components of the space-to-space baseline will be contaminated by the error $d\alpha_o$ of the initial reference right ascension in the following manner

$$\begin{bmatrix} d\Delta X_k^{\text{IJ}} \\ d\Delta Y_k^{\text{IJ}} \end{bmatrix} = \begin{bmatrix} d\Delta X_k^{\text{IJ}} \\ d\Delta Y_k^{\text{IJ}} \end{bmatrix} + \begin{bmatrix} 0 & d\alpha_o \\ -d\alpha_o & 0 \end{bmatrix} \begin{bmatrix} \Delta X_k^{\text{IJ}} \\ \Delta Y_k^{\text{IJ}} \end{bmatrix} \quad (179)$$

where $d\Delta X_k^{\text{IJ}}$ and $d\Delta Y_k^{\text{IJ}}$ are the differential changes in the equatorial components of the space-to-space baseline B^{IJ} .

3.6.2 Time Delay Rate Model. Differentiating Eq. (157) with respect to time we get

$$\dot{d}_{k\ell}^{\text{IJ}} = - \begin{bmatrix} \dot{X}_k^{\text{I}} - \dot{X}_k^{\text{J}} \\ \dot{Y}_k^{\text{I}} - \dot{Y}_k^{\text{J}} \\ \dot{Z}_k^{\text{I}} - \dot{Z}_k^{\text{J}} \end{bmatrix} \begin{bmatrix} \cos\delta_\ell \cos\alpha_\ell \\ \cos\delta_\ell \sin\alpha_\ell \\ \sin\delta_\ell \end{bmatrix} + c\Delta C_{\text{I}^{\text{IJ}}\text{r}^{\text{q}}} \quad (180)$$

where $\dot{X}_k^{\text{I}}, \dot{Y}_k^{\text{I}}, \dot{Z}_k^{\text{I}}$ and $\dot{X}_k^{\text{J}}, \dot{Y}_k^{\text{J}}, \dot{Z}_k^{\text{J}}$ are the velocity components of the orbiting radiotelescope S^{I} and S^{J} , respectively, in the true-of-date inertial reference frame at the epoch t_k . The model Eq. (180) contains nine unknown parameters

$$\{\dot{X}_k^{\text{I}}, \dot{Y}_k^{\text{I}}, \dot{Z}_k^{\text{I}}, \dot{X}_k^{\text{J}}, \dot{Y}_k^{\text{J}}, \dot{Z}_k^{\text{J}}, \alpha_\ell, \delta_\ell, \Delta C_{\text{I}^{\text{IJ}}\text{r}^{\text{q}}}\} \quad (181)$$

to be determined. Taking the differential of $\dot{d}_{k\ell}^{\text{IJ}}$ with respect to the parameters,

$$\dot{d}_{k\ell}^{\text{IJ}} = \sum_p F_p dP_p, \quad (182)$$

where the F_p 's are the required partial derivatives of the time delay with respect to the parameters of interest, indexed by p . They are as follows:

$$F_{\dot{X}_k^{\text{I}}} = -\cos\delta_\ell \cos\alpha_\ell \quad (183)$$

$$F_{\dot{Y}_k^{\text{I}}} = -\cos\delta_\ell \sin\alpha_\ell \quad (184)$$

$$F_{\dot{Z}_k^I} = -\sin \delta_\ell \quad (185)$$

$$F_{\dot{X}_k^J} = \cos \delta_\ell \cos \alpha_\ell \quad (186)$$

$$F_{\dot{Y}_k^J} = \cos \delta_\ell \sin \alpha_\ell \quad (187)$$

$$F_{\dot{Z}_k^I} = \sin \delta_\ell \quad (188)$$

$$F_{\alpha_\ell} = \cos \delta_\ell \left[(\dot{X}_k^I - \dot{X}_k^J) \sin \alpha_\ell - (\dot{Y}_k^I - \dot{Y}_k^J) \cos \alpha_\ell \right] \quad (189)$$

$$F_{\delta_\ell} = \sin \delta_\ell \left[(\dot{X}_k^I - \dot{X}_k^J) \cos \alpha_\ell + (\dot{Y}_k^I - \dot{Y}_k^J) \sin \alpha_\ell \right] - (\dot{Z}_k^I - \dot{Z}_k^J) \cos \delta_\ell \quad (190)$$

$$F_{\Delta C_{1rq}^{IJ}} = c \quad (191)$$

Eqs. (183) - (188) indicate a complete linear dependence ($F_{\dot{X}_k^I} = -F_{\dot{X}_k^J}$, $F_{\dot{Y}_k^I} = -F_{\dot{Y}_k^J}$, $F_{\dot{Z}_k^I} = -F_{\dot{Z}_k^J}$) between the corresponding velocity components of the orbiting radiotelescope S^I and S^J , respectively. Since they cannot be separated at all, the velocity component differences of the orbiting radiotelescope S^I and S^J

$$\Delta \dot{X}_k^{IJ} = \dot{X}_k^I - \dot{X}_k^J \quad (192)$$

$$\Delta \dot{Y}_k^{IJ} = \dot{Y}_k^I - \dot{Y}_k^J \quad (193)$$

$$\Delta \dot{Z}_k^{IJ} = \dot{Z}_k^I - \dot{Z}_k^J \quad (194)$$

are, therefore, introduced as new combined unknowns with the partial derivatives

$$F_{\Delta \dot{X}_k^{IJ}} = F_{\dot{X}_k^I} \quad (195)$$

$$F_{\Delta \dot{Y}_k^{IJ}} = F_{\dot{Y}_k^I} \quad (196)$$

$$F_{\Delta \dot{Z}_k^{IJ}} = F_{\dot{Z}_k^I} \quad (197)$$

Instead of satellite's Cartesian coordinates, Keplerian orbital parameters should be used in practice. For this purpose, the partials given in Eqs. (132) - (137) are useful. Using this parametrization, it can be shown that the ascending nodes (Ω_1 and Ω_2) of the orbiting radiotelescopes S^I and S^J and the right ascension (α_ℓ) of radio source are linearly interrelated, that is, they cannot be separated in a common adjustment. This inseparable situation for the parameters Ω_1 , Ω_2 and α_ℓ indicates that the space-to-space baseline time

delay rate observable is also insensitive to the orientation of the true-of-date inertial reference system in right ascension.

In order to circumvent to estimability problem, the right ascension of one (reference) source may be constrained to an adopted value and will not be included in the parameter set. Therefore, the corresponding estimable parameters are the right ascension differences $\Delta\alpha_{o\ell}$ similarly to that of the previous observation models. However, the equatorial velocity components of the space-to-space baseline will be contaminated by the error $d\alpha_o$ of the initial reference right ascension in the following manner:

$$\begin{bmatrix} d\Delta\dot{X}_k^{\text{IJ}} \\ d\Delta\dot{Y}_k^{\text{IJ}} \end{bmatrix} = \begin{bmatrix} d\Delta\dot{X}_k^{\text{IJ}} \\ d\Delta\dot{Y}_k^{\text{IJ}} \end{bmatrix} + \begin{bmatrix} 0 & d\alpha_o \\ -d\alpha_o & 0 \end{bmatrix} \begin{bmatrix} \Delta\dot{X}_k^{\text{IJ}} \\ \Delta\dot{Y}_k^{\text{IJ}} \end{bmatrix} \quad (198)$$

where $d\Delta\dot{X}_k^{\text{IJ}}$ and $d\Delta\dot{Y}_k^{\text{IJ}}$ are the differential changes in the equatorial velocity components of the space-to-space baseline B^{IJ} .

3.7 Differential VLBI Observations

Besides the usual VLBI observables (time delay, delay rate and phase delay), another very important tracking method is the differential VLBI (Δ VLBI) which can be applied to the determination of angular separation and its time derivative between a transponder on the orbiting radio observatory and natural extragalactic radio sources. As the name suggests, Δ VLBI is a measurement of the differential position between the satellite and a radio source; the measurement is made using VLBI and treating the satellite as a radio source. For an orbiting radiotelescope having a transponder on board, Δ VLBI observations from the ground of both the satellite and extragalactic radio sources in neighboring parts of the sky can serve to orient the satellite orbit with respect to the inertial frame formed by these sources. Since the positions of the natural radio sources are known with an accuracy of 0"001 as a result of astrometric programs, therefore the relative satellite coordinates are measurable with comparable accuracy if the satellite passes close to, and slowly by, one of the natural sources that constitute the inertial frame.

In the Δ VLBI technique signals received simultaneously from a satellite and a radio source located close together in the sky are compared. The difference between the phases of the signals received simultaneously from a radio source at the two ground stations, usually called the interferometric "fringe phase", is

$$\phi_{ij\ell}(t_k) = \frac{\omega}{c} B_{ij}(t_k) \cdot e_\ell, \quad (199)$$

where ω is the angular frequency of the transmitted signal, c is the speed of light, and $B_{ij} \cdot e_\ell$ is the pathlength difference [King, 1975].

If an orbiting transmitter and a natural radio source are observed simultaneously, the *differential fringe phase* is given by

$$\Delta\phi_{ij\ell}^I(t_k) = \phi_{ij\ell}^I(t_k) - \phi_{ij}^I(t_k) = \frac{\omega}{c} B_{ij}(t_k) \cdot (e_\ell - e_k^I) \quad (200)$$

where we have supposed that the two transmitter frequencies are the same. Because Δ VLBI involves taking differences not only between receiving points but also between transmitting points, it follows that any potential source of error will cancel if it is common either to all receivers or to all transmitters.

Expressing the small term $(e_\ell - e_k^I)$ in terms of the small differential right ascension, $\Delta\alpha_{k\ell}^I$ and differential declination, $\Delta\delta_{k\ell}^I$ between the directions to a natural radio source and an orbiting transmitter, we can write

$$\alpha_k^I = \alpha_\ell + \Delta\alpha_{k\ell}^I \quad (201a)$$

$$\delta_k^I = \delta_\ell + \Delta\delta_{k\ell}^I \quad (201b)$$

Whenever the orbiting radiotelescope passes close to the direction of an extragalactic radio source, Δ VLBI may be useful to determine the ground station— orbiting radiotelescope direction relative to that of the source to about 0."001. Such measurements could be used to determine precisely the orientation of the radio observatory orbits with respect to an inertial frame and to improve the estimates of the Earth's gravity model parameters.

4. ESTIMATION PROCESS AND SOME NUMERICAL TESTS

Some numerical computations were performed partly to check the mathematical models derived for space VLBI observables and partly to explore the overall applicability of space VLBI. In the following, first we discuss the datum definition problem in a general sense. This is followed by a short description of the numerical adjustment. Finally, results of the test computations are discussed.

4.1 Datum Definition for the Space/Ground VLBI Network

The general theory of network design including the geodetic datum problem and transformations is extensively discussed in the contributions, e.g., in Grafarend and Sansó (1985) and Teunissen (1985). In the following, the theoretical basis of the datum problem for a combined space/ground geodetic VLBI network will be shortly described.

The parameters in Eq. (82) we are looking for cannot be derived uniquely from the space VLBI time delay measurements. This typical situation in a combined space/ground VLBI geodetic network can be characterized by a rank defect of the design matrix C within the Gauss-Markov Model, see [Schaffrin, 1985 and 1990],

$$y - e = E(y) = Cx; \quad o(C) = n \times m, \quad rk(C) = q < m < n, \quad (202a)$$

$$D(y) = D(e) = \Sigma_{ee} = \sigma_e^2 P^{-1} \quad \text{positive-definite}, \quad (202b)$$

leading to the singular normal equations

$$N\hat{x} = b; \quad N := C^T P C, \quad rk N = q < m, \quad b := C^T P y. \quad (203)$$

Here y is the $n \times 1$ observation vector, e the $n \times 1$ error vector, C the $n \times m$ coefficient matrix (positive-semidefinite), x the $m \times 1$ parameter vector, Σ_{ee} the $n \times n$ positive-definite variance-covariance matrix of the observations, and P the respective $n \times n$ weight matrix (as well positive-definite). E denotes “expectation” and D “dispersion”.

Eq. (203) yields a whole class of different estimates \hat{x}

$$\hat{x} = (C^T P C)^- C^T P y = (C^T P C)_{rs}^- C^T P y \quad (204)$$

with accordingly different variance-covariance matrices

$$D(\hat{x}) = \sigma_\varepsilon^2 N_{rs}^- \text{ positive-semidefinite ,} \quad (205)$$

out of the class of reflexive symmetric g -inverses of N . Note that

$$C\hat{x} = C(C^T P C)^- C^T P y \quad (206)$$

is Best Linear Uniformly Unbiased Estimation (BLUUE) being unique for any choice of the g -inverse $(C^T P C)^-$.

From the rank deficiency of matrix C in (202a) [or (83)] follows that the unknowns x cannot be determined uniquely, even if $y \in R(C)$. Thus, the information contained in y is not sufficient to determine x uniquely. Note that the question of how to define vector x so that it is estimable from y involves the so-called *datum-problem*. The datum-problem deals with coordinate system definition, since coordinates are not estimable quantities from observations alone.

One can overcome this problem by adding the minimum information needed to determine x uniquely. Therefore, it is sufficient to introduce r additional minimal constraints

$$Kx = \kappa_o ; \quad o(K) = r \times m , \quad rk(K) = r := m - q \quad (207)$$

such that the space conditions

$$R(K^T) \cup R(C^T) = \mathbb{R}^m \quad (208a)$$

and

$$R(K^T) \cap R(C^T) = \{0\} \quad (208b)$$

hold, where $R(\cdot)$ denotes the range space of the corresponding matrices. From Eqs. (208a) and (208b) it follows that

$$R(K^T) \oplus R(C^T) = \mathbb{R}^m \quad (209a)$$

or

$$\mathbf{R}(\mathbf{K}^T) \oplus L(\mathbf{C}) = \mathbf{R}^m, \quad (209b)$$

where $L(\mathbf{C})$ is the nullspace of the design matrix \mathbf{C} . (Note that the concepts of nullspace and range space are very important for singular matrices.)

The famous restriction

$$r = m - q \quad (210)$$

is for the number of minimal constraints which are needed to overcome the rank deficiency of the matrix \mathbf{C} . Eq. (208b) says that the rows of the matrices \mathbf{C} and \mathbf{K} ought to be complementary (but not necessarily orthogonal). An equivalent condition with Eq. (208a) the

$$\bar{\mathbf{N}} := \mathbf{N} + \mathbf{K}^T\mathbf{K} \text{ regular}, \quad (211)$$

is fulfilled. In this case, we obtain the estimated parameters $\hat{\mathbf{x}}$ and their dispersion matrix $D(\hat{\mathbf{x}})$ from the extended normal equations

$$\begin{bmatrix} \mathbf{N} & \mathbf{K}^T \\ \mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \boldsymbol{\kappa}_0 \end{bmatrix} \quad (212)$$

where $\boldsymbol{\lambda}$ denotes some $r \times 1$ vector of “Langrange multipliers.”

In practice, special datum constraints

$$\mathbf{G}\mathbf{x} = \mathbf{0}; \quad \mathbf{o}(\mathbf{G}) = r \times m, \quad \text{rk}(\mathbf{G}) = r := m - q \quad (213)$$

can be obtained through the orthogonality relation

$$\mathbf{C}\mathbf{G}^T = \mathbf{0}. \quad (214)$$

The rows of matrix \mathbf{G} yield a basis of the nullspace $L(\mathbf{C})$ so that condition Eq. (209a) then reads as

$$\mathbf{R}(\mathbf{G}^T) \perp \mathbf{R}(\mathbf{C}^T) = \mathbf{R}^m, \quad (215)$$

i.e., the range spaces $R(C^T)$ and $R(G^T)$ are perpendicular to each other, thereby automatically implying complementarity.

If

$$K := G \tag{216}$$

is available for an arbitrary, but fixed $r \times m$ matrix G with $\text{rk}(G) = r$ and $CG^T = 0$, then the “free net adjustment” leads to the estimates

$$\hat{x} = N^+b \tag{217a}$$

$$D(\hat{x}) = \sigma_\epsilon^2 N^+ , \tag{217b}$$

where N^+ denotes the pseudo-inverse of $N = C^T P C$. N^+ is found directly by

$$N^+ = (N + G^T G)^{-1} - G^T (G G^T G G^T)^{-1} G , \tag{218}$$

which is independent of the choice of the matrix G .

If G matrix is taken, the so-called minimum norm solution is found, which has a minimal value for the trace of the variance-covariance matrix.

Adding the minimal constraints Eqs. (207) or (213) to the model Eq. (202a) removes the datum rank deficiency of the model. Applied to our combined space/ground VLBI geodetic network it means, that by adding the either minimal constraints the datum is established.

Note that if we collect the eigenvectors belonging to the r -multiple eigenvalue 0 of the normal equation matrix $C^T C$, we may even obtain an orthogonal basis (but it is not a necessary requirement). Therefore, we may apply a formal Singular Value Decomposition (SVD) to the positive semi-definite matrix $C^T C$. Since $C^T C$ is a real symmetric semi-positive definite matrix, it follows that its eigenvalues are non-negative. Denoting these eigenvalues by σ_i^2 , $i = 1, 2, 3, \dots, m$, we can arrange them in decreasing order that

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_q > 0 = \sigma_{q+1} = \dots = \sigma_m \tag{219}$$

For $C^T C$, there exist q different positive eigenvalues and a r -fold ($r = m - q$) eigenvalue zero. Associated with these eigenvalues exist m different mutual orthogonal eigenvectors.

The corresponding orthogonal eigenvectors are denoted by (v_1, v_2, \dots, v_m) and we separate them into

$$V_0 = (v_1, v_2, \dots, v_r) \quad (220a)$$

$$V_1 = (v_{q+1}, v_{q+2}, \dots, v_m). \quad (220b)$$

With $\Lambda_q^{1/2} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_q)$ we thus have

$$C^T C V_0 = V_0 \Lambda_q \quad (221)$$

and

$$C^T C V_1 = 0 \quad (222a)$$

or

$$C V_1 = 0 \quad (222b)$$

Since the coordinate system defining subspace $R(K^T)$ only needs to fulfill the complementarity condition Eq. (209b), it follows that there are many more coordinate frames possible, giving the same solutions in principle. Therefore, a relation exists between these solutions which is called S-transformation. For instance, a very special datum choice is given with Eq. (216). The condition Eq. (209b) is for practical purposes most manageable, since

$$R(V_1) = L(C) \quad (223)$$

is given by the linearized S-transformation [Teunissen, 1985] and K (or $K := G$) is chosen in order to define the coordinate system.

4.2 Numerical Adjustment

In the mathematical models of space VLBI time delay and delay rate observables different type of parameters are included. However, in practice we are primarily interested in some of them, that is, e.g., earth rotation parameters (ξ, η, κ) are of most interest for geodynamical studies, or in order to improve the Earth's gravity field, only the corrections to the Keplerian orbital elements are to be determined. Furthermore, supposedly we have

stochastic prior information on certain parameters. Therefore, a solution for a group of parameters in primary interest is often sought from a partitioned system of normal equations of the following form [Schaffrin, 1990]

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} P \ell \quad (224)$$

where

$$C = [C_1 \ C_2] \quad (225)$$

is the partitioning of the design matrix C of the space VLBI network and

$$x = [x_1 \ x_2] \quad (226)$$

is the partitioning of the unknown parameters into a sub-vector x_1 with stochastic prior information and a sub-vector x_2 of the parameters of interest without prior information.

The inversion of normal equation system (224) is

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}^{-1} = \begin{bmatrix} S_1^{-1} & -S_1^{-1} N_{12} N_{22}^{-1} \\ -N_{22}^{-1} N_{21} S_1^{-1} & N_{22}^{-1} N_{21} S_1^{-1} N_{12} N_{22}^{-1} + N_{22}^{-1} \end{bmatrix} \quad (227)$$

where

$$S_1 = N_{11} - N_{12} N_{22}^{-1} N_{21} \quad (228)$$

is the so-called "1st Schur complement". Consequently, the estimated dispersion of the estimated parameters \hat{x}_2 of interest is given by

$$\hat{D} \{ \hat{x}_2 \} = \hat{\sigma}_0^2 (N_{22}^{-1} + N_{22}^{-1} N_{21} S_1^{-1} N_{12} N_{22}^{-1}) . \quad (229)$$

However, the system of normal equations cannot be inverted because of the datum rank defect. In practice the datum rank defect may be overcome by using certain conventions for the coordinate system definition, that is, e.g., by the use of a minimum norm solution (see the previous paragraph). In that case, in order to get the dispersion of the estimated parameters \hat{x}_2 of interest, the pseudoinverse N^+ of $N = C^T P C$ given by Eq. (218) has to be used in partitioning in the practical computations.

4.3 Results of Test Computations

In Section 3.5, the analytical partials and linear dependency are derived for both space VLBI observables. In the case of time delay observable, the rank defect of the normal equation matrix $N = C^T C$ is discussed in detail. Numerical checking of rank defect by evaluating numerical values of partials has been done in the frame of test computations. For this purpose we computed the normal equation matrix $N = C^T C$. Eigenvalue analysis was used to check the rank defect of $N = C^T C$.

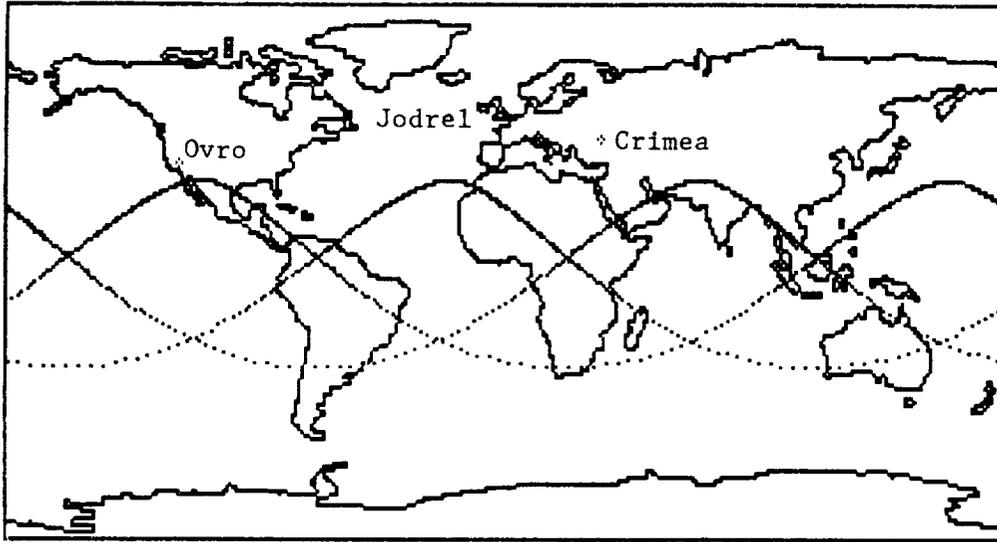
In order to derive the normal equation system matrix $N=C^T C$ numerically, generated data was used within a simulated ground/space VLBI network. For this purpose, minimum numbers were derived for participating stations, observed satellite positions and radio sources. The basic configuration consists of one satellite orbit for VSOP mission, three ground-based VLBI stations with three different radio source positions.

The stations used in test computations are: Crimea, Jodrel 2 and Ovro 130. Their coordinates are given in Table 2. The used source positions as published in [Carter et al., 1989] are collected in Table 3.

In order to compute satellite spatial coordinates, initial orbital parameters of the VSOP satellite were used. A standard orbit simulation parameter set for VSOP is given in Table 4. On the basis of these parameters, simulated orbit was generated for date of 01/01/1996 using a space VLBI related program package developed at the Satellite Geodetic Observatory in Hungary and made available by I Fejes. After computation of satellite Cartesian position and velocity component, a set of “approximate” Keplerian orbital parameters was generated to use it for the numerical evaluation of analytical partials.

The distribution of the three ground stations and the subsatellite track of the VSOP is shown in Fig. 17. An overall view of the stations and VSOP configuration is represented in Fig. 18.

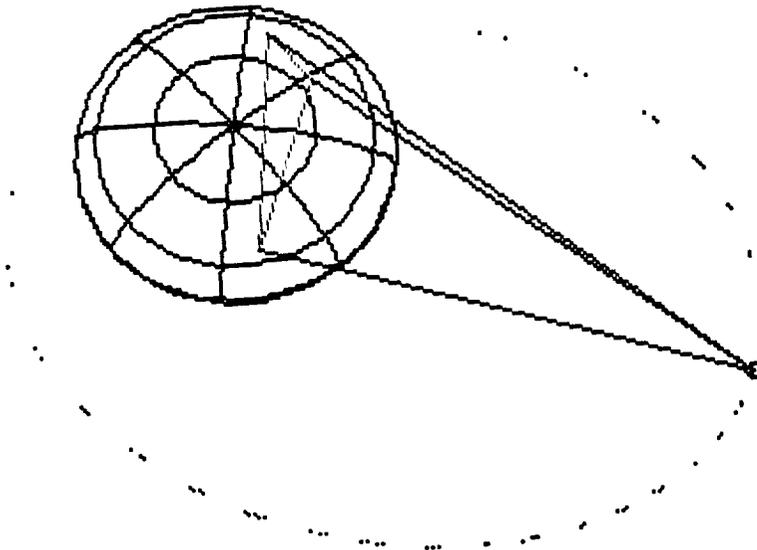
Simulated earth rotation parameters (ξ , η , κ) for epoch MJD50083 (01/01/1996) were calculated by using the prediction formulas given in [IERS Bulletin-A, 1990]. Their numerical values of $\xi = -0''142486$, $\eta = 0''193437$ and $\kappa = -0.17271$ ms were used.



Satellite:		VSOP	
Epoch:	Start	960101	00.00
	Stop	960102	00.00

Fig. 17 Subsatellite track line of VSOP and Location of the Ground-Based VLBI Stations Used in the Test Computations

Source: 1803+784
 ST: 56792d 2 h 24 m
 UT: 50083d 19 h 41 m



SATELLITE : VSOP

Fig. 18 Stations and VSOP Configuration Overall View from the Radio Source Direction

Table 2
Station Coordinates Used in Test Computations

Station	X (m)	Y (m)	Z(m)
Crimea	378 5227.20	255 1211.80	443 9806.93
Jodrel 2	382 2842.66	-15 3800.13	508 6287.22
Ovro 130	-240 9626.30	-447 8405.30	383 8606.70

Table 3
Source Positions Used in Test Computations

Source	Right Ascension			Declination of Arc		
	H	M	S	°	'	"
0212 + 735	2	17	30.813210	73	49	32.62230
1641 + 399	16	42	58.809930	39	48	36.99380
1803 + 784	18	0	45.683850	78	28	4.01790

Table 4
Initial Orbit Parameter Set for VSOP Satellite

Keplerian Orbital Elements of VSOP Satellite	
Epoch of orbital elements	47894.00000 [MJD]
Semi major axis	16878.00000 km
Eccentricity	0.56300
Inclination	31.000 deg
Argument of perigee	0.0000 deg
Longitude of ascending node	90.0000 deg
Mean anomaly	0.0000 deg

Table 5
Observations and Unknown Parameters of the Basic Configuration for Test Computations

Observations	Estimated Parameters
$\frac{12 \times 3}{36}$ time delays	3x3 station coordinates 3x2 source coordinates 1x6 Keplerian elements 1x3 ERP parameters $\frac{3 \times 2}{30}$ clock parameters

The Greenwich Apparent Sidereal Time ($GAST = \Theta_k$) was computed from observed Universal Time UT1 data in MJD for the observation time epochs t_k using formula given in [Obenson, 1970; p.25].

The relevant numbers which show the basic configuration for the test computation are shown in Table 5. On the basis of this configuration, there are 30 unknown parameters. Since we have three delay observations at each epoch, we need observations at ten different epoch for a unique solution. Instead of this minimum requirement, 12 satellite points were included to form our basic configuration for a space/ground VLBI network. Four satellite positions belong to one radio source observation period.

The input metric data (station coordinates and semi-major axis) were used in Mm unit, and the angle type data unit was in radian. The speed of light was used in unit of Mm/msec.

Using this generated data set, the partials of the time delay and delay rate with respect to the parameters involved have been calculated. Then the design matrix C for the time delay observable was found, and the elements of the normal equation system matrix $N = C^T C$ was computed. The dimension of the design matrix C and the normal equation system matrix $N = C^T C$ is 36×30 and 30×30 respectively.

Two cases were considered in the computations concerning the orbit length: 1) short-arc (1/5 part of a full orbit), and 2) full orbit was used. In the first case, satellite positions were taken every six minutes, and for the second case the integration time period was 30 minutes.

It is not surprising that the normal equations derived from observation equations results in a singular system. This is confirmed by an eigenvalue analysis and using LU and Cholesky decomposition.

The normal matrix $N = C^T C$ derived from the simulated data possesses a peculiar eigenvalue spectrum. The spectrum resembles that of a poorly conditioned system of equations with corresponding eigenvalues ranging from 10^{-7} to 10^{+5} in the case of "Short-arc" configuration. In the "full orbit" situation, the eigenvalue spectra is in the range of 10^{-11} to 10^{+4} . As a result, the condition number ($\lambda_{\max}/\lambda_{\min}$), representing a measure for the numerical stability of the system, is of the order of 10^{12} and 10^{15} respectively. In an

ideal situation, the condition number should lie in the range from 1 to 10^3 . This is not our case; that is, our normal equation system is poorly conditioned in both geometric configuration of the used space/ground VLBI network.

The only cause of the poor condition number can be the observation equations themselves and results in an unstable normal equation matrix having approximately five order of magnitude difference in the values of its diagonal elements. The eigenvalue spectrum of the normal equation matrix can be improved by transformation of the design matrix C . It can be done with multiplication of the design matrix by a diagonal matrix P resulting a normal matrix $\tilde{N} = \tilde{C}^T \tilde{C}$ having diagonal elements approximately the same order of magnitude in values. A multiplication of the design matrix C by a diagonal matrix \tilde{P} gives

$$\tilde{C} = C\tilde{P} \quad (230)$$

and the normal equation system matrix becomes

$$\tilde{N} = \tilde{P}^T C^T C \tilde{P} . \quad (231)$$

The inverse of this normal matrix \tilde{N} is (with $\tilde{P}^T = \tilde{P}$)

$$\tilde{N}^{-1} = (\tilde{P} C^T C \tilde{P})^{-1} = \tilde{P}^{-1} (C^T C)^{-1} \tilde{P}^{-1} . \quad (232)$$

The normal matrix $\tilde{N} = \tilde{C}^T \tilde{C}$, found after the transformation of the design matrix C , shows an eigenvalue spectrum having better properties. The results of this analysis are presented in Table 6. The computed eigenvalues of the normal matrix \tilde{N} show nine small eigenvalues indicating a considerable jump between the fourth and fifth eigenvalue, and in addition a system which is still poorly conditioned. The first four small eigenvalues may be due to true singularity. The eigenvalue spectra of the “full-orbit” network configuration is improved. It clearly shows four small eigenvalues which may be considered as zero with respect to the other 26 eigenvalues. The latter eigenvalues are in the range of 3.52 to 3.67×10^8 . The four zero eigenvalues clarify our results for linear dependencies derived analytically.

Table 6
Eigenvalues of the Normal Matrices in Case
“Short-arc” and “Full Orbit”

No.	“Short-arc”	“Full Orbit”
1	8.27×10^{-8}	2.88×10^{-7}
2	2.37×10^{-7}	3.24×10^{-7}
3	4.42×10^{-7}	5.33×10^{-7}
4	8.14×10^{-7}	7.01×10^{-7}
5	1.84×10^{-4}	3.52
6	1.59×10^{-3}	9.18
7	6.85×10^{-3}	6.24×10
8	2.91×10^{-2}	1.89×10^2
9	9.67×10^{-2}	7.47×10^2
10	3.99	6.13×10^3
11	1.98×10	1.43×10^4
12	9.60×10	3.36×10^4
...		
30	4.58×10^8	3.67×10^8

The eigen vectors belonging to the four zero eigenvalues are the nullspace base vectors of the design matrix. They can be used to form the column vectors of a G matrix to derive the pseudo-inverse N^+ in Eq. (218) for a minimum norm solution. Therefore, another numerical test is to check the orthogonality of these eigen vectors with respect to the column vectors of the design matrix, that is, to check the numerical validity of Eqs. (222a) and (222b). The results obtained for the elements of matrix multiplication expressed by Eqs. (222a) and (222b) are of the order of 10^{-6} to 10^{-4} .

As discussed in section 3.5.1 singularity of the normal equations is caused by linear dependency of the column vectors in the design matrix. In order to confirm our results given by Eqs. (108) and (109) in another way, the so-called LU-decomposition and Cholesky decomposition of the normal equation system matrix $\tilde{N} = \tilde{C}^T \tilde{C}$ have been performed as an independent procedure. For LU-decomposition a routine in [Press et al., 1986] has been used. The Cholesky reduction algorithm is given by Stewart (1973; p. 142). The parameters to be solved for are involved in the normal equation system in the following order:

$$\{X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3, a, e, i, \omega, \Omega, M, \xi, \eta, \kappa, \alpha_1, \delta_1, \alpha_2, \delta_2, \alpha_3, \delta_3, \Delta C_{0r1}^I, \Delta C_{1r1}^I, \Delta C_{0r2}^I, \Delta C_{1r2}^I, \Delta C_{0r3}^I, \Delta C_{1r3}^I\} \quad (223)$$

Both types of decomposition of normal matrix show small values for the diagonal elements corresponding to the earth rotation parameters and the right ascension of the third radio sources. In the case of LU-decomposition of $\tilde{C}^T \tilde{C}$, the small values in question are in the range of 10^{-7} to 10^{-5} . The corresponding values from Cholesky decomposition are of the order of 10^{-3} . They may be considered as zero with respect to the other diagonal elements in both decomposed forms of normal matrix. A zero value means complete dependence on (a linear combination of) other unknown parameters.

A simple checking for linear dependencies has also been performed by calculation of the numerical values of the Eqs. (108) - (109) and Eqs. (149) - (150). The validity of Eqs. (111) - (113) and Eqs. (151) - (153) derived for critical configuration study has been checked numerically as well.

5. SUMMARY AND RECOMMENDATIONS

Investigations presented in this study are summarized as follows:

(1) Qualitative expressions have been developed to demonstrate the relationship between the space VLBI observables (time delay and delay rate) and the solve for parameters, and to explore estimability problems inherent in the space VLBI system. Instead of Cartesian satellite coordinates, Keplerian orbital elements were introduced into the observation equations and partial derivatives of space VLBI observables with respect to the parameters of interest were derived. A sensitivity analysis using these partials has been carried out. The mathematical models for ground-to-ground baseline VLBI observables are also given for comparison. Simplified models were used for the space-to-space VLBI observation equations.

(2) Linear dependencies between partial derivatives have been derived analytically. Their expressions for both space VLBI (ground-to-space baseline time delay and delay rate) observables are formally similar. Four independent linear dependencies were found indicating that the number of the rank defect of the corresponding normal equation matrix is four. It has been shown through analytical derivation that the linear dependencies between partials of the ground-to-space baseline VLBI observables are formally very similar to ones of the ground-to-ground VLBI observables. However, in the case of both ground-to-space baseline VLBI observables the expression of linear dependency between earth rotation parameter κ and the right ascension of radio source α_l were extended by a new term for ascending node Ω of the satellite orbit indicating that these three parameters are nonseparable in a common adjustment.

(3) A geometric interpretation of datum rank defect (analytically derived linear dependencies) in the space VLBI time delay observation equations has been presented. The rank defects expressed by linear dependencies are confirmed using different numerical methods with simulated data set in the frame of test computations. The eigenvalue analysis and both decomposition (Cholesky and LU of a lower and an upper triangular matrix) of the normal equation matrix indicate that the number of datum rank defects is four concerning the mathematical model used for the space VLBI time delay observable.

the normal equation matrix indicate that the number of datum rank defects is four concerning the mathematical model used for the space VLBI time delay observable.

(4) In searching linear dependencies, analytical expressions were derived to study critical configuration in the case of both ground-based and space VLBI. They are useful in observation design work. A check of these relations was performed by test computation.

(5) Datum definition problem for space VLBI network is theoretically described within the framework of rank defect Gauss-Markov Model. Proper numerical adjustment procedure is given which can be used for covariance analysis of the parameters of geodetic interest.

In order to use successfully and efficiently the space VLBI observations in geodesy and geodynamics, several additional studies need to be made in order to investigate in full detail the potential advantages and the possible problems of the space VLBI technique. There have been no simulation studies so far to indicate the expected accuracies of the space VLBI system in the determination of the parameters of astrometric and geodetic-geodynamic interest. Therefore, a feasibility study should be undertaken in order to show whether or not space VLBI can provide real improvements in the potential application in geodesy and geodynamics. Hence, in order to give a feeling of what one should expect from such a system, a detailed sensitivity analysis needs to be carried out in order to estimate the accuracy achievable in determining various parameters. More emphasis should be given to the determination of earth rotation parameters for reference frame studies and to the estimation of orbit parameters to improve the gravity field determination. On the basis of this work, we would recommend that:

a) The mathematical model be augmented to include additional effects like variation of Keplerian orbital elements. Using the extended mathematical model, the rank defect analysis should be repeated.

b) Simulations in order to determine optimum global station network and orbit configuration, and accuracy estimates for the various parameters are needed. These would require extended simulation studies and error analysis using data from a realistic observational pattern.

c) Another area which needs to be developed in the near future is the software development for different tasks involved in space VLBI geodetic data analysis.

d) Such investigations should provide sufficient background for the inclusion of space VLBI observable in geodetic data processing programs (e.g., GEODYN).

REFERENCES

- Adam, J. (1989) "A Possible Application of the Space VLBI Observations for Establishment of a New Connection of Reference Frames," in *Theory of Satellite and Gravity Field Determination*, edited by F. Sansó and R. Rummel, Lecture Notes in Earth Sciences, No. 25, pp. 459-475, Springer-Verlag, Berlin Heidelberg.
- Adam, J. and I.I. Mueller (1989) "Space VLBI for Geodesy and Geodynamics," presented at American Geophysical Union Fall Meeting, San Francisco, December 4-8, 1989. Abstract published in *EOS*, Trans. AGU, 70, p. 1052, 1989.
- Arnold, K. (1974) "Geodetic Aspects of Laser Distance Measurements to the Moon and Radio-Interference Measurements to Quasars," *Gerlands Beitr. Geophysik*, 83/4, pp. 249-269.
- Bartel, N., T.A. Herring, M.I. Ratner, I.I. Shapiro and B.E. Corey (1986) "VLBI limits on the proper motion of the 'core' of the superluminal quasar 3C345," *Nature*, Vol. 319, No. 6056, pp. 733-738.
- Bartel, N. and 18 others (1988) "VLBI imaging with an angular resolution of 100 microarcseconds," *Nature*, Vol. 334, No. 6178, pp. 131-135.
- Bartel, N. (1989) "Differential Astrometry of Continuum Sources and Pulsars with (Space?) VLBI," pres. 8th RADIOASTRON Review Meeting, Green Bank, West Virginia, May 1-4.
- Bock, Y. (1980) *A VLBI Variance-Covariance Analysis Interactive Computer Program*, Report No. 298, Dept. of Geodetic Science, The Ohio State University, Columbus.
- Borza, T., I. Fejes and B.A.C. Ambrosius (1989) "On Orbit Determination Accuracy of Space-VLBI Satellites," *Adv. Space Res.*, 10, 3-4, pp. (3) 321-(3) 326, 1990.
- Boucher, C., M. Feissel and J.-F. Lestrade (1988) "Concepts and Methods of the Central Bureau of the International Earth Rotation Service," *Bull. Geod.*, 62 (1988), pp. 511-519.
- Brouwer, F.J.J. (1985) *On the Principles, Assumptions and Methods of Geodetic Very Long Baseline Interferometry*, Ph.D. Thesis, Publ. Netherlands Geodetic Commission, Vol. 7, No. 4, Delft.
- Burke, B.F. (1983) "Orbiting VLBI," *Adv. Space Res.*, 2, pp. 33-38.
- Campbell, J. (1987) "Very Long Baseline Interferometry," in *Applied Geodesy*, S. Turner (Ed.), pp. 67-87, Springer-Verlag, 1987.

- Carter, W.E., D.S. Robertson and F.W. Fallon (1989) "Polar Motion and UT1 Time Series Derived from VLBI Observations." IERS (1989): Technical Note No. 2, pp. 35-39.
- Counselman, C.C. (1976) "Radio Astrometry," Annual Review of Astronomy and Astrophysics, Vol. 14, pp. 197-214.
- Counselman, C.C. and I.I. Shapiro (1978) "Principles of VLBI Applied to Geodesy," Proc. of Conference VII: Stress and Strain Measurements Related to Earthquake Prediction, pp. 128-141, Menlo Park, California, 1978.
- Dermanis, A. (1980) "VLBI: Principles and Geodynamic Prospectives," *Quat. Geod.*, 1(3), pp. 213-230, 1980.
- Dermanis, A. and E.W. Grafarend (1981) "Estimability analysis of geodetic, astrometric and geodynamical quantities in very long baseline interferometry," *Geophys. J.R. astro. Soc.*, 64 (1981), pp. 31-56.
- Dermanis, A. and I.I. Mueller (1978) "Earth Rotation and Network Geometry Optimization for Very Long Baseline Interferometers," *Bull. Geod.*, 52 (1978), pp. 131-158.
- El'yasberg, P.E. (1967) "Introduction to the Theory of Flight of Artificial Earth Satellites," (translated from Russian). Published for the NASA and the NSF by the Israel Program for Scientific Translations, Jerusalem, 1967.
- Fejes, I., I. Almar, J. Adam and Sz. Mihaly (1986) "Space-VLBI: Potential Applications in Geodynamics," *Adv. Space Res.*, 6, pp. 205-209.
- Fejes, I., I. Almar, J. Adam and Sz. Mihaly (1987) "On Astrometric and Geodynamic Aspects of Space VLBI," *Nablyudeniya ISZ*, No. 25, pp. 502-516, Budapest.
- Fejes, I, ed. (1989) "Precise Orbit Determination of RADIOASTRON," report prepared by the RADIOASTRON NAG group for the 8th RADIOASTRON Review Meeting, Green Bank, West Virginia, May 1-4, 1989. (Avail. Satellite Geodetic Observatory, P.O. Box 546, H-1373 Budapest, Hungary.)
- Fejes, I., I. Almar, J. Adam and Sz. Mihaly (1989a) "Space VLBI: A New Technique for Unification of Reference Frames," presented at the IAG General Meeting, Symposium 105 "Earth Rotation and Coordinate Reference Frames," Edinburgh, Scotland, August 3-5, 1989.
- Fejes, I., T. Borza, Sz. Mihaly and L. Szanthy (1989b) "Orbit Determination Accuracy Improvement by Space VLBI Observables as Tracking Data," presented at the IAG General Meeting, Symposium 101 "Global and Regional Dynamics," Edinburgh, Scotland, August 3-5, 1989.
- Finkelstein, A.M. and Ya.S. Yatzkiv (1989) "Dedicated VLBI-Network QUASAR." Paper presented at the 18th Intercosmos Symp. Use of Artificial Satellite Observations for Geodesy and Geophysics, Cracow, Poland, June 12-17, 1989.
- Frisk, U., A. Hawkyard and J.W. Cornelisse (1988), "QUASAT—A 50,000 km Diameter Quasar Probe," *ESA Bulletin*, No. 55, 18-23.

- Grafarend, E.W. and E. Livieratos (1978) "Rank Defect Analysis of Satellite Geodetic Networks I.-Geometric and Semi-dynamic Mode," *manuscripta geodaetica*, Vol. 3 (1978), pp. 107-134.
- Grafarend, E.W. and F. Sansó (Editors) (1985) *Optimization and Design of Geodetic Networks*, Springer-Verlag, Berlin-Heidelberg, 1985.
- Harvey, B.R. (1985) "The Combination of VLBI and ground data for geodesy and geophysics," Ph.D. Thesis, UNISURV S-27, 1985.
- Hill, Ch. J. (1989) "Satellite Laser Ranging and Some Geophysical Applications," Ph.D. Thesis, Department of Civil Engineering, University of Nottingham, May 1989.
- Hirabayashi, H. (1984) "An Introductory Review for Japanese Space VLBI Mission," NRO Rep. 47, Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, Nobeyama, Japan.
- Hirabayashi, H. (1988) "VLBI Activities in Japan and a Projected Space-VLBI Program," in *The Impact of VLBI on Astrophysics and Geophysics*, M.J. Reid and J.M. Moran, eds., Kluwer Academic Publ., Dordrecht, Holland, 449-456.
- Hirabayashi, H. (1989a) "VSOP and the World Situations," Proc. of a domestic Workshop on "The Research of Space VLBI," Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, pp. 7-12, Nobeyama, Japan, March 14-15, 1989.
- Hirabayashi, H. (1989b) "VSOP and the Days After," Proc. of a domestic Workshop on "The Research of Space VLBI," Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, pp. 71-78, Nobeyama, Japan, March 14-15, 1989.
- Hirabayashi, H. (1989c) "VSOP Requirements for Astronomical Observing," Proc. of a domestic Workshop on "The Research of Space VLBI," Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, pp. 80-85, Nobeyama, Japan, March 14-15, 1989.
- IERS Bulletin A*, Vol. III, No. 3, 18 January 1990.
- Kardashev, N.S. and V.I. Slysh (1988) "The RADIOASTRON Project," in *The Impact of VLBI on Astrophysics and Geophysics*, M.J. Reid and J.M. Moran, eds., Kluwer Academic Publ., Dordrecht, Holland, pp. 433-440.
- Kardashev, N.S. (1989) "Space-Ground Radio Interferometer 'RADIOASTRON': Basic Specifications and Future Prospects," pres. 8th RADIOASTRON Review Meeting, Green Bank, West Virginia, May 1-4.
- Kawaguchi, N. (1989) "Measurement of the Earth's geocenter with VSOP," (in Japanese with English abstract). Proc. of a domestic Workshop on "The Research of Space VLBI," Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, pp. 45-49, Nobeyama, Japan, March 14-15, 1989.

- King, R.W. (1975) "Precision Selenodesy via Differential Very Long Baseline Interferometry," Ph.D. Thesis, Dept. of Aeronautics and Astronautics, M.I.T., Cambridge, 1975.
- Konopliv, A. (1989) "Preliminary Radioastron Orbit Determination with DSN Tracking," pres. 8th RADIOASTRON Review Meeting, Green Bank, West Virginia, May 1-4.
- Kovalevsky, J., I.I. Mueller and B. Kolaczek, eds. (1989) *Reference Frames in Astronomy and Geophysics*, Kluwer Academic Publishers, 1989.
- Koyama, Y. (1989) "A Study on Systematic Error Sources for the Geocenter Position Estimation," (in Japanese with English abstract). Proc. of a domestic Workshop on "The Research of Space VLBI," Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, pp. 50-54, Nobeyama, Japan, March 14-15, 1989.
- Levy, G.S. et al., (1986) "Very Long Baseline Interferometry Observations Using An Orbiting Radio Telescope," *Science*, 234, 187-189.
- Levy, G.S. et al., (1987) "Results and Communications Considerations of the Very Long Baseline Interferometry Demonstration Using the Tracking and Data Relay Satellite System," *Acta Astronautica*, 15, 481-487.
- Levy, G.S. and 31 others (1989) "VLBI Using a Telescope in Earth Orbit. I: The Observations," *The Astrophysical J.*, 336, 1098-1104.
- Linfield, R.P. and 28 others (1988), "Very Long Baseline Interferometry Observations Using the Tracking and Data Relay Satellite As an Orbiting Radio Telescope," in *The Impact of VLBI on Astrophysics and Geophysics*, M.J. Reid and J.M. Moran, eds., Kluwer Academic Publ., Dordrecht, Holland, 457-458.
- Linfield, R.P. and 14 others (1989), "VLBI Using a Telescope in Earth Orbit: Brightness Temperatures Exceeding the Inverse Compton Limit," (NRO Rep. 203, Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, Nobeyama, Japan), *The Astrophysical J.*, 336.
- Ma, C. (1978) "Very Long Baseline Interferometry Applied to Polar Motion, Relativity and Geodesy," Ph.D. Thesis, Dept. of Physics and Astronomy, University of Maryland (NASA Technical Memorandum 79582).
- Marsh, J.G. and 17 others (1988) "An Improved Model of the Earth's Gravitational Field: *GEM-T1*," NASA Technical Memorandum 4019, March 1988.
- Memorandum of the 8th RADIOASTRON Meeting, Green Bank, West Virginia, May 1-4, 1989.
- Moritz, H. and I.I. Mueller (1987) *Earth Rotation: Theory and Observation*. Ungar Publ., New York.
- Mueller, I.I. (1988) *Reference Coordinate Systems: An Update*, Report No. 394, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus.

- Murphy, D.W. (1989) "The Imaging Potential of RADIOASTRON," pres. 8th RADIOASTRON Review Meeting, Green Bank, West Virginia, May 1-4.
- Nishimura, T. and H. Hirabayashi (1988) "Space VLBI: Science and Technology," *Control Theory and Advanced Technology*, 4, 121-135 (publ. MITA Press, Tokyo).
- Obenson, G.F.T. (1970) "Direct Evaluation of the Earth's Gravity Anomaly Field from Orbital Analysis of Artificial Earth Satellites," Report No. 129, Department of Geodetic Science and Surveying, The Ohio State University, Columbus.
- Papo, H.B. and J. Saleh (1988) "Conventional Terrestrial System by VLBI - A Kinematic Approach." *Bull. Geod.*, 62 (1988), pp.499-510.
- Pavlis, N.K. (1986) "Orbiting Very Long Baseline Interferometry," term paper for Advanced Satellite Geodesy, Dept. of Geodetic Science and Surveying, The Ohio State Univ., Columbus, p. 32.
- Press, W.H., B.P. Flannery, S.A. Teukolsky and W.T. Vetterling (1986) *Numerical Recipes—The Art of Scientific Computing*, Cambridge Uni. Press, Cambridge.
- Preston, R.A. (1983) "A VLBI Observatory in Space—An Enhancement of the VLBA," Proc. of a Workshop on Multidisciplinary Use of the Very Long Baseline Array, National Academy Press, Washington, D.C., 132-142.
- Proc. of Workshop on "QUASAT," Gr. Enzersdorf, Austria, June 18-22, 1984 (ESA SP-213, Sept. 1984.
- Proc. of a Domestic Workshop on "The Research of Space VLBI," Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, Nobeyama, Japan, March 14-15, 1989.
- Rapp, R.H. (Ed.) (1989) *Progress in the Determination of the Earth's Gravity Field*, Report No. 397, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus.
- Robertson, D.S. (1975) "Geodetic and Astronomic Measurements with Very-Long Baseline Interferometry." Ph.D. Thesis, Dept. of Earth and Planetary Sciences, M.I.T., Cambridge.
- Romney, J.D. (1988) "The Very Long Baseline Array," in *The Impact of VLBI on Astrophysics and Geophysics*, M.J. Reid and J.M. Moran, eds., Kluwer Academic Publ., Dordrecht, Holland, pp. 461-468.
- Sagdeev, R.Z. (1984) "Some Prospects of Space VLBI," Proc. of Workshop on QUASAT, Gr. Enzersdorf, Austria, June 18-22, 1984 (ESA SP-213, Sept.), pp. 19-21.
- Schaffrin, B. (1985) "Aspects of Network Design," In: *Optimization and Design of Geodetic Networks*, Grafarend, E.W. and F. Sansó (Eds.), pp. 548-597, Springer-Verlag, Berlin-Heidelberg, 1985.
- Schaffrin, B. (1990) "Advanced Adjustment Computations," Lecture Notes, Department of Geodetic Science and Surveying, The Ohio State University, Columbus, Ohio, 1990.

- Schilizzi, R.T. (1988a) "QUASAT," in *The Impact of VLBI on Astrophysics and Geophysics*, M.J. Reid and J.M. Moran, eds., Kluwer Academic Publ., Dordrecht, Holland, pp. 441-446.
- Schilizzi, R.T. (1988b) "New Developments in European VLBI," in *The Impact of VLBI on Astrophysics and Geophysics*, M.J. Reid and J.M. Moran, eds., Kluwer Academic Publ., Dordrecht, Holland, pp. 469-473.
- Schilizzi, R.T., B.F. Burke, R.S. Booth, R.A. Preston, P.N. Wilkinson, J.F. Jordan, E. Preuss and D. Roberts (1984), "The QUASAT Project," in *VLBI and Compact Radio Sources*, R. Fanti, K. Kellermann and G. Setti, eds., Proc. of the IAU Symp. No. 110, Reidel Publ. Co., Dordrecht, Holland.
- Schilizzi, R.T. and 66 others (1989) "IVS (International VLBI Satellite)" a proposal to ESA, p. 32.
- Shapiro, I.I. (1978) *Principles of Very Long Baseline Interferometry*, Proc. of the 9th GEOP Research Conference on Applications of Geodesy to Geodynamics, Reports of the Dept. of Geodetic Science, The Ohio State University, No. 280, pp. 29-33.
- Sovers, O.J. and J.L. Fanselow (1987) "Observation Model and Parameter Partial for the JPL VLBI Parameter Estimation Software "MASTERFIT—1987," *JPL Publication 83-39*, Rev. 3, Jet Propulsion Laboratory, Pasadena, California.
- Stewart, G.W. (1973) *Introduction to Matrix Computations*, Academic Press, Inc., Orlando, Florida.
- Takahashi, Y. (1989) "The Importance of the celestial reference frame for VSOP and the possibility to make the celestial reference frame by VSOP," (in Japanese with English abstract). Proc. of a domestic Workshop on The Research of Space VLBI, Nobeyama Radio Observatory of the Tokyo Astronomical Observatory, Univ. of Tokyo, pp. 55-60, Nobeyama, Japan, March 14-15, 1989.
- Tang, G. (1984) "A Short Note on the High-Precision Navigation of QUASAT," Proc. of Workshop on QUASAT, Gr. Enzersdorf, Austria, June 18-22, 1984 (ESA SP-213, Sept.), pp. 185-186.
- Teunissen, P. (1985) "Zero Order Design: Generalized Inverses, Adjustment, the Datum Problem and S-Transformations," In: *Optimization and Design of Geodetic Networks*, Grafarend, E.W. and F. Sansó (Eds), pp. 11-55, Springer-Verlag, Berlin-Heidelberg, 1985.
- Wells, D.E., W. Lindlohr, B. Schaffrin and E.W. Grafarend (1987) "GPS Design: Undifferenced Carrier Beat Phase Observations and the Fundamental Differencing Theorem," Dept. of Surveying Engineering, University of New Brunswick, Fredericton, N.B., Canada, October, 1987.
- Whitney, A.R. (1974) "Precision Geodesy and Astrometry via Very Long Baseline Interferometry," Ph.D. Thesis, Dept. of Electrical Engineering, M.I.T., Cambridge.



APPENDIX A

Keplerian—Cartesian Transformations

First, the transformation between Keplerian orbital elements $\{a, e, i, \omega, \Omega, M\}$ and Cartesian orbital (rectangular) coordinates $\{X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}\}$ is considered. This is followed by the description of partial derivatives of the Cartesian orbital coordinates with respect to the Keplerian elements and partial derivatives of the Keplerian elements with respect to the Cartesian elements.

A 1. Kepler Transformation of Coordinate and Velocity Type

Transformation of Kepler elements $\{a, e, i, \omega, \Omega, M\}$ into inertial Cartesian position vector \mathbf{r} :

$$\mathbf{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = r \begin{bmatrix} \cos \Omega \cos u - \sin \Omega \sin u \cos i \\ \sin \Omega \cos u + \cos \Omega \sin u \cos i \\ \sin u \sin i \end{bmatrix}, \quad (\text{A.1})$$

where

$$r = a (1 - e \cos E) \quad (\text{A.2})$$

$$u = \omega + f \quad (\text{A.3})$$

$$\text{tg } f/2 = \sqrt{(1+e)/(1-e)} \text{ tg } E/2 \quad (\text{A.4})$$

$$M = E - e \sin E \quad (\text{A.5})$$

$$\sin f = \sqrt{1-e^2} \sin E / (1 - e \cos E) \quad (\text{A.6})$$

$$\cos f = (\cos E - e) / (1 - e \cos E) \quad (\text{A.7})$$

E is the eccentric anomaly.

Transformation of Kepler elements {a, e, i, ω, Ω, M} into inertial Cartesian velocity vector $\dot{\mathbf{r}}$:

$$\dot{\mathbf{r}} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = v_r \begin{bmatrix} \cos \Omega \cos u - \sin \Omega \sin u \cos i \\ \sin \Omega \cos u + \cos \Omega \sin u \cos i \\ \sin u \sin i \end{bmatrix} - v_u \begin{bmatrix} \cos \Omega \sin u + \sin \Omega \cos u \cos i \\ \sin \Omega \sin u - \cos \Omega \cos u \cos i \\ -\cos u \sin i \end{bmatrix} \quad (\text{A.8})$$

where

$$v_r = \sqrt{\mu/p} e \sin f \quad (\text{A.9})$$

$$v_u = \sqrt{\mu/p} (1 + e \cos f) \quad (\text{A.10})$$

$$p = a (1 - e^2) \quad (\text{A.11})$$

μ is the gravitational constant times the mass of the Earth.

An equivalent conversion from Keplerian elements to inertial Cartesian position and velocity vectors (\mathbf{r} and $\dot{\mathbf{r}}$):

$$\mathbf{q} = \frac{a(1-e^2)}{(1+e \cos f)} \begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix} = \begin{bmatrix} a \cos E - a e \\ a \sqrt{1-e^2} \sin E \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{q}} = \frac{n a}{(1-e \cos E)} \begin{bmatrix} -\sin E \\ \sqrt{1-e^2} \cos E \\ 0 \end{bmatrix}$$

$$\mathbf{R}_{x_q} = \mathbf{R}_3(-\Omega) \cdot \mathbf{R}_1(-i) \cdot \mathbf{R}_3(-\omega)$$

$$\mathbf{r} = \mathbf{R}_{x_q} \mathbf{q}$$

$$\dot{\mathbf{r}} = \mathbf{R}_{x_q} \dot{\mathbf{q}}$$

where n is the mean motion.

A 2. Conversion from Inertial Cartesian Position and Velocity Vectors to Keplerian Elements (Inverse Kepler Transformation)

Compute radial distance r and tangential velocity v :

$$r = |\mathbf{r}| = \sqrt{X^2 + Y^2 + Z^2} \quad (\text{A.17})$$

$$v = |\dot{\mathbf{r}}| = \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2} \quad (\text{A.18})$$

Compute angular momentum vector \mathbf{h} :

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = \begin{bmatrix} Y\dot{Z} - Z\dot{Y} \\ Z\dot{X} - X\dot{Z} \\ X\dot{Y} - Y\dot{X} \end{bmatrix} \quad \left\{ = \begin{bmatrix} h_X \\ h_Y \\ h_Z \end{bmatrix} \right\} \quad (\text{A.19})$$

$$h = \sqrt{h_X^2 + h_Y^2 + h_Z^2} \quad (\text{A.20})$$

Compute inclination angle i :

$$i = \cos^{-1} \left(\frac{h_Z}{h} \right) \quad (\text{A.21})$$

Compute ellipse semi-major axis a :

$$v^2 = GM \left[\frac{2}{r} - \frac{1}{a} \right] \quad (\text{A.22})$$

$$a = \left[\frac{2}{r} - \frac{v^2}{GM} \right]^{-1} \quad (\text{A.23})$$

Compute ellipse eccentricity e :

$$h^2 = GM a (1 - e^2) \quad (\text{A.24})$$

$$e = \left[1 - \left[\frac{h^2}{aGM} \right] \right]^{1/2} \quad (\text{A.25})$$

Compute longitude of ascending node Ω :

$$\Omega = \tan^{-1} \left(\frac{h_X}{-h_Y} \right) \quad (\text{A.26})$$

Compute true anomaly f :

$$f = \tan^{-1} \left(\frac{\sin f}{\cos f} \right), \quad (\text{A.27})$$

where

$$\cos f = \frac{a(1-e^2)}{er} - \frac{1}{e} \quad (\text{A.28})$$

$$\sin f = \frac{a(1-e^2)(\mathbf{r} \cdot \dot{\mathbf{r}})}{reh} \quad (\text{A.29})$$

Compute eccentric anomaly E :

$$E = \tan^{-1} \left(\frac{\sin E}{\cos E} \right) \quad (\text{A.30})$$

where

$$\cos E = \frac{\cos f + e}{1 + e \cos f} \quad (\text{A.31})$$

$$\sin E = \frac{\sqrt{1-e^2} \sin f}{1 + e \cos f} \quad (\text{A.32})$$

Use Kepler's equation

$$M = E - e \sin E \quad (\text{A.33})$$

to compute the mean anomaly M .

Compute geocentric angle u :

$$u = \tan^{-1} \left(\frac{\sin u}{\cos u} \right) \quad (\text{A.34})$$

where

$$\cos u = \frac{X \cos \Omega + Y \sin \Omega}{r} \quad (\text{A.35})$$

$$\sin u = \frac{(Y \cos \Omega - X \sin \Omega) \cos i + Z \sin i}{r} \quad (\text{A.36})$$

Compute argument of perigee ω :

$$\omega = u - f \quad (\text{A.37})$$

A3. Partial Derivatives of Inertial Cartesian Coordinates and Velocity Components with respect to Keplerian Elements (Linear Kepler transformation)

Elements of the linear Kepler transformation matrix of position type:

$$\frac{\partial X}{\partial a} = (1 - e \cos E) (\cos \Omega \cos u - \sin \Omega \sin u \cos i) \quad (\text{A.38})$$

$$\frac{\partial Y}{\partial a} = (1 - e \cos E) (\sin \Omega \cos u + \cos \Omega \sin u \cos i) \quad (\text{A.39})$$

$$\frac{\partial Z}{\partial a} = (1 - e \cos E) \sin u \sin i \quad (\text{A.40})$$

$$\begin{aligned} \frac{\partial X}{\partial e} = & [a(e - \cos E)/(1 - e \cos E)] (\cos \Omega \cos u - \sin \Omega \sin u \cos i) - \\ & - a \sin f \left(1 + \frac{1 - e \cos E}{1 - e^2} \right) (\cos \Omega \sin u + \sin \Omega \cos u \cos i) \end{aligned} \quad (\text{A.41})$$

$$\begin{aligned} \frac{\partial Y}{\partial e} = & [a(e - \cos E)/(1 - e \cos E)] (\cos \Omega \cos u + \sin \Omega \sin u \cos i) - \\ & - a \sin f \left(1 + \frac{1 - e \cos E}{1 - e^2} \right) (\sin \Omega \sin u - \cos \Omega \cos u \cos i) \end{aligned} \quad (\text{A.42})$$

$$\frac{\partial Z}{\partial e} = [a(e - \cos E)/(1 - e \cos E)] \sin u \sin i + a \sin f \left(1 + \frac{1 - e \cos E}{1 - e^2} \right) \cos u \sin i \quad (\text{A.43})$$

$$\frac{\partial X}{\partial \omega} = -r (\cos \Omega \sin u + \sin \Omega \cos u \cos i) \quad (\text{A.44})$$

$$\frac{\partial Y}{\partial \omega} = -r (\sin \Omega \sin u - \cos \Omega \cos u \cos i) \quad (\text{A.45})$$

$$\frac{\partial Z}{\partial \omega} = r \cos u \sin i \quad (\text{A.46})$$

$$\frac{\partial X}{\partial i} = r \sin \Omega \sin u \sin i \quad (\text{A.47})$$

$$\frac{\partial Y}{\partial i} = -r \cos \Omega \sin u \sin i \quad (\text{A.48})$$

$$\frac{\partial Z}{\partial i} = r \sin u \cos i \quad (\text{A.49})$$

$$\frac{\partial X}{\partial \Omega} = -r (\sin \Omega \cos u + \cos \Omega \sin u \cos i) \quad (\text{A.50})$$

$$\frac{\partial Y}{\partial \Omega} = r (\cos \Omega \cos u - \sin \Omega \sin u \cos i) \quad (\text{A.51})$$

$$\frac{\partial Z}{\partial \Omega} = \emptyset \quad (\text{A.52})$$

$$\begin{aligned} \frac{\partial X}{\partial M} &= [a e \sin E / (1 - e \cos E)] (\cos \Omega \cos u - \sin \Omega \sin u \cos i) - \\ &\quad - [a \sin f / \sin E] (\cos \Omega \sin u + \sin \Omega \cos u \cos i) \end{aligned} \quad (\text{A.53})$$

$$\begin{aligned} \frac{\partial Y}{\partial M} &= [a e \sin E / (1 - e \cos E)] (\sin \Omega \cos u + \cos \Omega \sin u \cos i) - \\ &\quad - [a \sin f / \sin E] (\sin \Omega \sin u - \cos \Omega \cos u \cos i) \end{aligned} \quad (\text{A.54})$$

$$\frac{\partial Z}{\partial M} = [a e \sin E / (1 - e \cos E)] \sin u \sin i + [a \sin f / \sin E] \cos u \sin i \quad (\text{A.55})$$

Elements of the linear Kepler transformation matrix of velocity type:

$$\frac{\partial \dot{X}}{\partial a} = -\frac{v_r}{2a} (\cos \Omega \cos u - \sin \Omega \sin u \cos i) + \frac{v_u}{2a} (\cos \Omega \sin u + \sin \Omega \cos u \cos i) \quad (\text{A.56})$$

$$\frac{\partial \dot{Y}}{\partial a} = -\frac{v_r}{2a} (\sin \Omega \cos u + \cos \Omega \sin u \cos i) + \frac{v_u}{2a} (\sin \Omega \sin u - \cos \Omega \cos u \cos i) \quad (\text{A.57})$$

$$\frac{\partial \dot{Z}}{\partial a} = -\left(\frac{v_r}{2a} \sin u \sin i + \frac{v_u}{2a} \cos u \sin i \right) \quad (\text{A.58})$$

$$\begin{aligned}\frac{\partial \dot{X}}{\partial e} = & -v_r \frac{1}{e(1-e \cos E)} (\cos \Omega \cos u - \sin \Omega \cos i \sin u) - \\ & -v_u \frac{e + \cos f}{(1-e^2)(1+e \cos f)} (\cos \Omega \sin u + \sin \Omega \cos i \cos u)\end{aligned}\quad (\text{A.59})$$

$$\begin{aligned}\frac{\partial \dot{Y}}{\partial e} = & -v_r \frac{1}{e(1-e \cos E)} (\sin \Omega \cos u + \cos \Omega \sin u \cos i) - \\ & -v_u \frac{e + \cos f}{(1-e^2)(1+e \cos f)} (\sin \Omega \sin u - \cos \Omega \cos u \cos i)\end{aligned}\quad (\text{A.60})$$

$$\begin{aligned}\frac{\partial \dot{Z}}{\partial e} = & -v_r \frac{1}{e(1-e \cos E)} \sin u \sin i \\ & + v_u \frac{e + \cos f}{(1-e^2)(1+e \cos f)} \cos u \sin i\end{aligned}\quad (\text{A.61})$$

$$\frac{\partial \dot{X}}{\partial \omega} = -v_r (\cos \Omega \sin u + \sin \Omega \cos u \cos i) - v_u (\cos \Omega \cos u - \sin \Omega \sin u \cos i) \quad (\text{A.62})$$

$$\frac{\partial \dot{Y}}{\partial \omega} = -v_r (\sin \Omega \sin u - \cos \Omega \cos u \cos i) - v_u (\sin \Omega \cos u + \cos \Omega \sin u \cos i) \quad (\text{A.63})$$

$$\frac{\partial \dot{Z}}{\partial \omega} = v_r \cos u \sin i - v_u \sin u \sin i \quad (\text{A.64})$$

$$\frac{\partial \dot{X}}{\partial i} = v_r \sin \Omega \sin u \sin i + v_u \sin \Omega \cos u \sin i \quad (\text{A.65})$$

$$\frac{\partial \dot{Y}}{\partial i} = -v_r \cos u \sin u \sin i - v_u \cos \Omega \cos u \sin i \quad (\text{A.66})$$

$$\frac{\partial \dot{Z}}{\partial i} = v_r \sin u \cos i + v_u \cos u \cos i \quad (\text{A.67})$$

$$\frac{\partial \dot{X}}{\partial \Omega} = -v_r (\sin \Omega \cos u + \cos \Omega \sin u \cos i) + v_u (\sin \Omega \sin u - \cos \Omega \cos u \cos i) \quad (\text{A.68})$$

$$\frac{\partial \dot{Y}}{\partial \Omega} = v_r (\cos \Omega \cos u - \sin \Omega \sin u \cos i) - v_u (\cos \Omega \sin u + \sin \Omega \cos u \cos i) \quad (\text{A.69})$$

$$\frac{\partial \dot{Z}}{\partial \Omega} = 0 \quad (\text{A.70})$$

$$\frac{\partial \dot{X}}{\partial M} = \frac{v_r \cos f - v_u \sin f}{\sin E (1 - e \cos E)} (\cos \Omega \cos u - \sin \Omega \sin u \cos i) \quad (\text{A.71})$$

$$\frac{\partial \dot{Y}}{\partial M} = \frac{v_r \cos f - v_u \sin f}{\sin E (1 - e \cos E)} (\sin \Omega \cos u + \cos \Omega \sin u \cos i) \quad (\text{A.72})$$

$$\frac{\partial \dot{Z}}{\partial M} = \frac{v_r \cos f - v_u \sin f}{\sin E (1 - e \cos E)} \sin u \sin i \quad (\text{A.73})$$

A 4. Partial Derivatives of Keplerian Elements with respect to Inertial Cartesian Coordinates and Velocity Components (Inverse Linear Kepler Transformation)

Notation:

$$s = X, Y, Z, \dot{X}, \dot{Y}, \dot{Z} \quad \text{respectively} \quad (\text{A.74})$$

Partial derivative of the radial component:

$$\frac{\partial r}{\partial s} = \frac{X}{r}, \frac{Y}{r}, \frac{Z}{r}, 0, 0, 0 \quad \text{respectively} \quad (\text{A.75})$$

Partial derivative of the velocity component:

$$\frac{\partial v}{\partial s} = 0, 0, 0, \frac{\dot{X}}{v}, \frac{\dot{Y}}{v}, \frac{\dot{Z}}{v} \quad \text{respectively} \quad (\text{A.76})$$

Partial derivative of the angular momentum h :

$$\frac{\partial h}{\partial s} = \frac{1}{h} \left(h_X \frac{\partial h_X}{\partial s} + h_Y \frac{\partial h_Y}{\partial s} + h_Z \frac{\partial h_Z}{\partial s} \right), \quad (\text{A.77})$$

where

$$\frac{\partial h_X}{\partial s} = 0, \dot{Z}, -\dot{Y}, 0, -Z, Y \quad \text{respectively,} \quad (\text{A.78})$$

$$\frac{\partial h_Y}{\partial s} = -\dot{Z}, 0, \dot{X}, Z, 0, -X \quad \text{respectively,} \quad (\text{A.79})$$

$$\frac{\partial h_Z}{\partial s} = \dot{Y}, -\dot{X}, 0, -Y, X, 0 \quad \text{respectively.} \quad (\text{A.80})$$

Partial derivative of $(\mathbf{r} \cdot \dot{\mathbf{r}})$:

$$\mathbf{r} \cdot \dot{\mathbf{r}} = X\dot{X} + Y\dot{Y} + Z\dot{Z} \quad (\text{A.81})$$

$$\frac{\partial (\mathbf{r} \cdot \dot{\mathbf{r}})}{\partial s} = \dot{X}, \dot{Y}, \dot{Z}, X, Y, Z \quad \text{respectively} \quad (\text{A.82})$$

Partial derivative of the inclination i :

$$\frac{\partial i}{\partial s} = \frac{1}{h^2} \left(\frac{h_X h_Z \frac{\partial h_X}{\partial s} + h_Y h_Z \frac{\partial h_Y}{\partial s} - (h_X^2 + h_Y^2) \frac{\partial h_Z}{\partial s}}{(h_X^2 + h_Y^2)^{1/2}} \right) \quad (\text{A.83})$$

Partial derivative of the semi-major axis a :

$$\frac{\partial a}{\partial s} = a^2 \left(\frac{2}{r^2} \frac{\partial r}{\partial s} + \frac{2v}{GM} \frac{\partial v}{\partial s} \right) \quad (\text{A.84})$$

$$\frac{\partial a}{\partial s} = \begin{cases} \frac{2a^2}{r^2} \frac{\partial r}{\partial s}, & s = X, Y, Z \\ \frac{2a^2}{GM} v \frac{\partial v}{\partial s}, & s = \dot{X}, \dot{Y}, \dot{Z} \end{cases} \quad (\text{A.85})$$

Partial derivative of the eccentricity e :

$$\frac{\partial e}{\partial s} = \frac{1 - e^2}{2ae} \frac{\partial a}{\partial s} - \frac{1 - e^2}{he} \frac{\partial h}{\partial s} \quad (\text{A.86})$$

Partial derivative of the ascending node Ω :

$$\frac{\partial \Omega}{\partial s} = \frac{1}{h_X^2 + h_Y^2} \left(h_X \frac{\partial h_Y}{\partial s} - h_Y \frac{\partial h_X}{\partial s} \right) \quad (\text{A.87})$$

Partial derivative of the true anomaly f :

$$\begin{aligned} \frac{\partial f}{\partial s} = & \frac{1 - e^2}{re} \left[\frac{\cos f}{h} (\mathbf{r} \cdot \dot{\mathbf{r}}) - \sin f \right] \frac{\partial a}{\partial s} \\ & - 2 \frac{a}{r} \left[\frac{\cos f}{h} (\mathbf{r} \cdot \dot{\mathbf{r}}) - \sin f \right] \frac{\partial e}{\partial s} \\ & - \frac{a(1 - e^2)}{reh} \left[\frac{\cos f}{h} (\mathbf{r} \cdot \dot{\mathbf{r}}) \right] \frac{\partial h}{\partial s} \\ & + \frac{a(1 - e^2)}{reh} [\cos f] \frac{\partial (\mathbf{r} \cdot \dot{\mathbf{r}})}{\partial s} \\ & + \frac{\sin f}{re} \frac{\partial r}{\partial s} \end{aligned} \quad (\text{A.88})$$

Partial derivative of the eccentric anomaly E :

$$\frac{\partial E}{\partial s} = \frac{\sqrt{1 - e^2}}{1 + e \cos f} \left[\frac{\partial f}{\partial s} - \frac{\sin f}{1 - e^2} \frac{\partial e}{\partial s} \right] \quad (\text{A.89})$$

Partial derivative of the mean anomaly M :

$$\frac{\partial M}{\partial s} = \frac{r^2}{a^2 \sqrt{1 - e^2}} \frac{\partial f}{\partial s} - \left[\frac{r}{a^2 \sqrt{1 - e^2}} + 1 \right] \sin E \frac{\partial e}{\partial s} \quad (\text{A.90})$$

Partial derivative of u :

$$\begin{aligned}
\frac{\partial u}{\partial s} = \frac{1}{r} \{ & [-Y \sin \Omega \cos i \cos u - X \cos \Omega \cos i \cos u \\
& + X \sin \Omega \sin u - Y \cos \Omega \sin u] \frac{\partial \Omega}{\partial s} \\
+ [X \sin \Omega \sin i \cos u - Y \cos \Omega \sin i \cos u + Z \cos i \cos u] \frac{\partial i}{\partial s} \\
& + [-\sin \Omega \cos i \cos u - \cos \Omega \sin u] \frac{\partial X}{\partial s} \\
& + [\cos \Omega \cos i \cos u - \sin \Omega \sin u] \frac{\partial Y}{\partial s} \\
& + [\sin i \cos u] \frac{\partial Z}{\partial s} \}
\end{aligned} \tag{A.91}$$

Partial derivative of the argument of perigee ω :

$$\frac{\partial \omega}{\partial s} = \frac{\partial u}{\partial s} - \frac{\partial f}{\partial s} \tag{A.92}$$

A5. Let the matrix \mathbf{K}_c representing the linear Kepler transformation matrix whose elements are the partial derivative of the Cartesian orbital elements with respect to the Keplerian elements:

$$\mathbf{K}_c = \begin{bmatrix}
\frac{\partial X}{\partial a} & \frac{\partial X}{\partial e} & \frac{\partial X}{\partial i} & \frac{\partial X}{\partial \omega} & \frac{\partial X}{\partial \Omega} & \frac{\partial X}{\partial M} \\
\frac{\partial Y}{\partial a} & \frac{\partial Y}{\partial e} & \frac{\partial Y}{\partial i} & \frac{\partial Y}{\partial \omega} & \frac{\partial Y}{\partial \Omega} & \frac{\partial Y}{\partial M} \\
\frac{\partial Z}{\partial a} & \frac{\partial Z}{\partial e} & \frac{\partial Z}{\partial i} & \frac{\partial Z}{\partial \omega} & \frac{\partial Z}{\partial \Omega} & \frac{\partial Z}{\partial M} \\
\frac{\partial \dot{X}}{\partial a} & \frac{\partial \dot{X}}{\partial e} & \frac{\partial \dot{X}}{\partial i} & \frac{\partial \dot{X}}{\partial \omega} & \frac{\partial \dot{X}}{\partial \Omega} & \frac{\partial \dot{X}}{\partial M} \\
\frac{\partial \dot{Y}}{\partial a} & \frac{\partial \dot{Y}}{\partial e} & \frac{\partial \dot{Y}}{\partial i} & \frac{\partial \dot{Y}}{\partial \omega} & \frac{\partial \dot{Y}}{\partial \Omega} & \frac{\partial \dot{Y}}{\partial M} \\
\frac{\partial \dot{Z}}{\partial a} & \frac{\partial \dot{Z}}{\partial e} & \frac{\partial \dot{Z}}{\partial i} & \frac{\partial \dot{Z}}{\partial \omega} & \frac{\partial \dot{Z}}{\partial \Omega} & \frac{\partial \dot{Z}}{\partial M}
\end{bmatrix} \tag{A.93}$$

Let the partial derivatives of the Keplerian orbital elements with respect to the Cartesian elements be the elements of the inverse linear Kepler transformation matrix \mathbf{K}^c in the following form:

$$\mathbf{K}^c = \begin{bmatrix} \frac{\partial a}{\partial X} & \frac{\partial a}{\partial Y} & \frac{\partial a}{\partial Z} & \frac{\partial a}{\partial \dot{X}} & \frac{\partial a}{\partial \dot{Y}} & \frac{\partial a}{\partial \dot{Z}} \\ \frac{\partial e}{\partial X} & \frac{\partial e}{\partial Y} & \frac{\partial e}{\partial Z} & \frac{\partial e}{\partial \dot{X}} & \frac{\partial e}{\partial \dot{Y}} & \frac{\partial e}{\partial \dot{Z}} \\ \frac{\partial i}{\partial X} & \frac{\partial i}{\partial Y} & \frac{\partial i}{\partial Z} & \frac{\partial i}{\partial \dot{X}} & \frac{\partial i}{\partial \dot{Y}} & \frac{\partial i}{\partial \dot{Z}} \\ \frac{\partial \omega}{\partial X} & \frac{\partial \omega}{\partial Y} & \frac{\partial \omega}{\partial Z} & \frac{\partial \omega}{\partial \dot{X}} & \frac{\partial \omega}{\partial \dot{Y}} & \frac{\partial \omega}{\partial \dot{Z}} \\ \frac{\partial \Omega}{\partial X} & \frac{\partial \Omega}{\partial Y} & \frac{\partial \Omega}{\partial Z} & \frac{\partial \Omega}{\partial \dot{X}} & \frac{\partial \Omega}{\partial \dot{Y}} & \frac{\partial \Omega}{\partial \dot{Z}} \\ \frac{\partial M}{\partial X} & \frac{\partial M}{\partial Y} & \frac{\partial M}{\partial Z} & \frac{\partial M}{\partial \dot{X}} & \frac{\partial M}{\partial \dot{Y}} & \frac{\partial M}{\partial \dot{Z}} \end{bmatrix} \quad (\text{A.94})$$

Then

$$\mathbf{K}_c \mathbf{K}^c = \mathbf{K}^c \mathbf{K}_c = \mathbf{I}. \quad (\text{A.95})$$

where \mathbf{K}_c and \mathbf{K}^c are the Jacobians of transformation from Keplerian orbital elements into Cartesian coordinates respectively from Cartesian coordinates into Keplerian orbital elements.

In order to be safe from the unavoidable typing errors in the relations given in this Appendix, the program listing of these relations is given in Appendix B for a comparison and practical use. The program was tested for numerical data. The output values of a test run are also given in Appendix B.

APPENDIX B

Fortran Program DOKEPLER for Keplerian-Cartesian Transformations

The program is written in Fortran 77 and consists, beside the main program unit, of two subroutines. The main program unit reads the input data, organizes the work of the subroutines and creates two new data files. The subroutines are as follows:

1) Subroutine LIKEPTRA calculates the Cartesian position and velocity components of the satellite and their partial derivatives with respect to the Keplerian orbital elements, provided the Keplerian orbital elements (a , e , i , ω , Ω , M , and f,E) and the Earth's gravitational constant, GM are known. The output values are the Cartesian satellite coordinates X,Y,Z and velocity components \dot{X},\dot{Y},\dot{Z} as well as the elements of the 6×6 Jacobian K_C of transformation from Keplerian orbital elements into Cartesian coordinates.

2) Subroutine INKEPTRA calculates the Keplerian orbital elements (a , e , i , ω , Ω , M , and f,E) together with the elements of the 6×6 Jacobian K^C of transformation from Cartesian coordinates into Keplerian orbital elements. The input data, which should be defined before calling the subroutine, are the rectangular components of position and velocity of the satellite and the Earth's gravitational constant.

The program calculates the elements of both multiplication matrices of the Jacobians, K_C and K^C in order to check Eq. (A.95) in Appendix A.

After the listing of the program and the subroutines, this Appendix contains numerical examples for Keplerian-Cartesian Transformations and for checking the multiplication products of the Jacobians. Instantaneous VSOP satellite position and velocity components respectively Keplerian orbital elements at epoch MJD 50083.000 (or GAST = 105854.567125788) have been used.

```

* *****
*          ***** PROGRAM DOKEPLER *****
* TO COMPUTE THE KEPLERIAN-CARTESIAN TRANSFORMATIONS AND THE
* ELEMENTS OF THE CORRESPONDING LINEAR KEPLER TRANSFORMATION
* MATRICES (KC,KCI) USING SUBROUTINES LIKEPTRA AND INKEPTRA
*          FINAL PROGRAM TESTED FOR DATA
* *****
IMPLICIT REAL*8(A-H,O-Z)
DOUBLE PRECISION KC(6,6),KCI(6,6)
DIMENSION UN(6,6),UNI(6,6)
OPEN(UNIT=1,FILE='A:¢DATA¢KEPLERCA.DAT')
OPEN(UNIT=2,FILE='A:¢DATA¢KEPLERCA.OUT')
OPEN(UNIT=3,FILE='A:¢DATA¢KEPLERUN.OUT')
READ(1,*)AO,EO,XIO
READ(1,*)OMO,OME0,XMO
READ(1,*)FO,EAO
READ(1,*)XO,YO,ZO
READ(1,*)XDO,YDO,ZDO
CALL LIKEPTRA(AO,EO,XIO,OMO,OME0,XMO,FO,EAO,XS,YS,ZS,XSD,
*YSD,ZSD,KC)
WRITE(2,6)XS,YS,ZS
WRITE(2,6)XSD,YSD,ZSD
6  FORMAT(3(1X,F23.11))
DO 1 I=1,6
WRITE(2,7) (KC(I,J),J=1,3)
WRITE(2,7) (KC(I,J),J=4,6)
1  CONTINUE
7  FORMAT(3(1X,F23.13))
CALL INKEPTRA(XO,YO,ZO,XDO,YDO,ZDO,AS,ES,XIS,
*OMS,OMES,XMS,FS,EAS,KCI)
WRITE(2,*)AS,ES,XIS
WRITE(2,*)OMS,OMES,XMS
WRITE(2,*)FS,EAS
DO 2 I=1,6
WRITE(2,8) (KCI(I,J),J=1,3)
WRITE(2,8) (KCI(I,J),J=4,6)
2  CONTINUE
8  FORMAT(3(1X,F23.17))
CALL MATMULT(KC,KCI,UN,6,6,6)
DO 3 I=1,6
WRITE(3,4) (UN(I,J),J=1,3)
WRITE(3,4) (UN(I,J),J=4,6)
3  CONTINUE
4  FORMAT(3(1X,F23.20))
CALL MATMULT(KCI,KC,UNI,6,6,6)
DO 5 I=1,6
WRITE(3,4) (UNI(I,J),J=1,3)
WRITE(3,4) (UNI(I,J),J=4,6)
5  CONTINUE
CLOSE(1)
CLOSE(2)
CLOSE(3)
STOP
END
CSINCLUDE A:¢SVLBI¢LIKEPTRA
CSINCLUDE A:¢SVLBI¢INKEPTRA
CSINCLUDE A:¢SVLBI¢MATMULT

```

```

* ***** SUBROUTINE LIKEPTRA ***** *
* TO CONVERT THE KEPLERIAN ELEMENTS INTO SATELLITE'S CARTESIAN *
* POSITION AND VELOCITY VECTORS AND TO COMPUTE THE ELEMENTS OF *
* THE LINEAR KEPLER TRANSFORMATION MATRIX (KC) *
* ***** *
SUBROUTINE LIKEPTRA(A,E,XI,OM,OME,XM,F,EA,X,Y,Z,XD,YD,ZD,KC)
IMPLICIT REAL*8(A-H,O-Z)
DOUBLE PRECISION KC(6,6)
GM=398600.436D9
C TO COMPUTE THE COEFFICIENTS
U=OM+F
COM=DCOS(OME)
SOM=DSIN(OME)
SU=DSIN(U)
CU=DCOS(U)
SI=DSIN(XI)
CI=DCOS(XI)
SF=DSIN(F)
CF=DCOS(F)
SE=DSIN(EA)
CE=DCOS(EA)
P=A*(1.0-E*E)
R=A*(1.0-E*CE)
VR=E*SF*DSQRT(GM/P)
VU=(1.0+E*CF)*DSQRT(GM/P)
C TO COMPUTE SATELLITE COORDINATES
X=R*(COM*CU-SOM*SU*CI)
Y=R*(SOM*CU+COM*SU*CI)
Z=R*SU*SI
C TO COMPUTE SATELLITE VELOCITY COMPONENTS
XD=VR*(COM*CU-SOM*SU*CI)-VU*(COM*SU+SOM*CU*CI)
YD=VR*(SOM*CU+COM*SU*CI)-VU*(SOM*SU-COM*CU*CI)
ZD=VR*SU*SI+VU*CU*SI
C TO COMPUTE THE COEFFICIENTS OF THE PARTIALS
A1=(A*(E-CE))/(1.0-E*CE)
A2=1.0+(1.0-E*CE)/(1.0-E*E)
A3=(A*E*SE)/(1.0-E*CE)
A4=-1.0/(E*(1.0-E*CE))
A5=(E+CF)/((1.0-E*E)*(1.0+E*CF))
A6=(VR*CF-VU*SF)/(SE*(1.0-E*CE))
C TO COMPUTE THE PARTIALS of X
KC(1,1)=(1.0-E*CE)*(COM*CU-SOM*SU*CI)
KC(1,2)=A1*(COM*CU-SOM*SU*CI)-A*SF*A2*(COM*SU+SOM*CU*CI)
KC(1,3)=R*SOM*SU*SI
KC(1,4)=-R*(COM*SU+SOM*CU*CI)
KC(1,5)=-R*(SOM*CU+COM*SU*CI)
KC(1,6)=A3*(COM*CU-SOM*SU*CI)-(A*SF/SE)*(COM*SU+SOM*CU*CI)
C TO COMPUTE THE PARTIALS of Y
KC(2,1)=(1.0-E*CE)*(SOM*CU+COM*SU*CI)
KC(2,2)=A1*(SOM*CU+COM*SU*CI)-A*SF*A2*(SOM*SU-COM*CU*CI)
KC(2,3)=-R*COM*SU*SI
KC(2,4)=-R*(SOM*SU-COM*CU*CI)
KC(2,5)=R*(COM*CU-SOM*SU*CI)
KC(2,6)=A3*(SOM*CU+COM*SU*CI)-(A*SF/SE)*(SOM*SU-COM*CU*CI)
C TO COMPUTE THE PARTIALS of Z
KC(3,1)=(1.0-E*CE)*SU*SI
KC(3,2)=A1*SU*SI+A*SF*A2*CU*SI
KC(3,3)=R*SU*CI
KC(3,4)=R*CU*SI

```

```

      KC(3,5)=0.0
      KC(3,6)=A3*SU*SI+(A*SF/SE)*CU*SI
C TO COMPUTE THE PARTIALS of XD
      KC(4,1)=(-VR*(COM*CU-SOM*SU*CI)+VU*(COM*SU+SOM*CU*CI))/(2.0*A)
      KC(4,2)=VR*A4*(COM*CU-SOM*CI*SU)-VU*A5*(COM*SU+SOM*CI*CU)
      KC(4,3)=VR*SOM*SU*SI+VU*SOM*CU*SI
      KC(4,4)=-VR*(COM*SU+SOM*CU*CI)-VU*(COM*CU-SOM*SU*CI)
      KC(4,5)=-VR*(SOM*CU+COM*SU*CI)+VU*(SOM*SU-COM*CU*CI)
      KC(4,6)=A6*(COM*CU-SOM*SU*CI)
C TO COMPUTE THE PARTIALS of YD
      KC(5,1)=(-VR*(SOM*CU+COM*SU*CI)+VU*(SOM*SU-COM*CU*CI))/(2.0*A)
      KC(5,2)=VR*A4*(SOM*CU+COM*CI*SU)-VU*A5*(SOM*SU-COM*CI*CU)
      KC(5,3)=-VR*COM*SU*SI-VU*COM*CU*SI
      KC(5,4)=-VR*(SOM*SU-COM*CU*CI)-VU*(SOM*CU+COM*SU*CI)
      KC(5,5)=VR*(COM*CU-SOM*SU*CI)-VU*(COM*SU+SOM*CU*CI)
      KC(5,6)=A6*(SOM*CU+COM*SU*CI)
C TO COMPUTE THE PARTIALS of ZD
      KC(6,1)=(-VR*SU*SI-VU*CU*SI)/(2.0*A)
      KC(6,2)=VR*A4*SU*SI+VU*A5*CU*SI
      KC(6,3)=VR*SU*CI+VU*CU*CI
      KC(6,4)=VR*CU*SI-VU*SU*SI
      KC(6,5)=0.0
      KC(6,6)=A6*SU*SI
      RETURN
      END

```

```

* *****
*          ***** SUBROUTINE INKEPTRA *****
* TO CONVERT THE SATELLITE'S CARTESIAN POSITION AND VELOCITY
* VECTORS INTO KEPLERIAN ORBITAL ELEMENTS AND TO COMPUTE THE
* ELEMENTS OF THE INVERZ LINEAR KEPLER TRANSFORMATION MATRIX
* *****
      SUBROUTINE INKEPTRA(X,Y,Z,XD,YD,ZD,A,E,XI,OM,OME,XM,F,EA,KCI)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION KCI(6,6)
      GM=398600.436D9
      R=DSQRT(X**2+Y**2+Z**2)
      V=DSQRT(XD**2+YD**2+ZD**2)
      RRD=X*XD+Y*YD+Z*ZD
C MOMENTUM VECTOR
      HX=Y*ZD-Z*YD
      HY=Z*XD-X*ZD
      HZ=X*YD-Y*XD
      H=DSQRT(HX**2+HY**2+HZ**2)
C INCLINATION (i)
      XI=DACOS(HZ/H)
C SEMI-MAJOR AXIS (a)
      A=(R*GM)/(2.0D0*GM-R*V*V)
C ECCENTRICITY (e)
      E=DSQRT(1.0D0-(H*H)/(A*GM))
C LONGITUDE OF ASCENDING NODE (OMEGA)
      OME=DATAN2(HX,-HY)
C TRUE ANOMALY (f)
      COSF=(A*(1.0-E*E)/(E*R))-(1.0D0/E)
      SINF=A*(1.0D0-E*E)*RRD/(R*E*H)
      F=DATAN2(SINF,COSF)
C ECCENTRIC ANOMALY (E)
      COSEA=(COSF+E)/(1.0D0+E*COSF)
      SINEA=(DSQRT(1.0D0-E*E))*SINF/(1.0D0+E*COSF)

```

```

EA=DATAN2(SINEA, COSEA)
C MEAN ANOMALY (M)
XM=EA-E*SINEA
C GEOCENTRIC ANGLE (U)
COSU=(X*DCOS(OME)+Y*DSIN(OME))/R
SINU=((Y*DCOS(OME)-X*DSIN(OME))*DCOS(XI)+Z*DSIN(XI))/R
U=DATAN2(SINU, COSU)
C ARGUMENT OF PERIGEE (omega)
OM=U-F
C PARTIALS OF A (semi-major axis)
KCI(1,1)=2.0*(A**2)*X/(R**3)
KCI(1,2)=2.0*(A**2)*Y/(R**3)
KCI(1,3)=2.0*(A**2)*Z/(R**3)
KCI(1,4)=2.0*(A**2)*XD/GM
KCI(1,5)=2.0*(A**2)*YD/GM
KCI(1,6)=2.0*(A**2)*ZD/GM
C PARTIALS OF E (eccentricity)
KCI(2,1)=(1.0-E*E)*KCI(1,1)/((2.0*A*E)-(1.0-E*E))*(HZ*YD-HY*ZD)/
*(E*H*H)
KCI(2,2)=(1.0-E*E)*KCI(1,2)/((2.0*A*E)-(1.0-E*E))*(HX*ZD-HZ*XD)/
*(E*H*H)
KCI(2,3)=(1.0-E*E)*KCI(1,3)/((2.0*A*E)-(1.0-E*E))*(HY*XD-HX*YD)/
*(E*H*H)
KCI(2,4)=(1.0-E*E)*KCI(1,4)/((2.0*A*E)-(1.0-E*E))*(HY*Z-HZ*Y)/
*(E*H*H)
KCI(2,5)=(1.0-E*E)*KCI(1,5)/((2.0*A*E)-(1.0-E*E))*(HZ*X-HX*Z)/
*(E*H*H)
KCI(2,6)=(1.0-E*E)*KCI(1,6)/((2.0*A*E)-(1.0-E*E))*(HX*Y-HY*X)/
*(E*H*H)
C PARTIALS OF I (inclination)
KCI(3,1)=(-HY*HZ*ZD-(HX*HX+HY*HY)*YD)/(H*H*DSQRT(HX*HX+HY*HY))
KCI(3,2)=(HX*HZ*ZD+(HX*HX+HY*HY)*XD)/(H*H*DSQRT(HX*HX+HY*HY))
KCI(3,3)=(-HX*HZ*YD+HY*HZ*XD)/(H*H*DSQRT(HX*HX+HY*HY))
KCI(3,4)=(HY*HZ*Z+(HX*HX+HY*HY)*Y)/(H*H*DSQRT(HX*HX+HY*HY))
KCI(3,5)=(-HX*HZ*Z-(HX*HX+HY*HY)*X)/(H*H*DSQRT(HX*HX+HY*HY))
KCI(3,6)=(-HY*HZ*X+HX*HZ*Y)/(H*H*DSQRT(HX*HX+HY*HY))
C PARTIALS of OMEGA (ascending node)
KCI(5,1)=(-HX*ZD)/(HX*HX+HY*HY)
KCI(5,2)=(-HY*ZD)/(HX*HX+HY*HY)
KCI(5,3)=(HX*XD+HY*YD)/(HX*HX+HY*HY)
KCI(5,4)=(HX*Z)/(HX*HX+HY*HY)
KCI(5,5)=(HY*Z)/(HX*HX+HY*HY)
KCI(5,6)=(-HX*X-HY*Y)/(HX*HX+HY*HY)
C PARTIALS of M (mean anomaly)
F1=(1.0-E*E)*(COSF*RRD/H-SINF)*KCI(1,1)/(R*E)-2.*A*(COSF*RRD/H-
*SINF)*KCI(2,1)/R-(A*(1.0-E*E)/(R*E*H))*(COSF*RRD/H)*(HZ*YD-HY*ZD
)/H+(A*(1.0-E*E)/(R*E*H))*COSF*XD+(SINF*X/(R*R*E))
F2=(1.0-E*E)*(COSF*RRD/H-SINF)*KCI(1,2)/(R*E)-2.*A*(COSF*RRD/H-
*SINF)*KCI(2,2)/R-(A*(1.0-E*E)/(R*E*H))*(COSF*RRD/H)*(HX*ZD-HZ*XD
)/H+(A*(1.0-E*E)/(R*E*H))*COSF*YD+(SINF*Y/(R*R*E))
F3=(1.0-E*E)*(COSF*RRD/H-SINF)*KCI(1,3)/(R*E)-2.*A*(COSF*RRD/H-
*SINF)*KCI(2,3)/R-(A*(1.0-E*E)/(R*E*H))*(COSF*RRD/H)*(HY*XD-HX*YD
)/H+(A*(1.0-E*E)/(R*E*H))*COSF*ZD+(SINF*Z/(R*R*E))
F4=(1.0-E*E)*(COSF*RRD/H-SINF)*KCI(1,4)/(R*E)-2.*A*(COSF*RRD/H-
*SINF)*KCI(2,4)/R-(A*(1.0-E*E)/(R*E*H))*(COSF*RRD/H)*(HY*Z-HZ*Y)
*/H+(A*(1.0-E*E)/(R*E*H))*COSF*X
F5=(1.0-E*E)*(COSF*RRD/H-SINF)*KCI(1,5)/(R*E)-2.*A*(COSF*RRD/H-
*SINF)*KCI(2,5)/R-(A*(1.0-E*E)/(R*E*H))*(COSF*RRD/H)*(HZ*X-HX*Z)
*/H+(A*(1.0-E*E)/(R*E*H))*COSF*Y
F6=(1.0-E*E)*(COSF*RRD/H-SINF)*KCI(1,6)/(R*E)-2.*A*(COSF*RRD/H-

```

```

*SINF)*KCI(2,6)/R-(A*(1.0-E*E)/(R*E*H))*(COSF*RRD/H)*(HX*Y-HY*X)
*/H+(A*(1.0-E*E)/(R*E*H))*COSF*Z
E1=((DSQRT(1.0-E*E))/(1.0+E*COSF))*(F1-SINF*KCI(2,1)/(1.0-E*E))
E2=((DSQRT(1.0-E*E))/(1.0+E*COSF))*(F2-SINF*KCI(2,2)/(1.0-E*E))
E3=((DSQRT(1.0-E*E))/(1.0+E*COSF))*(F3-SINF*KCI(2,3)/(1.0-E*E))
E4=((DSQRT(1.0-E*E))/(1.0+E*COSF))*(F4-SINF*KCI(2,4)/(1.0-E*E))
E5=((DSQRT(1.0-E*E))/(1.0+E*COSF))*(F5-SINF*KCI(2,5)/(1.0-E*E))
E6=((DSQRT(1.0-E*E))/(1.0+E*COSF))*(F6-SINF*KCI(2,6)/(1.0-E*E))
KCI(6,1)=(1.0-E*COSEA)*E1-SINEA*KCI(2,1)
KCI(6,2)=(1.0-E*COSEA)*E2-SINEA*KCI(2,2)
KCI(6,3)=(1.0-E*COSEA)*E3-SINEA*KCI(2,3)
KCI(6,4)=(1.0-E*COSEA)*E4-SINEA*KCI(2,4)
KCI(6,5)=(1.0-E*COSEA)*E5-SINEA*KCI(2,5)
KCI(6,6)=(1.0-E*COSEA)*E6-SINEA*KCI(2,6)

```

C PARTIALS of omega

```

U1=(1.0/R)*((-Y*DSIN(OME)*DCOS(XI)*COSU-X*DCOS(OME)*DCOS(XI)*
*COSU+X*DSIN(OME)*SINU-Y*DCOS(OME)*SINU)*KCI(5,1)+(X*DSIN(OME)*
*DSIN(XI)*COSU-Y*DCOS(OME)*DSIN(XI)*COSU+Z*DCOS(XI)*COSU)*KCI(3,1
*)+(-DSIN(OME)*DCOS(XI)*COSU-DCOS(OME)*SINU)
U2=(1.0/R)*((-Y*DSIN(OME)*DCOS(XI)*COSU-X*DCOS(OME)*DCOS(XI)*
*COSU+X*DSIN(OME)*SINU-Y*DCOS(OME)*SINU)*KCI(5,2)+(X*DSIN(OME)*
*DSIN(XI)*COSU-Y*DCOS(OME)*DSIN(XI)*COSU+Z*DCOS(XI)*COSU)*KCI(3,2
*)+(DCOS(OME)*DCOS(XI)*COSU-DSIN(OME)*SINU)
U3=(1.0/R)*((-Y*DSIN(OME)*DCOS(XI)*COSU-X*DCOS(OME)*DCOS(XI)*
*COSU+X*DSIN(OME)*SINU-Y*DCOS(OME)*SINU)*KCI(5,3)+(X*DSIN(OME)*
*DSIN(XI)*COSU-Y*DCOS(OME)*DSIN(XI)*COSU+Z*DCOS(XI)*COSU)*KCI(3,3
*)+DSIN(XI)*COSU)
U4=(1.0/R)*((-Y*DSIN(OME)*DCOS(XI)*COSU-X*DCOS(OME)*DCOS(XI)*
*COSU+X*DSIN(OME)*SINU-Y*DCOS(OME)*SINU)*KCI(5,4)+(X*DSIN(OME)*
*DSIN(XI)*COSU-Y*DCOS(OME)*DSIN(XI)*COSU+Z*DCOS(XI)*COSU)*KCI(3,4
*)
U5=(1.0/R)*((-Y*DSIN(OME)*DCOS(XI)*COSU-X*DCOS(OME)*DCOS(XI)*
*COSU+X*DSIN(OME)*SINU-Y*DCOS(OME)*SINU)*KCI(5,5)+(X*DSIN(OME)*
*DSIN(XI)*COSU-Y*DCOS(OME)*DSIN(XI)*COSU+Z*DCOS(XI)*COSU)*KCI(3,5
*)
U6=(1.0/R)*((-Y*DSIN(OME)*DCOS(XI)*COSU-X*DCOS(OME)*DCOS(XI)*
*COSU+X*DSIN(OME)*SINU-Y*DCOS(OME)*SINU)*KCI(5,6)+(X*DSIN(OME)*
*DSIN(XI)*COSU-Y*DCOS(OME)*DSIN(XI)*COSU+Z*DCOS(XI)*COSU)*KCI(3,6
*)
KCI(4,1)=U1-F1
KCI(4,2)=U2-F2
KCI(4,3)=U3-F3
KCI(4,4)=U4-F4
KCI(4,5)=U5-F5
KCI(4,6)=U6-F6
RETURN
END

```

```

***INPUT DATA:AO,EO,XIO,OMO,OME0,XMO,FO,EAO,XO,YO,ZO,XDO,YDO,ZDO****
*****
* 17570925.77037010      0.568848843824957      0.541052068118242*
*-0.881968672701392    3.124086383674180      0.747478091842417*
* 1.929121467522410      1.294798081992500      *
*-7615946.972994897    -10889534.81787064      6622196.773840833*
* 2396.747150406745     -4319.688274669979      2569.923171671754*
*****

```

```

**** ELEMENTS OF THE SATELLITE STATE VECTOR (XS,YS,ZS,XSD,YSD,ZSD) ****
-7615946.97299500000    -10889534.81787060000    6622196.77384083000
2396.74715040671      -4319.68827467000      2569.92317167176

```

```

***** ELEMENTS OF THE LINEAR KEPLER TRANSFORMATION MATRIX KC *****
-0.4334402792730      28607803.1683634000000    115924.0427484690000
12744314.0219510000000    10889534.8178706000000    8841893.9377193300000
-0.6197473576625     -20941688.6224014000000    6621182.0491344800000
-6587846.0049455100000    -7615946.9729950000000    -15935859.3843162000000
0.3768837715431      12280202.6842647000000    11021186.2187911000000
3823722.2809170100000    0.000000000000000    9480761.5013344300000
-0.0000682020737      4895.4299974719600      44.9874707423364
5026.1010036825600    4319.6882746700000      3421.8010333312000
0.0001229214764      3878.7401494792500      2569.5293802118000
2031.2430250534000    2396.7471504067100      4892.6051644543400
-0.0000731299877     -2381.7166470278200      4277.0704058304800
-1273.1729165956700    0.000000000000000      -2975.3148024795100
*****

```

```

***** KEPLERIAN ORBITAL ELEMENTS (AS,ES,XIS,OMS,OMES,XMS,FS,EAS) *****
1.757092577037010D+007    0.568848843824957      0.541052068118242
-0.881968672701392      3.124086383674180      0.747478091842417
1.929121467522410      1.294798081992500

```

```

***** INVERSE LINEAR KEPLER TRANSFORMATION MATRIX KCI *****
-1.43685580235904000    -2.05446431593198000    1.24937081266399000
3712.81862519266000000    -6691.66085312722000000    3981.08687246287000000
0.00000003821214436     -0.00000003442479720     0.00000002027940900
-0.00009454042695639    -0.00011261293567051     0.00006864871363555
0.00000000032380779     0.00000001849478572     0.00000003078520965
0.00000097249246146     0.00005554541988079     0.00009245726995033
-0.00000008348546989    -0.00000000487797231     0.00000014456563439
0.00049142428946580     -0.00052188742069056     -0.00005433900188108
-0.00000000126905576    -0.00000007248409427     -0.00000012065227854
0.00000327011214863     0.00018677754281016     0.00031089767137094
0.00000004091773235     0.00000010672721602     -0.00000006454873928
-0.00022446651641490    0.00055778897052638     -0.00033274106616196
*****

```

FULFILMENT OF KC*KCI=I

```

*****
1.00000000000000000000 0.00000000000000591861 -0.00000000000000149414
-0.000000000000725564053 -0.00000000000042099657 0.00000000000037148062
*****
-0.000000000000001013875 1.00000000000000000000 0.00000000000000232355
0.000000000000916000609 -0.000000000001096189806 -0.00000000000075761619
*****
0.00000000000000400881 0.00000000000000253020 1.00000000000000000000
-0.000000000000583844084 -0.00000000000010880186 -0.000000000001283706474
*****
0.000000000000000000072 -0.000000000000000000120 0.000000000000000000070
1.000000000000000000000 0.0000000000000000301040 0.00000000000000069714
*****
-0.000000000000000000062 0.000000000000000000096 0.000000000000000000043
-0.0000000000000000469340 1.000000000000000000000 -0.00000000000000116671
*****
0.000000000000000000041 0.000000000000000000052 0.000000000000000000138
0.000000000000000055576 -0.000000000000000099508 1.000000000000000000000
*****

```

FULFILMENT OF KCI*KC=I

```

*****
1.00000000000000000000 0.00000137576353154145 0.00000000985528458841
0.00000062681783674634 0.00000055624150263611 0.00000072245620685862
*****
0.00000000000000000000 1.00000000000001000000 0.00000000000000018922
0.000000000000000450163 0.00000000000000328440 -0.00000000000000115492
*****
0.00000000000000000000 -0.00000000000000293900 1.00000000000000000000
-0.00000000000000185891 -0.00000000000000358525 -0.00000000000000122450
*****
0.00000000000000000000 0.000000000000007482006 -0.00000000000000375618
1.00000000000004000000 0.000000000000003101946 0.00000000000004706265
*****
0.00000000000000000000 0.00000000000000450378 0.00000000000000818540
0.0000000000000009552 1.00000000000000000000 0.0000000000000027067
*****
0.00000000000000000000 -0.000000000000008301422 -0.00000000000000178362
-0.000000000000003955419 -0.000000000000003490144 0.99999999999996000000
*****

```