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EXPONENTIAL GROUND IMPEDANCE MODELS AND THEIR INTERPRETATION

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SUMMARY

In this paper we compare the results of Donato's exponentially varying ground model, Attenborough's exponentially varying ground model and the rigid backed thin layer model. We show that these models produce similar results for slow variations. For rapid variations the results are quite different but the basic theory used is only correct for the thin layer model. These results suggest that the exponentially varying models are not necessary for fitting ground impedance data.

INTRODUCTION

Donato proposed an exponentially varying ground model to be used for the interpretation of ground impedance data.[1] Attenborough has demonstrated that the exponential variation chosen by Donato results in model grounds with increasing porosity with depth and has derived a ground model which has a decreasing porosity with depth.[2]

In this paper we examine the behavior of both these models in the limit of large and small variation and compare the results to the rigid backed layer model.[3] To facilitate this we have reduced the solutions to their simplest forms and have employed Attenborough's low frequency/high flow resistivity results for numerical comparison.

I. GROUND MODELS

A. Rigid Backed Layer

A layer of porous material of thickness d overlying an acoustically rigid surface has a surface impedance of the form:

$$Z(0) = i\, Z_c \cot (kd) \qquad (1)$$

where Z_c is the impedance of a seminfinite half space of the porous material and k is the complex wave number in the porous material.

B. Donato's Exponential Model

Donato has derived a impedance model for a material whose porosity times wave number decreases exponentially with depth. Attenborough has demonstrated that for natural grounds this implies that the porosity increases exponentially with depth and the wave number decreases exponentially with depth. This will not commonly occur in natural ground surfaces but may be a useful model in some cases. With the notation above Donato's formula becomes

$$Z(0) = i\, Z_c \frac{J_0(2k/\alpha)}{J_1(2k/\alpha)}\ ; \tag{2}$$

α is the exponential varation of the square of the complex wave number

$$k(z)^2 = k(0)^2\, e^{-\alpha z}. \tag{3}$$

C. Attenborough's Exponential Model

Attenborough's solution for a porous material whose porosity decreases exponentially with depth and wave number increases exponentially with depth is given by

$$Z(0) = i\, Z_c \frac{H_0^{(2)}(2k/\alpha)}{H_1^{(2)}(2k/\alpha)}\ ; \tag{4}$$

where

$$k(z)^2 = k(0)^2\, e^{\alpha z}. \tag{5}$$

II. BEHAVIOR OF THE IMPEDANCE AND WAVE NUMBER

It will be useful in the interpretation of these models to have a specific formulae for the wave number and impedance of a homogeneous porous material. For this paper we will use Attenborough's low frequency approximation:

$$Z_c = \frac{kc}{\gamma\, \Omega \omega} = .218\left(\frac{\sigma_e}{f}\right)^{1/2} (1 + i). \tag{6}$$

σ_e is the effective flow resistivity of the material, γ is the ratio of specific heats and c is the speed of sound in air.[2]

III. BEHAVIOR OF THE GROUND MODELS IN THE LIMIT OF LARGE AND SMALL ARGUMENTS

A. Rigid Backed Layer

i) Limit as $d \rightarrow 0$.

For a thin layer $d \rightarrow 0$ and Eq. (1) becomes

$$Z(0) = \lim_{d \to 0} i\, Z_c \cot(kd) = i\,\frac{Z_c}{kd} - i\,\frac{Z_c kd}{3} \tag{7}$$

If we use Eq. (6) to relate Z_c and k for low frequency we find

$$Z(0) = \frac{4\pi(.218)^2\, \gamma\Omega d\sigma_e}{3c} + i\,\frac{1}{\gamma\Omega k_0 d}\,. \tag{8}$$

where k_0 is ω/c. Note that the imaginary term approaches infinity as $k_0 d$ goes to zero, while the real part depends only on the layer thickness and the surface flow resistance. This form is displayed by Attenborough.[2]

ii) Limit as $d \rightarrow \infty$

As $d \rightarrow \infty$ the model should recover the result for the homogeneous half space. The cotangent can be expanded in terms of the exponents of the real and imaginary parts of kd.

$$\lim_{d \to \infty} \cot(kd) = \lim_{d \to \infty} = \frac{e^{i k_1 d}\, e^{-k_2 d} + e^{-ik_1 d}\, e^{+k_2 d}}{2} \Big/ \frac{e^{i k_1 d}\, e^{-k_2 d} - e^{-ik_1 d}\, e^{k_2 d}}{2i} \tag{9}$$

where

$k = k_1 + i\, k_2$.

k_2 must be positive so that

$$Z(0) = i\, Z_c\, (-\, i) = Z_c, \tag{10}$$

and the original condition is recovered.

B. Donato's Exponential Model

i) Limit as α becomes small

As α becomes small the medium approaches a homogeneous media. If we take the limit of Eq. (2) for small α and large $2k/\alpha$ we find

$$Z(0) \approx i\,Z_c \frac{\sqrt{\frac{2\pi\alpha}{2k}}\cos\left(\frac{2k}{\alpha} - \frac{\pi}{4}\right)}{\sqrt{\frac{2\pi\alpha}{2k}}\cos\left(\frac{2k}{\alpha} - \frac{\pi}{2} - \frac{\pi}{4}\right)} = i\,Z_c \cot\left(\frac{2k}{\alpha} - \frac{\pi}{4}\right). \tag{11}$$

This is like the impedance of a thin layer of thickness $2/\alpha$ with an additional $-\pi/4$ phase change. The next correction term is of order $\alpha/2k$. A pressure release backed thin layer would have a phase change of $-\pi/2$. As $\alpha \to 0$, the cotangent term will approach $-i$ as in Section A-ii) and $Z(0) = Z_c$ as expected.

ii) Limit as $\alpha \to \infty$

In the limit as $\alpha \to \infty$, the argument becomes small and the ascending series may be used to evaluate the Bessel functions.

$$Z(0) \approx i\,Z_c \frac{1}{\frac{k}{\alpha}} - i\,\frac{Z_c k}{2\alpha}. \tag{12}$$

The behavior of this solution is very similar to Eq. 7. We have a rapidly increasing imaginary part and a constant real part as the frequency decreases for fixed d and σ_e. The imaginary parts are identical if the rigid backed layer has a thickness $1/\alpha$, while the real parts are equal if the rigid backed layer thickness is given by $1.5/\alpha$.

C. Attenborough's Exponential Model

i) Limit as $\alpha \to 0$

The asymptotic expansions can be employed for the Hankel functions giving

$$Z(0) \approx i\,Z_c \frac{\sqrt{\frac{2}{\pi}\frac{\alpha}{2k}}\, e^{-i(2k/\alpha - \pi/4)}}{\sqrt{\frac{2}{\pi}\frac{\alpha}{2k}}\, e^{i\frac{\pi}{2}}\, e^{-i(2k/\alpha - \pi/4)}} \approx i\,Z_c\, e^{-i\pi/2} = Z_c. \tag{13}$$

The Attenborough model recovers the homogeneous half space surface impedance as $\alpha \to 0$.

ii) Limit as $\alpha \to \infty$

The small argument formulae for the Hankel functions are inserted in Eq. (4) to give

$$Z(0) = Z_c\left[\left(\frac{\pi}{2} - i\varepsilon\right)\frac{2k}{\alpha} - i\,\frac{2k}{\alpha}\ln\left(\frac{k}{a}\right)\right], \tag{14}$$

where $\varepsilon = .5772$.

This result is not easily interpreted in terms of a layered model. The behavior of this solution is best illustrated by use of Eq. (6) to yield

$$Z(0) = 5.923\ \ln\left|\frac{k}{\alpha}\right| + 3.419 + i\ 13.955 \tag{15}$$

As $\alpha \to \infty$ the impedance of the Attenborough model has a large negative real part tending to $-\infty$ and a constant imaginary part. This puzzling result indicates that the surface is not absorbing energy and has a reflection coefficient greater than one! In a gross sense the behavior is physical. The reflection coefficient approaches one as the impedance becomes infinite. The only problem is that the surface cannot be generating acoustic energy.

iii) Limit for $2k/\alpha > 1$, α not infinite

A third limit is developed by Attenborough as useful for computation and comparison with data. This form is developed for α small enough that the leading term in asymptotic series for the Hankel functions may be used. For $2k/\alpha > 1$

$$\frac{H_0^{(2)}(2k/\alpha)}{H_1^{(2)}(2k/\alpha)} \approx -i\,\frac{\left(1 + \frac{i}{8}\frac{\alpha}{2k}\right)}{\left(1 - \frac{i}{8}3\frac{\alpha}{2k}\right)} \cong -i\left\{1 + \frac{i\alpha}{4k}\right\} \tag{16}$$

and

$$Z(0) = Z_c\left\{1 + \frac{i\alpha}{4k}\right\} \tag{17}$$

Using Eq. (6) to relate k and Z_c gives us

$$Z(0) = Z_c + \frac{ic}{4\gamma\omega}\left[\frac{\alpha}{\Omega}\right]. \tag{18}$$

Defining $\alpha_e = \alpha/\Omega$ and inserting numerical values from Eq. (6) gives us Attenborough's form:

$$Z(0) = .218\left(\frac{\sigma_e}{f}\right)^{1/2} + i\left[.218\left(\frac{\sigma_e}{f}\right)^{1/2} + 9.74\left(\frac{\alpha_e}{f}\right)\right] \qquad (19)$$

The next terms in the asymptotic series are on the order of 7% of the last term in Eq. (19) when the argument of the Hankel function is one.

Note that we can recover Eq. (13) by letting α approach zero. Also note that the second term in Eq. (17) is very similar to the form for the imaginary part of the impedance of a thin rigid backed layer. Compare

$$i\,Z_c\frac{\alpha}{4k} \text{ and } \frac{i\,Z_c}{kd}\ . \qquad (20)$$

The second term in Eq. (17) is the imaginary part of the impedance of a thin layer of effective thickness $d_e = 4/\alpha$. The imaginary parts dominate the impedance for large α.

IV. NUMERICAL RESULTS

To calculate numerical values for the three impedance models we set

$$kd = 2k/\alpha = x(1 + i). \qquad (21)$$

Then, using Eq. (6), we solve for f and Z_c in terms of x:

$$f = \left[\frac{\alpha x c}{4\pi\gamma\Omega\,(.218)}\right]^2 \frac{1}{\sigma_e}\ , \qquad (22)$$

and

$$Z_c = \frac{4\pi\gamma\Omega(.218)^2\sigma_e}{\alpha\, x\, c}\,(1 + i)\ . \qquad (23)$$

We use the following typical values of γ, Ω, σ_e, and α based on our experience and that of Attenborough:

$\gamma = 1.4$

$\Omega = 0.4$

$\sigma_e = 120{,}000$ MKS rayls

$\alpha = 40.\ m^{-1}$; $d = 5$ cm.

Then, we calculate impedances using Eqs. (1), (2) and (4) for x = 0 to 5. The results are plotted in

Fig. 1 (rigid backed layer), Fig. 2 (Donato's formula), and Fig. 3 (Attenborough's solution). The imaginary parts of the impedance are multiplied by -1 so the plots of the real part are usually on the positive side of the vertical axis and the imaginary parts are on the negative side. The plots are nearly identical for values of x greater than one. For the variables above, x = 1.0 corresponds to 654 Hz.

Figure 4 displays the normal reflection coefficient calculated from Eqs. (1,2, and 4). The behavior is similar for all the models. Better agreement can be achieved between any two models by the choice of the equivalent depth of the exponential variation.

V. DISCUSSIONS AND CONCLUSIONS

The surface impedance predicted by each of the three models above approaches the homogeneous half-space impedance as the variation of wave number becomes small or the layer depth becomes large in the rigid backed model.

As the exponential variations become larger the impedance formula can be approximated as a constant or slowly varying real and imaginary part plus an imaginary term which is proportional to α/ω or $1/\omega d$.

For very rapid variations, the expansion of Attenborough's solution results in a non-physical solution (Eq. 13).

The basic assumption in the derivation of Eq. (6) and it's more exact analogues, is that the gradients of the variables with respect to the propagation direction are much smaller than gradients of the variables normal to the direction of propagation.[4] The result that the reflection coefficient is greater than one for small variable x is probably due to the error in Eq. (6) rather than any physical error in the theory leading to Eq. (4).

By the same reasoning, Donato's formula should be inaccurate for small values of the variable x. There is no physical problem with the thin rigid backed layer since the porous layer is homogeneous and Eq. (6) should hold. For the variables we have chosen, there appears to be little practical reason to employ the exponential models to fit ground data, while there appears to be a significant theoretical reason for not using the exponential models in the region where they vary significantly from the rigid backed layer.

At very low frequencies, the impedance translation theorem can be employed to calculate the impedance of an impedance backed layer. This model has sufficient flexibility to fit most data without the theoretical difficulties of the Donato or Attenborough models.

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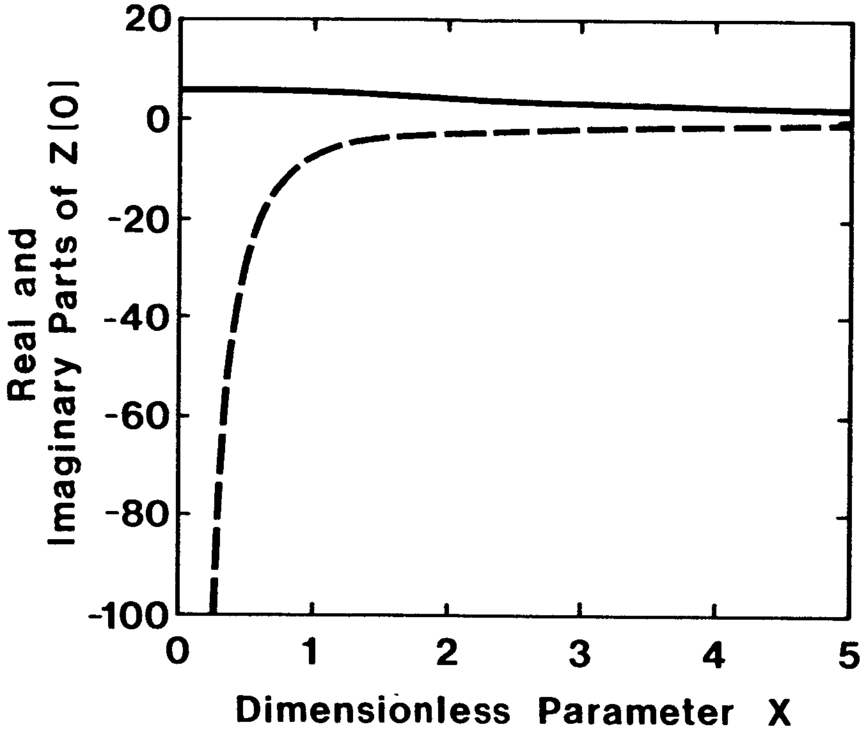

Figure 1. The real and imaginary part of the ground impedance versus the parameter x for the thin rigid backed layer. The imaginary part is multiplied by negative one for display purposes.

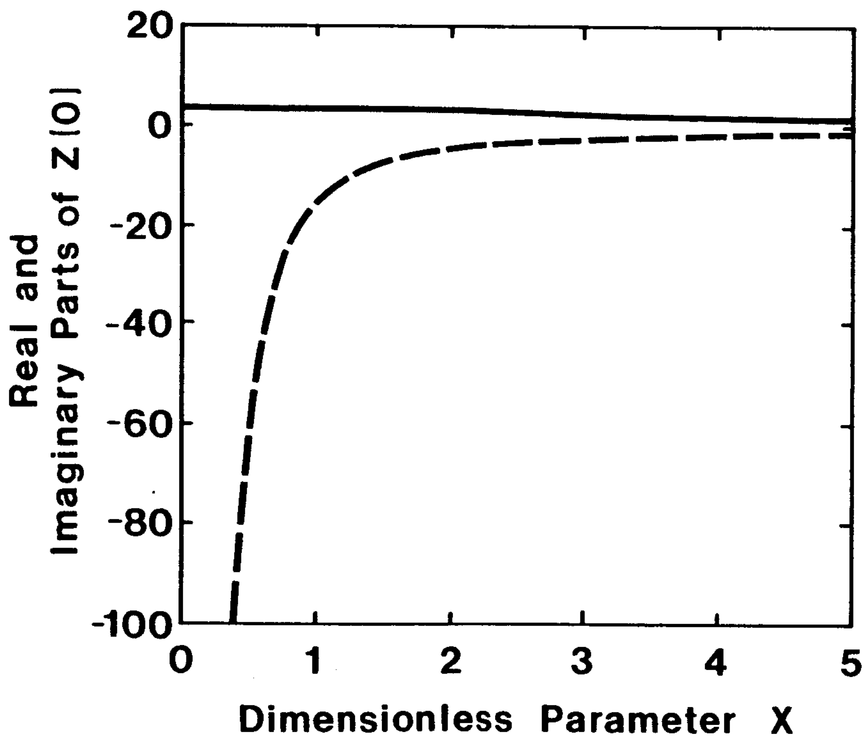

Figure 2. The real and imaginary parts of the ground impedance versus the parameter x for Donato's exponentially varying model.

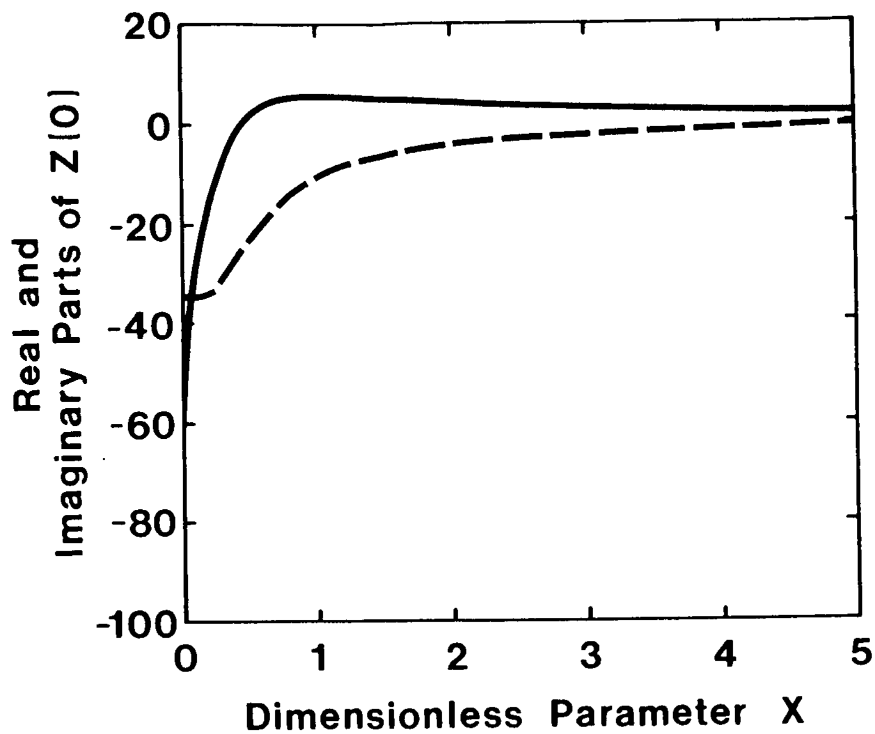

Figure 3. The real and imaginary parts of the ground impedance versus the parameter x for Attenborough's exponentially varying model.

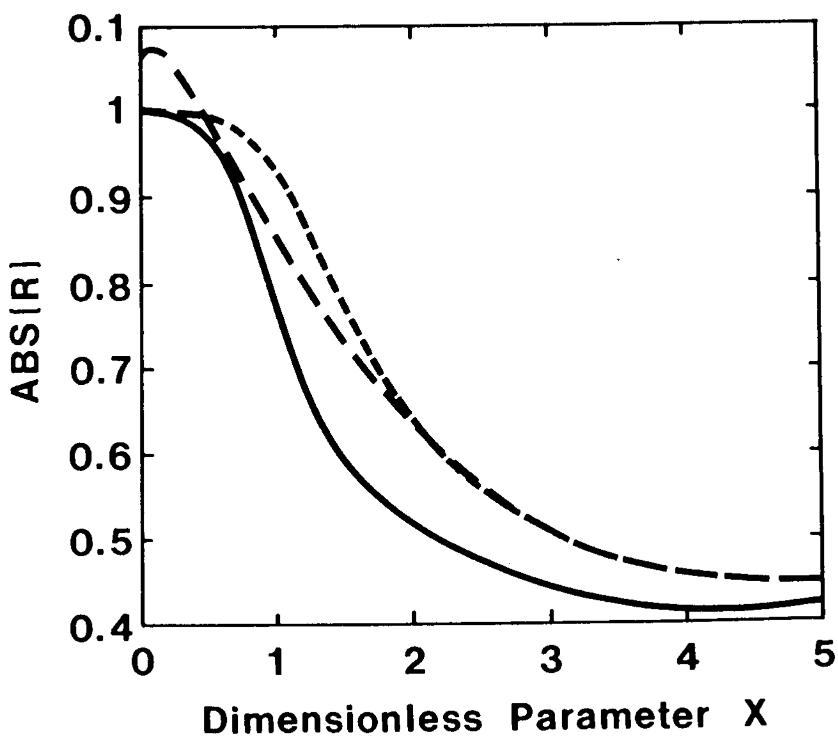

Figure 4. The absolute value of the reflection factor for normal reflection for the three models versus the parameter x—thin layer,-----Donato,— —Attenborough.)