N91-16701

WAVE PROPAGATION THROUGH RANDOM MEDIA:

A Local Method of Small Perturbations based on the Helmholtz Equation

Ralf Große, Universität Oldenburg, Postfach 2503, D-2900 Oldenburg, FRG

1 Introduction

Propagation of sound through the turbulent atmosphere is a statistical problem. The randomness of the refractive index field causes sound pressure fluctuations. Although no general theory to predict sound pressure statistics from given refractive index statistics exists, there are several approximate solutions to the problem. The most common approximation is the parabolic equation method. Results obtained by this method are restricted to small refractive index fluctuations and to small wave lengths. While the first condition is generally met in the atmosphere, it is desirable to overcome the second. This paper presents a generalization of the parabolic equation method with respect to the small wave length restriction.

2 Parabolic Equation Method

For the small wave length limit[1] the Helmholtz equation can be converted into a parabolic form (main propagation direction $\vec{e}_z$) /1/:

$$\left(2ik\frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\mu(\vec{r})\right) \quad \psi(\vec{r}) = 0 \tag{1}$$

k = *wave number;* μ = *refractive index deviation;* ψ = *complex sound pressure*

The refractive index deviation μ is considered as a random function. Therefore equation (1) is a stochastic differential equation and the sound pressure ψ becomes a random function, too. Stochastically this equation is nonlinear, e.g. it contains a product of two random variables. For this reason the stochastic parabolic equation cannot be solved exactly. Further approximations are necessary. Several mathematical tools were applied to remove the stochastical nonlinearity, i.e. path integrals /2/, functional derivatives /3/, perturbation expansion methods /1/. Despite approximations used in the calculations looking different, the results are all the same.

The physical meaning of all these approximations becomes evident in the *local method of small perturbations* /1/. By this method the scattering volume is divided into slabs perpendicular to the main propagation direction. Each slab is chosen as thin as required by the validity limit of the first order perturbation expansion term (single scattering approximation, Born approximation). This distance clearly depends on the strength of the refractive index fluctuations. If these fluctuations are sufficiently small, the slabs are much thicker than one correlation length of the random medium. Therefore the slabs can be regarded as uncorrelated. Based on both assumptions - small refractive index fluctuations and a small wave length compared to the correlation length - the statistical independence of subsequent slabs can be proofed mathematically. Wave propagation through random media is described here as a Markov process.

[1]The wave length must be small compared to the size of a typical inhomogeneity of the medium. In statistical terms this size is expressed by the correlation length of the refractive index autocorrelation- function.

As a consequence of the Markov property slabs of finite thickness are no longer necessary. This results in a differential equation for the mean sound pressure which is linear in the stochastical sense as well[2]/1/:

$$\left(2ik\frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik\alpha\right) \quad < \psi(\vec{r}) >= 0 \tag{2}$$

$$\alpha = \frac{< \mu^2 > k^2 l}{4} \tag{3}$$

l = correlation length

Equation (2) can be solved easily:

$$< \psi(\vec{r}) >= \psi_0(\vec{r})\ exp\{-\alpha z\} \tag{4}$$

ψ_0 = *free propagated incident wave*

The mean complex sound pressure decreases exponentially while the wave propagates through the random medium. This is an effect of decorrelation of the sound wave due to phase fluctuations. Different members of the statistical assembly interfere destructively because of their different phases.

The validity of result (4) - and of any other result obtained by the parabolic equation method - is restricted first by the validity of the parabolic wave equation (1) and second by the validity of the Markov assumption. Necessary conditions are the smallness of refractive index fluctuations and the smallness of the wave length (compared with the correlation length). In the next section the first condition is also assumed to be true. The small wave length assumption, however, will be replaced by the weaker condition of negligible backscattering. This will lead to a generalized Markov process and, consequently, to generalized results with respect to the wave length - correlation length ratio.

3 Generalized Local Method of Small Perturbations

While generalizing the parabolic equation method the main idea of the local method of small perturbations will be retained. The refractive index fluctuations are assumed to be small enough to justify the application of a single scattering approximation within a distance Δz in the scattering volume which is large compared to the correlation length. Again the scattering volume is divided into slabs of this size. Therefore subsequent slabs are uncorrelated as well. But contrary to the parabolic equation method the contribution of one slab will not be calculated from the parabolic equation but from the Helmholtz equation:

$$\left(\Delta + k^2(1 + \mu(\vec{r}))\right)\ \psi(\vec{r}) = 0 \tag{5}$$

Δ = *Laplace operator*

Neglecting backscattering[3] yields a difference equation for the mean sound pressure:

$$< \psi(\vec{\rho}_n, n\Delta z) >= \left(\hat{G}_0 + \hat{S}\right)\ < \psi(\vec{\rho}_{n-1}, (n-1)\Delta z) > \tag{6}$$

[2]It is possible to derive similar equations for the higher statistical moments of ψ by the parabolic equation method, too. Only the first moment equation and its solution are presented here in order to compare them with the generalized results derived in section 3 of this paper. The decorrelation coeffecient α is calculated for an exponential autocorrelation function for comparison, too.

[3]Neglecting of backscattering is not an *assumption* of the parabolic equation method but a *consequence* of the small wave length limit. The parabolic equation implies that there are small scattering angles only and no backscattering at all.

$\vec{\rho} = (x, y)$; n = *number of slab*

$\hat{G}_0$ is the integral operator of the homogeneous Helmholtz equation ($\mu = 0$, free propagation) and $\hat{S}$ is an integral operator for the scattering within one slab. Since double scattering is the lowest order non-vanishing term, the kernel of $\hat{S}$ contains the autocorrelation function of the refractive index field:

$$\hat{S}(\vec{r}_n, \vec{r}) < \psi(\vec{r}) >= -k^4 \int d^3\vec{r}' \int d^3\vec{r}\, G(\vec{r}_n, \vec{r}')G(\vec{r}', \vec{r}) < \mu(\vec{r}')\mu(\vec{r}) >< \psi(\vec{r}) > \tag{7}$$

G = *Greens function of the Helmholtz equation*

The solving of equation (6) is somewhat different from that of the related parabolic equation problem. Here no wave length approximation helps to calculate the integrals. But for the case of a homogeneous refractive index autocorrelation function, equation (7) becomes a convolution product. It can be Fourier-transformed with respect to the variable $\vec{\rho}$, and this operation turns the operators $\hat{S}$ and $\hat{G}_0$ into simple functions $\tilde{S}$ and $\tilde{G}_0$. In the Fourier representation equation (6) reads:

$$<\tilde{\psi}(\vec{\kappa}, n\Delta z) >= \left(\tilde{G}_0 + \tilde{S}\right) <\tilde{\psi}(\vec{\kappa}, (n-1)\Delta z) > \tag{8}$$

$\vec{\kappa}$ = *2-dimensional spatial frequency*

By Fourier-transformation the $\vec{\rho}$-integrations are already performed and only the z-integrations are left. They can be performed, too, if the z-dependence of the medium autocorrelation function is known. For the sake of simplicity this dependence is assumed to be exponential[4]. After z-integration the scattering contribution of one slab is seen to be proportional to the slabs thickness Δz. Therefore equation (8) can be written as ($\tilde{S} = \tilde{s}\ \Delta z$):

$$<\tilde{\psi}(\vec{\kappa}, n\Delta z) >= \left(\tilde{G}_0 + \tilde{s}\ \Delta z\right) <\tilde{\psi}(\vec{\kappa}, (n-1)\Delta z) > \tag{9}$$

The effect of all slabs is obtained by iteration:

$$<\tilde{\psi}(\vec{\kappa}, n\Delta z) >= \left(\tilde{G}_0 + \tilde{s}\ \Delta z\right)^n \tilde{\psi}_0(\vec{\kappa}, 0) \tag{10}$$

Regarding a sufficiently large number of slabs yields ($z = n\,\Delta z$):

$$<\tilde{\psi}(\vec{\kappa}, z) >= \tilde{\psi}_0(\vec{\kappa}, z)\ exp\{\tilde{s}(\vec{\kappa})\, z\} \tag{11}$$

For the special case of an isotropic exponential autocorrelation function $\tilde{s}$ becomes:

$$\tilde{s}(\kappa) = \frac{<\mu^2> k^4}{4aa_l(a_l - a)} \tag{12}$$

$$a = \sqrt{k^2 - \kappa^2} \quad , \quad a_l = \sqrt{(k + i/l)^2 - \kappa^2} \tag{13}$$

To compare (11) and (12) with the corresponding parabolic equation method results (4) and (3), the incident wave ψ_0 is assumed to be a plane one. Then the Fourier-transformation of (11) results in:

[4]For more complicated functions the z-integration can be performed numerically.

$$< \psi(\vec{r}) >= \psi_0(\vec{r})\, exp\{\tilde{s}\,(0)\, z\} \tag{14}$$

$$\tilde{s}\,(0) = \frac{< \mu^2 > k^2 l}{4} \frac{(ikl - k^2 l^2)}{(1 + k^2 l^2)} \tag{15}$$

The real part of $\tilde{s}(0)$ describes the decorrelation of the sound wave. It is a more general expression than (3) - only in the small wave length limit they are equal. Because of $\tilde{s}(0)$ being a complex number, a second effect is predicted by this method, which cannot be seen in the parabolic results. The imaginary part is a stochastic correction to the wave number k due to an increase of the mean propagation distance in the random medium. Only in the small wave length limit the scattering angles are small and the mean propagation distance corresponds to the z-extension of the scattering volume.

4 Conclusions

A generalized form of the local method of small perturbations has been presented in this paper. Working directly from the Helmholtz equation instead of the parabolic equation the *small angle scattering method* was replaced by a *forward scattering method.* By this method only one result was derived here: The first statistical moment of an incident plane wave scattered by a very weakly statistical homogeneous random medium with an exponential autocorrelation function. This result shows corrections to the corresponding parabolic equation method result.

It is possible to apply the method to more complicated problems, i.e. a difference equation for the second statistical moment can be derived by the same idea.

If the medium fluctuations become stronger, the thickness of one slab decreases. The slabs might be thicker than the correlation length, but not as much as assumed before. Then correlations between two slabs have to be taken into account. This leads to difference equations which connect statistical moments not only from one slab to the following, but to the next following, too.

The most valuable advantage of this method might be its suitability for numerical calculations. For any given medium autocorrelation function the scattering function $\tilde{s}$ can be obtained by Fourier-transformation. The incident wave is also Fourier-transformed. Then the algorithm given by equation (9) is applied iteratively until the desired propagation distance is covered. The final result is obtained by Fourier-transformation again.

The author wishes to thank Prof. K. Haubold, Prof. V. Mellert, Dr. M. Schultz-von Glahn and A. Sill for the valuable discussions during the whole work.

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