Contribution of Zonal Harmonics to Gravitational Moment

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May 23, 1990

Abstract

A celestial body produces a gravitational moment about the mass center of a small orbiting body, which affects the orientation of the smaller body. Each zonal harmonic in the gravitational potential of a celestial body is shown to make a contribution to the gravitational moment which can be expressed in a recursive vector-dyadic form. A formal derivation is presented, followed by an example in which the result is employed in obtaining the contribution of the zonal harmonic of 2nd degree. The contribution of the zonal harmonic of 3rd degree is also reported.

Introduction

The gravitational moment about the mass center of a body in orbit about a celestial body has an important effect on the orientation of the orbiting body. The more misshapen the celestial body, and the less uniform its mass distribution, the more involved is the calculation of the gravitational moment (and

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force) it exerts. Situations in which it might be important to calculate accurately the gravitational moment include the design of spacecraft for expeditions to asteroids, comets, and the moons of Mars.

In Ref [1], a method for obtaining a vector-dyadic expression for the moment exerted about a small body's mass center by an oblate spheroid was set forth. The derivation of that expression made use of a gravitational potential written in terms of the zonal harmonic of 2nd degree. When gravitational potentials containing zonal harmonics of degree 2 or greater are considered, each zonal harmonic makes a contribution to the gravitational moment.

Recorded below is a vector-dyadic expression for the contribution of a zonal harmonic of degree n to the gravitational moment, produced by a body, about the mass center of a small body. As is the case with all vector-dyadic expressions, this result is basis independent— that is, the vectors and dyadics can be expressed in any convenient vector basis.

The equation given below is recursive: the contribution to the gravitational moment from the zonal harmonic of degree n is a function of the moment contributions from the zonal harmonics of degree $n-1$ and $n-2$. The equation contains Legendre polynomials and derivatives of Legendre polynomials of degree $n-1$ and $n-2$. The Legendre polynomials, as well as their derivatives, can, themselves, be generated by means of recursion formulae.

As an example, the contribution to the gravitational moment from the zonal harmonic of degree 2 is worked out. The contribution of the zonal harmonic of degree 3 is also given.

Assertion

Figure 1 shows a small body B in the presence of an axisymmetric body E. The distance between B^*, the mass center of B, and E^*, the mass center of

E, is assumed to exceed the greatest distance from B^* to any point of B. The system of gravitational forces exerted by E on B produces a moment $\boldsymbol{M}$ about B^*, and $\boldsymbol{M}$ can be written as

$$\boldsymbol{M} = \frac{3\mu}{R^3}\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} + \sum_{n=2}^{\infty} \boldsymbol{M}_n \tag{1}$$

where $\boldsymbol{M}_n$ is the contribution of the zonal harmonic of degree n and can be obtained by using the recursion relation

$$\begin{aligned}
\boldsymbol{M}_n = \frac{\mu J_n {R_E}^n}{n R^{n+3}} \Big\{ & \Big[(2n-1)\left[(n+2)P_{n-1} + 3(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})P'_{n-1}\right] - (2n-2)P'_{n-2}\Big] \\
& (\hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{r}} + \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{n}}) \\
+ & \Big[(4n-4)\left[(n+1)P_{n-2} + (\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})P'_{n-2}\right] \\
& -(2n-1)(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})\left[(4n+8)P_{n-1} + 4(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})P'_{n-1}\right]\Big]\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} \\
- & \Big[2(2n-1)P'_{n-1}\Big]\hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{n}} \Big\} \\
+ & \frac{2n-1}{n}\frac{J_n}{J_{n-1}}\frac{R_E}{R}(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})\boldsymbol{M}_{n-1} - \frac{n-1}{n}\frac{J_n}{J_{n-2}}\left(\frac{R_E}{R}\right)^2 \boldsymbol{M}_{n-2}
\end{aligned} \tag{2}$$

where; μ is the gravitational parameter of E, J_n is the zonal harmonic coefficient of degree n, R_E is the mean equatorial radius of E, R is the distance from E^* to B^*, P_n is the Legendre polynomial of degree n and argument S_λ, S_λ is the Sine of λ, the latitude of B^* ($\mathrm{S}_\lambda = \hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}}$), P'_n is the first derivative, with respect to its argument, of P_n, $\hat{\boldsymbol{r}}$ is the unit position vector from E^* to B^*, $\hat{\boldsymbol{n}}$ is the unit vector in the direction of the axis of symmetry of E, and $\underline{I}$ is the inertia dyadic of B relative to B^*.

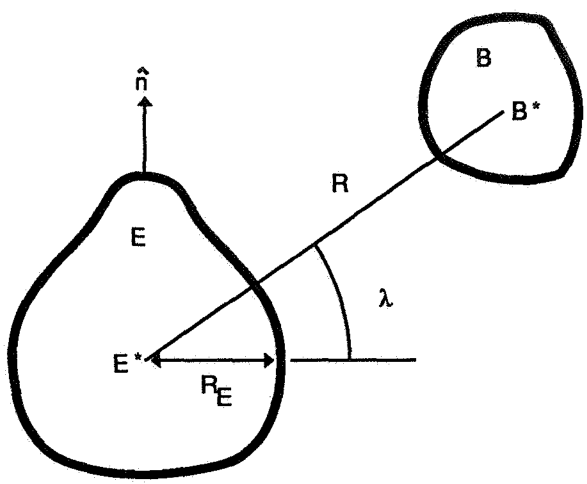

Figure 1: Body B in the Presence of Body E

Derivation

The gravitational forces exerted by a body E on a small body produce a moment $\boldsymbol{M}$ about the mass center B^* of B. $\boldsymbol{M}$ is given approximately by equation (2.18.1) of Ref [2],

$$\boldsymbol{M} = -\underline{I} \overset{\times}{\bullet} \nabla\nabla V(\boldsymbol{R}) \tag{3}$$

where $\boldsymbol{R}$ is the position vector from E^* to B^* and ∇ denotes differentiation with respect to the vector $\boldsymbol{R}$. Section 2.9 of Ref [2] contains a thorough explanation of how one differentiates with respect to a vector. The definition of the cross-dot product, $\overset{\times}{\bullet}$, appears on p. 156 of Ref [2]. The gravitational potential of E is symbolized by V.

Equation (2.13.14) in Ref [2] deals with the gravitational potential of an axisymmetric body and contains an infinite series of zonal harmonics. For a particle of unit mass coincident with B^*,

$$V = \frac{\mu}{R}\left[1 - \sum_{n=2}^{\infty}\left(\frac{R_E}{R}\right)^n J_n P_n(\mathrm{S}_\lambda)\right] \tag{4}$$

where S_λ, the argument of P_n, is equal to the sine of the geographic latitude of B^*. Eq. (4) can be simplified to

$$V = \frac{\mu}{R} + \sum_{n=2}^{\infty} V_n \tag{5}$$

when the contribution to the gravitational potential of the zonal harmonic of degree n, V_n, is defined as

$$V_n \triangleq -\frac{\mu {R_E}^n}{R^{n+1}} J_n P_n \tag{6}$$

Ref [1] shows that $-\underline{I} \overset{\times}{\bullet} \nabla\nabla(\mu/R) = (3\mu/R^3)\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}}$. Hence, eq. (3) can be rewritten as eq. (1),

$$\boldsymbol{M} = \frac{3\mu}{R^3}\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} + \sum_{n=2}^{\infty} \boldsymbol{M}_n \tag{7}$$

so long as the contribution to the gravitational moment of the zonal harmonic of degree n, $\boldsymbol{M}_n$, is defined as

$$\boldsymbol{M}_n \triangleq -\underline{I} \overset{\times}{\bullet} \nabla\nabla V_n \tag{8}$$

The Legendre polynomial of degree n, $P_n(x)$, is expressed recursively in equation (8.71) of Ref [3] in terms of Legendre polynomials of degree $n-1$, $n-2$, and their argument, x, for $n \geq 2$, as follows:

$$P_n(x) = \frac{1}{n}[(2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)] \tag{9}$$

Eq. (9) can also be produced with $m = 0$ in formula I of Table 1 in Ref [4]. Substituting from this recursion relation for the Legendre polynomials into eq. (6) leads to a recursive expression for V_n:

$$V_n = \frac{2n-1}{n}\frac{J_n}{J_{n-1}}\frac{R_E}{R}\mathrm{S}_\lambda V_{n-1} - \frac{n-1}{n}\frac{J_n}{J_{n-2}}\left(\frac{R_E}{R}\right)^2 V_{n-2} \tag{10}$$

Eq. (8) requires that a dyadic be formed by differentiating V_n twice with respect to $\boldsymbol{R}$. The first derivative of V_n with respect to $\boldsymbol{R}$ yields a recursion relation for the vector ∇V_n, the contribution of the zonal harmonic of degree n to the gravitational force exerted by E on a particle of unit mass coincident with B^*.

$$\nabla V_n = \frac{2n-1}{n}\frac{J_n}{J_{n-1}}R_E\left[\frac{V_{n-1}}{R}\nabla \mathrm{S}_\lambda + \frac{\mathrm{S}_\lambda}{R}\nabla V_{n-1} - \frac{\mathrm{S}_\lambda V_{n-1}}{R^3}\boldsymbol{R}\right] - \frac{n-1}{n}\frac{J_n}{J_{n-2}}{R_E}^2\left[\frac{\nabla V_{n-2}}{R^2} - \frac{2V_{n-2}}{R^4}\boldsymbol{R}\right] \tag{11}$$

The second derivative of V_n with respect to $\boldsymbol{R}$ yields a recursion relation for the symmetric dyadic $\nabla\nabla V_n$,

$$\begin{aligned}\nabla\nabla V_n \;=\; & \frac{2n-1}{n}\frac{J_n}{J_{n-1}}R_E\Big\{\frac{V_{n-1}}{R}\nabla\nabla \mathrm{S}_\lambda + \frac{\nabla \mathrm{S}_\lambda \nabla V_{n-1}}{R} - \frac{V_{n-1}\nabla \mathrm{S}_\lambda \boldsymbol{R}}{R^3} \\ + & \frac{\nabla V_{n-1}\nabla \mathrm{S}_\lambda}{R} + \frac{\mathrm{S}_\lambda}{R}\nabla\nabla V_{n-1} - \frac{\mathrm{S}_\lambda \nabla V_{n-1}\boldsymbol{R}}{R^3} \\ - & \frac{V_{n-1}\boldsymbol{R}\nabla \mathrm{S}_\lambda}{R^3} - \frac{\mathrm{S}_\lambda \boldsymbol{R}\nabla V_{n-1}}{R^3} + \frac{3\mathrm{S}_\lambda V_{n-1}\boldsymbol{R}\boldsymbol{R}}{R^5} - \frac{\mathrm{S}_\lambda V_{n-1}}{R^3}\underline{U}\Big\} \\ - & \frac{n-1}{n}\frac{J_n}{J_{n-2}}{R_E}^2\Big\{\frac{\nabla\nabla V_{n-2}}{R^2} - \frac{2\nabla V_{n-2}\boldsymbol{R}}{R^4} \\ - & \frac{2\boldsymbol{R}\nabla V_{n-2}}{R^4} + \frac{8V_{n-2}\boldsymbol{R}\boldsymbol{R}}{R^6} - \frac{2V_{n-2}}{R^4}\underline{U}\Big\}\end{aligned} \tag{12}$$

where $\underline{U}$ is the unit dyadic.

Cross-Dot Products

We now perform the cross-dot product with dyadics $\underline{I}$ and $\nabla\nabla V_n$, making use of the right hand side of eq. (12) and the cross-dot identity $\underline{I} \overset{\times}{\bullet} \underline{U} = \mathbf{0}$, which is set forth in eq. (19) of Ref [1], obtaining

$$\begin{aligned}
\underline{I} \overset{\times}{\bullet} \nabla\nabla V_n &= \frac{2n-1}{n}\frac{J_n}{J_{n-1}}R_E\Big\{\frac{V_{n-1}}{R}\underline{I} \overset{\times}{\bullet} \nabla\nabla \mathrm{S}_\lambda - \frac{V_{n-1}}{R^3}\underline{I} \overset{\times}{\bullet} (\nabla \mathrm{S}_\lambda \boldsymbol{R} + \boldsymbol{R}\nabla \mathrm{S}_\lambda) \\
&+ \frac{\mathrm{S}_\lambda}{R}\underline{I} \overset{\times}{\bullet} \nabla\nabla V_{n-1} + \frac{1}{R}\underline{I} \overset{\times}{\bullet} (\nabla \mathrm{S}_\lambda \nabla V_{n-1} + \nabla V_{n-1}\nabla \mathrm{S}_\lambda) \\
&- \frac{\mathrm{S}_\lambda}{R^3}\underline{I} \overset{\times}{\bullet} (\nabla V_{n-1}\boldsymbol{R} + \boldsymbol{R}\nabla V_{n-1}) + \frac{3\mathrm{S}_\lambda V_{n-1}}{R^5}\underline{I} \overset{\times}{\bullet} \boldsymbol{RR} - 0\Big\} \\
&- \frac{n-1}{n}\frac{J_n}{J_{n-2}}{R_E}^2\Big\{\frac{1}{R^2}\underline{I} \overset{\times}{\bullet} \nabla\nabla V_{n-2} + \frac{8V_{n-2}}{R^6}\underline{I} \overset{\times}{\bullet} \boldsymbol{RR} - 0 \\
&- \frac{2}{R^4}\underline{I} \overset{\times}{\bullet} (\nabla V_{n-2}\boldsymbol{R} + \boldsymbol{R}\nabla V_{n-2})\Big\} \qquad (13)
\end{aligned}$$

In order to carry out the cross-dot products with $\underline{I}$ and the other dyadics on the right side of eq. (13), we will express these dyadics in terms of $\boldsymbol{R}$ and $\hat{\boldsymbol{n}}$, a unit vector parallel to the axis of symmetry of E.

The sine of λ can be expressed as $\sin\lambda = (\boldsymbol{R}\bullet\hat{\boldsymbol{n}})/R = \hat{\boldsymbol{r}}\bullet\hat{\boldsymbol{n}}$ so that the first derivative of S_λ with respect to $\boldsymbol{R}$ is

$$\nabla \mathrm{S}_\lambda = \nabla\frac{\boldsymbol{R}\bullet\hat{\boldsymbol{n}}}{R} = \frac{\hat{\boldsymbol{n}}}{R} - \frac{(\boldsymbol{R}\bullet\hat{\boldsymbol{n}})\boldsymbol{R}}{R^3} \qquad (14)$$

and the second derivative of S_λ with respect to $\boldsymbol{R}$ is

$$\nabla\nabla \mathrm{S}_\lambda = \frac{3(\boldsymbol{R}\bullet\hat{\boldsymbol{n}})\boldsymbol{RR}}{R^5} - \frac{1}{R^3}[\hat{\boldsymbol{n}}\boldsymbol{R} + \boldsymbol{R}\hat{\boldsymbol{n}} + (\boldsymbol{R}\bullet\hat{\boldsymbol{n}})\underline{U}] \qquad (15)$$

Eq. (20) of Ref [1] is a derivation of a cross-dot identity which will be used repeatedly throughout the sequel: For any dyad $\boldsymbol{uv}$ composed of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, it is shown that $\underline{I} \overset{\times}{\bullet} \boldsymbol{uv} = -\boldsymbol{u}\times\underline{I}\bullet\boldsymbol{v}$. By making use of this identity, and eq. (15) above, one can write

$$\underline{I} \overset{\times}{\bullet} \nabla\nabla \mathrm{S}_\lambda = \frac{1}{R^2}(\hat{\boldsymbol{n}}\times\underline{I}\bullet\hat{\boldsymbol{r}} + \hat{\boldsymbol{r}}\times\underline{I}\bullet\hat{\boldsymbol{n}}) + 0 - \frac{3(\boldsymbol{R}\bullet\hat{\boldsymbol{n}})}{R^3}\hat{\boldsymbol{r}}\times\underline{I}\bullet\hat{\boldsymbol{r}} \qquad (16)$$

where $\hat{\boldsymbol{r}}$ is a unit vector in the direction of $\boldsymbol{R}$. By recalling the definition in eq. (6), one can evaluate the first cross-dot product on the right side of eq. (13).

$$\frac{V_{n-1}}{R}\underline{I} \overset{\times}{\bullet} \nabla\nabla S_\lambda =$$
$$-\mu J_{n-1} {R_E}^{n-1} \frac{P_{n-1}}{R^{n+3}} [\hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{r}} + \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{n}} - 3(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}}] \quad (17)$$

The dyadic required for the second cross-dot product on the right side of eq. (13) is easily constructed by using eq. (14):

$$\nabla S_\lambda \boldsymbol{R} + \boldsymbol{R} \nabla S_\lambda = \hat{\boldsymbol{n}}\hat{\boldsymbol{r}} + \hat{\boldsymbol{r}}\hat{\boldsymbol{n}} - 2(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} \quad (18)$$

Thus,

$$\frac{V_{n-1}}{R^3}\underline{I} \overset{\times}{\bullet} (\nabla S_\lambda \boldsymbol{R} + \boldsymbol{R}\nabla S_\lambda) =$$
$$-\mu J_{n-1} {R_E}^{n-1} \frac{P_{n-1}}{R^{n+3}} [2(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} - \hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{n}}] \quad (19)$$

Replacing n with $n-1$ in eq. (8) allows the third cross-dot product on the right side of eq. (13) to be immediately expressed in terms of $\boldsymbol{M}_{n-1}$.

$$\frac{S_\lambda}{R}\underline{I} \overset{\times}{\bullet} \nabla\nabla V_{n-1} = -\frac{\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}}}{R}\boldsymbol{M}_{n-1} \quad (20)$$

The fourth cross-dot product on the right side of eq. (13) contains the quantity ∇V_{n-1}. Temporarily allow x to be the argument of P_{n-1} [See eq. (6)] and write the derivative of V_{n-1} with respect to $\boldsymbol{R}$ as

$$\nabla V_{n-1}(x) = -\mu J_{n-1} {R_E}^{n-1} \left[\frac{1}{R^n}\frac{\mathrm{d}}{\mathrm{dx}}P_{n-1}(x)\nabla x - \frac{nP_{n-1}(x)}{R^{n+2}}\boldsymbol{R}\right] \quad (21)$$

The first derivative of a Legendre polynomial P_n with respect to its argument is often denoted by P_n'. For $n \geq 2$, a useful recurrence formula for P_n' can be found in problem 8-9 of Ref [3], p. 393, or in formula I of Table 1 in Ref [4] (with $m = 1$):

$$P_n'(x) = \frac{1}{n-1}\left[(2n-1)xP_{n-1}'(x) - nP_{n-2}'(x)\right] \quad (22)$$

Since the argument of P_{n-1} is known to be S_λ, we can make use of eq. (14) to rewrite eq. (21) as

$$\nabla V_{n-1} = -\mu J_{n-1} {R_E}^{n-1} \left\{ \frac{P'_{n-1}}{R^n} \left[\frac{\hat{\boldsymbol{n}}}{R} - \frac{(\boldsymbol{R} \bullet \hat{\boldsymbol{n}})\boldsymbol{R}}{R^3} \right] - \frac{n P_{n-1}}{R^{n+2}} \boldsymbol{R} \right\} \tag{23}$$

The sum of two dyadics, formed by juxtaposing the vectors ∇S_λ and ∇V_{n-1} in opposite order, yields the symmetric dyadic

$$\nabla S_\lambda \nabla V_{n-1} + \nabla V_{n-1} \nabla S_\lambda = -\mu J_{n-1} {R_E}^{n-1} \left\{ \frac{2P'_{n-1}}{R^n} \left[\frac{\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}}{R^2} + \frac{(\boldsymbol{R} \bullet \hat{\boldsymbol{n}})^2}{R^6} \boldsymbol{R}\boldsymbol{R} \right.\right.$$
$$\left.\left. - \frac{(\boldsymbol{R} \bullet \hat{\boldsymbol{n}})}{R^4} (\boldsymbol{R}\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}}\boldsymbol{R}) \right] - \frac{n P_{n-1}}{R^{n+2}} \left[\frac{1}{R} (\boldsymbol{R}\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}}\boldsymbol{R}) - \frac{2(\boldsymbol{R} \bullet \hat{\boldsymbol{n}})}{R^3} \boldsymbol{R}\boldsymbol{R} \right] \right\} \tag{24}$$

Consequently, the fourth cross-dot product on the right side of eq. (13) can be expressed as

$$\frac{1}{R} \underline{I} \overset{\times}{\bullet} (\nabla S_\lambda \nabla V_{n-1} + \nabla V_{n-1} \nabla S_\lambda) = -\mu J_{n-1} {R_E}^{n-1}$$
$$\left\{ \frac{2P'_{n-1}}{R^{n+3}} \left[(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})(\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{n}} + \hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{r}}) - \hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{n}} - (\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})^2 \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} \right] \right.$$
$$\left. - \frac{n P_{n-1}}{R^{n+3}} \left[2(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}}) \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} - \hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{n}} \right] \right\} \tag{25}$$

The dyadic required for the fifth cross-dot product on the right side of eq. (13) can be constructed rather easily by employing eq. (23), which yields

$$\nabla V_{n-1} \boldsymbol{R} + \boldsymbol{R} \nabla V_{n-1} =$$
$$-\mu J_{n-1} {R_E}^{n-1} \left\{ \frac{P'_{n-1}}{R^n} \left[\hat{\boldsymbol{n}}\hat{\boldsymbol{r}} + \hat{\boldsymbol{r}}\hat{\boldsymbol{n}} - 2(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}}) \hat{\boldsymbol{r}}\hat{\boldsymbol{r}} \right] - \frac{2n P_{n-1}}{R^n} \hat{\boldsymbol{r}}\hat{\boldsymbol{r}} \right\} \tag{26}$$

so that

$$\frac{S_\lambda}{R^3} \underline{I} \overset{\times}{\bullet} (\nabla V_{n-1} \boldsymbol{R} + \boldsymbol{R} \nabla V_{n-1}) =$$
$$-\mu J_{n-1} {R_E}^{n-1} \left\{ \frac{(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}}) P'_{n-1}}{R^{n+3}} \left[2(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}}) \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} - \hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{n}} \right] \right.$$
$$\left. + \frac{2n (\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}}) P_{n-1}}{R^{n+3}} \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} \right\} \tag{27}$$

The sixth cross-dot multiplication which must be performed in order to obtain $\underline{I} \overset{\times}{\bullet} \nabla\nabla V_n$ is one of the easiest to carry out. That is,

$$\frac{3S_\lambda V_{n-1}}{R^5}\underline{I} \overset{\times}{\bullet} \boldsymbol{RR} = 3\mu J_{n-1}{R_E}^{n-1}(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})\frac{P_{n-1}}{R^{n+3}}\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} \tag{28}$$

The seventh cross-dot product on the right side of eq. (13) can be expressed in terms of $\boldsymbol{M}_{n-2}$. Replacing n with $n-2$ in eq. (8), we get

$$\frac{1}{R^2}\underline{I} \overset{\times}{\bullet} \nabla\nabla V_{n-2} = -\frac{1}{R^2}\boldsymbol{M}_{n-2} \tag{29}$$

The eighth cross-dot product to be evaluated is simply

$$\frac{8V_{n-2}}{R^6}\underline{I} \overset{\times}{\bullet} \boldsymbol{RR} = 8\mu J_{n-2}{R_E}^{n-2}\frac{P_{n-2}}{R^{n+3}}\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} \tag{30}$$

The dyadic required for the final cross-dot product is similar to that needed for the fifth cross-dot product,

$$\begin{aligned}&\nabla V_{n-2}\boldsymbol{R} + \boldsymbol{R}\nabla V_{n-2} = \\ &\quad -\mu J_{n-2}{R_E}^{n-2}\left\{\frac{P'_{n-2}}{R^{n-1}}\left[\hat{\boldsymbol{n}}\hat{\boldsymbol{r}} + \hat{\boldsymbol{r}}\hat{\boldsymbol{n}} - 2(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right] - \frac{2(n-1)P_{n-2}}{R^{n-1}}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right\}\end{aligned} \tag{31}$$

so that

$$\begin{aligned}&\frac{2}{R^4}\underline{I} \overset{\times}{\bullet} (\nabla V_{n-2}\boldsymbol{R} + \boldsymbol{R}\nabla V_{n-2}) = \\ &\quad -\mu J_{n-2}{R_E}^{n-2}\left\{\frac{2P'_{n-2}}{R^{n+3}}\left[2(\hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}})\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} - \hat{\boldsymbol{n}} \times \underline{I} \bullet \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{n}}\right]\right. \\ &\quad + \left.\frac{4(n-1)P_{n-2}}{R^{n+3}}\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}}\right\}\end{aligned} \tag{32}$$

Substituting from eqs. (17), (19), (20), (25), (27) - (30), and (32) into (13) and then into (8) leads to eq. (2), which is a recursive vector-dyadic expression for the contribution of the nth zonal harmonic to the gravitational moment.

Examples

In order to demonstrate the use of eq. (2), we will use it to obtain $\boldsymbol{M}_2$, the contribution to the gravitational moment from the zonal harmonic of degree 2.

The two required Legendre polynomials are $P_1(S_\lambda) = S_\lambda = \hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{n}}$ and $P_0(S_\lambda) = 1$. Legendre polynomials of degree greater than or equal to 2 can be obtained recursively by using eq. (9). The two derivatives of Legendre polynomials which will be needed are $P_1'(S_\lambda) = 1$ and $P_0'(S_\lambda) = 0$. Derivatives with respect to the argument of Legendre polynomials can be generated with the recursion formula (22) for $n \geq 2$.

Eq. (2) also requires knowledge of $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ in order to produce $\boldsymbol{M}_2$. Eq. (8) is helpful in developing expressions for $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$.

The Legendre polynomial of degree zero is equal to 1, regardless of its argument, and the scalar V_0 [See eq. (6)] is

$$V_0 = -\mu J_0 P_0/R = -\mu J_0/R \tag{33}$$

The dyadic formed with V_0 is then

$$\nabla\nabla V_0 = \nabla\nabla(-\mu J_0/R) = \nabla(\mu J_0 \boldsymbol{R}/R^3) = \frac{\mu J_0}{R^3}(\underline{U} - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}) \tag{34}$$

Eq. (8) tells us that

$$\boldsymbol{M}_0 = -\frac{3\mu J_0}{R^3}\hat{\boldsymbol{r}} \times \underline{I} \bullet \hat{\boldsymbol{r}} \tag{35}$$

J_0 is an undefined constant, but the coefficient of $\boldsymbol{M}_0$ in eq. (2) contains J_0 in the denominator. Hence, a numerical value of J_0 is not required for constructing $\boldsymbol{M}_2$.

A similar process leads to $\boldsymbol{M}_1$. The value of the Legendre polynomial of degree 1 is identical to the argument, so the scalar V_1 is

$$V_1 = -\mu J_1 R_E \frac{P_1}{R^2} = -\mu J_1 R_E \frac{(\boldsymbol{R} \bullet \hat{\boldsymbol{n}})}{R^3} \tag{36}$$

The dyadic formed with V_1 is

$$\nabla\nabla V_1 = -\mu J_1 R_E \nabla\nabla \frac{(\boldsymbol{R} \bullet \hat{\boldsymbol{n}})}{R^3} = -\mu J_1 R_E \nabla\left[\frac{\hat{\boldsymbol{n}}}{R^3} - \frac{3(\boldsymbol{R} \bullet \hat{\boldsymbol{n}})}{R^5}\boldsymbol{R}\right] =$$
$$-\mu J_1 R_E \left\{\frac{15(\boldsymbol{R} \bullet \hat{\boldsymbol{n}})}{R^7}\boldsymbol{R}\boldsymbol{R} - \frac{3}{R^5}\left[\boldsymbol{R}\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}}\boldsymbol{R} + (\boldsymbol{R} \bullet \hat{\boldsymbol{n}})\underline{U}\right]\right\} \tag{37}$$

so that

$$M_1 = \frac{3\mu J_1 R_E}{R^4}\left[\hat{n} \times \underline{I} \bullet \hat{r} + \hat{r} \times \underline{I} \bullet \hat{n} - 5(\hat{r} \bullet \hat{n})\hat{r} \times \underline{I} \bullet \hat{r}\right] \tag{38}$$

Like J_0, the constant J_1 is undefined, and unneeded for the purpose of obtaining $\boldsymbol{M}_2$. Note that $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ do not represent contributions to the gravitational moment, but are required to begin the process of recursion which will generate moment contributions beginning with $\boldsymbol{M}_2$.

By substituting from eqs (35) and (38) into (2), we arrive at the following result with $n = 2$:

$$\begin{aligned} M_2 &= \frac{\mu J_2 {R_E}^2}{2R^5}\Big\{[30(\hat{r} \bullet \hat{n})]\,(\hat{n} \times \underline{I} \bullet \hat{r} + \hat{r} \times \underline{I} \bullet \hat{n}) \\ &+ \left[15 - 105(\hat{r} \bullet \hat{n})^2\right] \hat{r} \times \underline{I} \bullet \hat{r} - 6\hat{n} \times \underline{I} \bullet \hat{n}\Big\} \end{aligned} \tag{39}$$

If eq. (1) of Ref [1] is expressed as $\boldsymbol{M} = (3\mu/R^3)\hat{r} \times \underline{I} \bullet \hat{r} + \boldsymbol{M}_2$, it can be seen that $\boldsymbol{M}_2$ from Ref [1] is identical to eq. (39), above.

The contribution $\boldsymbol{M}_3$ can be obtained in a similar manner, using the values of $P_2(\mathrm{S}_\lambda)$, $P_1(\mathrm{S}_\lambda)$, $P_2'(\mathrm{S}_\lambda)$, $P_1'(\mathrm{S}_\lambda)$, $\boldsymbol{M}_2$, and $\boldsymbol{M}_1$.

$$\begin{aligned} M_3 &= \frac{\mu J_3 {R_E}^3}{6R^6}\Big\{\left[315(\hat{r} \bullet \hat{n})^2 - 45\right](\hat{n} \times \underline{I} \bullet \hat{r} + \hat{r} \times \underline{I} \bullet \hat{n}) \\ &+ \left[315(\hat{r} \bullet \hat{n}) - 945(\hat{r} \bullet \hat{n})^3\right] \hat{r} \times \underline{I} \bullet \hat{r} - 90(\hat{r} \bullet \hat{n})\hat{n} \times \underline{I} \bullet \hat{n}\Big\} \end{aligned} \tag{40}$$

Conclusions

A recursive vector-dyadic expression for the contribution of a zonal harmonic of degree n to the gravitational moment about the mass center of a small body can be obtained by a procedure which involves differentiating a celestial body's gravitational potential twice with respect to a vector. The recursive property of the result is a consequence of taking advantage of a recursion relation for Legendre polynomials that appear in the gravitational potential. When a celestial body's gravitational potential includes zonal harmonics, the vector-dyadic expression above is useful for calculating their contributions to the gravitational moment. The contribution of the zonal harmonic of degree 2 is consistent with the gravitational moment exerted by an oblate spheroid.

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