

The Generalized Tsiolkovsky Equation

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Abstract

The performance of a rocket, say a chemical rocket, can be greatly improved by dynamically transferring energy from one part of the propellant to another. Although with present technology the achievable degree of energy concentration is low, payload increases on the order of 20% are theoretically possible. With unlimited energy redistribution increases of a factor of two are possible.

Introduction

Some eighty-five years have passed since K.E. Tsiolkovsky published the equation for the ideal-velocity-gain of a rocket:

$$V_i = V_e \ln(m_0 / m_1) \quad (\text{Tsiolkovsky}) \quad \text{.....(1)}$$

Here V_i is the ideal-velocity-gain.

V_e is the effective exhaust velocity of the mass ejecta.

m_0 and m_1 are the initial and final rocket masses, respectively.

It is customary to introduce two derived quantities, the mass-ratio (μ) and the specific-impulse (I_{sp}):

$$\mu = m_0 / m_1, \quad I_{sp} = V_e / g \quad \text{.....(2)}$$

where g is the earth's gravitational constant.

Typical values for I_{sp} are 200 to 450, corresponding to V_e of 1.96 to 4.41 km/sec. The upper range is achieved by high-performance LOX-LH (liquid-oxygen/liquid-hydrogen) propulsion systems such as the Space Shuttle. Exotic oxydizers such as ozone or fluorine can push this into the low 500's, but this seems to be about the chemical propulsion limit. Usable and practical mass-ratios are in the range 5 to 10. For structural reasons values beyond 20 are very difficult to achieve.

The important thing about (1) (and this was most emphatically noted by the early space pioneers) is that the mass-ratio, and therefore the size of the rocket, increases exponentially with the required ideal-velocity-gain. A single-stage-to-orbit using LOX-LH requires a mass-ratio of about 10 (after performance losses are taken into account). This is difficult but achievable. But a low performance propellant ($I_{sp} = 200$) would require a mass-ratio of 150, which is essentially impossible.

Let us now look at an apparently unrelated situation: the fission of a particle (m_0) into two particles (m_1, m_2) with a release of energy (E), which goes wholly into kinetic energy. The result, familiar from undergraduate physics, is

$$V_1 = \sqrt{\frac{2E/m_1}{1 + m_1/m_2}} \quad \text{.....(4)}$$

where V_1 is the velocity of particle one.

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Suppose now that all the energy resides initially in particle two, and that this energy has a specific energy (energy per unit mass) of

$$\sigma = E/m_2 = V_e^2/2 \quad \dots(5)$$

Substituting this into (4) and introducing the mass-ratio we obtain

$$V_1 = V_e(\sqrt{\mu} - 1/\sqrt{\mu}) \quad \dots(6a)$$

as opposed to the classical (Tsiolkovsky) equation

$$V_1 = V_e \ln(\mu) \quad \dots(6b)$$

As expected, (6a) always produces a greater value than (6b), as shown in Figure 1.

A natural question is the reason for this discrepancy. The usual explanation is that the classical rocket suffers by having to drag its propellant along. But this is only partly correct. A better answer lies in the inability of the classical rocket to transfer energy from one part of the propellant to another.

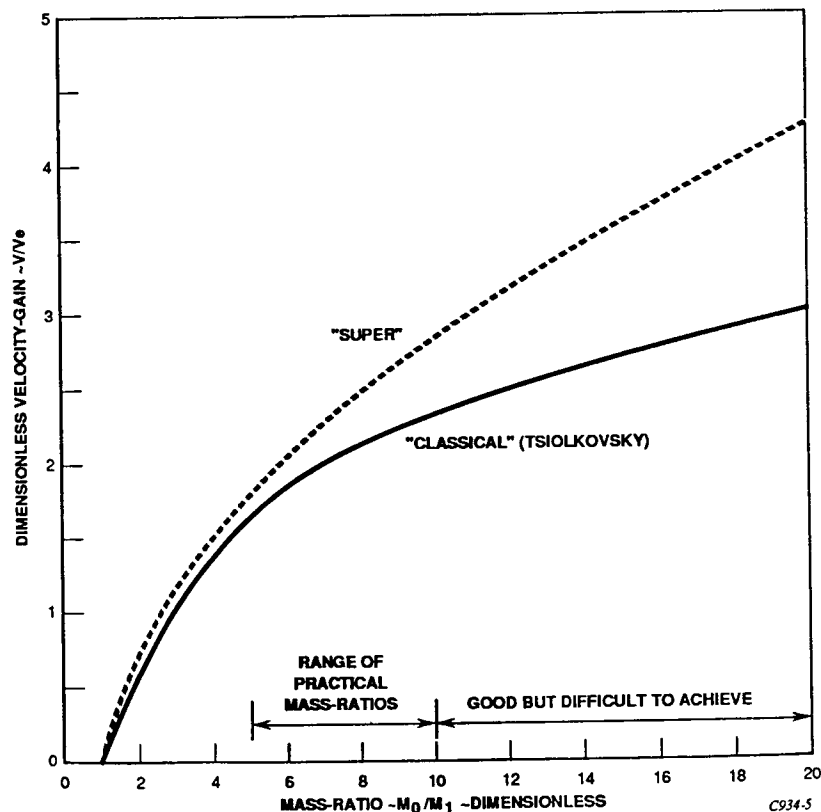


Figure 1. Performance Comparison of Classical and "Super" Rockets

emphatically, it does not need to occur instantaneously. Let us agree to call the rocket which obeys (6a) the "super" rocket. Figure 2 provides additional performance comparisons between the classical and super rocket. The first curve shows how much additional specific impulse is needed by the classical rocket to match the super, as a function of mass-ratio. The second curve shows how much the payload (final mass) is increased on the super with the same initial energy. For a mass-ratio of 20 the super has an 84% higher payload, which is equivalent to increasing the specific impulse in the classical rocket by 42%.

Methods of Energy Modulation

Do practical methods exist for transferring energy from one part of the propellant to another? The answer is, yes and no. Ways certainly exist, some may even be "practical". The problem is that it is not possible to achieve the high exhaust-velocity modulation required by the previous theory. Let us consider some of the ways available:

1. mechanical -- the energy is directly stored as rotational kinetic energy.
2. thermal -- the energy is stored as heat.
3. chemical -- the energy is concentrated by creating certain high-energy compounds or species; for instance, making ozone out of oxygen.
4. mixture-ratio shift
 - the relative ratio of two substances is changed. This may be considered to be a special case of the "chemical" method.

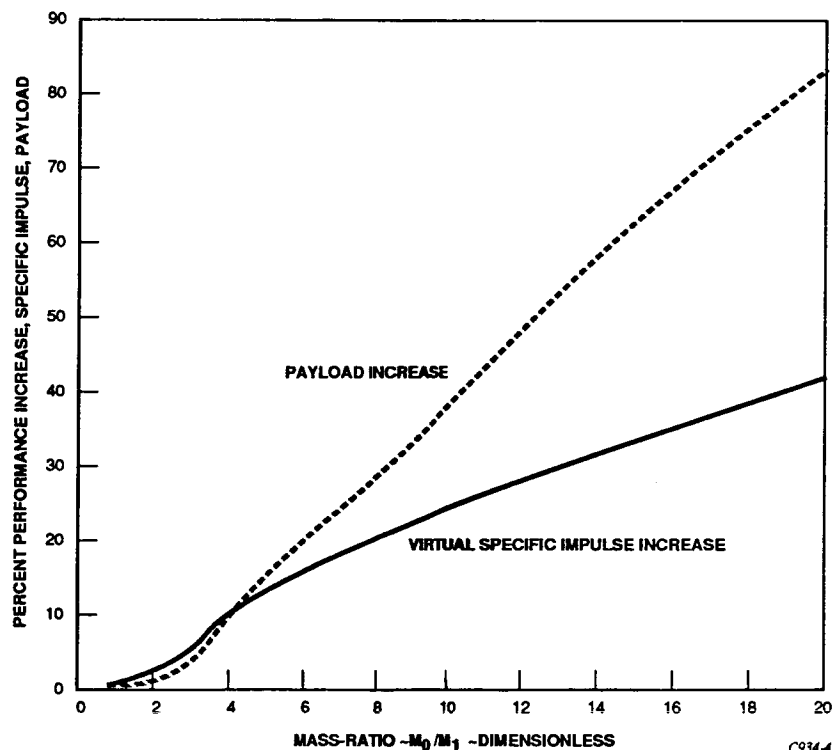


Figure 2. Percent Performance Increase for "Super"

We now show two things: 1) that (6a) is the upper limit of the velocity attainable by mass ejection of mass m_2 and expended energy E , and 2), that this limit is in fact attainable with unlimited energy redistribution.

Let $V_e(m)$ be the variable exhaust velocity, taken as a function of the propellant mass remaining, and measured with respect to the space-fixed system in which the rocket is initially at rest. The total propellant mass is m_2 (as before) and the mass of the all-burned rocket is m_1 . As before, we assume the total energy expended is E .

Then we have from the momentum and energy conservation equations :

$$\int_0^{m_2} V_e(m) dm = V_1 m_1 \quad \dots(7a)$$

and

$$\int_0^{m_2} V_e^2(m) dm + V_1^2 m_1 = 2E \quad \dots(7b)$$

Let

$$V_e = V_1 m_1 / m_2 \quad \dots(7c)$$

and introduce the "loss function", L , via

$$L = \int_0^{m_2} (V_e(m) - V_e)^2 dm \quad \dots(7d)$$

From these we derive :

$$V_1 = V_e (\sqrt{\mu} - 1/\sqrt{\mu}) \sqrt{1 - L/(V_e^2 m_2)} \quad \dots(8)$$

where V_e is essentially as given by (5), the star having been added to differentiate it from other uses of V_e .

Since $L \geq 0$ it is clear that the velocity is maximized when $L=0$, in which case (8) becomes identical to (6a). Equation (7d) shows that to achieve the condition $L=0$ we must have the exhaust velocity constant in a space-fixed frame, rather than with respect to the rocket. The exhaust velocity must therefore be of the form

$$|V_e(t)| = |V_{e,0}| + |V_1(t)| \quad \dots(9)$$

where $V_1(t)$ is the instantaneous rocket velocity and $V_{e,0}$ is the initial exhaust velocity.

Note that, as in the classical equation, there are no restrictions on the rate at which mass is expended. We can take all the time we want. We can even have bursts of mass ejection and pauses. Most

5. "exotic" -- this is a catch-all to include everything else, such as, electrical energy in capacitors, plasma, etc.

Surprisingly a conceptually simple mechanical method exists. Figure 3 shows the basic concept. The rocket is "flying-saucer-shaped", with rocket engines mounted on booms at the periphery. These are fired sequentially as the rocket rotates, so that the thrust maintains a space-fixed direction. During the initial phase energy is transferred into kinetic energy of rotation. During the subsequent phase opposing rockets are fired so that the energy is returned to the mass-ejecta stream. Because of structural limitations it is difficult to achieve peripheral rotation speeds beyond about 300 m/sec. Thus the velocity modulation is limited to about 7%, corresponding to about 14% modulation in energy density.

Comparable energy modulation is achievable using heat storage. Here the problem is not only a structural one -- high temperatures cause high pressures, requiring strong vessels -- but also a thermodynamic one: as the temperature is elevated disassociation causes energy extraction to drop.

Chemical methods do not provide a much higher energy modulation than the previous, mainly because the types of reactions in rockets are already among the most energetic. Using reactions such as the previously mentioned oxygen to ozone conversion a modulation of 20 to 30% might be possible.

Mixture-ratio shift is interesting because it is comparable in its simplicity to the mechanical method. The basic idea is to mix the ratio of two "reactants". These need not be fuel and oxidizer. Indeed the separation is a conceptual one, and they need not be physically isolated. The two species could be, 1) propellant, and 2) an unreactive tamper; or else 1) the stoichiometric part, and 2) the excess part. The modulation is generally lower than any of the previous.

As for "exotic" methods, I know of no practical ones. Perhaps electrical plasmas, in which atoms are stripped of their electrons, provide some potential.

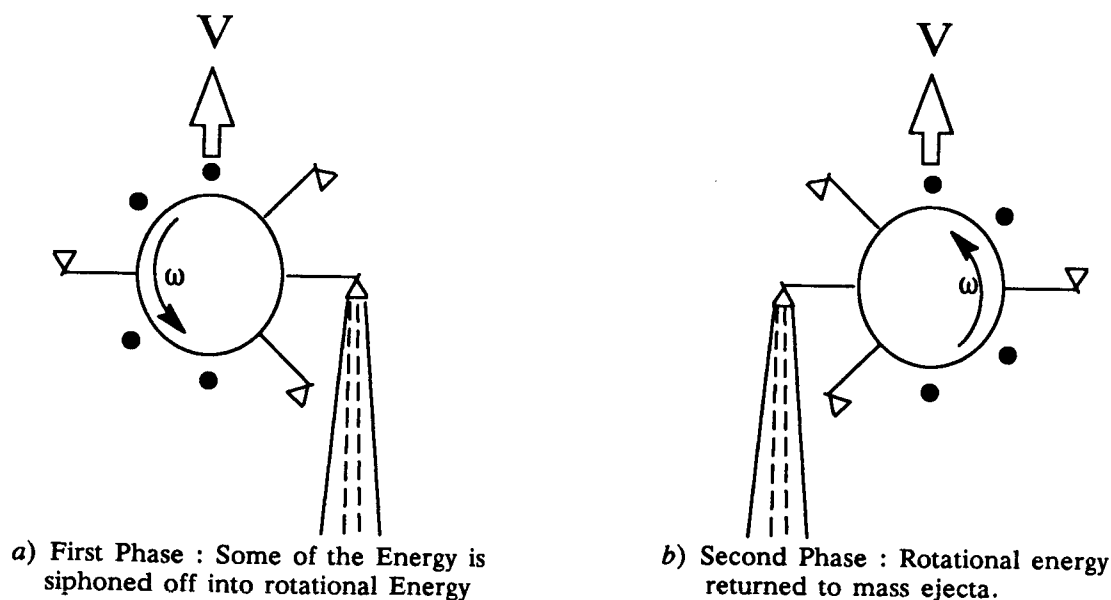


Figure 3 - Mechanical Means of Energy Modulation

The picture which emerges is this: the kinds of energy density modulation feasible are in the order of a few percent to a maximum of some 30%, rather than the factors of 25 to 100 required by the simple theory expounded in the first part of this paper. We thus need to modify the theory.

Case of Limited Energy Modulation

Suppose the total propellant (m_2) consists of two parts (m_3, m_4), where m_3 is the lower energy portion (say), and m_4 the higher. It is not necessary that these two parts be physically separated, as in different tanks; the separation may be a virtual one, in which m_3 represents that part of the propellant which is depleted in energy, while m_4 represents the energy-enhanced part. Let σ_3 and σ_4 be the corresponding specific energies. If an amount $dm_2 = dm_3 + dm_4$ of mass is ejected the amount of energy available is :

$$dE = \sigma_3 dm_3 + \sigma_4 dm_4 \quad \dots (10)$$

and the total momentum of the (differential) mass ejecta will be

$$dp = \sqrt{2dE * dm_2} \quad \dots (11)$$

and this amount of momentum will be imparted to the rocket.

Introduce now the mixture-ratio

$$r = \frac{dm_4}{dm_3 + dm_4} \quad \dots (12)$$

The velocity-gain of the rocket is now given by :

$$V_i = \sqrt{2} \int_0^{m_2} \sqrt{(1-r)\sigma_3 + r\sigma_4} dm/m \quad \dots (13)$$

where dm is the change in mass of the rocket and is the same as dm_2 .

Our objective is to find the optimal mixture-ratio profile which maximizes (13). To do this we need to add the constraint

$$m_4 = \int_0^{m_2} r dm \quad \dots (14)$$

which says nothing more than at all-burned all of m_4 is used up.

In addition we need to observe the restraint

$$0 \leq r \leq 1 \quad \dots (15)$$

The optimization of (14) can now be accomplished using either the optimal control theory formulation or the older calculus of variations. The gist of the solution is that there are three phases:

1. during the first phase $r=0$.

almost totally self-taught. By profession he was a school teacher in the city of Kaluga. His scientific studies were his avocation. At an early age he was influenced by Jules Verne toward speculations on space exploration. In this he was spectacularly successful.

Equation 1 -- the Tsiolkovsky equation -- appears to have been derived by him as early as 1895, though it was not published until 1903. He was apparently the first to have derived it, though the Soviet literature gives credit to a certain Meshchersky (1859-1935) for having done work in the dynamics of variable-mass bodies. It does not appear, though, that he derived eq. 1. (A very early rocket theoretician was the English scientist Benjamin Robins (1707-1751). He belongs to the generation following Newton. Interestingly enough, he was also largely self-taught. He did much good work on ballistics, including the determination that the exhaust velocity of blackpowder was in the order of 7000 fps.)

When a man achieves a very great station his mistakes become noteworthy. Perhaps Tsiolkovsky's greatest error occurred when he initially discounted the step or stage principle, announced by Goddard. He soon realized his mistake and became an avowed supporter. It is interesting to note that his reason for discounting staging was a too literal interpretation of eq. (1)! A more venial mistake came about as follows. Tsiolkovsky had the ingenious idea of circling the exhaust gasses, via looped pipes, prior to ejection. This, he reasoned, would provide desired stability through gyroscopic action. Realizing that a gyroscope can stabilize in only two axis, he considered two such orthogonal loops. But of course two gyroscopes rigidly tied together act as a single (or no) gyro. He died in his native country, greatly honored and revered, in 1935.

References:

1. K. E. Tsiolkovsky Selected Works, compiled by V. N. Sokolsky, MIR Publishers (Moscow), English Translation, 1968.
2. Sloop, John L., Liquid Hydrogen as a Propulsion Fuel, 1945-1959, NASA History Series, NASA Publication SP-4404, 1978. This monograph cannot be praised too highly: besides giving an excellent summary of the performance and history of liquid hydrogen, it provides excellent short biographics of the space pioneers.

