CALCULATION OF DOUBLE-LUNAR SWINGBY TRAJECTORIES: II. NUMERICAL SOLUTIONS IN THE RESTRICTED PROBLEM OF THREE BODIES*

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ABSTRACT

The double-lunar swingby trajectory is a method for maintaining alignment of an Earth satellite's line of apsides with the Sun-Earth line. From a Keplerian point of view, successive close encounters with the Moon cause discrete, instantaneous changes in the satellite's eccentricity and semimajor axis. This paper identifies numerical solutions to the planar, restricted problem of three bodies as double-lunar swingby trajectories. The method of solution is described and the results compared to the Keplerian formulation.
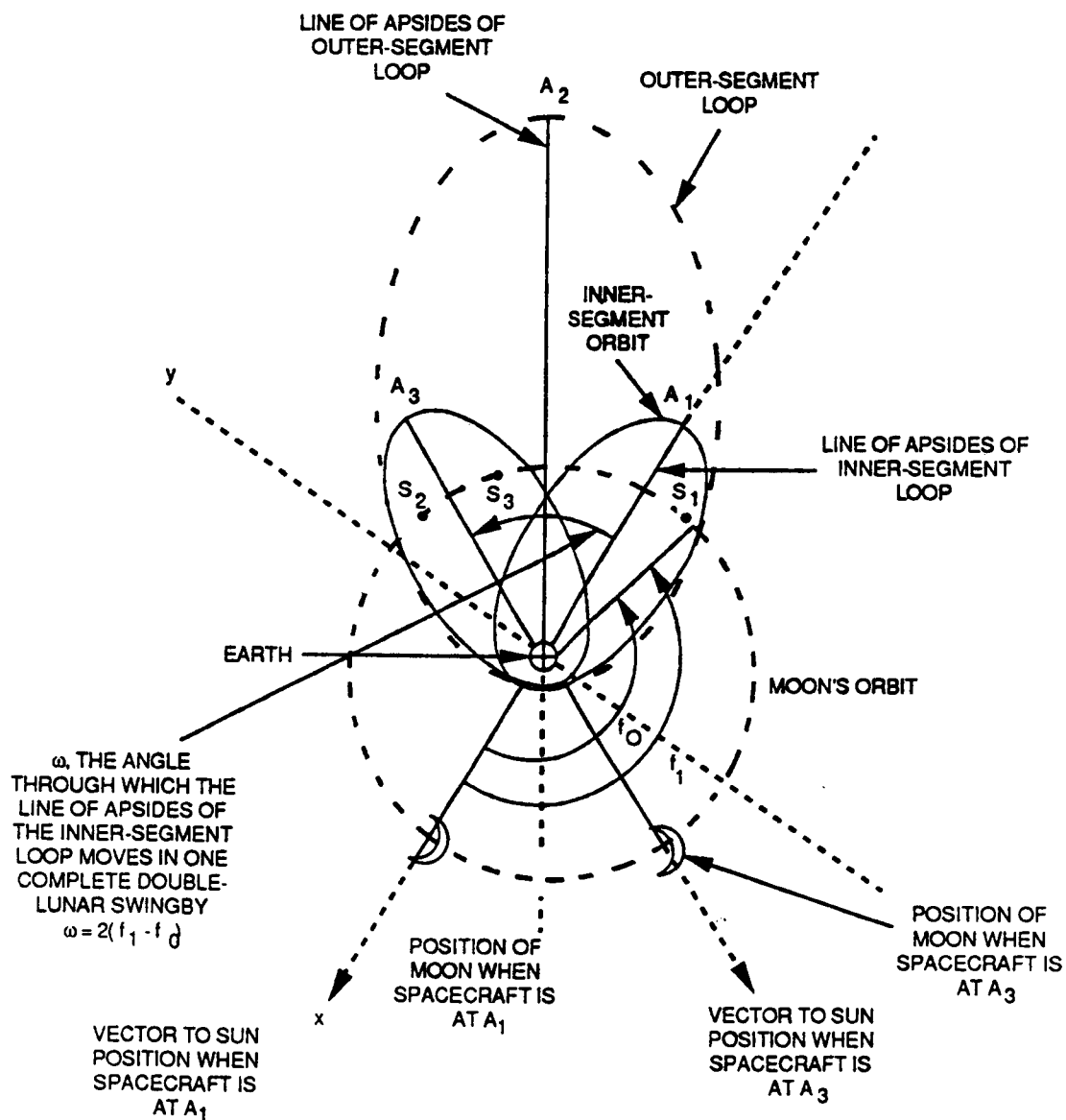
I. FORMULATION OF THE PROBLEM

Farquhar and Dunham (Reference 1) first recognized the utility of the double-lunar swingby trajectory. This trajectory uses encounters with the Moon to "rotate the line of apsides" at a rate equal to that of the Earth about the Sun (Figure 1). At the end of a complete double-lunar swingby cycle, the spacecraft, Sun, Earth, and Moon have the same relative position as at the beginning of the cycle. Hence, the trajectory is said to be periodic in the "lunar-rotating frame" as well as in the "solar-rotating frame." (The lunar-rotating frame is that coordinate system in which the Earth and Moon appear to be at rest with each other. That is, the coordinate axes rotate at the same rate as the rotation of the Moon about the Earth. Similarly, for the solar-rotating frame, the Sun and Earth are at rest with respect to each other. The coordinate axes then rotate at the rate of the mean solar motion.) Figure 1 depicts a typical double-lunar swingby trajectory in an "Earth-inertial" coordinate system; the Moon, Sun, and spacecraft motions are shown relative to axes centered on the Earth and at rest with respect to the stars. Figure 2 shows the same spacecraft trajectory viewed in the solar-rotating frame. Figure 3 gives the same spacecraft trajectory viewed in the lunar-rotating frame.

Dunham and Davis, in Reference 2, presented extensive tables describing double-lunar swingby trajectories in terms of Keplerian orbital elements. Reference 3 documented the method used to calculate these elements. In each case, the trajectory is viewed as two elliptic Earth orbits: an inner-segment loop orbit and an outer-segment loop orbit. Transfer is considered to be instantaneous between the two orbits and occurs at the lunar radius. Physically, of course, the change is not instantaneous. Other simplifications include the neglect of the Sun's gravity, the assumption of a circular lunar orbit, and the assumption that the Sun, Moon, and Earth are coplanar. How well, then, does the Keplerian formulation describe the double-lunar swingby trajectory? Further, how may we conveniently characterize such trajectories without resorting to Keplerian elements? Howell (Reference 4) addressed several of these issues and obtained numerical solutions, discussed in the following paragraphs.

One might integrate the equations of motion of the Earth, Moon, Sun, and spacecraft. A double-lunar swingby trajectory would then depend only on the spacecraft state vector (position and velocity) at a given time. This, in fact, is the procedure for a real mission. But how is that state vector determined? By satisfying the necessary equations given in Reference 3. How does one guess an initial state vector that might satisfy those equations? The answer is to build the state vector from the parameters given in Reference 2. Thus, the utility of the Keplerian formulation is seen: the model does not, however, depend on the Moon's true anomaly or argument of perigee or on the Earth's true anomaly in its orbit around the Sun. Little utility might be found in preparing massive tables that add these variables to our model.

We still may examine the Keplerian formulation's assumption of an instantaneous transfer between inner-segment and outer-segment orbits. We know that the transfer is not instantaneous. In fact, no Keplerian elements describe the possible behavior of a spacecraft under the influence of two gravitational forces. The simplest model that will describe such behavior is referred to as "the restricted problem of three bodies" (References 5

LINE OF APSIDES OF
OUTER-SEGMENT
LOOP

A_2

OUTER-SEGMENT
LOOP

INNER-
SEGMENT
ORBIT

y

A_3

A_1

LINE OF APSIDES OF
INNER-SEGMENT
LOOP

S_2 S_3 S_1

EARTH

MOON'S ORBIT

ω, THE ANGLE
THROUGH WHICH THE
LINE OF APSIDES OF
THE INNER-SEGMENT
LOOP MOVES IN ONE
COMPLETE DOUBLE-
LUNAR SWINGBY

$\omega = 2(f_1 - f_d)$

POSITION OF
MOON WHEN
SPACECRAFT IS
AT A_1

POSITION OF
MOON WHEN
SPACECRAFT IS
AT A_3

x

VECTOR TO SUN
POSITION WHEN
SPACECRAFT IS
AT A_1

VECTOR TO SUN
POSITION WHEN
SPACECRAFT IS
AT A_3

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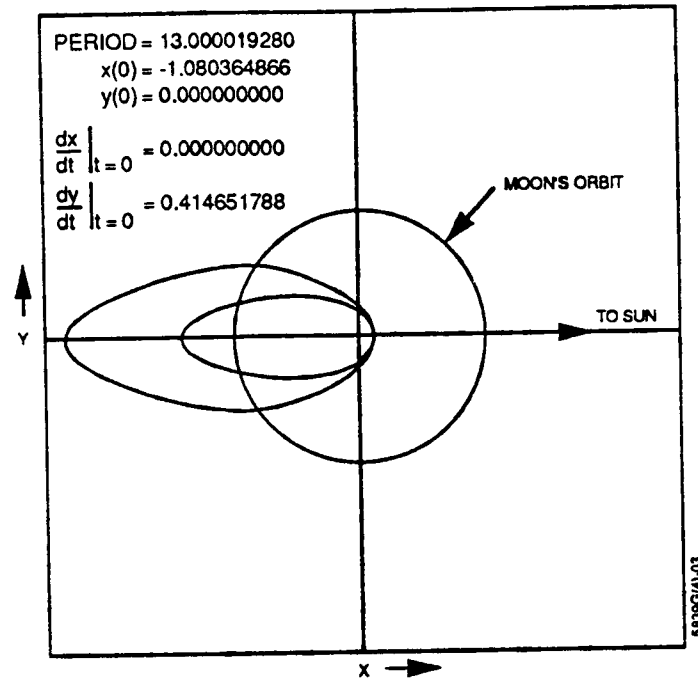**Figure 1. A Double-Lunar Swingby Trajectory**

Figure 2. Double-Lunar Swingby Trajectory As Viewed in the Solar-Rotating Frame
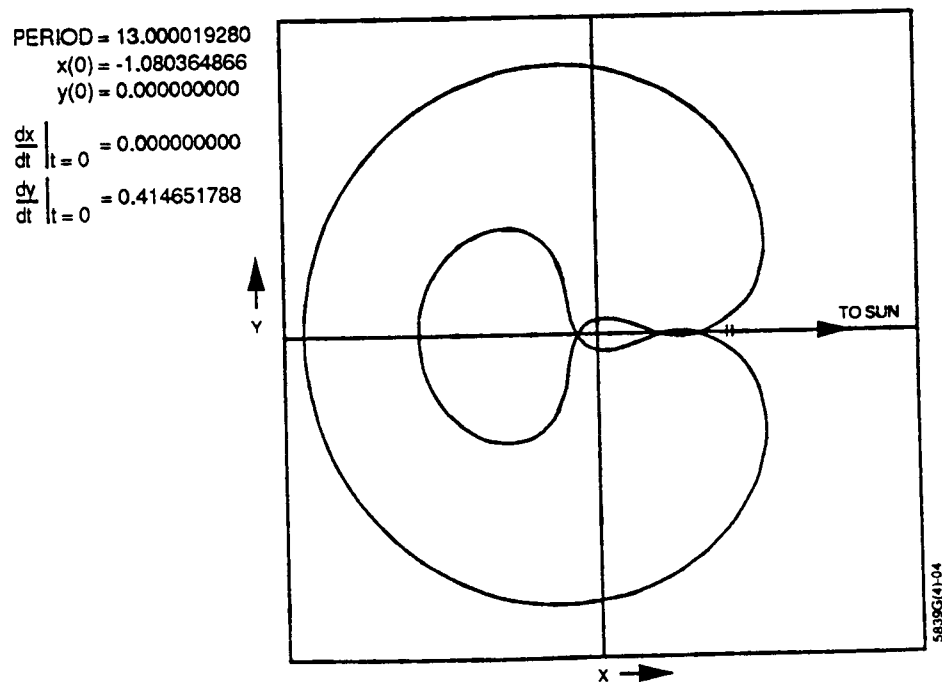

Figure 3. Double-Lunar Swingby Trajectory As Viewed in the Lunar-Rotating Frame

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through 8). In this model, a body of zero mass (the spacecraft) is acted upon by the gravitational attraction of two bodies (here, the Earth and the Moon). The two bodies of finite mass are considered to be in a circular orbit about their center of mass, and the body of zero mass moves in the plane of that orbit. That is, we require that the initial out-of-plane position and velocity be zero. Hence, any trajectory depends only on five numbers: the spacecraft position and velocity and the relative masses of the finite bodies.

Our coordinate system is shown in Figure 4. Since the x and y axes rotate in the direction of the Moon's orbit about the Earth, both the Earth and the Moon appear motionless in this coordinate system: the lunar-rotating frame. The coordinate system's origin is at the center of mass of the Earth and the Moon. The equations of motion for the spacecraft, in this coordinate system, are

$$\frac{d^2x}{dt^2} = x + 2\frac{dy}{dt} - \frac{(1 - \mu)(x + \mu)}{r_E^3} - \mu\frac{(x - 1 + \mu)}{r_M^3}$$

$$\frac{d^2y}{dt^2} = y - 2\frac{dx}{dt} - \frac{(1 - \mu)y}{r_E^3} - \frac{\mu y}{r_M^3}$$

(1-1)

where

r_E = spacecraft-Earth distance = $[(x + \mu)^2 + y^2]^{1/2}$

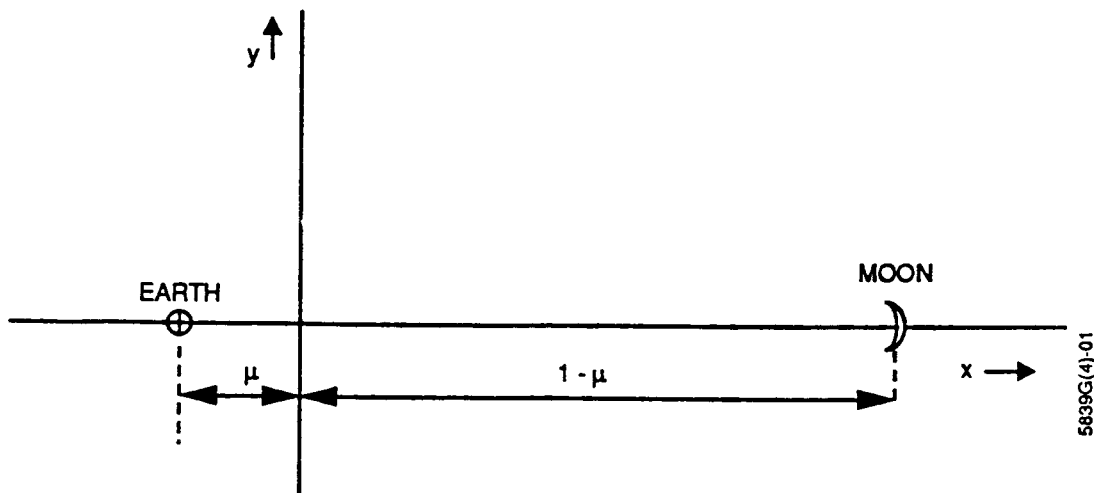r_M = spacecraft-Moon distance = $[(x - 1 + \mu)^2 + y^2]^{1/2}$

Figure 4. Lunar-Rotating Frame Used for Three-Body Equations

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These equations are derived in pages 277-280 of Reference 6. Here, the mass unit is chosen such that the sum of the Earth's and Moon's masses is 1. The length unit is chosen such that the distance between the Earth and Moon is 1. The time unit is chosen such that the mean angular motion of the Moon about the Earth is 1. The Moon's mass is denoted by μ, and the Earth's mass is then $(1 - \mu)$. The (x, y) coordinates of the Earth are $(-\mu, 0)$. The coordinates for the Moon are $(1 - \mu, 0)$.

2. DEFINITION OF PERIODIC SOLUTIONS AND METHOD OF SOLUTION

Let the period of a double-lunar swingby be T_{DLS}. Then, by the definition of periodicity, the spacecraft coordinates and velocity at time t are equal to those at time $t + T_{DLS}$. For simplicity, we choose our time $t = 0$ to occur at apogee of the outer-segment loop. From Figure 3, we note that the symmetry of the orbit requires

$$y(0) = y(T_{DLS}/2) = y(T_{DLS}) = 0$$

$$\left.\frac{dx}{dt}\right|_{t=0} = \left.\frac{dx}{dt}\right|_{t=T_{DLS}/2} = \left.\frac{dx}{dt}\right|_{t=T_{DLS}} = 0$$

(2-1)

Suppose we integrate the equations of motion from $t = 0$ to $t = T_{DLS}/2$. Since y and the x velocity are known at $t = 0$, the values of y and of the x velocity at $t = T_{DLS}/2$ are a function of only two variables: the values of x and of the y velocity at $t = 0$. Thus, we must find the solution of a set of two equations (the values of y and of the x velocity integrated from $t = 0$ to $t = T_{DLS}/2$) in two unknowns (the initial x and the initial y velocity), or

$$y(T_{DLS}/2) = \int_0^{T_{DLS}/2} \frac{dy}{dt}\, dt \equiv f_1\left(x(0), \left.\frac{dy}{dt}\right|_{t=0}\right) = 0$$

$$\left.\frac{dx}{dt}\right|_{t=T_{DLS}/2} = \int_0^{T_{DLS}/2} \frac{d^2x}{dt^2}\, dt \equiv f_2\left(x(0), \left.\frac{dy}{dt}\right|_{t=0}\right) = 0$$

(2-2)

Any method used for the iterative solution of nonlinear equations may be applied to Equation (2-2). Both the Newton-Raphson method used in Reference 3 and the multivariate Halley method (Reference 9) were used to solve Equation (2-2). Note that Equation (2-1) is the same for $t = T_{DLS}/2$ or for $t = T_{DLS}$, so we may also have approached the problem by integrating from $t = 0$ to $t = T_{DLS}$. Another approach, suitable
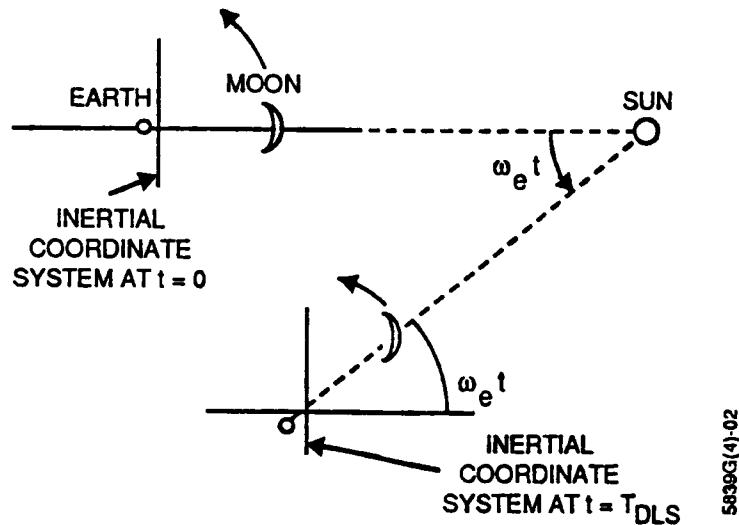
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Figure 5. Determination of the Period of a Double-Lunar Swingby, T_{DLS}

even for asymmetric periodic solutions, is to form the norm of all four variables evaluated at $t = 0$ and at $t = T_{DLS}$

$$F\left(x(0), \; y(0), \; \frac{dx}{dt}\bigg|_{t=0}, \; \frac{dy}{dt}\bigg|_{t=0}\right) \equiv \left\{ [x(0) - x(T_{DLS})]^2 + [y(0) - y(T_{DLS})]^2 \right.$$

$$\left. + \left(\frac{dx}{dt}\bigg|_{t=0} - \frac{dx}{dt}\bigg|_{t=T_{DLS}}\right)^2 + \left(\frac{dy}{dt}\bigg|_{t=0} - \frac{dy}{dt}\bigg|_{t=T_{DLS}}\right)^2 \right\}$$

$$(2\text{-}3)$$

One then seeks to minimize this norm by any nonlinear optimization algorithm. Both a discrete Newton and the Davidon-Fletcher-Powell algorithms were implemented (References 9 and 10).

We now determine T_{DLS}. Consider Figure 5, which shows the Earth-inertial coordinate system at two different times. The Earth, moving with constant angular velocity ω_e around the Sun, has traveled the angle $\omega_e t$. The Moon, moving with constant angular

velocity ω_m around the Earth, returns to its original position with respect to the Sun and the Earth when it moves through the angle

$$\omega_m t = 2\pi n + \omega_e t \tag{2-4}$$

in the time

$$t = \frac{2\pi n}{\omega_m - \omega_e} \tag{2-5}$$

For the case n = 1

$$T_{DLS} = \frac{2\pi}{\dfrac{2\pi}{27.32} - \dfrac{2\pi}{365.25}} = 29.5287 \text{ days} \tag{2-6}$$

For the case n = 2

$$T_{DLS} = \frac{4\pi}{\dfrac{2\pi}{27.32} - \dfrac{2\pi}{365.25}} = 59.057 \text{ days} \tag{2-7}$$

Knowing T_{DLS}, we may now seek numerical solutions to Equation (2-2).

To find the three-body solution for the double-lunar swingby of Figure 1, we use initial values of x and the y velocity taken from Reference 1. The inner-segment loop has an apogee of 549889 kilometers (km) and a perigee of 37436 km. Using these values, the initial x and the y velocity values in the lunar-rotating frame are calculated to be

$$x(0) = -1.41836564, \quad \left.\frac{dy}{dt}\right|_{t=0} = +1.1216628 \tag{2-8}$$

These initial conditions are then used to begin the iterative solution of Equation (2-2). The three-body solution is found to be in close agreement with the Keplerian solution. In the units of the lunar-rotating frame, the solution is

$$x(0) = -1.42050598244, \quad \left.\frac{dy}{dt}\right|_{t=0} = +1.09755070684 \qquad (2\text{-}9)$$

(The period of the trajectory, in the units of the lunar-rotating frame, is 13.600878. This corresponds to 59.057 days. We consistently use a distance unit of 384399 km and μ = 0.0121505649405.)

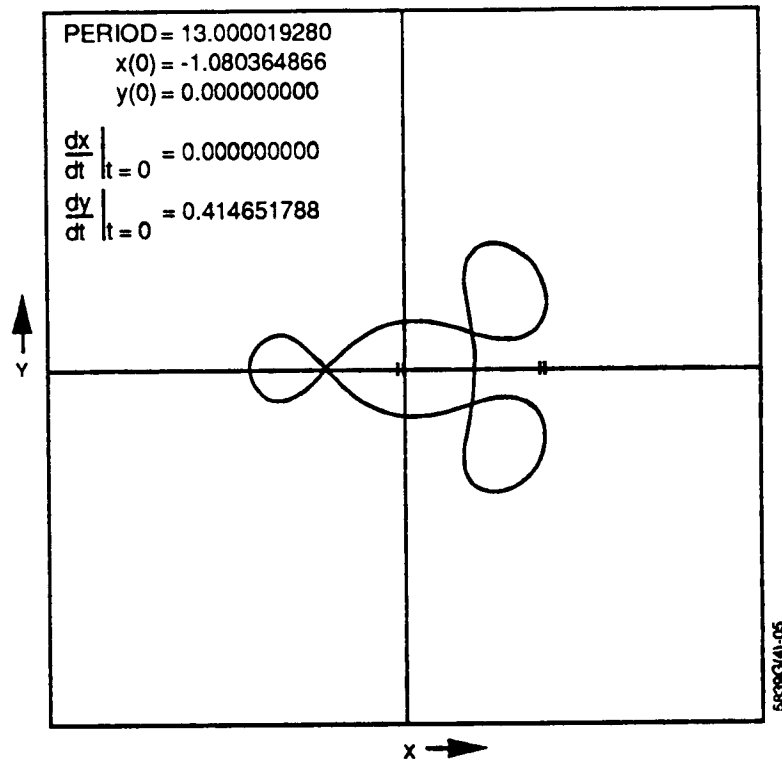This solution of the three-body problem was first obtained by Howell in Reference 4, for μ = 0.012.

Table 1 compares the three-body solution with the Keplerian solution. The values are so close, one may question the necessity for the three-body calculation. However, the three-body trajectory has no instantaneous jump in the velocity at the lunar orbit, as does the Keplerian solution. Further, of course, no simple parameterization akin to the Keplerian elements is possible for the general solution to three-body motion.

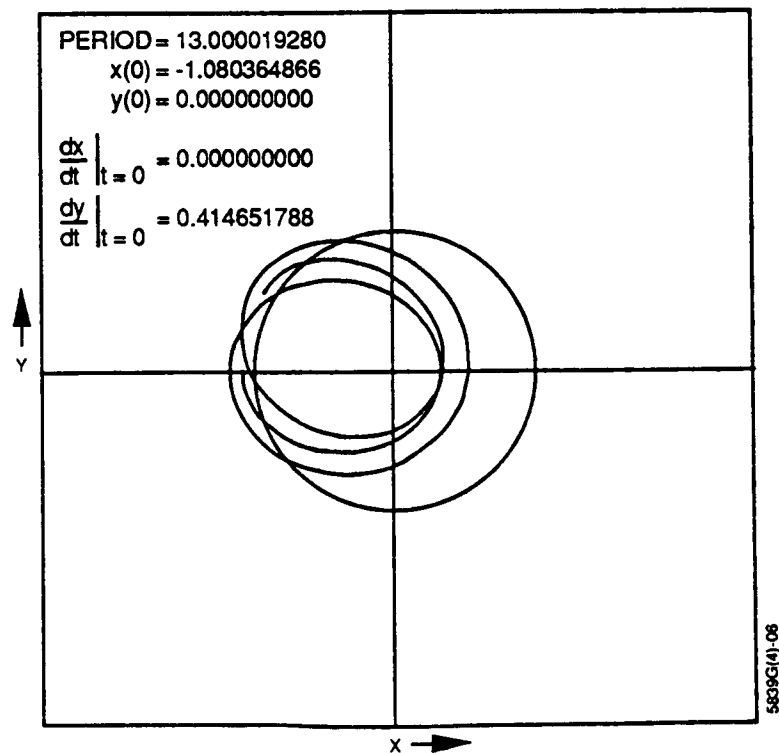### Table 1. Comparison of Three-Body Solution With Keplerian Solution of Reference 1

	Three-Body	Reference 1
Inner-segment loop apogee	541,370 km	549,889
Inner-segment loop perigee	42,350 km	37,438
Outer-segment loop apogee	892,710 km	898,915

Not every solution to the equations given in Reference 3 has a corresponding solution in the three-body problem. For example, the [1, 1, 1, 1] double-lunar trajectory shown in Figure 6, while certainly periodic in the lunar-rotating frame, is not quite periodic in the solar-rotating frame (Figure 7).

Also, the Keplerian formulation is not amenable to analysis of orbits such as that of Figure 8. Here the line of apsides is rotated *within* the lunar orbit. The apogee and perigee of this orbit are 248400 km and 120983 km, giving a period of approximately 9.14 days. The periodicity in the solar- and lunar-rotating frames is 29.5287 days for this orbit (n = 1).

PERIOD = 13.000019280
x(0) = -1.080364866
y(0) = 0.000000000

$$\frac{dx}{dt}\Big|_{t=0} = 0.000000000$$

$$\frac{dy}{dt}\Big|_{t=0} = 0.414651788$$

Figure 6. A Double-Lunar Swingby Trajectory Periodic Only in the Lunar-Rotating Frame

PERIOD = 13.000019280
x(0) = -1.080364866
y(0) = 0.000000000

$$\frac{dx}{dt}\Big|_{t=0} = 0.000000000$$

$$\frac{dy}{dt}\Big|_{t=0} = 0.414651788$$

Figure 7. The Trajectory of Figure 6, Viewed in the Solar-Rotating Frame

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PERIOD = 6.50000966
x(0) = 0.302584580
y(0) = 0.000000000

$$\left.\frac{dx}{dt}\right|_{t=0} = 0.000000000$$

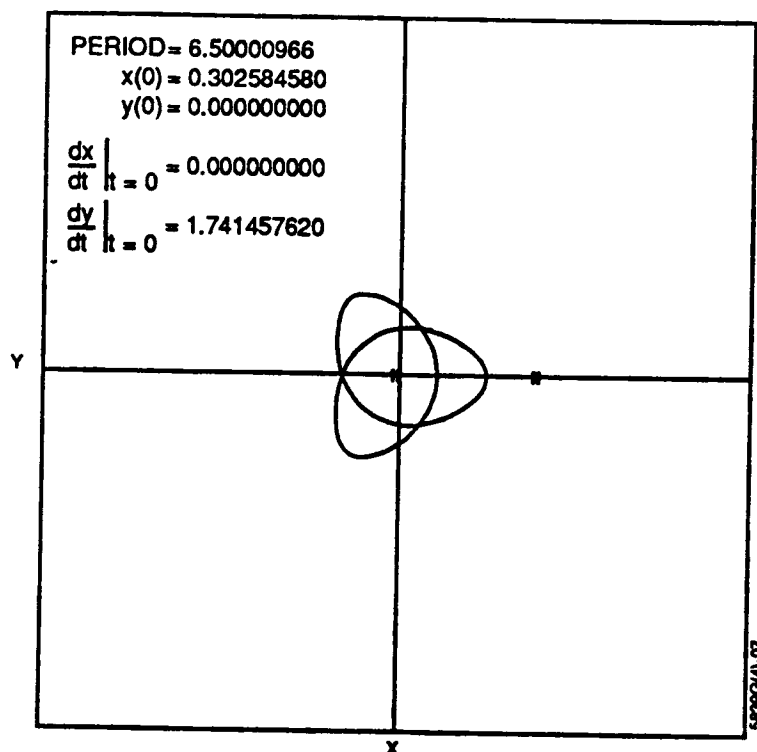$$\left.\frac{dy}{dt}\right|_{t=0} = 1.741457620$$

Figure 8. A Cislunar Periodic Solution of the Three-Body Problem

REFERENCES

1. R. W. Farquhar and D. W. Dunham, "A New Trajectory Concept for Exploring the Earth's Geomagnetic Tail," *Journal of Guidance and Control*, vol. 4, 1981, pp. 192-196

2. D. W. Dunham and S. A. Davis, "Catalog of Double-Lunar Swingby Orbits for Exploring the Earth's Geomagnetic Tail," Computer Sciences Corporation, CSC/TM-80/6322, October 1980

3. S. Stalos, "Calculation of Double-Lunar Swingby Trajectories: I. Keplerian Formulation," Flight Mechanics/Estimation Theory Symposium, May 23-24, 1989, Goddard Space Flight Center, Greenbelt, Maryland

4. K. C. Howell, "Halo Orbit Studies, Final Report," Purdue University, June 1985, pp. 56-60

5. V. Szebehely, *Theory of Orbits*, Academic Press, 1967

6. F. R. Moulton, *An Introduction to Celestial Mechanics*, Second Revised Edition, Dover Press, 1970

7. J. M. A. Danby, *Fundamentals of Celestial Mechanics*, Second Edition, Willman-Bell, 1988

8. E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Fourth Edition, Cambridge University Press, 1988

9. A. M. Cuyt, "Computational Implementation of the Multivariate Halley Method for Solving Nonlinear Systems of Equations," *ACM Transactions on Mathematical Software*, vol. 11, no. 1, March 1985, pp. 20-36

10. R. W. Daniels, *An Introduction to Numerical Methods and Optimization Techniques*, Elsevier/North Holland, 1978, pp. 226-232

11. W. E. Gill, R. T. Murray, and C. Wright, *Practical Optimization*, Academic Press, 1984