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# An Analysis of the Least-Squares Problem for the DSN Systematic Pointing Error Model

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*A systematic pointing error model is used to calibrate antennas in the Deep Space Network. This article describes and analyzes the least-squares problem and the solution methods used to determine the model's parameters. Specifically studied are the rank-degeneracy problems resulting from beam-pointing error measurement sets that incorporate inadequate sky coverage. A least-squares parameter subset selection method is described and its applicability to the systematic error modeling process is demonstrated on a Voyager 2 measurement distribution.*

## I. Introduction

A pointing error model is used in the Deep Space Network's (DSN's) antenna-calibration process. With the exception of environmental effects, the major sources of errors in an antenna-pointing system are systematic and repetitive and therefore can be closely modeled. Examples of parameters in the model are residual errors in the geometric alignment of the mount axes and fixed-angle encoder offsets. Data collected from spacecraft and radio star observations are used to determine the parameters in the model and are then entered into the pointing system to accurately point the antenna. The origins of the pointing error modeling approach for radio-frequency (RF) antennas can be found in [1,2] while its development within the DSN is discussed in [3].[1]

[1] R. L. Riggs, "Antenna Pointing Angle Corrections," DSN Antenna Seminar, Videotapes 49–54, Jet Propulsion Laboratory, Pasadena, California, May 1986.

The complete pointing error model is the sum of its separate error components. Table 1 shows individual error sources and the elevation/cross-elevation (or declination/cross-declination, depending on antenna mount) regressor variables used to estimate parameters. See [1,2,4] for a more thorough discussion of these parameters. Currently, this entire model is set in motion in the antenna-pointing system by entering parameter values manually. The DSN 70-m antennas track targets in both the computer command and precision modes of operation, each defined by a set of axis position transducers. (See [5] for a discussion of the axis servos and controllers.) The 34-m antennas employ only the computer command mode. In [3,6] recommended model parameter sets are given that apply to each tracking mode of these antennas; they are also repeated in Table 2. As can be seen, nine error parameters are used to estimate in the precision mode and eight in the computer command mode. In practice, the model parameters are determined by performing a least-squares fit on the pointing offset data collected from the spacecraft and/or from radio star observations. In this article, reference will only be made to the particular combinations of parameters in Table 2.

This article explores the numerical properties of the systematic error modeling process. Specifically, the analysis focuses on the numerical properties of the matrix formed by the pointing model regressor variables evaluated over the beam-pointing error data sets. These measurement sets may not cover enough points in the sky to accurately estimate all of the parameters. This is due to the finite number of targets and to other practical operational constraints, such as lack of antenna time. On the other hand, the objective of particular calibrations may be to optimize pointing in a particular region of the sky, such as along a constant declination. In practice, however, the limited measurement sets lead to rank deficiency in the least-squares measurement distribution matrix. This study of the problem will lead to a more objective approach to parameter selection and parameter estimate interpretation. In addition, the analytical techniques provided here may be used to predict which directions in the sky will yield optimal estimation.

The remainder of this article will formulate the systematic error parameter estimation problem and then establish a hypothetical performance index for matrix conditioning. In addition, the numerical tools presented will be used to analyze practical sky distributions in the context of the least-squares approximation and the current solution method will be reviewed. The article concludes with a proposed algorithm for parameter selection.

## II. Model Generation

In order to accurately point the antenna, pointing error correction models must be generated from radio star or spacecraft pointing offset data. This section deals with the model-fitting process, which uses the least-squares algorithm and assumes that the measurement data sets are accurate. At this time, the estimation process does not deal with uncertainties in the conical scan pointing offsets and radio star boresights except in human filtering of very large nonrepeatable and unexplainable offsets.

### A. Least-Squares Problem Formulation

The parameter vector $\mathbf{p}$ of the systematic pointing error model is determined by performing a linear least-squares fit on the offset data. The estimation problem is formulated from $m$ observations as

$$\begin{bmatrix} \delta xel_1 \\ \vdots \\ \delta xel_m \\ \delta el_1 \\ \vdots \\ \delta el_m \end{bmatrix} = \begin{bmatrix} Axel_1(el_1, az_1) \\ \vdots \\ Axel_m(el_m, az_m) \\ Ael_1(el_1, az_1) \\ \vdots \\ Ael_m(el_m, az_m) \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \quad (1)$$

or

$$\mathbf{y} = \mathbf{A}\mathbf{p} \quad (2)$$

where the offset vector $\mathbf{y}$ is $2m \times 1$, the measurement distribution matrix $\mathbf{A}$ is $2m \times n$, and the parameter vector $\mathbf{p}$ is $n \times 1$. As can be seen, equations representing both the cross-elevation and elevation error functions are obtained for each single observation point in the sky. Let the least-squares estimator be $\hat{\mathbf{p}}$ and satisfy the following matrix equation

$$\hat{\mathbf{y}} = \mathbf{A}\hat{\mathbf{p}} \quad (3)$$

where the vector $\hat{\mathbf{y}}$ contains the estimated (or fitted) values to the cross-elevation and elevation offsets of Eq. (1). The difference between the individual elements, or residuals, is defined as

$$r_i = y_i - \hat{y}_i \quad (4)$$

The method of least squares chooses the parameter estimate $\hat{\mathbf{p}}$, such that the following quantity is minimized

$$\sum_{i=1}^{2m} r_i^2 \quad (5)$$

The estimate satisfies the following matrix equation [7,9]

$$\hat{\mathbf{p}} = (\mathbf{A}^t\mathbf{A})^{-1}\mathbf{A}^t\mathbf{y} \quad (6)$$

where $\mathbf{A}^t$ is the transpose of $\mathbf{A}$. Caution must be given to least-squares problems in which the regressor variables, or basis terms of $\mathbf{A}$, are not truly independent. In such cases the measurement distribution matrix $\mathbf{A}$ may be close to, or is, rank deficient. If $\mathbf{A}$ is rank deficient, then there are an infinite number of solutions to the least-squares problem and no conclusion can be drawn as to the role of the individual regressor variables [9].

During the systematic pointing error estimation process, limited data sets and inherent correlations in the pointing error model have led to rank deficiency and its associated problems. This situation was discussed in [8] where the condition of empirically correlated regressor variables was termed "multicolinearity." It was pointed out in [8] that regressor variables of the model are not truly independent. However, this is not accurate when a proper combination of parameters is selected, as recommended in Table 2. The degree of linear independence in the columns of matrix **A** for various antenna configurations is strongly dependent on the distribution of the observation points over the sky. This situation and its effect on the pointing error estimation is discussed below.

## B. All-Sky Model Analysis

An analytical approach was taken to obtain a performance index for the numerical conditioning of the systematic error least-squares problem and to compare it with results from practical measurement sets. One such performance index can be determined by examining a hypothetical all-sky uniform distribution of pointing offset data. These measurement points are used to generate measurement distribution matrices for different combinations of parameters. Intuitively, it would make sense to obtain pointing error offsets uniformly throughout the field of view of the antenna and conclude that this is the optimal distribution for input into the parameter estimation problem. However, observing the basis terms of the pointing model given in Table 2, it can be seen that not all terms are simultaneously functions of both azimuth and elevation. This condition will tend to result in redundant column elements of **A**; thus, optimal matrix conditioning will most likely not be obtained with the all-sky distribution. However, as will be shown, all-sky matrices do have acceptable conditioning and can be used for a suitable performance index. Singular value decomposition (SVD) was used to analyze the linear independence of the columns of **A** and is defined in the following theorem.

**Theorem 1.** Let $\mathbf{A}$ be a real $m \times n$ matrix with $m \geq n$. Then there is an orthogonal matrix $\mathbf{U} = [\mathbf{u_1}, \ldots, \mathbf{u_m}]$ of order $m$ and an orthogonal matrix $\mathbf{V} = [\mathbf{v_1}, \ldots, \mathbf{v_n}]$ of order $n$ such that

$$\mathbf{U^t A V} = \begin{bmatrix} \mathbf{\Sigma} \\ 0 \end{bmatrix} \tag{7}$$

where

$$\mathbf{\Sigma} = diag(\sigma_1, \ldots, \sigma_n) \tag{8}$$

and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \tag{9}$$

The theorem is taken from [10] and the more general SVD is proven in [9]. The numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$, which are unique, are called the singular values of $\mathbf{A}$. The columns $[\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}]$ of $\mathbf{U}$ are called the left singular vectors of $\mathbf{A}$, and the columns $[\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}]$ of $\mathbf{V}$ are called the right singular vectors of $\mathbf{A}$. SVD is extremely useful in analyzing numerical rank deficiency because the singular values indicate how near $\mathbf{A}$ is to a matrix of lower rank. The matrix $\mathbf{A}$ has rank $r$ if and only if

$$\sigma_r > 0 = \sigma_{r+1} \tag{10}$$

Mathematically speaking, the smallest singular value of $\mathbf{A}$ is the 2-norm distance of $\mathbf{A}$ to the set of all rank-deficient matrices [9]. The ratio of the largest to smallest singular value is termed the condition number of $\mathbf{A}$. This number quantifies the sensitivity of the least-squares solution $\hat{\mathbf{p}}$ of Eq. (2). Large condition numbers indicate that relatively small changes in $\mathbf{A}$ or the offset vector $\mathbf{y}$ can induce large changes in the computed least-squares solution $\hat{\mathbf{p}}$. This is undesirable since parameter vector estimates computed from such ill-conditioned measurement distribution matrices can lead to erroneous pointing offset corrections that will be applied in future antenna tracks. The pointing model corresponding to the DSS 14 70-m antenna (i.e., latitude = 35.426) was used to generate the full $\mathbf{A}$ matrix of relevant error parameters. The uniform distribution consists of 9-deg increments in elevation and 20-deg increments in azimuth. This full measurement set is illustrated in Fig. 1. The singular values of the $\mathbf{A}$ matrices corresponding to the precision mode and computer command mode of operation are presented in Table 3.

As can be seen, the numerical conditioning for the least-squares problem resulting from this hypothetical all-sky distribution is well behaved. Both the precision mode and computer command mode parameter sets yield $\mathbf{A}$ matrices with reasonably nonzero singular values and small condition numbers, implying full-column rank. Another quantity commonly considered in least-squares analysis is the correlation matrix derived from $(\mathbf{A^t A})^{-1}$ of Eq. (6), which is numerically shown in Table 4. The matrix $(\mathbf{A^t A})^{-1}$ is an estimate of the covariance matrix for the solution vector of the least-squares problem. Values near one in the correlation matrix indicate high pairwise correlation between the estimated parameters.

It is evident that such high correlation is implied between the first three parameters (az collimation, fixed az encoder offset, and az/el axis skew) of the computer command mode set. Evaluation of the basis terms corresponding to these parameters in the limited elevation range of 0 to 90 deg results in the pairwise correlation and cannot be avoided regardless of the azimuth distribution. The implication of inherent correlation to parameter estimate stability was investigated through Monte Carlo simulations. The empirical estimation covariance matrix was computed and found to be in very close agreement with the theoretical covariance matrix computed from $(\mathbf{A}^t\mathbf{A})^{-1}$, thus illustrating that stable parameter estimates will result from an all-sky distribution. It was noted in the simulations that individual estimates of the first three elements always varied in the same direction of magnitude, but that differences never exceeded the bounds predicted in the theoretical standard-deviation vector given by

$$\sigma_{\mathbf{p}} = \sum_{i=1}^{n} \frac{\mathbf{v_i}}{\sigma_i} \tag{11}$$

where the $\mathbf{v_i}$ and $\sigma_i$ are defined in Theorem 1. The above equation is obtained by solving for $\mathbf{A}$ in Eq. (7), substituting it into $(\mathbf{A}^t\mathbf{A})^{-1}$, and then taking the square root of the diagonal of the resulting matrix.

The numerical conditioning of the least-squares estimation of antenna precision and computer mode systematic error parameter sets was evaluated above for an all-sky distribution. The resulting measurement distribution matrix for each mode of operation was found to have full rank, thus ensuring unique least-squares solutions for $\hat{\mathbf{p}}$. Also, the large values in the correlation matrix were not seen to degrade the stability of repeated parameter estimates. The linear dependence of the parameters implied by the correlation matrix is due to their mathematical definitions and selecting them simultaneously will not degrade the estimate of the measurement vector $\mathbf{y}$. Such rich offset distributions can never be obtained in practice, thus it is inevitable that poorer matrix conditioning will lead to least-squares estimates of poorer quality. As shown by this analysis, the singular values and condition number of the distribution matrix $\mathbf{A}$ are key parameters in evaluating ill-conditioned least-squares problems.

## C. Reduced and Sparse Data Sets

Current practices dictate that systematic error models be generated from antenna-pointing error-correction data taken from as much of the sky as possible or from an area defined by one or two declination angles. The first is used to generate all-sky pointing models, while the second computes model parameters applicable only in limited directions of the sky. Both situations typically diverge from the hypothetical all-sky example since the basis terms of the pointing model are evaluated in fewer, and perhaps more redundant, directions. Their effect on the least-squares estimation process will be illustrated with examples.

Figure 2 shows the sky trajectory for the Voyager 2 spacecraft. Conical-scan offset data collected at a declination of −22.5 deg clearly represent only a small portion of the total sky measurement space. Tables 5 and 6 show the singular values, condition numbers, and theoretical standard deviations in millidegrees (mdeg) for the least-squares estimate using the $\mathbf{A}$ matrices generated for precision and computer command operation. As implied in the tables, matrix condition deteriorates in both parameter sets because of reduced measurement space. The theoretical standard deviations of the all-sky parameter sets are shown in mdeg in Table 7. Comparison with those of Table 6 illustrate the degradation of the least-squares parameter estimation. In [8], least-squares parameter fits were done on Voyager 1 conical-scan data obtained from the DSS 14 64-m antenna. The results were parameters that were too large in magnitude to be realistic or practical and that were unstable on a day-to-day basis. It has been shown through SVD analysis that such ill conditioning of the systematic error least-squares problem can, in general, be inferred a priori for any constant declination measurement set.

Figure 3 shows sky distribution for a radio source boresight offset file taken at the DSS 13 26-m antenna. The distribution is typical of data gathered during planetary radio astronomy experiments—here, for four radio sources. The pointing model regressor values were once again evaluated at the source coordinates and results of the SVD analysis are shown in Tables 8 and 9. Condition numbers for precision and computer command mode parameter sets are comparably small in magnitude to those from an all-sky distribution. The smallest singular values are also reasonably nonzero. Only minimal estimate degradation is predicted by the increase in theoretical standard deviations. Furthermore, the magnitude of this uncertainty is still reasonably small in the context of parameter estimates, which are usually in the tens of mdeg. This example illustrates that rank deficient measurement distribution matrices can be avoided by using recommended parameter sets and by evaluating the regressor variables with adequate sky distribution of pointing offsets. Concluding that such a measurement set is adequate for the least-squares model fitting is essentially putting emphasis on the norm of the resulting solution vector $\hat{\mathbf{p}}$ instead of minimizing the norm of the residual vector $\mathbf{r}$ of Eq. (4). This approach appears to be the most logical given that the measurement uncertainties are not modeled. It has also been shown that, for radio source pointing calibrations, this matrix condition analysis can be done during pretrack activities, thus influencing the scheduling of calibrators.

It must be stressed that these results hold only for the sets of parameters recommended in Table 2. Different combinations of 21 error coefficients in the current pointing model will yield different, and in some cases disastrous, numerical properties of the matrices involved in the computation of the least-squares solution.

## III. Solution Methods

### A. Parameter Selection

The two goals of the modeling process are to quantify contributors to the antenna's systematic pointing error so that pointing can be corrected and so that knowledge of the antenna's mechanical and structural characteristics can be acquired. To achieve both of these objectives simultaneously, identical parameter vectors must be chosen for estimation on a consistent basis. These parameters for the 70-m and 34-m antennas in the applicable mode of operation have been given in Table 2. Subsets of these vectors should be chosen either when parameter values are physically known a priori or when they are consistently estimated with small magnitudes. In practice, however, the goal of correct pointing can be achieved without accurate knowledge of actual antenna error characteristics. Optimization may be based on any random set of parameters that minimizes the sum of the squares of the residuals given in Eq. (5) without regard for physical interpretation.

Regardless of the estimation philosophy practiced, problems always arise when building models for particular regions of the sky—for example, along a line or band of constant declination for one or more sources—or for a particular spacecraft. The rank deficiency that plagues least-squares problems in these cases generates uncertainty in parameter selection and interpretation. However, such models for locally optimized pointing are needed for critical spacecraft and holography tracks and for those tracks of single sources known as strong, reliable antenna calibrators. The current least-squares solution method described next uses the SVD to accommodate ill-conditioned measurement distribution matrices.

### B. Singular Value Decomposition

The SVD subroutines in the systematic error modeling software that were used to solve the least-squares problem were taken from [11]. A key feature of the SVD method is its ability to handle rank deficiency. Ill-conditioned $\mathbf{A}$ matrices result in the rank $(\mathbf{A}) = r$ being less than the parameter dimension $n$. This results in a rank-deficient least-squares problem that has an infinite number of solutions, for if the vector $\mathbf{p}$ is a minimizer and the vector $\mathbf{z} \in null(\mathbf{A})$, then $\mathbf{p}+\mathbf{z}$ is also a minimizer. The SVD method is useful in such situations since it is a revealing and complete orthogonal decomposition. The routines from [11] basically implement the following theorem taken from [9], given here without proof.

**Theorem 2.** Suppose $\mathbf{U^tAV} = \Sigma$ is the SVD of $\mathbf{A} \in \Re^{\mathbf{m}\times\mathbf{n}}$ with $r = \text{rank}(\mathbf{A})$. If $\mathbf{U} = [\mathbf{u_1}, \ldots, \mathbf{u_m}]$ and $\mathbf{V} = [\mathbf{v_1}, \ldots, \mathbf{v_n}]$ are column partitionings and $\mathbf{y} \in \Re^{\mathbf{m}}$ then

$$\mathbf{p_{LS}} = \sum_{i=1}^{r} \frac{\mathbf{u_i^t y}}{\sigma_i}\mathbf{v_i} \tag{12}$$

minimizes $\| \mathbf{Ap} - \mathbf{y} \|_2$ and has the smallest 2-norm of all minimizers. Moreover

$$\| \mathbf{Ap_{LS}} - \mathbf{y} \|_2^2 = \sum_{i=r+1}^{m} (\mathbf{u_i^t y})^2 \tag{13}$$

Note that if $r < n$, this corresponds to simply adding a zero multiple to the solution vector $\mathbf{p_{LS}}$ rather than adding random large-valued multiples produced by the near-zero singular values. This may reduce uncertainty in the estimated coefficients, as in Eq. (11), but increases the residual norm, as in the increased summation index of Eq. (13). This point was touched on earlier. In practice, one must still come up with a numerical estimate $\hat{r}$ of $r$. The systematic error modeling software estimates the numerical rank $\hat{r}$ of $\mathbf{A}$ as

$$\sigma_1 \geq \cdots \sigma_{\hat{r}} \geq \delta > \sigma_{\hat{r}+1} \geq \cdots \sigma_n \tag{14}$$

where the tolerance $\delta$ is chosen to be $\sigma_1$, scaled by a machine-precision dependent factor. The selection of $\hat{r}$ results in

$$\mathbf{p_{\hat{r}}} = \sum_{i=1}^{\hat{r}} \frac{\mathbf{u_i^t y}}{\sigma_i}\mathbf{v_i} \tag{15}$$

as an approximation to $\mathbf{p_{LS}}$. If $\sigma_{\hat{r}} >> \delta$, then $\mathbf{p_{\hat{r}}}$ is a very close approximation to the true minimizer $\mathbf{p_{LS}}$ since

$\mathbf{A}$ can be unambiguously regarded as a matrix with rank $\hat{r}$ [9]. When $[\sigma_1, \ldots, \sigma_n]$ do not clearly split into small and large values, rank determination may be somewhat arbitrary.

### C. A New Algorithm for Parameter Selection

As has been shown, the SVD solution currently alleviates the rank-deficiency problems associated with limited pointing offset distributions. This means filtering out small singular values of $\mathbf{A}$ and replacing them with the matrix $\mathbf{A}_{\hat{r}}$ defined as

$$\mathbf{A}_{\hat{\mathbf{r}}} = \sum_{i=1}^{\hat{r}} \sigma_i \mathbf{u}_i \mathbf{v}_i^t \tag{16}$$

where $\hat{r}$ is the numerically determined estimate of the rank of $\mathbf{A}$. As discussed in [9] such a cutoff makes sense when the measurement distribution matrix is derived from noisy data. However, in this case, $\mathbf{A}$ is being evaluated using accurate ephemeris from observed targets, as in Eq. (1). In other applications, rank deficiency is an indication of redundancy among factors that comprise the model. As has been shown in previous sections, redundancy among systematic error regressor variables occurs in the estimation process only when dealing with limited and reduced pointing measurement sets. In these cases, the systematic error predictor $\mathbf{A}_{\hat{r}}\mathbf{p}_{\hat{r}}$ used in subsequent tracking will involve all $n$ redundant factors that may have been chosen as a result of random parameter selection. Although such solutions may correct future pointing, parameter estimates can obscure physical interpretation of true antenna mechanical characteristics. In such instances, it is argued that the least-squares solution vector should contain at most $\hat{r}$ nonzero systematic error parameters, which in turn dictate which columns of $\mathbf{A}$ will be used in approximating the observation vector $\mathbf{y}$. The problem of choosing the appropriate columns of the measurement distribution matrix is termed subset selection. The SVD-based subset selection procedure that has been chosen for this least-squares application is summarized below:

(1) Compute the SVD $\mathbf{A} = \mathbf{U\Sigma V^t}$ and use it to determine a rank estimate $\hat{r}$.

(2) Calculate a permutation matrix $\mathbf{P}$ such that the columns of the matrix $\mathbf{B_1} \in \Re^{m\times\hat{r}}$ in $\mathbf{AP} = [\mathbf{B_1}\ \mathbf{B_2}]$ are "sufficiently independent."

(3) Predict $\mathbf{y}$ with the vector $\mathbf{Ap}_{\mathrm{sub}}$ where $\mathbf{p}_{\mathrm{sub}} = [\mathbf{z}\ \mathbf{0}]^t$ and $\mathbf{z} \in \Re^{\hat{r}}$ minimizes $\|\ \mathbf{B_1 z} - \mathbf{y}\ \|_2$.

Using systematic error modeling, the rank determination in the first step can be chosen with more heuristic criteria instead of those used in Eq. (14). The new criteria are based on the matrix condition number and the magnitude of the theoretical standard-deviation vector given by Eq. (11). Given $\hat{r}$, the first $\hat{r}$ columns of permutation matrix $\mathbf{P}$ give the column indices of $\mathbf{A}$ for use in the least-squares estimation. These are equivalent to the parameters from which the model is selected. A thorough discussion of the various approaches to this problem can be found in [9,10]. Below is a summary of the algorithm to compute $\mathbf{P}$ that was chosen and implemented in the systematic error modeling software. It is based on both the SVD and on QR factorization with the column-pivoting algorithm. For $\mathbf{A} \in \Re^{m\times n}$, QR factorization with column pivoting from [9] produces $\mathbf{AP} = \mathbf{QR}$ where

$$\mathbf{R} = \begin{array}{cc} \begin{bmatrix} \mathbf{R_{11}} & \mathbf{R_{12}} \\ 0 & 0 \end{bmatrix} & \begin{array}{c} \hat{r} \\ m-\hat{r} \end{array} \\ \begin{array}{cc} \hat{r} & \quad n-\hat{r} \end{array} & \end{array} \tag{17}$$

where $\hat{r}$ is the rank $(\mathbf{A})$, $\mathbf{Q}$ is orthogonal, $\mathbf{R_{11}}$ is upper triangular and nonsingular, and $\mathbf{P}$ is a permutation matrix. This factorization implies that the first $\hat{r}$ columns of $\mathbf{Q}$ form an orthonormal basis for range $(\mathbf{A})$. It is the desired result since the measurement vector $\mathbf{y}$ in the least-squares problem may be approximated by the first $\hat{r}$ columns of the matrix $\mathbf{AP}$, which is just $\mathbf{B_1}$ of the second step above. This is equivalent to choosing the first $\hat{r}$ parameters of $\mathbf{Pp}$ for estimation, which is equal to the vector $\mathbf{z}$ in step three above. As in a previous section where the case for cutting off singular values in the SVD method was presented, reducing the order of the parameter solution vector will also increase the residual norm.

Unfortunately, QR factorization with column pivoting alone is not a totally robust method for computing the permutation matrix $\mathbf{P}$ [9]. The preferred algorithm implemented in the software that uses both SVD and QR factorization is presented in the revised steps below:

(1) Compute the SVD $\mathbf{A} = \mathbf{U\Sigma V^t}$ and use it to determine a rank estimate $\hat{r}$. Save the matrix $\mathbf{V}$.

(2) Apply the QR factorization with column pivoting to the subset of $\mathbf{V^t}$ : $\mathbf{Q^t V}(:, 1 : \hat{r})^t\mathbf{P} = [R_{11}\ R_{12}]$ and set $\mathbf{AP} = [\mathbf{B_1}\ \mathbf{B_2}]$ with $\mathbf{B_1} \in \Re^{m\times\hat{r}}$ and $\mathbf{B_2} \in \Re^{m\times(n-\hat{r})}$.

(3) Determine $\mathbf{z} \in \Re^{\hat{r}}$, which minimizes $\|\ \mathbf{B_1 z} - \mathbf{y}\ \|_2$.

The main contribution of this algorithm is facilitating parameter selection for reduced and constant declination-pointing measurement sets. In the latter case, its application will ensure consistent parameter selection for particular radio sources and spacecraft tracks. This subset selection procedure essentially eliminates parameters that the algorithm has deemed unobservable in the given measurement distribution. The next step is to decide how to deal with these excluded parameters. One approach is to simply ignore them and proceed as usual with the least-squares estimation with the reduced vector **z**, as determined above.

A different approach, when possible, is to use physically known or accurate a priori estimates for unobservable parameters and subtract their contributions from the measurement vector **y** before estimating **z**. Such an option is available in the current software. The resultant solution vector should be more consistent with all-sky models. Finally, it should be noted that the tools presented here can be used for the opposite effect (e.g., predicting matrix condition and rank or for least-squares estimate accuracy) when the measurement vector **y** is augmented with pointing offsets taken in new directions.

To illustrate, this algorithm is applied to the **A** matrix which resulted from the Voyager 2 trajectory, as shown in Fig. 2. Referring to Table 5, one can base the rank determination of the **A** matrix on the smallest of the singular values. For example, choosing 0.1 as a singular value cutoff results in precision and computer command mode parameter selections and matrix conditions that are summarized in Tables 10 and 11. Eliminating parameters 11 and 21 from the precision mode and 1 from the computer command mode results in reduced matrix condition and smaller estimation standard deviations for some elements of the solution vector. In practice, the actual systematic error estimated values are generally less than 100 mdeg. Thus, estimation accuracy for some of the remaining parameters in Table 10 will be a certain percentage of the estimated values.

Depending on the antenna's frequency band, this may or may not meet the pointing requirements. (A detailed description of errors will not be covered here.) Estimation errors will always be larger in practice because of uncertainties in the measurement vector **y**, so one may decide to increase the singular value cutoff and apply the subset selection algorithm. Using cutoffs 1.0 for the precision mode and 0.2 for the computer command mode yields the results summarized in Tables 12 and 13. To achieve accuracy comparable to the hypothetical all-sky models, the parameters to be excluded are 1, 7, 11, and 21 from the precision-mode set and 1 and 7 from the computer command-mode set. It is advised that whenever the fixed angular encoder error parameters (for example, 7 and 21) are excluded in the subset selection procedure, their values should be determined directly from the pointing offset data and contributions to **y** should be removed before making an estimation.

## IV. Summary

This article has described and analyzed the least-squares problem encountered in the DSN systematic pointing error modeling process. Specifically investigated is the relationship between rank degeneracy of measurement distribution matrices and limited-sky distributions of the pointing error offsets. Using a hypothetical all-sky performance index and an SVD analysis, it is shown that an acceptable matrix condition of the least-squares problem can be obtained by evaluating the pointing model regressor variables with adequate sky distributions of the pointing measurements. In addition to matrix condition, the theoretical standard deviations of the least-squares estimate are used to evaluate accuracy. It is shown through an example that redundancy among the systematic error model regressor variables occurs when dealing with limited and sparse data sets. In practice, rank-degenerate matrices are encountered when building models for particular regions of the sky, such as along a band of constant declination.

The key feature of the analysis presented is its predictive capability. Matrix condition and least-squares estimate accuracy based on measurement distribution may be predicted before actual pointing calibration activities commence. The current least-squares solution method based on singular-value decomposition is also presented. This method can handle ill-conditioned measurement distribution matrices encountered in the model-building process. For limited measurement sets, it was argued that it may be preferred to estimate only observable parameters. Systematically eliminating redundant parameters will facilitate the parameter selection process and make it consistent. A recommended subset selection algorithm based on singular-value decomposition and QR factorization is illustrated with a Voyager 2 measurement set.

## V. Future Work

This article has presented an analytical approach using mathematical tools to answer fundamental numerical questions arising from the systematic error-modeling process. Such general but consistent procedures are needed in the modeling process because of the many antenna-specific mechanical and other practical considerations encompassed by the problem. Once past this juncture, one may begin to address the deficiencies and look for possible refinements in the estimation process. The most obvious is that of recursive estimation. Methods must be devised to handle data sets spanning many weeks or years and incorporating many a priori models and model uncertainties into the estimation algorithms. If and when uncertainties in pointing measurements can be accurately modeled, including those from natural or manmade sources as well as from antenna-system imperfections, then the algorithm should also be modified to allow for weighted observations.

All these enhancements must be worked into the existing modeling software. This package should also enable the functional form of the model to change relatively often. This will allow for the addition of newly discovered error terms and for enhancements to accommodate new antenna architectures such as the DSS 13 beam-waveguide antenna.

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**Table 1. Systematic pointing error sources and model terms**

| Error source | Model function | |
|---|---|---|
| | Cross-elevation error | Elevation error |
| Az[a] collimation | $P_1$ [e] | – |
| Az encoder fixed offset | $P_2$ cos (el) | – |
| Az/el skew | $P_3$ sin (el) | – |
| Az axis tilt | $P_4$ sin (el) cos (az) | $-P_4$ sin (az) |
| Az axis tilt | $P_5$ sin (el) sin (az) | $P_5$ cos (az) |
| Source dec[b] | $P_6$ sin (az) | $P_6$ sin (el) cos (az) |
| El[c] encoder fixed offset | – | $P_7$ |
| Gravitational flexure | – | $P_8$ cos (el) |
| Residual refraction | – | $P_9$ cot (el) |
| Az encoder scale error | $P_{10}$ (az/360) cos (el) | – |
| | Cross-declination error | Declination error |
| HA[d]/dec axis skew | $-P_{11}$ sin (dec) | – |
| HA axis tilt | $P_{12}$ sin (HA) sin (dec) | $P_{12}$ cos (HA) |
| HA axis tilt | $-P_{13}$ cos (HA) sin (dec) | $P_{13}$ sin (HA) |
| HA feed offset | $-P_{14}$ | – |
| Gravitational flexure | $P_{15}$ cos $(p)$[f] cos (el) | $-P_{15}$ sin $(p)$ cos (el) |
| Declination feed offset | – | $P_{16}$ |
| Gravitational flexure | $P_{17}$ sin $(p)$ cos (el) | – |
| Gravitational flexure | – | $-P_{18}$ cos $(p)$ cos (el) |
| Gravitational flexure | $-P_{19}$ sin (el) | – |
| Gravitational flexure | – | $P_{20}$ sin (el) (el) |
| HA encoder bias | $P_{21}$ cos (dec) | – |

[a] Az refers to azimuth angle.
[b] Dec refers to declination angle.
[c] El refers to elevation angle.
[d] HA refers to hour angle.
[e] Uppercase $P$ refers to parameter value.
[f] Lowercase $p$ refers to paralectic angle.

**Table 2. Applicable parameter sets to DSN 70-m and 34-m antennas**

| Precision mode | Computer command mode |
|---|---|
| 1 | 1 |
| 7 | 2 |
| 8 | 3 |
| 9 | 4 |
| 11 | 5 |
| 12 | 7 |
| 13 | 8 |
| 14 | 9 |
| 21 | – |

**Table 3. Singular values for all-sky distributions**

| Precision mode | Computer command mode |
|---|---|
| 34.252 | 34.252 |
| 18.108 | 18.096 |
| 15.785 | 11.814 |
| 11.811 | 11.807 |
| 10.342 | 10.342 |
| 6.161 | 5.644 |
| 4.131 | 2.623 |
| 2.925 | 0.819 |
| 2.623 | – |
| Condition number | Condition number |
| 13.06 | 41.80 |

**Table 4. Computer command mode correlation matrix for all-sky distribution**

| Parameter | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.00 | −0.97 | −0.98 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 2 | −0.97 | 1.00 | 0.92 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 3 | −0.98 | 0.92 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 4 | 0.00 | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 5 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 | 0.00 |
| 7 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 | −0.81 | 0.36 |
| 8 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | −0.81 | 1.00 | −0.75 |
| 9 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.36 | −0.75 | 1.00 |

**Table 5. Singular values for the Voyager 2 data set**

| Precision mode | Computer command mode |
|---|---|
| 57.226 | 57.226 |
| 14.530 | 12.171 |
| 10.282 | 10.282 |
| 5.912 | 5.042 |
| 1.483 | 1.483 |
| 0.334 | 1.315 |
| 0.199 | 0.199 |
| 0.0033 | 0.0671 |
| 0.0028 | – |
| Condition number | Condition number |
| 20,438 | 853 |

**Table 6. Theoretical standard deviations for the Voyager 2 data set in millidegrees**

| Parameter | Precision mode | Computer command mode |
|---|---|---|
| 1 | 2.19 | 10.46 |
| 2 | – | 10.06 |
| 3 | – | 3.46 |
| 4 | – | 0.20 |
| 5 | – | 0.72 |
| 7 | 3.55 | 3.55 |
| 8 | 3.53 | 3.53 |
| 9 | 0.03 | 0.03 |
| 11 | 289.48 | – |
| 12 | 0.72 | – |
| 13 | 1.89 | – |
| 14 | 252.03 | – |
| 21 | 269.20 | – |

**Table 7. Theoretical standard deviations for the all-sky data set in millidegrees**

| Parameter | Precision mode | Computer command mode |
|---|---|---|
| 1 | 0.16 | 0.82 |
| 2 | – | 0.61 |
| 3 | – | 0.69 |
| 4 | – | 0.08 |
| 5 | – | 0.08 |
| 7 | 0.17 | 0.17 |
| 8 | 0.35 | 0.35 |
| 9 | 0.06 | 0.06 |
| 11 | 0.23 | – |
| 12 | 0.08 | – |
| 13 | 0.21 | – |
| 14 | 0.20 | – |
| 21 | 0.22 | – |

**Table 8. Singular values for radio source distribution**

| Precision mode | Computer command mode |
|---|---|
| 15.951 | 15.452 |
| 13.006 | 13.157 |
| 9.885 | 8.176 |
| 7.660 | 7.504 |
| 4.326 | 4.238 |
| 3.550 | 2.603 |
| 2.020 | 0.826 |
| 1.156 | 0.283 |
| 0.825 | – |
| Condition number | Condition number |
| 19.335 | 54.600 |

Table 9. Theoretical standard deviations for the radio source data set in millidegrees

| Parameter | Precision mode | Computer command mode |
|---|---|---|
| 1 | 0.33 | 2.45 |
| 2 | – | 1.68 |
| 3 | – | 1.96 |
| 4 | – | 0.13 |
| 5 | – | 0.14 |
| 7 | 0.42 | 0.42 |
| 8 | 1.11 | 1.11 |
| 9 | 0.36 | 0.35 |
| 11 | 0.44 | – |
| 12 | 0.14 | – |
| 13 | 0.44 | – |
| 14 | 0.49 | – |
| 21 | 0.57 | – |

Table 10. Reduced parameter standard deviations for the Voyager 2 data set in millidegrees

| Parameter | Precision mode | Computer command mode |
|---|---|---|
| 1 | 2.16 | – |
| 2 | – | 0.29 |
| 3 | – | 0.72 |
| 4 | – | 0.20 |
| 5 | – | 0.72 |
| 7 | 3.55 | 3.55 |
| 8 | 3.52 | 3.53 |
| 9 | 0.03 | 0.03 |
| 12 | 0.72 | – |
| 13 | 1.87 | – |
| 14 | 1.30 | – |

Table 11. Singular values for reduced Voyager 2 parameter set

| Precision mode | Computer command mode |
|---|---|
| 57.226 | 57.226 |
| 12.157 | 10.282 |
| 10.282 | 8.893 |
| 5.221 | 4.845 |
| 1.483 | 1.483 |
| 0.319 | 1.315 |
| 0.199 | 0.199 |
| Condition number | Condition number |
| 288 | 288 |

Table 12. Reduced parameter standard deviations for the Voyager 2 data set in millidegrees

| Parameter | Precision mode | Computer command mode |
|---|---|---|
| 2 | – | 0.29 |
| 3 | – | 0.72 |
| 4 | – | 0.20 |
| 5 | – | 0.53 |
| 7 | – | – |
| 8 | 0.55 | 0.55 |
| 9 | 0.03 | 0.03 |
| 12 | 0.53 | – |
| 13 | 0.19 | – |
| 14 | 0.13 | – |

Table 13. Singular values for reduced Voyager 2 parameter set

| Precision mode | Computer command mode |
|---|---|
| 56.955 | 56.955 |
| 8.952 | 8.893 |
| 8.137 | 8.137 |
| 4.900 | 4.845 |
| 1.335 | 1.335 |
| – | 1.315 |
| Condition number | Condition number |
| 43 | 43 |

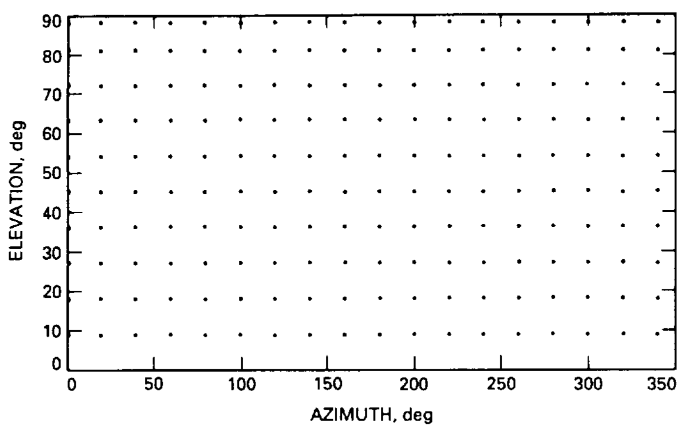


Fig. 1. Hypothetical all-sky measurement set.

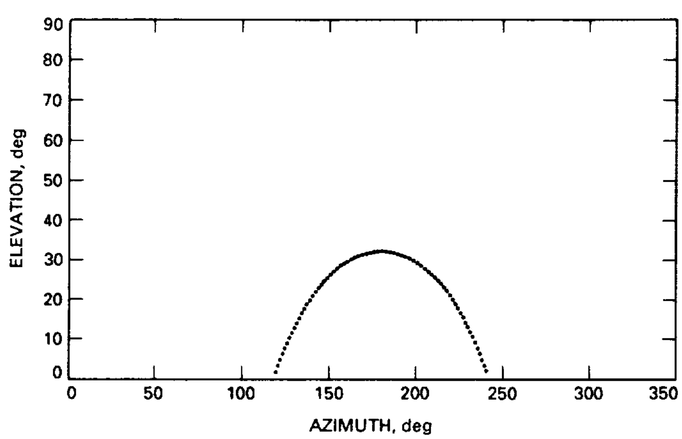


Fig. 2. Voyager 2 measurement set.

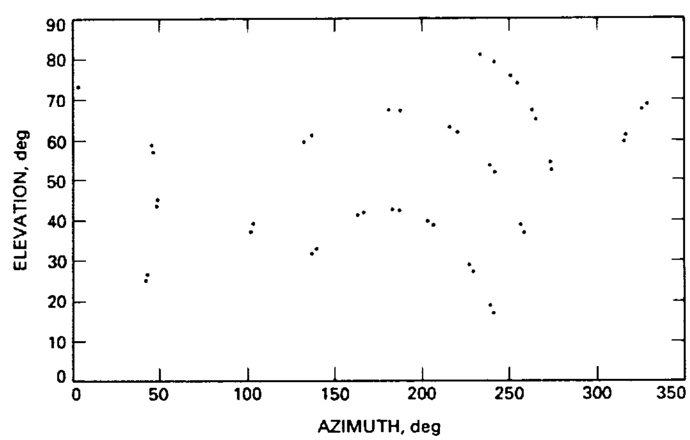


Fig. 3. Radio source boresight measurement set.