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On Estimating the Phase of a Periodic Waveform in Additive Gaussian Noise— Part III

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Motivated by advances in signal processing technology that support more complex algorithms, researchers have taken a new look at the problem of estimating the phase and other parameters of a nearly periodic waveform in additive Gaussian noise, based on observation during a given time interval. In Part I, the general problem was introduced and the maximum a posteriori probability criterion with signal space interpretation was used to obtain the structures of optimum and some suboptimum phase estimators for known constant frequency and unknown phase with an a priori distribution. In Part II, optimal algorithms were obtained for some cases where the phase (and frequency) is a parameterized function of time with the unknown parameters having a joint a priori distribution. The intrinsic and extrinsic geometry of hypersurfaces was introduced to provide insight to the estimation problem for the small-noise and large-noise cases. In Part III, the actual performances of some of the highly nonlinear estimation algorithms of Parts I and II are evaluated by numerical simulation using Monte Carlo techniques.

I. Introduction

The work of Part I [1] and Part II [2] is limited to analytical results which, although they provide the structure of nonlinear optimum estimators, can give the performance of these estimators only in the case of small noise—and then not always in the case of some suboptimum estimators such as phase-locked loops. In what follows, certain nonlinear estimators are exactly simulated by numerical methods and their performance is evaluated

by Monte Carlo techniques. These include nonsinusoidal waveforms with unknown phase and known frequency and sinusoidal waveforms with unknown phase and frequency.

II. Performance of Phase Estimators With Known Constant Frequency

For a sinusoid of known frequency and unknown phase, the probability density function given in footnote 2 on

page 157 of [1] is easily numerically integrated, after multiplication by error squared, to obtain the mean-square (ms) phase error of the well-known optimum estimator of Eq. (33) of [1]. For a nonsinusoid consisting of a fundamental and one or more harmonics, the optimum estimator is the maximization of Eq. (61) of [1]. Since the probability distribution of the error is not available, the ms error of this estimator is found by numerical methods using a Monte Carlo technique. The results for (a) a sinusoid, (b) a sinusoid plus third square-wave harmonic, and (c) a sinusoid plus third, fifth, and seventh square-wave harmonics are plotted in Fig. 1. As the maximum slope of the periodic waveform increases with additional harmonics, the large-noise threshold of the optimum estimator becomes steeper and moves to higher input signal-to-noise ratios, as would be expected. Some insight is given by Fig. 2 of [2] and Sections VI of [1] and VIII of [2].

It is interesting to compare the performance of a second-order phase-locked loop (PLL) with that of the previous optimum estimator for the phase of a sinusoid with known frequency. The PLL filter, preceding the voltage-controlled oscillator, is taken to be of the form $(\tau_1 s + \tau_2)/s$, where τ_1 and τ_2 are chosen to give a damping ratio of $1/\sqrt{2}$. This is essentially what is used in the carrier tracking loops of Deep Space Network (DSN) receivers.[1] The noise bandwidth of the optimum estimator is $1/2T$, where T is the duration of the observation interval.

For a valid comparison between the PLL and the optimum estimator, the PLL must be observed at a time interval T after the PLL is turned on, with an initial phase error uniformly distributed over one cycle (and no initial frequency error). If the PLL filter is chosen to give a PLL noise bandwidth equal to that of the optimum estimator, this turns out to be too small. In this case, the ms phase error of the PLL is dominated by the transient responses of the loop to the initial phase errors. This is much larger than the ms phase error contribution resulting from the additive noise for any useful signal-to-noise ratio. The observation interval T is only $\sqrt{2}/3\tau \approx 0.150$ of the undamped period of the PLL when its noise bandwidth is equal to that of the optimum estimator $(1/2T)$. As the loop noise bandwidth is increased (period is decreased) the ms phase error contribution from the initial transients

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[1] Actually, the denominator of the DSN PLL filter is of the form $s+\tau$, where τ is on the order of 0.001 at the smaller loop bandwidths.

decreases while the contribution from the additive noise increases. For each input signal-to-noise ratio there is an optimum loop noise bandwidth which minimizes the total ms phase error of the PLL at the end of the observation interval T.

In Fig. 2 these PLL minimum ms phase error values are plotted together with the ms phase error of the optimum estimator. Even at large input signal-to-noise ratios (small noise) the PLL performance is about 9 dB worse than the optimum estimator. The PLL results are obtained by numerical solution (fourth-order Runge-Kutta) of the baseband second-order nonlinear differential equation for the loop phase error.

III. Performance of Optimum Estimator With Unknown Constant Frequency

In this case the estimation algorithm consists of choosing the frequency f in Eq. (102) of [2] to maximize Eq. (113), supported by Eq. (103). The phase is then given by Eq. (111). For this numerical simulation, the a priori distribution of phase is uniform over one cycle and the independent a priori distribution of frequency is taken to be uniform over the interval $(-5.5/T$ to $+5.5/T)$ centered around a given frequency. [The index in Eq. (103) runs from -5 to $+5$.]

The ms phase error of the optimum estimator is given in Fig. 3, with the ms phase error for known frequency as reference. In accordance with the analytical result of Eq. (96) of [2], the small-noise performance is the same as that for known frequency. However, the steeper large-noise threshold occurs at a higher signal-to-noise ratio.

The ms frequency error of the optimum estimator is given in Fig. 4. It is evident that the large-noise threshold is more abrupt than that for phase error. In the case of unknown phase and frequency, the PLL performance is much worse than for the case of unknown phase and known frequency, shown in the previous section.

It should be kept in mind that the results of this section depend on the a priori distribution, above, chosen for the unknown frequency.

References

[1] Rauch, L. L., "On Estimating the Phase of a Periodic Waveform in Additive Gaussian Noise—Part I," *DSN Progress Report 42-45*, vol. March and April 1978, Jet Propulsion Laboratory, Pasadena, California, pp. 152–164, June 15, 1978.

[2] Rauch, L. L., "On Estimating the Phase of a Periodic Waveform in Additive Gaussian Noise—Part II," *TDA Progress Report 42-79*, vol. July–September 1984, Jet Propulsion Laboratory, Pasadena, California, pp. 17–24, November 15, 1984.
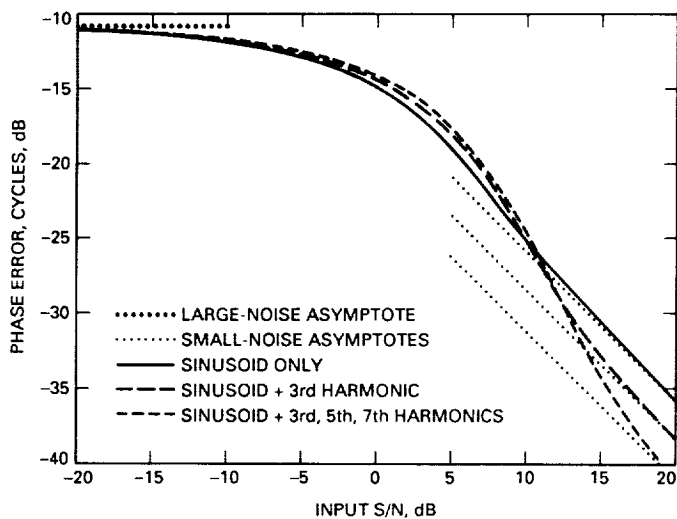
Fig. 1. Phase-error performance of three optimum estimation algorithms for the case of known constant frequency. The large-noise asymptote corresponds to phase error uniformly distributed over one cycle. The three small-noise asymptotes come from Eqs. (54), (59), and a similar equation for the first four components of a square wave, obtained by using Eq. (56) of [1].
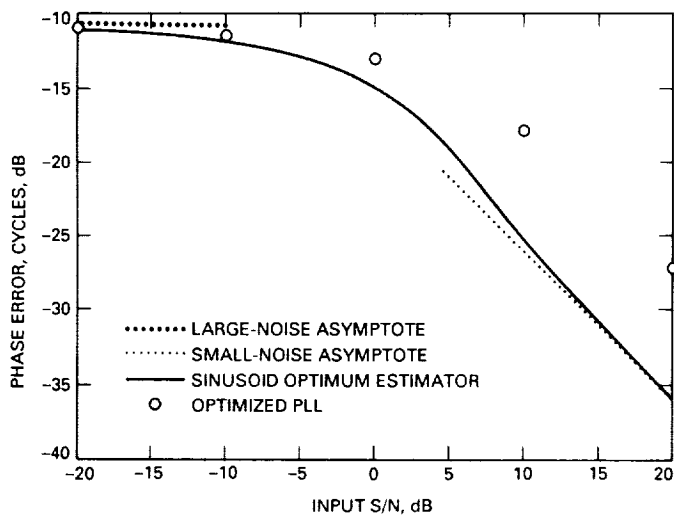

Fig. 2. Phase-error performance of an optimized phase-locked loop subject to the same signal-to-noise ratio and observation interval as the optimum estimator.
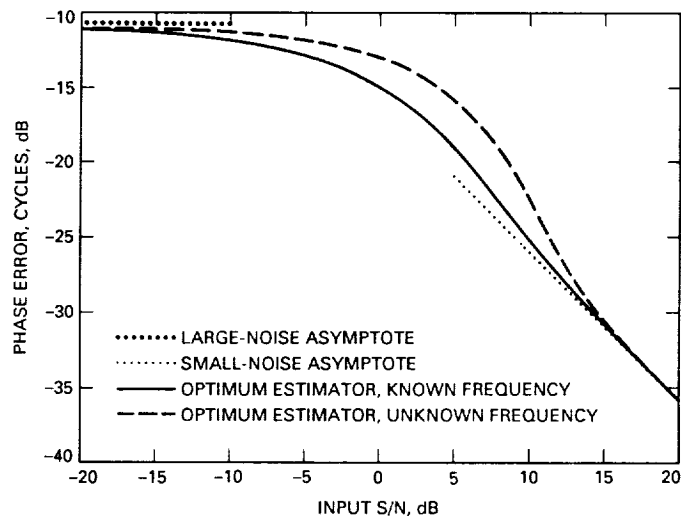

Fig. 3. Phase-error performance of the optimum estimator for the case of unknown constant frequency. The small-noise asymptote comes from Eq. (96) of [2].
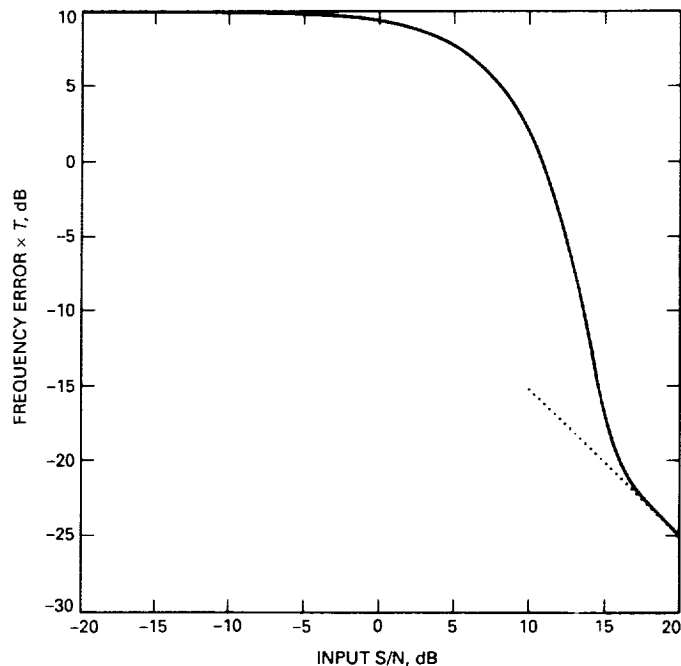

Fig. 4. Frequency error performance of the optimum estimator for the case of unknown constant frequency. The small-noise asymptote comes from Eq. (99) of [2]. The large-noise asymptote (not plotted) is at 10.04 dB. This corresponds to a frequency error uniformly distributed from $-5.5/T$ to $+5.5/T$.