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R. A. Nicolaides

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Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, Virginia 23665–5225

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Langley Research Center
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ANALYSIS AND CONVERGENCE OF THE MAC SCHEME. 1. THE LINEAR PROBLEM

R. A. Nicolaides[1]
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

ABSTRACT

The MAC discretization of fluid flow is analyzed for the stationary Stokes equations. It is proved that the discrete approximations do in fact converge to the exact solutions of the flow equations. Estimates using mesh dependent norms analogous to the standard $\mathbf{H}^1$ and L^2 norms are given for the velocity and pressure respectively.

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1. Introduction

The original MAC (Marker and Cell) scheme of Harlow and Welch is now over thirty years old. Many evolutionary changes have occurred and the MAC approach is still a widely used finite difference method for incompressible flow computation.

The MAC scheme is described in [Hirt 1979] and in most CFD books, for example [Fletcher 1989] and [Peyret and Taylor 1983]. It also forms the basis of several flow packages including SOLA [Hirt 1979] and SIMPLE [Patankar 1980].

More recently, it is finite element techniques which have dominated incompressible CFD. Although highly successful in many ways, the absence of fully satisfactory lower order schemes continues to be a stimulus to the development of new approaches. Recent work on finite elements may be found in [Girault and Raviart 1986], [Gunzburger 1989] and [Pironneau 1989].

The stability problems of low order finite elements [e.g Boland and Nicolaides 1985] suggest looking for stable schemes outside the finite element framework. The MAC scheme is one possibility. The standard MAC method suffers from being limited to rectangular meshes. A second difficulty is the apparent absence of any rigorous analysis for the stationary problem - the subject of this report. ([Porsching 1979] discusses the evolutionary problem.)

On the first point, a generalization of the MAC ideas to triangular and other more general meshes was presented in [Nicolaides 1989]. In this *complementary volume* or *covolume* framework the MAC ideas are given a control volume interpretation. Complementary pairs of control volumes are used, and they can conveniently be taken as the triangles and polygons of a Delaunay-Voronoi mesh system. When specialized to a standard triangulation of, for example, a rectangular domain the precise MAC equations are recovered. [Nicolaides and Wu 1991] contains an implementation for a specific problem. [Hall, Cavendish, and Frey 1991] contains a more general implementation.

The covolume formulation is based on discretization of the vector operators grad, div, and curl and unlike the usual MAC derivation is coordinate free. The error estimates that are proved here follow from this formulation of the MAC scheme. A related approach was used in [Nicolaides 1991] to discretize and estimate some div-curl systems and several results from there will be used below.

Before proceeding, a few technical comments are in order. First, we will present the analysis for two dimensional problems although the methods used are not inherently two dimensional. Second, the estimates for the velocity are in a discrete $\mathbf{H}^1$ norm on $\bar{\Omega}$. This implies in particular, estimates for the vorticity on the boundary. While such estimates are desirable from a practical point of view, we have assumed that $\omega \in H^2$ and $p \in H^2$. This is more regularity than one would like to see. In our analysis, the MAC scheme itself is obtained as a discretization of the "strong" second order form of the equations. This already suggests the need for extra regularity. The third point is that while we use constant mesh spacings in the x and y directions, the domain need not be rectangular or convex. Coordinate free methods make this possible. Summation by parts formulas for example become inherently multidimensional and not limited to summations along grid lines.

2. Preliminaries

For clarity, we assume a rectangular domain Ω and a primal Cartesian mesh having x and y spacings equal to h and h'. To avoid unnecessary complications, we will suppose that $h = h'$. It is easy to modify the results to cover the contrary case. In addition, by joining the centers of adjacent mesh rectangles (cells) a dual ("staggered") mesh is made. At boundaries, nodes of the dual mesh are joined to the midpoints of the adjacent boundary edges.

The N nodes of the primal mesh are numbered from left to right, and from bottom to top. The T nodes of the dual mesh are numbered in a similar way. There are T mesh cells in all, and there are N dual mesh cells including the half size cells at the boundary. Similarly, the E edges of the primal mesh are labelled in some convenient way. There is a bijective correspondence between dual and primal edges, each edge crossing exactly one edge of the other set. The numbering of the dual edges reflects this, each being numbered the same as its primal companion. The cells, edges and nodes of the primal mesh are denoted generically by τ_i, σ_j and ν_k respectively. Those of the dual mesh are similarly denoted by primed quantities such as σ'_j. The lengths of the dual mesh edges are equal to h except near boundaries where they may be $h/2$. A direction is assigned to each primal edge according to the rule that positive is from low to high node number. The dual edges are directed by the convention that (σ'_k, σ_k) are oriented like the (x, y) axes of the coordinate system.

Defined on each primal edge σ_j is a component of the velocity field, directed in the positive direction of σ'_j. This velocity component is computed as $\mathbf{n} \cdot \mathbf{u}$ or its average, where $\mathbf{u}$ denotes the velocity vector at the midpoint of the primal edge, and $\mathbf{n}$ is the normal to the edge. Components of other vector fields are defined similarly. Such sets of normal components defined on the primal edges can be identified with R^E. We will introduce an inner product into R^E by

$$(u, v)_W := \sum_{\sigma_j \in \Omega} u_j v_j W_j.$$

In this, W_j equals twice the area of the figure obtained by joining the nodes of a primal edge to the adjacent dual nodes. The sum is to be taken over all edges of the primal mesh. The associated norm is denoted by $\| \cdot \|_W$. Clearly, it is twice a discrete L^2 norm. This inner product space is denoted by $\mathcal{U}$, or by $\mathcal{U}_0$ if the normal components assigned to the boundary edges are all zero.

Various scalar fields including pressures are defined at the dual nodes. They can be identified with elements of R^T. An inner product on R^T is defined by

$$(\phi, \theta)_A := \sum_{\tau_i \in \Omega} \phi_i \theta_i A_i.$$

Here, the sum is over all mesh cells and A_i denotes the area of the i^{th} cell. The associated norm is denoted by $\| \cdot \|_A$. This inner product space is denoted by $\mathcal{P}$.

Analogous to $\mathcal{P}$, scalar fields defined on the primal nodes can be identified with elements of R^N, and an inner product defined by

$$(\psi, \chi)_{A'} := \sum_{\tau'_k \in \Omega} \psi_k \chi_k A'_k,$$

where the sum is over all dual cells including dual boundary cells, and A'_k denotes the area of the k^{th} dual cell. The norm is denoted by $\|\cdot\|_{A'}$, and the inner product space by $\mathcal{S}$, or by $\mathcal{S}_0$ if the boundary values are all zero. Sometimes, it will be convenient to extend this norm and inner product over the interior or boundary nodes separately. A subscript will be used to denote such dependence thus: $\|\cdot\|_{A',\Omega}$.

For each primal cell τ_i, discrete flux and divergence operators are defined on $\mathcal{U}$ by

$$(\hat{D}u)_i := \sum_{\sigma_j \in \partial\tau_i} u_j \tilde{h}_j$$

and

$$(Du)_i = (\hat{D}u)_i / A_i.$$

By $\tilde{h}_j$ we mean h_j negatively signed if the corresponding velocity component is directed towards the inside of τ_i, and positively signed otherwise.

For each interior dual cell τ'_j discrete circulation and curl operators are defined by

$$(\hat{C}u)_j := \sum_{\sigma'_k \in \partial\tau'_j} u_k \tilde{h}'_k$$

and

$$(Cu)_j = (\hat{C}u)_j / A'_j.$$

This time the tilde produces a negative sign if the corresponding dual edge is directed against the positive sense of description of $\partial\tau'_j$ and a positive sign otherwise.

This definition cannot be used in boundary dual cells. To extend it, we must require that tangential velocity components are specified along boundary segments defined by the intersections of consecutive dual mesh edges with Γ. Then we can define discrete circulations and curls in the same way even for the boundary dual cells. These extensions of C and $\hat{C}$ to the boundary are denoted by C_b and $\hat{C}_b$. We will consider that the components of Cu are associated with interior nodes ν_i and that the components of $C_b u$ are associated with interior or boundary nodes as appropriate.

Normal boundary components of $u \in \mathcal{U}$ are denoted by $u|_\Gamma$. Sometimes, it will be convenient to use the same notation to include tangential components in the sense of the previous paragraph as well as normal components. It will be explicitly stated whenever such tangential boundary components are included.

Slope operators are defined on $\mathcal{P}$ and $\mathcal{S}$ by

$$(G\phi)_i := \frac{\phi_{i_2} - \phi_{i_1}}{h}$$

where i_1 and i_2 are the nodes defining σ'_i and the positive direction is from i_1 to i_2, and

$$(R\psi)_i := \frac{\psi_{i_2} - \psi_{i_1}}{h}, \; i_2 > i_1$$

where σ_i is a primal edge with endpoints i_1 and i_2.

We will use the summation by parts formulas which follow:

$$(i) \qquad (u, R\psi)_W = (C_b u, \psi)_{A'} \quad \forall u \in \mathcal{U} \; \forall \psi \in \mathcal{S}$$

where tangential components of u are zero. This notation is consistent with the definition of C_b only if the tangential components of u are zero. We will require a variant of this to cover the case when $R\psi$ is restricted to interior edges - those with at most one node on Γ. This is

$$(i)' \qquad (u,\, R\psi)_W = (C_b u,\, \psi)_{A'} \qquad \forall u \in \mathcal{U}_0 \ \forall \psi \in \mathcal{S}$$

so that the normal boundary values of u are zero as well.

$$(ii) \qquad (u,\, G\phi)_W = -(Du,\, \phi)_A \qquad \forall u \in \mathcal{U}_0 \ \forall \phi \in \mathcal{P}.$$

These are easily proved by direct computation.

3. Div-curl Results

A covolume method and analysis was given in [Nicolaides 1991] for the div-curl system

$$\begin{aligned} \operatorname{div} \mathbf{u} &= \rho \\ \operatorname{curl} \mathbf{u} &= \omega \\ \mathbf{u} \cdot \mathbf{n}|_\Gamma &= f. \end{aligned}$$

We will quote some results of [Nicolaides 1991] which are used below.

In the notation introduced in section 2 the discrete div-curl approximation is

$$\begin{aligned} Du &= \bar{\rho} \\ Cu &= \bar{\omega} \\ u|_\Gamma &= \bar{f} \end{aligned}$$

where in each case the data is computed by simple averaging over the appropriate geometrical element. For example,

$$\bar{\omega}_i := \frac{1}{|\tau_i'|} \int_{\tau_i'} \omega \, dx\, dy$$

and

$$\bar{f}_j := \frac{1}{|\sigma_j|} \int_{\sigma_j} f \, ds$$

where σ_j is numbered anticlockwise around Γ.

It is worth remembering that Cu are the discrete curls (normalized circulations) taken around interior nodes only.

If Ω is multiply connected there is an additional condition [Nicolaides 1991], but we are considering rectangular domains here. The results stated below are valid generally.

Define

$$u_k^{(2)} := \frac{1}{|\sigma_k'|} \int_{\sigma_k'} \mathbf{u} \cdot \mathbf{t} \, ds \qquad \sigma_k \in \Omega \tag{1}$$

where the integration is along the positive direction $\mathbf{t}$ of σ_k'. The superscript is chosen for consistency with [Nicolaides 1991]. If $\sigma_k \in \Gamma$ then we define

$$u_k^{(2)} := \bar{f}_k.$$

An error estimate follows:

Theorem 1. *Assume that the div-curl equations have a unique solution* $\mathbf{u}$ *with* $\mathbf{u} \in H^2(\Omega)$. *Then (a) the discrete div-curl equations have unique solution* u, *and (b) the estimate*

$$\|u - u^{(2)}\|_W \leq Kh^2 |\mathbf{u}|_{H^2(\Omega)}$$

holds where K *is independent of* $\mathbf{u}$ *and* h.

Proof. This is proved in the remarks following Theorem 6.1 in [Nicolaides 1991].

We will also need the following bound on the discrete solution by its data.

Theorem 2. *The solution of the system*

$$\begin{aligned} Du &= 0 \\ Cu &= \omega \\ u|_\Gamma &= 0 \end{aligned}$$

satisfies

$$\|u\|_W \leq K\|\omega\|_{A'}$$

where K *is a constant independent of* ω *and* h.

The norm on the right here refers to the summation over interior nodes of the primal mesh.

Proof. See section 7 of [Nicolaides 1991].

4. Stokes Equations

The inhomogeneous stationary Stokes equations are

$$\begin{aligned} \triangle\mathbf{u} - \nabla p &= \mathbf{f} \qquad \text{in}\,\Omega \\ \operatorname{div}\mathbf{u} &= 0 \qquad \text{in}\,\Omega \\ \mathbf{u}|_\Gamma &= \mathbf{g}. \end{aligned}$$

Here, Ω denotes a polygonal domain in R^2, and Γ denotes its boundary.

The standard weak form of the Stokes equations has unique solution $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $p \in L^2_0(\Omega)$ if $\mathbf{f} \in L^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ [Girault and Raviart 1986].

The vorticity ω is defined to be $\operatorname{curl}\mathbf{u} := \partial_x v - \partial_y u$ where the velocity field $\mathbf{u} := (u, v)$. In terms of ω and using the incompressibility constraint, the momentum equation becomes

$$-\mathbf{curl}\,\omega - \nabla p = \mathbf{f} \tag{2}$$

where **curl** denotes the operator $(\partial_y, -\partial_x)$.

Let σ_i denote an interior mesh segment, and let $\mathbf{t}$ and $\mathbf{n}$ be the unit tangent and unit normal directed along the positive directions of σ_i and σ_i' respectively. σ_i may have at most one node on Γ. Let A and B denote σ_i's nodes where the positive direction is from A to B. Taking the inner product of the last equation with $\mathbf{n}$ and integrating along σ_i from A to B gives

$$-\omega(B) + \omega(A) = \int_{\sigma_i} \frac{\partial p}{\partial n}\, dt + \int_{\sigma_i} \mathbf{f} \cdot \mathbf{n}\, ds. \tag{3}$$

If, say, B is a boundary node we will use this formula to extend the vorticity to B. The extended value is well defined and independent of the particular line segment through B (or the point A on it) under the assumptions $\omega \in H^2(\Omega)$, $\nabla p \in \mathbf{H}^1(\Omega)$, $\mathbf{f} \in \mathbf{H}^1(\Omega)$. To prove this, denote the extension to B just defined by $\omega_1(B)$, and let another one based on a positively directed line segment from C to B be called $\omega_2(B)$. Then it follows that

$$[\omega_2(B) - \omega_1(B)] + [\omega(A) - \omega(C)] = \int_{AB} (\frac{\partial p}{\partial n} + \mathbf{f} \cdot \mathbf{n})\, dt - \int_{CB} (\frac{\partial p}{\partial n} + \mathbf{f} \cdot \mathbf{n})\, dt$$

so that

$$\begin{aligned}[\omega_2(B) - \omega_1(B)] &= \int_{\partial(ABC)} (\nabla p - \mathbf{f}) \cdot \mathbf{n}\, ds \\ &= 0\end{aligned}$$

where now $\mathbf{n}$ denotes the outer unit normal to the boundary $\partial(ABC)$ of the triangle ABC and the trace theorem was used in association with (2).

Note that since $\omega \in H^2(\Omega)$, by Sobolev's lemma ω is uniformly continuous on Ω and so can be uniquely extended to be continuous on $\bar{\Omega}$. The extension of ω onto $\bar{\Omega}$ using (3) and the Sobolev extension coincide: for denoting by $\omega_S(B)$ the Sobolev value it follows that with $A \in \Omega$

$$\begin{aligned}|\omega(B) - \omega_S(B)| &\leq |\omega(B) - \omega(A)| + |\omega_S(B) - \omega(A)| \\ &\leq \int_{AB} |\frac{\partial p}{\partial n}|\, ds + |\omega_S(B) - \omega_S(A)| \\ &\leq |AB|^{1/2} \|p\|_{H^2(\Omega)} + |\omega_S(B) - \omega_S(A)|\end{aligned}$$

where the integration is performed along the line segment joining A and B, and Cauchy's inequality and a trace theorem were used. The result follows from this.

5. Discrete Stokes Equations

The discretization of the Stokes equations is based on (3) obtained above. The momentum equation may be written as

$$-R\omega - \mu(\frac{\partial p}{\partial n}) = F$$

where

$$\mu(\frac{\partial p}{\partial n})_i := \frac{1}{h} \int_{\sigma_i} \frac{\partial p}{\partial n}\, dt$$

and $\mathbf{n}$ denotes the unit normal to σ_i, and F denotes the vector of average normal components of $\mathbf{f}$. In terms of the operators of section 2, the discrete equations are by definition

$$\begin{aligned} -R\omega' - Gp' &= F \qquad \sigma_i \in \Omega \\ Du' &= 0 \\ C_b u' &= \omega'. \end{aligned}$$

Boundary values for u are defined as simple averages of the normal and tangential components of $\mathbf{g}$ along the boundary edges of the mesh, so that a typical normal boundary value is

$$u' = \frac{1}{h}\int_{\sigma_i} \mathbf{g}\cdot\mathbf{n}\,ds \qquad \sigma_i \in \Gamma$$

where $\mathbf{n}$ denotes the unit normal directed positively for the edge and h denotes the edge length. The typical tangential boundary value, associated to a primal boundary node ν_i is similarly the mean of the tangential component of $\mathbf{g}$ between the dual edge intersections enclosing ν_i.

To normalize the pressure approximation, we may impose

$$(1, p')_A = 0.$$

The discrete momentum equation has one component per interior edge, and there is one incompressibility constraint for each primal cell. Since there is one unknown velocity component per interior edge and one pressure variable per cell there is a match between equations and unknowns, excluding the pressure normalization.

It can be proved by direct calculation [Nicolaides 1988] that this approximation to the Stokes *operator* is precisely the MAC approximation. The *equations* would be identical were it not for our treatment of the data by exact integration in place of the quadrature likely to be used in reality.

Explicit in the equations above is an unexpected connection with the velocity-vorticity formulation of the Stokes equations. From our discrete equations above we see that the MAC scheme can reasonably be described as a discretization of that formulation. But in the standard finite difference way, they are derived as an approximation to the primitive variable equations. So the MAC scheme simultaneously discretizes the two forms of the governing equations. Whatever advantages or disadvantages are perceived in either formulation are therefore present in the discrete scheme. From a slightly different viewpoint it follows that there are discrete analogs in the MAC framework (and in the covolume framework in general) of the transformations which enable us to go from one formulation to the other [Choudhury and Nicolaides 1991].

Including the pressure normalization there is one more equation than unknowns. This is not always convenient. We can avoid it by subtracting the mean pressure in the momentum equation. For this, let $e \in R^T$ be the vector with ones in all its positions, and let $\hat{A} \in R^T$ have $\hat{A}_j := A_j/|\Omega|$. In place of Gp we write $G(I - e\hat{A}^t)p$. The discrete Stokes equations become

$$\begin{aligned} -RC_b u' - G(I - e\hat{A}^t)p' &= F \\ Du' &= 0 \end{aligned}$$

together with the boundary equations. Now the pressure normalization is built in and the number of equations and unknowns and equations is the same.

Theorem 1. *The equations*

$$\begin{aligned} -RC_b u' - G(I - e\hat{A}^t)p' &= F \\ Du' &= 0 \end{aligned}$$

with prescribed normal and tangential boundary values $u|_\Gamma$ *have a unique solution.*

Proof. Consider the homogeneous equations

$$\begin{aligned} -RC_b u' - G(I - e\hat{A}^t)p' &= 0 \\ Du' &= 0 \\ u'|_\Gamma &= 0. \end{aligned}$$

Included in the homogeneous boundary condition are both the normal and tangential values.
Taking the inner product with $u' \in \mathcal{U}_0$ we obtain

$$(u', RC_b u')_W + (u', G(I - e\hat{A}^t)p')_W = 0.$$

Using the summation formulas $(i)'$ and (ii) reduces this to

$$||C_b u'||^2_{A'} = 0$$

and in particular $Cu' = 0$. By Theorem 3.1 (a) it follows that $u' = 0$. Then $G(I - e\hat{A}^t)p' = 0$ and $(I - e\hat{A}^t)p'$ is constant since the mesh is connected. But p' can differ from its mean by a constant only if the constant is zero, and so $p' = 0$. Uniqueness follows, and since the coefficient matrix is square so does existence.

6. Error Bounds

We will define

$$(\bar{\omega}_v)_i := \frac{1}{A'_i}\int_{\tau'_i} \omega\, dx\, dy \qquad \nu_i \in \Omega. \tag{4}$$

By Stokes' theorem we also have

$$(\bar{\omega}_v)_i = \frac{1}{A'_i}\int_{\partial\tau'_i} \mathbf{u}\cdot\mathbf{t}\, ds \qquad \nu_i \in \Omega. \tag{5}$$

Associated with $\bar{\omega}_v$ is a unique discrete velocity field v defined by

$$\begin{aligned} Cv &= \bar{\omega}_v \\ Dv &= 0 \\ v|_\Gamma &= u' \end{aligned}$$

where we recall that $u'|_\Gamma$ is defined as the simple average of the normal component $\mathbf{g}\cdot\mathbf{n}$ on each boundary edge.

The field v defined in this way has no tangential components on Γ associated with it. We will now define them to be equal to those of u'. It is convenient to continue to call the augmented field v. Recall that the tangential components of u' on Γ are defined as averages of the prescribed tangential velocity. It follows that the tangential components of $u'-v$ are all zero.

The last equations are the standard covolume approximation to the system

$$\begin{aligned} \operatorname{curl}\mathbf{u} &= \omega \\ \operatorname{div}\mathbf{u} &= 0 \\ \mathbf{u}\cdot\mathbf{n}|_\Gamma &= \mathbf{g}\cdot\mathbf{n}. \end{aligned}$$

The difference $u'-v$ satisfies the discrete div-curl equations

$$\begin{aligned} D(u'-v) &= 0 \\ C(u'-v) &= \omega'-\bar{\omega} \\ (u'-v)|_\Gamma &= 0 \end{aligned}$$

and by Theorem 3.2, it follows that

$$\|u'-v\|_W \le K\|\omega'-\bar{\omega}_v\|_{A'}. \tag{6}$$

In addition to the interior values, boundary values for $\bar{\omega}_v$ are defined by

$$(\bar{\omega}_v)_i := (C_b v)_i \quad \nu_i \in \Gamma.$$

Subtracting the exact and approximate momentum equations and introducing $\bar{p}\in\mathcal{P}$ equal to the mean pressure in the primal mesh cells we obtain

$$-R(\omega'-\bar{\omega}_v) - G(p'-\bar{p}) = -R(\omega-\bar{\omega}_v) - (\mu(\frac{\partial p}{\partial n}) - G\bar{p}) \qquad \sigma\in\Omega. \tag{7}$$

The meaning of the notation $\sigma\in\Omega$ is that the equation has one component for each edge of the primal mesh which is not on the boundary Γ.

It follows that

$$\begin{aligned} -(u'-v,\, R(\omega'-\bar{\omega}_v))_W - (u'-v,\, G(p'-\bar{p}))_W \quad &= \\ (u'-v,\, R(\omega-\bar{\omega}_v))_W - (u'-v,\, \mu(\frac{\partial p}{\partial n}) - G\bar{p})_W \qquad & \sigma\in\Omega \end{aligned}$$

and using both of the boundary conditions on $u'-v$ and the summation formulas $(i)'$ and (ii) that

$$\begin{aligned} -(C_b(u'-v),\, \omega'-\bar{\omega}_v)_{A'} + (D(u'-v),\, p'-\bar{p})_A \quad &= \\ -(C_b(u'-v),\, \omega-\bar{\omega}_v)_{A'} \quad - \quad & (u'-v,\, \mu(\frac{\partial p}{\partial n}) - G\bar{p})_W. \end{aligned}$$

Thus,

$$(\omega' - \bar{\omega}_v,\, \omega' - \bar{\omega}_v)_{A'} = (\omega - \bar{\omega}_v,\, \omega - \bar{\omega}_v)_{A'} + (u' - v,\, \mu(\frac{\partial p}{\partial n}) - G\bar{p})_W$$

from which it follows by Cauchy's inequality and bounding $\|u' - v\|_W$ as in (6) that

$$\|\omega' - \bar{\omega}_v\|_{A'} \leq \|\omega - \bar{\omega}_v\|_{A'} + K\|\mu(\frac{\partial p}{\partial n}) - G\bar{p}\|_W. \tag{8}$$

In this equation, the A' norms are extended over all nodes including those on Γ.

We will define a new field $u^* \in \mathcal{U}$ as follows: on interior dual edges it is given by (1), and on dual boundary edges by a similar average. It also has tangential boundary values equal to those of u'. By definition of C_b it follows from (4) and (5) extended to Γ that

$$C_b u^* =: \bar{\omega} = \frac{1}{A_i'} \int_{\tau_i'} \omega \, dx\, dy \qquad \nu_i \in \bar{\Omega}.$$

$\bar{\omega}$ gives the mean vorticity in each dual cell, including those on Γ. Notice that for interior dual cells $\bar{\omega}_v$ and $\bar{\omega}$ coincide.

Using (8) we now have with $K\epsilon(p) := \|\mu(\frac{\partial p}{\partial n}) - G\bar{p}\|_W$

$$\begin{aligned}
\|\omega' - \bar{\omega}\|_{A'} &\leq \|\omega' - \bar{\omega}_v\|_{A'} + \|\bar{\omega}_v - \bar{\omega}\|_{A'} \\
&\leq \|\omega - \bar{\omega}_v\|_{A'} + \|\bar{\omega} - \bar{\omega}_v\|_{A'} + K\epsilon(p) \\
&\leq \|\omega - \bar{\omega}\|_{A'} + 2\|\bar{\omega} - \bar{\omega}_v\|_{A'} + K\epsilon(p) \\
&\leq \|\omega - \bar{\omega}\|_{A'} + 2\|\bar{\omega} - \bar{\omega}_v\|_{A',\Gamma} + K\epsilon(p) \\
&\leq \|\omega - \bar{\omega}\|_{A',\Omega} + \|\omega - \bar{\omega}\|_{A',\Gamma} + 2\|\bar{\omega} - \bar{\omega}_v\|_{A',\Gamma} + K\epsilon(p).
\end{aligned} \tag{9}$$

The terms bounding the error can be estimated by approximation theory and Theorem 1.

6. Estimates

To begin with we will estimate $\|\omega - \bar{\omega}\|_{A',\Omega}$ in (9). For this, consider the linear functional

$$l_1(\omega) := \omega(0,0) - \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \omega(\xi,\eta)\, d\xi\, d\eta$$

on $H^2(Q_1)$, where Q_1 denotes the square $-1/2 < \xi < 1/2$, $-1/2 < \eta < 1/2$. It is clear that the functional is bounded and is zero on linear functions. It follows that

$$|l_1(\omega)| \leq K|\omega|_{H^2(Q_1)}.$$

Changing the variables in the integrals to $x := \xi h$ and $y := \eta h$ gives eventually

$$|(\omega - \bar{\omega})_j| \leq Kh|\omega|_{H^2(\tau_j')} \qquad \tau_j' \in \Omega.$$

Squaring and summing over all $\tau_j' \in \Omega$ gives

$$\|\omega - \bar{\omega}\|_{A',\Omega} \leq Kh^2|\omega|_{H^2(\Omega)}.$$

For h sufficiently small ($\leq |\omega|_{H^1(\Omega)}|/|\omega|_{H^2(\Omega)}|$) it follows that

$$\|\omega - \bar{\omega}\|_{A',\Omega} \leq Kh|\mathbf{u}|_{H^2(\Omega)}$$

which is the form we will use below.

For the corresponding boundary term we consider the cell $-1/2 < \xi < 1/2$, $0 < \eta < 1/2$. Then denoting this by Q_2, we have

$$|\omega(0,0) - \frac{1}{|Q_2|}\int_{Q_2} \omega\, d\xi\, d\eta| \leq |\omega(0,0) - \omega(0,\frac{1}{4})| + |\omega(0,\frac{1}{4}) - \frac{1}{|Q_2|}\int_{Q_2} \omega\, d\xi\, d\eta|.$$

The second term can be estimated by the same method used above for Q_1. For the first term,

$$\begin{aligned} |\int_0^{\frac{1}{4}} \omega_\eta(0,\eta)\, d\eta| &\leq K\|\omega_\eta(0,\cdot)\|_{(0,\frac{1}{4})} \\ &\leq K\|\omega_\eta\|_{H^1(Q_2)} \\ &\leq K(|\omega|_{H^1(Q_2)} + |\omega|_{H^2(Q_2)}) \end{aligned}$$

using the trace theorem. Changing variables to $x := \xi h$ and $y := \eta h$ gives

$$|\int_0^{\frac{h}{4}} \omega_y(0,y)\, dy| \leq K(|\omega|_{H^1(Q_{2,h})}) + h|\omega|_{H^2(Q_{2,h})}).$$

Squaring, multiplying by h^2 and summing over the boundary nodes and incorporating the second boundary term yields

$$\|\omega - \bar{\omega}\|^2_{A',\Gamma} \leq K^2(h^4|\omega|^2_{H^2(\Omega)} + h^2|\omega|^2_{H^1(\Omega)})$$

so that for h sufficiently small we have the estimate

$$\|\omega - \bar{\omega}\|_{A',\Gamma} \leq Kh|\omega|_{H^1(\Omega)}.$$

We shall amalgamate the interior and boundary estimates to get

$$\begin{aligned} \|\omega - \bar{\omega}\|_{A',\Omega} &\leq Kh|\omega|_{H^1(\Omega)} \\ &\leq Kh|\mathbf{u}|_{H^2(\Omega)} \end{aligned}$$

for h sufficiently small.

Next we will estimate $\|\bar{\omega}_v - \bar{\omega}\|^2_{A',\Gamma}$. The contribution to this sum associated with a fixed boundary node i has the form $|(\bar{\omega}_v)_i - (\bar{\omega})_i|^2\frac{h^2}{2}$. Suppressing the dependence on i we have

$$\begin{aligned} |\bar{\omega}_v - \bar{\omega}| &= |(u^* - v)_1\frac{h}{2} - (u^* - v)_2\frac{h}{2} + (u^* - v)_3 h|/\frac{h^2}{2} \\ &\leq |(u^* - v)_1 - (u^* - v)_2|/h + |(u^* - v)_3|/\frac{h}{2}. \end{aligned}$$

In this, the subscripts refer to the two dual edges of length $\frac{h}{2}$ and the interior dual edge of length h which are used to compute the discrete curl around the boundary node. There is no contribution from the boundary itself because the tangential component of $u^* - v$ is zero there. The term $(u^* - v)_1$, for example, is the difference between the average normal velocity component measured along the normal from the boundary to the dual node of the boundary cell in question and the average of the (prescribed) normal velocity component measured along the boundary edge of the same boundary cell. $(u^* - v)_2$ denotes the similar difference of averages on the opposite side of the node i. The union of these two boundary cells is the cell $Q_{2,h}$ which was encountered in the previous estimate.

To estimate the first term, we note that the functional $(u^* - v)_1 - (u^* - v)_2$ is bounded on $\mathbf{H}^2(Q_2)$ and it can be checked that it vanishes on linear vector fields. Using the familiar argument and a scale change we obtain the estimate

$$|(u^* - v)_1 - (u^* - v)_2| \leq Kh|\mathbf{u}|_{H^2(Q_{2,h})}.$$

Summing the corresponding terms over Γ gives

$$\sum_{\Gamma} |(u^* - v)_1 - (u^* - v)_2|^2 \leq K^2 h^2 |\mathbf{u}|^2_{H^2(\Omega)}.$$

For the remaining part, using Theorem 3.1 we obtain

$$\begin{aligned}
\sum_{\Gamma} |(u^* - v)|^2 &= h^{-2} \sum_{\Gamma} |(u^* - v)| h^2 \\
&\leq h^{-2} \sum_{\Omega} |(u^* - v)| h^2 \\
&\leq K^2 h^{-2} h^4 |\mathbf{u}|_{H^2(\Omega)} \\
&\leq K^2 h^2 |\mathbf{u}|_{H^2(\Omega)}.
\end{aligned}$$

In this, the sum on the left is over all dual edges parallel to Γ appearing in the definition of the boundary circulations (in fact, the 'dual boundary' of Ω) and the right hand sum in the second line is over interior edges of the primal mesh. Assembling the last two estimates we obtain

$$\|\bar{\omega}_v - \bar{\omega}\|_{A',\Gamma} \leq Kh|\mathbf{u}|_{H^2(\Omega)}.$$

Estimation of the pressure term follows similar lines. The basic functional for this case is

$$l_3(p) := \int_{-1/2}^{1/2} \frac{\partial p(\xi, 0)}{\partial \xi} \, d\eta - \left(\int_{-1/2}^{1/2} \int_{-1}^{0} p \, d\xi \, d\eta - \int_{-1/2}^{1/2} \int_{0}^{1} p \, d\xi \, d\eta\right).$$

$l_3(\cdot)$ is bounded on $H^2(Q_3)$, where Q_3 denotes the rectangle with corners at $(-1, \pm 1/2)$ and $(1, \pm 1/2)$, and is zero for linear functions. It follows that

$$|l_3(p)| \leq K|p|_{H^2(Q_3)}.$$

Introducing the scale changes $x := \xi h$, $y := \eta h$ we obtain

$$\left|\frac{1}{h} \int_{\sigma_k} \frac{\partial p(x, y)}{\partial x} \, dy - \frac{\bar{p}_{k_2} - \bar{p}_{k_1}}{h}\right| \leq K|p|_{H^2(Q_3(h))}$$

where the integration is along the positive direction of σ_k and $Q_3(h)$ denotes the scaled region with σ_k separating the primal mesh squares τ_{k_2} and τ_{k_1}. A similar result holds when σ_k is horizontal. Squaring and summing, it follows that

$$\|\mu(\frac{\partial p}{\partial n}) - G\bar{p}\|_W \leq Kh|p|_{H^2(\Omega)}. \tag{10}$$

These estimates give the first part of the main result:

Theorem 1. *Under the regularity assumptions* $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $\omega \in H^2(\Omega)$ *and* $p \in H^2(\Omega)$, *the estimate*

$$\|\omega' - \bar{\omega}\|_{A'} \leq Kh(|p|_{H^2(\Omega)} + |\mathbf{u}|_{H^2(\Omega)})$$

holds for all h sufficiently small.

Recalling that $\|\mathbf{u}\|^2_{H^1_0(\Omega)} = \|\operatorname{div}\mathbf{u}\|^2_{L^2(\Omega)} + \|\operatorname{curl}\mathbf{u}\|^2_{L^2(\Omega)}$ it follows that the norm of the error is indeed a discrete (mesh dependent) $\mathbf{H}^1_0(\Omega)$ norm.

8. Pressure Error

We begin this section by recalling the following standard result:

Lemma. *The equation*

$$\operatorname{div}\mathbf{v} = f \quad \in L^2_0(\Omega)$$

has a solution $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ *satisfying*

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq K\|f\|_{L^2(\Omega)}.$$

Proofs may be found in [Girault and Raviart 1986], and in [Temam 1984].

We will apply this result with

$$f := \bar{p} - p' \quad \in L^2_0(\Omega)$$

where the right side denotes the piecewise constant function with these values in each primal mesh square. Clearly

$$\|\bar{p} - p'\|_{L^2(\Omega)} = \|\bar{p} - p'\|_A$$

so that

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq K\|\bar{p} - p'\|_A. \tag{11}$$

Next, introduce $v^{(1)} \in \mathcal{U}_0$ defined on each edge of the mesh as the mean of the normal component of $\mathbf{v}$ on the associated primal edge, i.e.

$$v^{(1)}|_{\sigma_k} := \frac{1}{h}\int_{\sigma_k} \mathbf{v}\cdot\mathbf{n}\,ds \qquad \sigma_k \in \bar{\Omega}.$$

Use of the divergence theorem shows that

$$Dv^{(1)} = \bar{p} - p'. \tag{12}$$

In addition, we have

$$\|v^{(1)}\|_W \le K\|\mathbf{v}\|_{H^1(\Omega)} \le K\|\bar{p} - p'\|_A. \tag{13}$$

Only the first inequality is new. It is a consequence of the fact that the functional

$$B\mathbf{v} := \int_{-1/2}^{1/2} v_1(0, y)\, dy$$

is bounded on $\mathbf{H}^1(Q_4)$, where Q_4 denotes the square with corners $(\pm 1/2, 0)$ and $(0, \pm 1/2)$. Then changing the scale to h in both coordinate directions and summing over all the dual edges gives the result.

Similar to $v^{(1)}$ we will need v^* which is defined on the dual edges, as the mean of the tangential component along the dual edge in question:

$$v^*|_{\sigma'_k} := \frac{1}{|\sigma'_k|}\int_{\sigma'} \mathbf{v}\cdot\mathbf{n}\, ds \qquad \sigma'_k \in \bar{\Omega}.$$

$\mathbf{n}$ points along σ' in this. We will define tangential components for v^* on the boundary segments delineated by dual mesh lines. These tangential values are defined to be zero.

Now we have

$$(C_b v^*)^j = \frac{1}{\tau'_j}\int_{\partial\tau'_j} \mathbf{v}\cdot\mathbf{t}\, dt = \frac{1}{\tau'_j}\int_{\tau'_j} \operatorname{curl}\mathbf{v}\, dx\, dy$$

from which it follows that

$$\|C_b v^*\|^2_{A'} \le \|\operatorname{curl}\mathbf{v}\|^2_{L^2(\Omega)} \le \|\mathbf{v}\|^2_{H^1(\Omega)} \le K\|\bar{p} - p'\|^2_A. \tag{14}$$

We will need an estimate for $\|C_b v^{(1)}\|_{A'}$. To obtain it, we first note the estimate

$$\|v^* - v^{(1)}\|_W \le Kh\|\mathbf{v}\|_{H^1(\Omega)}. \tag{15}$$

The technique for proving this is given in [Nicolaides 1991, Section 6]. We will omit the details here. Then

$$\begin{aligned} \|C_b v^{(1)}\|_{A'} &\le \|C_b(v^* - v^{(1)})\|_{A'} + \|C_b v^*\|_{A'} \\ &\le \frac{K}{h}\|v^* - v^{(1)}\|_W + \|C_b v^*\|_{A'} \\ &\le K\|\bar{p} - p'\|_A \end{aligned} \tag{16}$$

by (15), (11) and (14) .

Now taking the inner product of $v^{(1)}$ with the basic error equation gives

$$(v^{(1)}, R(\omega' - \omega))_W + (v^{(1)}, G(\bar{p} - p'))_W = (v^{(1)}, \mu(\frac{\partial p}{\partial n}) - G\bar{p})_W$$

and using the summation formula (ii) gives

$$\|\bar{p}-p'\|_A^2 \leq \|C_b v^{(1)}\|_{A'}\|\omega'-\omega\|_{A',\bar{\Omega}} + \|v^{(1)}\|_W \|\mu(\frac{\partial p}{\partial n}) - G\bar{p}\|_W.$$

Then using (16) and (13) we finally obtain

$$\|\bar{p}-p'\|_A \leq K(\|\omega'-\omega\|_{A',\bar{\Omega}} + \|\mu(\frac{\partial p}{\partial n}) - G\bar{p}\|_W.$$

Using Theorem 6.1 and the approximation theory estimates we now have the second part of the main result:

Theorem 1. *Under the regularity assumptions* $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $\omega \in H^2(\Omega)$ *and* $p \in H^2(\Omega)$, *the estimate*

$$\|p'-\bar{p}\|_A \leq Kh(|p|_{H^2(\Omega)} + |\mathbf{u}|_{H^2(\Omega)})$$

holds for all h sufficiently small.

Thus the MAC scheme is first order accurate for the vorticity and pressure. It is not known yet whether the velocity is of higher order accuracy than the vorticity. Computations suggest that it is one order greater. [Nicolaides and Wu 1991].

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16. Abstract

The MAC discretization of fluid flow is analyzed for the stationary Stokes equations. It is proved that the discrete approximations do in fact converge to the exact solutions of the flow equations. Estimates using mesh dependent norms analogous to the standard H^1 and L^2 norms are given for the velocity and pressure respectively.

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