N91-20819

FUZZY GEOMETRY, ENTROPY AND IMAGE INFORMATION

Sankar K. Pal*
Software Technology Branch/PT4
National Aeronautics and Space Administration
Lyndon B. Johnson Space Center
Houston, Texas 77058, U.S.A.

INTRODUCTION

A gray tone picture possesses some ambiguity within the pixels due to the possible multivalued levels of brightness. The incertitude in an image may arise from grayness ambiguity or spatial (geometrical) ambiguity or both. Grayness ambiguity means "indefiniteness" in deciding a pixel as white or black. Spatial ambiguity refers to "indefiniteness" in shape and geometry of a region e.g., where is the boundary or edge of a region? or is this contour "sharp"?

When the regions in a image are ill-defined (fuzzy), it is natural and also appropriate to avoid committing ourselves to a specific (hard) decision e.g., segmentation/thresholding and skeletonization by allowing the segments or skeletons or contours, to be fuzzy subsets of the image. Similarly, for describing and interpreting ill-defined structural information in a pattern (when the pattern in-determinary is due to inherent vagueness rather than randomness), it is natural to define primitives and relation among them using labels of fuzzy set.

The present article provides various uncertainty measures arising from grayness ambiguity and spatial ambiguity in an image, and their possible applications as image information measures. The first part of the article consists of definitions of an image in the light of fuzzy set theory, and of information measures (arising from fuzziness) and tools relevant for processing/ analysis e.g., fuzzy geometrical properties, correlation, bound functions and entropy measures. The second part provides formulation of algorithms along with management of uncertainties (ambiguities) for segmentation and object extraction, and edge detection. The output obtained here is both fuzzy and nonfuzzy. Ambiguity in evaluation and assessment of membership function has also been described here.

* Dr. Pal is on leave from the post of Professor in the Electronics and Communication Sciences Unit, Indian Statistical Institute, Calcutta 700035, India.

IMAGE DEFINITION

An image X of size MxN and L levels can be considered as an array of fuzzy singletons, each having a value of membership denoting its degree of brightness relative to some brightness level ℓ, $\ell = 0, 1, 2, \ldots L-1$. In the notation of fuzzy sets, we may therefore write

$$X = \{\mu_X(x_{mn}) = \mu_{mn}/x_{mn};\ m = 1, 2 \ldots M;\ n = 1, 2, \ldots N\} \quad (1)$$

where $\mu_X(x_{mn})$ or μ_{mn}/x_{mn}, $(0 \le \mu_{mn} \le 1)$

denotes the grade of possessing some property μ_{mn} (e.g., brightness, edginess, smoothness) by the (m,n)th pixel intensity x_{mn}. In other words, a fuzzy subset of an image X is a mapping μ from X into [0, 1]. For any point $p \in X$, $\mu(p)$ is called the degree of membership of p in μ.

One may use either global or local information of an image in defining a membership function characterizing some property. For example, brightness or darkness property can be defined only in terms of gray value of a pixel x_{mn} whereas, edginess, darkness or textural property need the neighborhood information of a pixel to define their membership functions. Similarly, positional or co-ordinate information is necessary, in addition to gray level and neighborhood information to characterize a dynamic property of an image.

Again, the aforesaid information can be used in a number of ways (in their various functional forms), depending on individuals opinion and/or the problem to his hand, to define a requisite membership function for an image property.

MEASURES OF FUZZINESS AND IMAGE INFORMATION

The definitions of various entropy and other related measures which represent grayness ambiguity in an image (based on individual pixel as well as a collection of pixels) are listed below.

Linear Index of Fuzziness

$$\begin{aligned}\gamma_l(X) &= (2/MN)\sum_m\sum_n |\mu_{mn} - \underset{\sim}{\mu}_{mn}| \\ &= (2/MN)\sum_m\sum_n \min(\mu_{mn}, 1-\mu_{mn})\end{aligned} \quad (2)$$

$m = 1, 2, \ldots M;\ n = 1, 2, \ldots N$

Quadratic Index of Fuzziness

$$\gamma_q(X) = (2/\sqrt{MN})\left[\sum_m\sum_n \{\mu_{mn} - \underset{\sim}{\mu}_{mn}\}^2\right]^{0.5} \quad (3)$$

$m = 1, 2, \ldots M;\ n = 1, 2, \ldots N$

Entropy

$$H(X) = (1/MN \ln 2)\sum_m \sum_n Sn(\mu_{mn}) \tag{4}$$

with $S_n(\mu_{mn}) = -\mu_{mn} \ln \mu_{mn} - (1 - \mu_{mn}) \ln(1 - \mu_{mn})$

$m = 1, 2, \ldots M;\ n = 1, 2, \ldots N$

μ_{mn} denotes the degree of possessing some property μ by the (m, n)th pixel x_{mn}. $\underset{\sim}{\mu}_{mn}$ denotes the nearest two tone version of μ_{mn}

rth Order Entropy

$$H^r(X) = (-1/k)\sum_i \left\{\mu\left(s_i^\gamma\right)\log\left\{\mu\left(s_i^\gamma\right)\right\} + \left\{1 - \mu\left(s_i^\gamma\right)\right\}\log\left\{1 - \mu\left(s_i^\gamma\right)\right\}\right\} \tag{5}$$

$i = 1, 2, \ldots k$

s_i^r denotes the ith combination (sequence) of r pixels in X. k is the number of such sequences. $\mu(s_i^r)$ denotes the degree to which the combination s_i^r, as a whole, possesses the property μ.

Hybrid Entropy

$$H_{hy}(X) = -P_w \log E_w - P_b \log E_b \tag{6}$$

with
$$E_w = (1/MN)\sum_m \sum_n \mu_{mn} \exp(1 - \mu_{mn})$$
$$E_b = (1/MN)\sum_m \sum_n (1 - \mu_{mn}) \exp(\mu_{mn})$$

$m = 1, 2, \ldots M;\ n = 1, 2, \ldots N$

μ_{mn} denotes the degree of "whiteness" of (m, n)th pixel. P_w and P_b denote probability of occurrences of white ($\mu_{mn} = 1$) and black ($\mu_{mn} = 0$) pixels respectively. E_w and E_b denote the average likeliness (possibility) of interpreting a pixel as white and black respectively.

Correlation

$$C(\mu_1, \mu_2) = 1 - 4\left[\sum_m \sum_n \{\mu_{1mn} - \mu_{2mn}\}^2\right] \Big/ (X_1 + X_2) \tag{7}$$

$C(\mu_1, \mu_2) = 1 \quad \text{if } X_1 + X_2 = 0$

with $X_1 = \sum_m \sum_n \{2\mu_{1mn} - 1\}^2$

and $X_2 = \sum_m \sum_n \{2\mu_{2mn} - 1\}^2$, $m = 1, 2, \ldots M;\ n = 1, 2, \ldots N$

$C(\mu_1, \mu_2)$ denotes the correlation between two properties μ_1 and μ_2 (defined over the same domain). μ_{1mn} and μ_{2mn} denote the degree of possessing the properties μ_1 and μ_2 respectively by the (m, n)th pixel.

These expressions (equations 2-7) are the versions extended to two dimensional image plane from those defined for a fuzzy set. For example, index of fuzziness was defined by Kaufmann [1], entropy by DeLuca and Termini [2], rth order entropy and hybrid entropy by Pal and Pal [3], and correlation by Murthy, Pal and Dutta Majumdar [4].

Interpretation

Let us describe the properties of these measures along with their relevance to image processing/ analysis problems. Index of fuzziness reflects the ambiguity present in an image by measuring the distance between its fuzzy property plane and the nearest ordinary plane. The term "entropy", on the other hand, uses Shannon's function in the property plane but its meaning is quite different from the one of classical entropy because no probabilistic concept is needed to define it. $H^r(X)$ gives a measure of the average amount of difficulty in taking a decision on any subset of size r with respect to an image property. If $r = 1$, $H^r(X)$ reduces to (unnormalized) $H(X)$ of equation (4). Equation (5) is formulated based on the logarithmic behavior of gain function. Similar expression using exponential gain function [14] can also be defined. $H_{hy}(X)$ represents an amount of difficulty in deciding whether a pixel possesses certain properties or not by making a prevision on its probability of occurrence. In absence of fuzziness (i.e.,with proper defuzzification), H_{hy} reduces to two state classical entropy of Shannon, the states being black and white. Since a fuzzy set is a generalized version of an ordinary set, the entropy of a fuzzy set deserves to be a generalized version of classical entropy by taking into account not only the fuzziness of the set but also the underlying probability structure. In that respect, H_{hy} can be regarded as a generalized entropy such that classical entropy becomes its special case when fuzziness is properly removed.

All these terms, which give an idea of 'indefiniteness' or fuzziness of an image may be regarded as the measures of average intrinsic information which is received when one has to make a decision (as in pattern analysis) in order to classify the ensembles of patterns described by a fuzzy set.

$\gamma(X)$ and $H(X)$ are normalized in the interval [0, 1] such that

$$\text{Pr 1: } \gamma_{min} = H_{min} = 0 \text{ for } \mu_{mn} = 0 \text{ for all } (m,n)(X) \tag{8a}$$

$$\text{Pr 2: } \gamma_{max} = H_{max} = 1 \text{ for } \mu_{mn} = 0.5 \text{ for all } (m,n) \tag{8b}$$

$$\text{Pr 3: } \gamma(X) \geq \gamma(X^*)(\text{or, } H(X) \geq H(X^*)) \tag{8c}$$

and

$$\text{Pr 4: } \gamma(X) = \gamma(\overline{X})(\text{or, } H(X) \geq H(\overline{X})) \tag{8d}$$

where X* is the 'sharpened' or 'intensified' version of X such that

$$\mu_{X^*}(x_{mn}) \geq \mu_X(x_{mn}) \text{ if } \mu_X(x_{mn}) \geq 0.5$$
$$\text{and } \mu_{X^*}(x_{mn}) \leq \mu_X(x_{mn}) \text{ if } \mu_X(x_{mn}) \leq 0.5 \qquad (9)$$

In other words, $\gamma(X)$ or H(X) increases monotonically with μ, reaches a maximum at $\mu = 0.5$ and then decreases monotonically. This is explained in Fig. 1. It is to be mentioned here that the definition of fuzzy entropy [14,16] based on exponential gain function also satisfies the aforesaid properties.

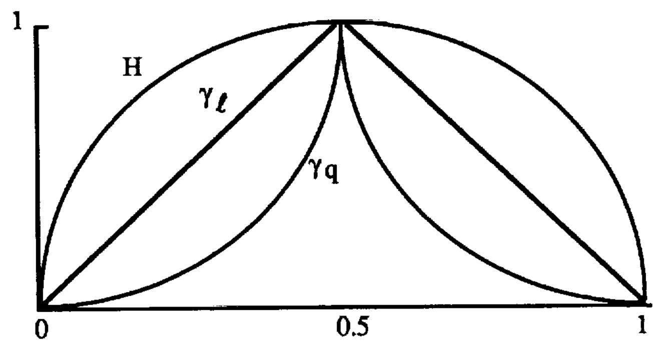

Figure 1 Variation of Fuzziness with μ.

According to property 8(c), these parameters decrease with contrast enhancement of an image. Now through processing, if we can partially remove the uncertainty on the grey levels of X, we say that we have obtained an average amount of information given by $\delta\gamma = \gamma(X) - \gamma(X^*)$ or $\delta H = H(X) - H(X^*)$ by taking a decision bright or dark on the pixels of X. The criteria $\gamma(X^*) \leq \gamma(X)$ and $H(X^*) \leq H(X)$, in order to have positive $\delta\gamma$ and δH-values, follow from Eq. (8c). If the uncertainty is completely removed, then $\gamma(X^*) = H(X^*) = 0$. In other words, $\gamma(X)$ and H(X) can be regarded as measures of the average amount of information (about the grey levels of pixels) which has been lost by transforming the classical pattern (two-tone) into a fuzzy pattern X.

It is to be noted that $\gamma(X)$ or H(X) reduces to zero as long as μ_{mn} is made 0 or 1 for all (m, n), no matter whether the resulting defuzzification (or transforming process) is correct or not. In the following discussion it will be clear how H_{hy}, takes care of this situation.

$H^r(X)$ has the following properties:

Pr 1: H^r attains a maximum if $\mu_i = 0.5$ for all i.

Pr 2: H^r attains a minimum if $\mu_i = 0$ or 1 for all i.

Pr 3: $H^r \geq H^{*r}$, where H^{*r} is the rth order entropy of a sharpened version of the fuzzy set.

Pr 4: H^r is, in general, not equal to $\overline{H}^r$, where $\overline{H}^r$ is the rth order entropy of the complement set.

Pr 5: $H^r \leq H^{r+1}$ when all $\mu_i \in [0.5, 1]$.

$H^r \geq H^{r+1}$ when all $\mu_i \in [0, 0.5]$.

Note that the property P4 of equation 8(d) is not, in general, valid here. The additional property Pr 5 implies that H^r is a monotonically nonincreasing function of r for $\mu_i \in [0, 0.5]$ and a monotonically nondecreasing function of r for $\mu_i \in [0.5, 1]$ (when 'min' operator has been used to get the group membership value).

When all the μ_i values are same, $H^1(X) = H^2(X) = \ldots = H^r(X)$. This is because of the fact that the difficulty in taking a decision regarding possession of a property on an individual is same as that of a group selected therefrom. The value of H^r would, of course, be dependent on the μ_i values.

Again, the higher the similarity among singletons the quicker is the convergence to the limiting value of H^r. Based on this observation, let us define an index of similarity of supports of a fuzzy set as $S = H^1/H^2$ (when $H^2 = 0$, H^1 is also zero and S is taken as 1). Obviously, when $\mu_i \in [0.5, 1]$ and the min operator is used to assign the degree of possession of the property by a collection of supports, S will lie in [0, 1] as $H^r \leq H^{r+1}$. Similarly, when $\mu_i \in [0, 0.5]$ S may be defined as H^2/H^1 so that S lies in [0, 1]. Higher the value of S the more alike (similar) are the supports of the fuzzy set with respect to the property P. This index of similarity can therefore be regarded as a measure of the degree to which the members of a fuzzy set are alike.

Therefore, the value of conventional fuzzy entropy (H^1 or Eq. 4) can only indicate whether the fuzziness in a set is low or high. In addition to this, the value of H^r also enables one to infer whether the fuzzy set contains similar supports (or elements) or not. The similarity index thus defined can be successfully used for measuring interclass and intraclass ambiguity (i.e., class homogeneity and contrast) in pattern recognition and image processing problems.

The aforesaid features are explained in Table 1 when $\mu_i \in [0.5, 1]$, min operator is used to compute group membership and k in Eq. 5 is considered to be $^{10}C_r$, $r = 1, 2, \ldots 6$.

$H_{hy}(X)$ has the following properties. In the absence of fuzziness when MNP_b pixels become completely black ($\mu_{mn} = 0$) and MNP_w pixels become completely white ($\mu_{mn} = 1$) then $E_w = P_w$, $E_b = P_b$ and H_{hy} boils down to two state classical entropy

$$H_c = -P_w \log P_w - P_b \log P_b, \qquad (10)$$

the states being black and white. Thus, H_{hy} reduces to H_c only when a proper

Table 1: Higher Order Entropy

Case	μ_X	H1	H2	H3	H4	H5	H6	S
1	{1,1,1,1,1,1,1,1,1,1}	0	0	0	0	0	0	1
2	{.5,.5,.5,.5,.5,.5,.5,.5,.5,.5}	1	1	1	1	1	1	1
3	{1,1,1,1,1,.5,.5,.5,.5,.5}	.5	.777	.916	.976	.996	1	.642
4	{.5,.5,.5,.5,.5,.6,.6,.6,.6,.6}	.980	.991	.996	.999	.999	1	.989
5	{.6,.6,.65,.9,.9,.9,.9,.9,.9,.915}	.538	.678	.781	.855	.905	.937	.793
6	{.8,.8,.8,.8,.8,.8,.9,.9,.9,.9}	.538	.613	.641	.649	.650	.650	.878
7	{.5,.5,.5,.5,.5,.5,.9,.9,.9,.9}	.748	.916	.979	.997	1	1	.816
8	{.7,.7,.7,.7,.7,.8,.8,.8,.8,.8}	.748	.802	.830	.841	.845	.846	.932

defuzzification process is applied to detect (restore) the pixels. $|H_{hy} - H_c|$ can therefore be acted as an objective function for enhancement and noise reduction. The lower the difference, the lesser is the fuzziness associated with the individual symbol and higher will be the accuracy in classifying them as their original value (white or black). (This property was lacking with $\gamma(X)$ and $H(X)$ measures (equations 2-4) which always reduce to zero irrespective of the defuzzification process). In other words, $|H_{hy} - H_c|$ represents an amount of information which was lost by transforming a two tone image to a gray tone.

For a given P_w and P_b $(P_w + P_b = 1,\ 0 \le P_w, P_b \le 1)$, of all possible defuzzified versions, H_{hy} is minimum for the one with properly defuzzified.

If $\quad \mu_{mn} = 0.5$ for all (m, n) then $E_w = E_b$

and $\quad H_{hy} = -\log(0.5 \exp 0.5) \qquad (11)$

i.e., H_{hy} takes a constant value and becomes independent of P_w and P_b. This is logical in the sense that the machine is unable to take decision on the pixels since all μ_{mn} values are 0.5.

Let us consider an example of a digital image in which, say, 70% pixels look white, while the remaining 30% look dark. Thus the probability of a white pixel P_w is 0.7 and that of a dark pixel P_b is 0.3. Suppose, the whiteness of the pixels is not constant, i.e., there is a variation (grayness) and similar is the case with the black pixels.

Let us now consider the effect of improper defuzzification on the pattern shown in case 1 of the Table 2. Two types of defuzzifications are considered here. In cases 2-4 all the symbols with $\mu = 0.5$ are transformed to zero when some of them were actually generated from symbol '1'. In cases 5-6 of Table 2 some of the μ values greater than 0.5 which were generated from symbol 1 (or belong to the white portion of the image) are wrongly defuzzified and brought down towards zero (instead of 1).

In both situations, it is to be noted that $|H - H_{hy}|$ does not reduce to zero. The case 7, on the other hand, has all its elements properly defuzzified. As a result, E_1 and E_0 become 0.3 and 0.7 respectively and $|H_{hy} - H_c|$ reduces to zero.

Table 2: Effect of wrong defuzzification(with $p_0 = 0.3$ and $p_1 = 0.7$)

Case	μ_X	E_1	E_0	H_{hy}	$\lvert H - H_{hy} \rvert$
1	{.9,.9,.8,.8,.7,.6,.5,.5,.4,.3}	.620	.876	.235	.375
2	{.999,.999,.9,.8,.7,.7,.3,.3,.2,.1}	.576	.776	.342	.268
3	{1,1,1,.99,.9,.9,.1,.1,0,0}	.450	.648	.542	.068
4	{1,1,1,1,1,1,0,0,0,0}	.400	.600	.632	.021
5	{.99,.99,.1,.1,.9,.8,.7,.2,.1,.1}	.630	.634	.456	.154
6	{1,1,0,0,1,1,1,0,0,0}	.500	.500	.693	.082
7	{1,1,1,1,1,1,1,0,0,0}	.300	.700	.611	0

There had been some attempts [2, 15] to combine probabilistic entropy and possibilistic entropy, they failed to have the aforesaid property of the effect of wrong defuzzification. The details of classical entropy measures (e.g., higher order, conditional and positional) of an image are available in [14, 16].

$C(\mu_1, \mu_2)$ of equation (7) has the following properties.

a) If for higher values of $\mu_1(X)$, $\mu_2(X)$ takes higher values and the converse is also true then $C(\mu_1, \mu_2)$ must be very high.

b) If with increase of x, both μ_1 and μ_2 increase then $C(\mu_1, \mu_2) > 0$.

c) If with increase of x, μ_1 increases and μ_2 decreases or vice versa then $C(\mu_1, \mu_2) < 0$.

d) $C(\mu_1, \mu_1) = 1$

e) $C(\mu_1, \mu_1) \geq C(\mu_1, \mu_2)$

f) $C(\mu_1, 1-\mu_1) = -1$

g) $C(\mu_1, \mu_2) = C(\mu_2, \mu_1)$

h) $-1 \leq C(\mu_1, \mu_2) \leq 1$

i) $C(\mu_1, \mu_2) = -C(1-\mu_1, \mu_2)$

j) $C(\mu_1, \mu_2) = C(1-\mu_1, 1-\mu_2)$

IMAGE GEOMETRY

The various geometrical properties of a fuzzy image subset (characterized by $\mu_X(x_{mn})$ or simply by μ) as defined by Rosenfeld [5,6] and Pal and Ghosh [7] are given below with illustration. These provide measures of ambiguity in geometry (spatial domain) of an image.

Area The area of a fuzzy subset μ is defined as [5]

$$a(\mu) = \int \mu \tag{12}$$

where the integration is taken over a region outside which μ=0. For μ being piecewise constant (in case of digital image) the area is

$$a(\mu) = \Sigma \mu \tag{13}$$

where the summation is over a region outside which μ=0. Note from equation (13) that area is the weighted sum of the regions on which μ has constant value weighted by these values.

Example 1 Let μ be of the form

$$\begin{matrix} 0.2 & 0.4 & 0.3 \\ 0.2 & 0.7 & 0.6 \\ 0.6 & 0.5 & 0.6 \end{matrix}$$

Area $a(\mu)$ = (0.2+0.4+0.3+0.2+0.7+0.6+0.6+0.5+0.6) = 4.1

Perimeter If μ is piecewise constant, the perimeter of μ is defined as [5]

$$p(\mu) = \sum_{i,\,j,\,k} |\mu(i) - \mu(j)| * |A(i, j, k)| \tag{14}$$

This is just the weighted sum of the lengths of the arcs A(i, j, k) along which the regions having constant μ values μ(i) and μ(j) meet, weighted by the absolute difference of these values. In case of an image if we consider the pixels as the piecewise constant regions, and the common arc length for adjacent pixels as unity then the perimeter of an image is defined by

$$p(\mu) = \sum_{i,\,j} |\mu(i) - \mu(j)| \tag{15}$$

where μ(i) and μ(j) are the membership values of two adjacent pixels.

For the fuzzy subset μ of example 1, perimeter is

$$\begin{aligned} p(\mu) = {} & |0.2-0.4|+|0.2-0.2|+|0.4-0.3|+|0.4-0.7| \\ & +|0.3-0.6|+|0.2-0.6|+|0.2-0.7|+|0.7-0.6| \\ & +|0.7-0.5|+|0.6-0.6|+|0.6-0.5|+|0.5-0.6| \\ & =2.3 \end{aligned}$$

Compactness The compactness of a fuzzy set μ having an area of a (μ) and a perimeter of p(μ) is defined as [5]

$$\mathrm{comp}(\mu) = \frac{a(\mu)}{(p(\mu))^2} \tag{16}$$

Physically, compactness means the fraction of maximum area (that can be encircled by the perimeter) actually occupied by the object. In non fuzzy case the value of compactness is maximum for a circle and is equal to $\pi/4$. In case of fuzzy disc, where the membership value is only dependent on its distance from the center, this compactness value is $\geq \pi/4$ [6]. Of all possible fuzzy discs compactness is therefore minimum for its crisp version.

For the fuzzy subset μ of example 1, comp(μ) = 4.1/(2.3*2.3) = 0.775.

Height and Width The height of a fuzzy set μ is defined as [5]

$$h(\mu) = \int \max_{m} \mu_{mn} dn \qquad [17]$$

where the integration is taken over a region outside which $\mu_{mn} = 0$.

Similarly, the width of the fuzzy set is defined by

$$w(\mu) = \int \max_{n} \mu_{mn} dm \qquad (18)$$

with the same condition over integration as above. For digital pictures m and n can take only discrete values, and since $\mu = 0$ outside the bounded region, the max operators are taken over a finite set. In this case the definitions take the form

$$h(\mu) = \sum_{n} \max_{m} \mu_{mn} \qquad (19)$$

and

$$w(\mu) = \sum_{m} \max_{n} \mu_{mn} \qquad (20)$$

$m = 1, 2, \ldots M;\ n = 1, 2, \ldots N$

So physically, in case of a digital picture, height is the sum of the maximum membership values of each row. Similarly, by width we mean the sum of the maximum membership values of each column.

For the fuzzy subset μ of example 1, height is h(μ) = 0.4+0.7+0.6 = 1.7 and width is w(μ) = 0.6+0.7+0.6 = 1.9.

Length and Breadth The length of a fuzzy set μ is defined as [7]

$$l(\mu) = \max_{m} \left(\int \mu_{mn} dn \right) \qquad (21)$$

where the integration is taken over the region outside which $\mu_{mn} = 0$. In case of a digital picture where m and n can take only discrete values the expression takes the form

$$l(\mu) = \max_{m} \left(\sum_{n} \mu_{mn} \right) \qquad (22)$$

Physically speaking, the length of an image fuzzy subset gives its longest expansion in the column direction. If μ is crisp, $\mu_{mn} = 0$ or 1; in this case length is the maximum number of pixels in a column. Comparing equation (22) with (19) we notice that the length is different from height in the sense, the former takes the summation of the entries in a column first and then maximizes over different columns whereas, the later maximizes the entries in a column and then sums over different columns.

The breadth of a fuzzy set μ is defined as

$$b(\mu) = \max_n \left(\int \mu_{mn} dm\right) \tag{23}$$

where the integration is taken over the region outside which $\mu_{mn} = 0$. In case of a digital picture the expression takes the form

$$b(\mu) = \max_n \left(\sum_m \mu_{mn}\right) \tag{24}$$

Physically speaking, the breadth of an image fuzzy subset gives its longest expansion in the row direction. If μ is crisp, $\mu_{mn} = 0$ or 1; in this case breadth is the maximum number of pixels in a row. The difference between width and breadth is same as that between height and length.

For the fuzzy subset μ in example 1, length is $l(\mu) = 0.4 + 0.7 + 0.5 = 1.6$ and breadth is $b(\mu) = 0.6 + 0.5 + 0.6 = 1.7$.

Index of Area Coverage The index of area coverage of a fuzzy set may be defined as [7]

$$\mathrm{IOAC}(\mu) = \frac{\mathrm{area}(\mu)}{l(\mu) * b(\mu)} \tag{25}$$

In nonfuzzy case, the IOAC has value of 1 for a rectangle (placed along the axes of measurement). For a circle this value is $\pi r^2 / (2r * 2r) = \pi / 4$. Physically by IOAC of a fuzzy image we mean the fraction (which may be improper also) of the maximum area (that can be covered by the length and breadth of the image) actually covered by the image.

For the fuzzy subset μ of example 1, the maximum area that can be covered by its length and breadth is $1.6 * 1.7 = 2.72$ whereas, the actual area is 4.1, so the IOAC $= 4.1 / 2.72 = 1.51$.

It is to be noted that

$$l(X)/h(X) \le 1 \tag{26}$$

$$b(X)/w(X) \le 1 \tag{27}$$

When equality holds for (26) or (27) the object is either vertically or horizontally oriented.

Degree of Adjacency The degree to which two regions S and T of an image are adjacent is defined as

$$a(S,T) = \sum_{p \in BP(S)} \frac{1}{1 + |\mu(p) - r(q)|} * \frac{1}{1 + d(p)} \tag{28}$$

Here d(p) is the shortest distance between p and q, q is a border pixel (BP) of T and p is a border pixel of S. The other symbols are having their same meaning as in the previous discussion.

The degree of adjacency of two regions is maximum (=1) only when they are physically adjacent i.e., d(p)=0 and their membership values are also equal i.e., $\mu(p) = r(q)$. If two regions are physically adjacent then their degree of adjacency is determined only by the difference of their membership values. Similarly, if the membership values of two regions are equal their degree of adjacency is determined by their physical distance only.

In the following sections, we will be explaining how the aforesaid measures can be used for image segmentation and edge detection problems.

SEGMENTATION AND OBJECT EXTRACTION

The problem of grey level thresholding plays an important role in image processing and vision problems. For example, in enhancing contrast in a image we need to select proper threshold levels from its histogram so that some suitable non-linear transformation can highlight a desirable set of pixel intensities compared to others. Similarly, in image segmentation one needs proper histogram thresholding whose objective is to establish boundaries in order to partition the image spaces into meaningful regions. This Section illustrates an algorithm where these various information measures can be used to make this selection task automatic so that an optimum threshold (or set of thresholds) may be estimated without the need to refer directly to the histogram.

Algorithm 1

Given an L level image X of dimension MxN with minimum and maximum gray vales ℓ_{min} and ℓ_{max} respectively,

Step 1: Construct the membership plane using the standard S function (equation (29)) as

$$\mu_{mn} = \mu(\ell) = S(\ell;\ a,\ b,\ c)$$

(called bright image plane if the object regions possess higher gray values)

or $\quad \mu_{mn} = \mu(\ell) = 1 - S(\ell;\ a,\ b,\ c)$

(called dark image plane if the object regions possess lower gray values) with cross-over point b and a bandwidth Δ b. The S function as given below in equation (29) is shown graphically in Fig. 2 for an L-level image.

$$P_{mn} = \mu_X(x_{mn}) = S(x_{mn};\ a,\ b,\ c) = 0,\ x_{mn} \le a \tag{29a}$$

$$= 2\left[(x_{mn} - a)/(c - a)\right]^2, a \le x_{mn} \le b \tag{29b}$$

$$= 1 - 2\left[(x_{mn} - c)/(c - a)\right]^2, b \le x_{mn} \le c \tag{29c}$$

$$= 1,\ x_{mn} \ge c \tag{29d}$$

with $\quad b = (a+c)/2,\ b-a = c-b = \Delta b$.

The parameter b is the cross-over point, i.e., S(b; a, b ,c) = 0.5. Δb is the bandwidth.

Step 2: Compute $\gamma(X)$,H(X), Comp(X) and IOAC(X)

Step 3: Vary b between ℓ_{min} and ℓ_{max} and select those b for which I(X) (where I(X)) denotes one of the aforesaid measures or a combination of them) has local minima. Among the local minima let the global one have a cross over point s.

The level s, therefore, denotes the cross over point of the fuzzy image plane μ_{mn}, which has minimum grayness and/or geometrical ambiguity. The μ_{mn} plane

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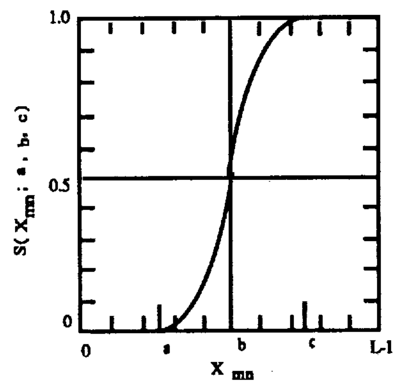

Figure 2 Standard S function for an L-level image.

then can be viewed as a fuzzy segmented version of the image X. For the purpose of nonfuzzy segmentation, we can take s as the threshold or boundary for classifying or segmenting image into object and background. (For images having multiple regions, one would have a set of such optimum $\mu(X)$ planes).

The measure I(X) in Step 3 can represent either grayness ambiguity (i.e., $\gamma(X)$ or H(X)) or geometrical ambiguity (i.e., comp(X) or IOAC(X) or a(S,T)) or both (i.e., product of grayness and geometrical ambiguities).

Faster Method of Computation

From the algorithm 1 it appears that one needs to scan an L level image L times (corresponding to L cross over points of the membership function) for computing the parameters for detecting its threshold. The time of computation can be reduced significantly by scanning it only once for computing its co-occurrence matrix, row histogram and column histogram, and by computing $\mu(l)$, l = 1, 2, . .. L every time with the membership function of a particular cross over point.

The computations of $\gamma(X)$ (or H(X)), a(X), p(X), 1(X) and b(X) can be made faster in the following way. Let h(i), i=1,2..L be the number of occurrences of the level i, c[i,j], i = 1, 2 . . L, j = 1, 2 . . L the co-occurrence matrix and $\mu(i)$, i = 1, 2.. L the membership vector for a fixed cross over point of an L level image X.

Determine $\gamma(X)$, area and perimeter as

$$\gamma(X) = \frac{2}{MN} \sum_{i=1}^{L} T(i)\, h(i) \tag{30a}$$

$$T(i) = \min\{\mu(i), 1-\mu(i)\} \tag{30b}$$

For calculating length and breadth following steps can be used. Compute the row histogram R[m, 1], m = 1, . . . M, l = 1 . . L, where R[m, 1] represents the number of occurrences of the gray level 1 in the mth row of the image. Find the column histogram C[n, 1], n = 1. . N, l = 1 . . L, where C[n, 1] represents the number of occurrences of the gray level 1 in the nth column of the image. Calculate length and breadth as

$$l(X) = \max_{n} \sum_{l=1}^{L} C[n, l].\ \mu(l) \tag{33}$$

$$b(X) = \max_{m} \sum_{l=1}^{L} R[m, l].\ \mu(l) \tag{34}$$

Some Remarks

The grayness ambiguity measure e.g., $\gamma(X)$ or H(X) basically sharpens the histogram of X using its global information only and it detects a single threshold in its valley region. Therefore, if the histogram does not have a valley, the above measures will not be able to select a threshold for partitioning the histogram. This can readily be seen from Equation (30) which shows that the minima of $\gamma(X)$ measure will only correspond to those regions of gray level which has minimum occurrences (i.e., valley region). Comp (X) or IOAC(X), on the other hand, uses local information to determine the fuzziness in spatial domain of an image. As a result, these are expected to result better segmentation by detecting thresholds even in the absence of a valley in the histogram.

Again, comp(X) measure attempts to make a circular approximation of the object region for its extraction, whereas, the IOAC(X) goes by the rectangular approximation. Their suitability to an image should therefore be guided by this criterion.

Choice of Membership Function

In the aforesaid algorithm $w = 2\Delta b$ is the length of the interval which is shifted over the entire dynamic range of gray scale. As w decreases, the $\mu(x_{mn})$ plane would have more intensified contrast around the cross-over point resulting in decrease of ambiguity in X. As a result, the possibility of detecting some undesirable thresholds (spurious minima) increases because of the smaller value of Δ b. On the other hand, increase of w results in a higher value of fuzziness and thus leads towards the possibility of losing some of the weak minima.

The criteria regarding the selection of membership function and the length of window (i.e., w) have been reported recently by Murthy and Pal [10] assuming continuous function for both histogram and membership function. For a fuzzy set "bright image plane", the membership function μ: $[0, w] \rightarrow [0,1]$ should be such that

i) μ is continuous, $\mu(0) = 0$, $\mu(w) = 1$

ii) μ is montominally non-decreasing, and

iii) $\mu(x) = 1 - \mu(w-x)$ for all $x \in [0, w]$ where w>0 is the length of the window.

Furthermore, μ should satisfy the bound criteria derived based on the correlation measure (equation 7). The main properties on which correlation was formulated are

P_1: If for higher values of μ_1, μ_2 takes higher values and for lower values of μ_1, μ_2 also takes lower values then $C(\mu_1, \mu_2) > 0$

P_2: If $\mu_1\uparrow$ and $\mu_2\uparrow$ then $C(\mu_1, \mu_2) > 0$

P_3: If $\mu_1\uparrow$ and $\mu_2\downarrow$ then $C(\mu_1, \mu_2) < 0$

[$\uparrow$ denotes increases and $\downarrow$ denotes decreases].

It is to be mentioned that P2 and P3 should not be considered in isolation of P_1. Had this been the case, one can cite several examples when $\mu_1\uparrow$ and $\mu_2\uparrow$ but $C(\mu_1, \mu_2) < 0$ and $\mu_1\uparrow$ and $\mu_2\downarrow$ but $C(\mu_1, \mu_2) > 0$. Subsequently, the type of membership functions which should not be considered in fuzzy set theory are categorized with the help of correlation. Bound functions h_1 and h_2 are accordingly derived [11]. They are

$$h_1(x) = 0, \quad 0 \le x \le \epsilon \qquad (35)$$
$$= x - \epsilon, \quad \epsilon \le x \le 1$$

$$h_2(x) = x + \epsilon, \quad 0 \le x \le 1 - \epsilon \qquad (36)$$
$$= 1, \quad 1 - \epsilon \le x \le 1$$

where $\epsilon = 0.25$. The bounds for membership function μ are such that

$$h_1(x) \le \mu(x) \le h_2(x) \text{ for } x \in [0,1].$$

For x belonging to any arbitrary interval, the bound functions will be changed proportionately. For $h_1 \le \mu \le h_2$, $C(h_1, h_2) \ge 0$, $C(h_1, \mu) \ge 0$ and $C(h_2, \mu) \ge 0$.

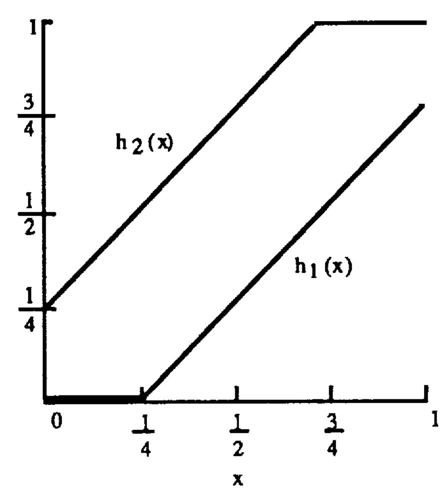

Figure 3 Bound Functions for $\mu(x)$.

The function μ lying in between h_1 and h_2 does not have most of its variation concentrated (i) in a very small interval, (ii) towards one of the end points of the interval under consideration and (iii) towards both the end points of the interval under consideration.

Figure 3 shows such bound functions. It is to be noted that Zadeh's standard S function (equation 29) satisfies these bounds.

It has been shown [10] that for detecting a minimum in the valley region of a histogram, the window length w of the μ function should be less that the distance between two peaks around that valley region.

H^r as an Objective Criterion

Let us now explain another way of extracting object by minimizing higher order fuzzy entropy (equation 5) of both object and background regions. Before explaining the algorithm, let us describe the membership function and its selection procedure.

Let s be an assumed threshold which partitions the image X into two parts namely, object and background. Suppose the gray level ranges [1 - s] and [s + 1 - L] denote, respectively, the object and background of the image X. An inverse π-type function as shown by the solid line in the Figure 4 is used here to obtain μ_{mn} values of X. The inverse π-type function is seen (from Fig. 4) to be generated by taking union of $S(x ; (s - (L - s)), s, L)$ and $1 - S(x; 1, s, (s + s - 1))$, where S denotes the standard S function defined by Zadeh (equation 29).

The resulting function as shown by the solid line, makes μ lie in [0.5,1]. Since the ambiguity (difficulty) in deciding a level as a member of the object or the background is maximum for the boundary level S, it has been assigned a membership value of 0.5 (i.e., cross-over point). Ambiguity decreases (i.e., degree of belongingness to either object or background increases) as the gray value moves away from s on either side. The μ_{mn} thus obtained denotes the degree of belongingness of a pixel x_{mn} to either object or background.

Since s is not necessarily the mid point of the entire gray scale, the membership function (solid line if Fig. 4) may not be a symmmetric one. It is further to be noted that one may use any linear or nonlinear equation (instead of Zadeh's standard S function) to represent the membership function in Fig. 4. Unlike the Algorithm-1, the membership function does not need any parameter selection to control the output.

Algorithm 2

Assume a threshold s, $1 \leq s \leq L$ and execute the following steps.

Step 1: Apply an inverse π - type function [Fig. 4] to get the fuzzy μ_{mn} plane, with $\mu_{mn} \in$ [0.5, 1]. (The membership function is in general asymmetric).

Step 2: Compute the rth order fuzzy entropy of the object H_O^r and the background

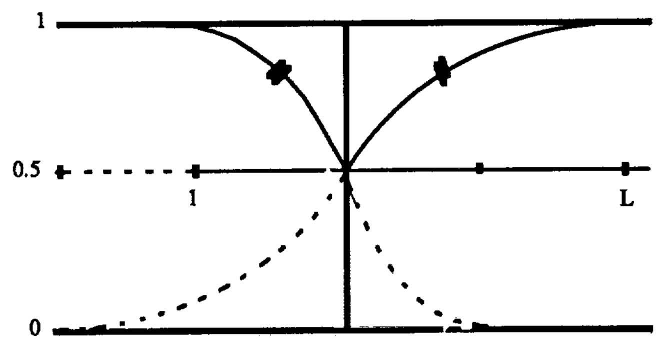

Figure 4 Inverse π function (solid line) for computing object and background entropy.

H^r_B considering only the spatially adjacent sequences of pixels present within the object and background respectively. Use the 'min' operator to get the membership value of a sequence of pixels.

Step 3: Compute the total rth order fuzzy entropy of the partitioned image as

$$H^r_s = H^r_O + H^r_B.$$

Step 4: Minimize H^r_s with respect to s to get the threshold for object background classification.

Referring back to the Table 1, we have seen that H^2 reflects the homogeneity among the supports in a set, in a better way than H^1 does. Higher the value of r, the stronger is the validity of this fact. Thus, considering the problem of object-background classification, H^r seems to be more sensitive (as r increases) to the selection of appropriate threshold; i.e., the improper selection of the threshold is more strongly reflected by H^r than H^{r-1} For example, the thresholds obtained by H^2 measure has more validity than those by H^1 (which only takes into account the histogram information). Similar arguments hold good for even higher order ($r > 2$) entropy.

Example 2

Figures 5 and 6 show the images of Lincoln and blurred chromosome along with the histogram. Table 3 shows the thresholds obtained by comp (X) and IOAC (X) measures for various window sizes w when Zadeh's S function is used as membership function. Lincoln image is of 64x64 with 32 gray levels whereas, chromosome image is of 64x64 with 64 gray levels.

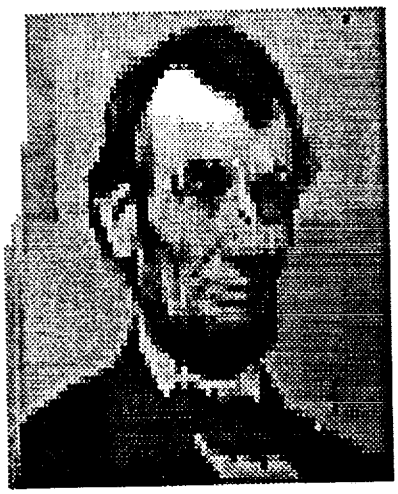

FIgure 5(a) Input.

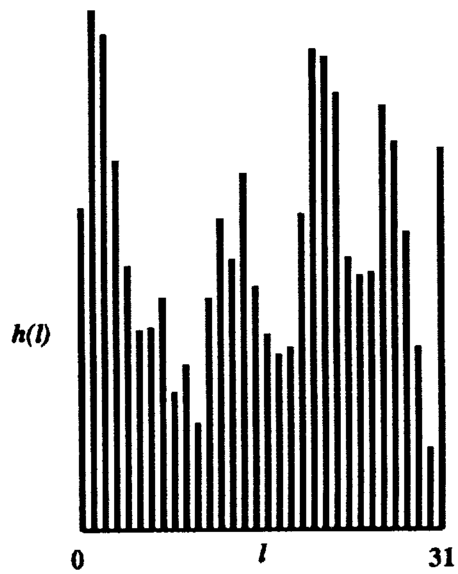

Figure 5(b) Histogram

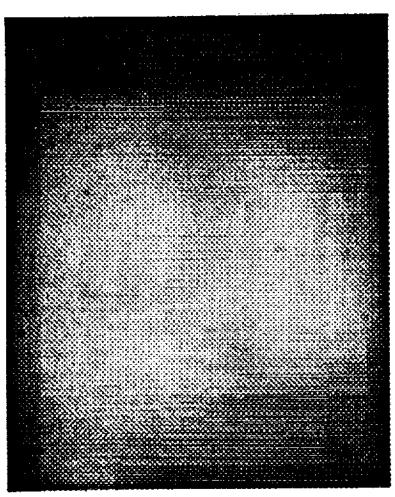

Figure 6(a) Input.

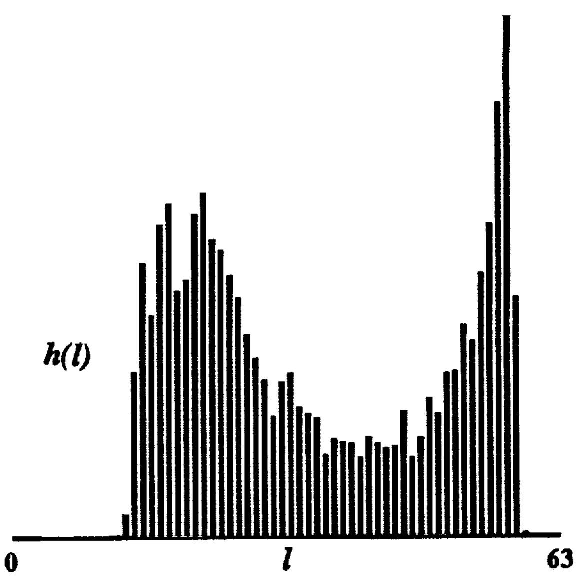

Figure 6(b) Histogram.

Figure 7(a) Threshold = 10.

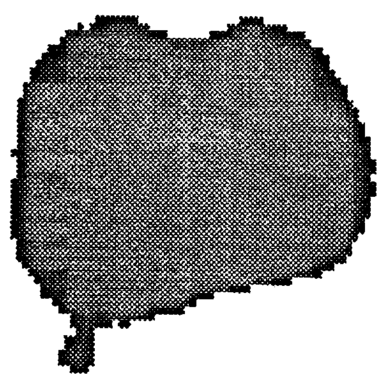

Figure 7(b) Threshold = 32.

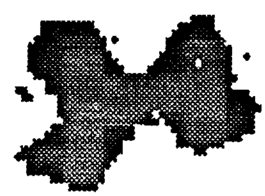

Figure 7(c) Threshold = 56.

Table 3 Various Thresholds (* denotes global minimum)

W	Lincoln	
	Comp	IOAC
8	10	11 * 23
10	10	11 * 23
12	10	11 * 23
16	9	11

W	Chromosome	
	Comp	IOAC
12	33 56	* 30 * 51
16	55	31 * 49
20	54	32 * 46
24	52	34

Threshold produced by H^2 measure (Algorithm 2) is 8 for Lincoln image. Some typical nonfuzzy thresholded outputs of these images are shown in Figure 7. Recently, transitional correlation and within class correlation have been defined [12] based on equation (7) for image segmentation which takes both local and global information into account. Automatic selection of an appropriate enhancement operator is available in [17].

EDGINESS MEASURE

Let us now describe an edginess measure [18,19] based on H^1(Equation 5) which denotes an amount of difficulty is deciding whether a pixel can be called an edge or not. Let $N^3_{x,y}$ be a 3 x 3 neighborhood of a pixel at (x, y) such that

$$N^3_{x,y} = \{(x, y), (x-1,y), (x+1,y), (x,y-1), (x,y+1), (x-1, y-1), (x-1, y+1), (x+1,y-1), (x+1,y+1)\} \tag{37}$$

The edge-entropy, $H^E_{x,y}$ of the pixel (x, y), giving a measure of edginess at (x, y) may be computed as follows. For every pixel (x, y), compute the average, maximum and minimum values of gray levels over $N^3_{x,y}$. Let us denote the average, maximum and minimum values by Avg, Max, Min respectively. Now define the following parameters.

$$D = \max \{ \text{Max} - \text{Avg}, \text{Avg} - \text{Min}\} \tag{38}$$

$$B = \text{Avg} \tag{39}$$

$$A = B - D \tag{40}$$

$$C = B + D \tag{41}$$

A π-type membership function is then used to compute μ_{xy} for all (x, y) $\in N^3_{x,y}$, such that $\mu(A) = \mu(C) = 0.5$ and $\mu(B) = 1$. It is to be noted that $\mu_{xy} \geq 0.5$. Such a μ_{xy}, therefore, gives the degree to which a gray level is close to the average value computed over $N^3_{x,y}$. In other words, it represents a fuzzy set "pixel intensity close to its average value", averaged over $N^3_{x,y}$. When all pixel values over $N^3_{x,y}$ are either equal or close to each other (i.e., they are within the same region), such a transformation will make all $\mu_{xy} = 1$ or close to 1. In other words, if there is no edge, pixel values will be close to each other and the μ values will be close to one(1); thus resulting in a low value of H^1. On the other hand, if there is an edge (dissimilarity in gray values over $N^3_{x,y}$), then the μ values will be more away from unity; thus resulting in a high value of H^1. Therefore, the entropy H^1 over $N^3_{x,y}$ can be viewed as a measure of edginess ($H^E_{x,y}$) at the point (x, y). The higher the value of $H^E_{x,y}$, the stronger is the edge intensity and the easier is its detection. As mentioned before, there are several ways in which one can define a π-type function

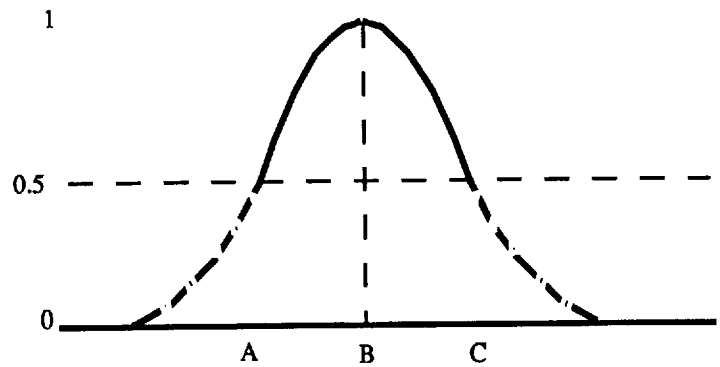

Figure 8 π function (solid line)for computing edge entropy.

(solid line) as shown in Fig. 8.

The proposed entropic measure is less sensitive to noise because of the use of a dynamic membership function based on a local neighborhood. The method is also not sensitive to the direction of edges. Other edginess measures are available in [13,20].

CONCLUSIONS

Various uncertainty and image information measures, as conveyed by entropy and fuzzy geometry, have been explained. The problems of object extraction and edge detection have been considered, as an example, to demonstrate their applications. Uncertainty in membership evaluation in these problems and its management have also been explained through bound functions. The measures comp(X) and IOAC(X) can also be used for skeleton extraction and medial axis transformation of a gray tone image [21].

Acknowledgements

This work was done while the author held an NRC-NASA research Associateship at the Johnson Space Center, Houston, Texas. The author gratefully acknowledges Dr. Robert N. Lea for his interest in this work, Ms. Dianne Rader, Ms. Kim Herhold, and Mr. Todd Carlson for typing the manuscript and Mr. Albert Leigh for his assistance in getting some results. The author is also grateful to his Institute for sanctioning him leave.

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