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N91-21206

MICROGRAVITY VIBRATION ISOLATION: An OPTIMAL CONTROL LAW for the ONE-DIMENSIONAL CASE

Richard D. Hampton

University of Virginia

Thornton Hall, McCormick Road

Charlottesville

VA 22901

Carlos M. Grodsinsky

NASA Lewis Research Center

Mail Stop 23-3

Cleveland

OH 44135

Paul E. Allaire, David W. Lewis, Carl R. Knospe

University of Virginia

Thornton Hall, McCormick Road

Charlottesville

VA 22901

ABSTRACT

Certain experiments contemplated for space platforms must be isolated from the accelerations of the platform. In this paper an optimal active control is developed for microgravity vibration isolation, using constant state feedback gains (identical to those obtained from the Linear Quadratic Regulator [LQR] approach) along with constant feedforward (preview) gains.

The quadratic cost function for this control algorithm effectively weights external accelerations of the platform disturbances by a factor proportional to $(1/\omega)^4$. Low frequency accelerations (less than 50 Hz) are attenuated by greater than two orders of magnitude. The control relies on the absolute position and velocity feedback of the experiment and the absolute position and velocity feed–forward of the platform, and generally derives the stability robustness characteristics guaranteed by the LQR approach to optimality.

The method as derived is extendable to the case in which only the relative positions and velocities and the absolute accelerations of the experiment and space platform are available.

1. INTRODUCTION

A space platform experiences local, low frequency accelerations (0.01–30 Hz) due to equipment motions and vibrations, and to crew activity [1]. Certain experiments, such as the growth of isotropic crystals, require an environment in which the accelerations amount to only a few micro–g's [2]. Such an environment is not presently available on manned space platforms.

Since the experiment and space platform centers of gravity do not coincide, a means is needed to prevent the experiment from drifting into its own orbital motion and into the space platform wall. Additionally, some experiments require umbilicals to provide power, experiment control, coolant flow, communications linkage, or other services. Unfortunately, such measures also mean that unwanted platform accelerations will be transmitted to the experiments. This necessitates experiment isolation. Passive isolators, however, cannot compensate for umbilical stiffness, nor can they achieve low enough corner frequencies even if umbilicals are absent. Active isolation is therefore essential.

The problem, then, is to design an active isolation system to minimize these undesired acceleration transmissions, while achieving adequate stability margins and system robustness. Spatial and control energy limitations must also be accommodated.

2. MATHEMATICAL MODEL

The general problem has three translational and three rotational degrees of freedom. For simplicity, however, this analysis will consider only the one-dimensional problem. The general problem could be treated in an analogous manner. Let the experiment be modeled as a mass m, with position x(t). Assume that the space station has position d(t), and that umbilicals with stiffness k and damping c connect the experiment and space station. Suppose further that a magnetic actuator applies a control force proportional to the applied current i(t), with proportionality constant α. Such a model is shown in Figure 1.

The system equation of motion is

$$m\ddot{x} + c(\dot{x}-\dot{d}) + k(x-d) + \alpha i = 0 \tag{1}$$

Division by m and rearrangement yields

$$\ddot{x} = -\frac{k}{m}(x-d) - \frac{c}{m}(\dot{x}-\dot{d}) - \frac{\alpha}{m} i \tag{2}$$

In state space notation this becomes

$$\dot{\underline{x}} = A\,\underline{x} + \underline{b}\,u + \underline{f} \tag{3}$$

where

$$\underline{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}, \quad \dot{\underline{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix}.$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \underline{b} = \begin{Bmatrix} 0 \\ -\frac{\alpha}{m} \end{Bmatrix}.$$

$$u = i \quad , \quad \underline{f} = \left\{ \begin{matrix} 0 \\ \frac{k}{m} d + \frac{c}{m} \dot{d} \end{matrix} \right\}$$

The objective is to minimize the acceleration $\ddot{x}(t)$.

3. OPTIMAL CONTROL PROBLEM

The optimal control problem is that of determining the control current $u(t) = i$ which minimizes a suitable performance index

$$J = J(\underline{x}, u, t) \tag{4}$$

for the system described by Eqn. (3) subject to the state variable conditions

$$\underline{x}(0) = \underline{x}_0 \tag{5a}$$

$$\lim_{t \to \infty} \underline{x}(t) = \underline{0} \tag{5b}$$

Another reasonable assumption is that $\underline{f}(t)$ is bounded, and it will be found mathematically advantageous (and only minimally restrictive) to assume that $\underline{f}(t)$ is also a dwindling function:

$$\lim_{t \to \infty} \underline{f}(t) = \underline{0} \tag{5c}$$

A quadratic performance index

$$J = \frac{1}{2} \int_0^\infty [\underline{x}^T W_1 \underline{x} + w_3 u^2] \, dt \tag{6}$$

has been chosen, as one that lends itself well to the variational approach to optimal controls, since an analytical solution is desired. The upper limit of the definite integral has been selected so as to yield a time-invariant controller. Here W_1 is a square 2x2 constant weighting matrix while w_3 is a weighting constant.

Although, W_1 could be a full 2x2 matrix, for this problem a diagonal form has been employed for the sake of simplicity.

$$W_1 = \begin{bmatrix} w_{1a} & 0 \\ 0 & w_{1b} \end{bmatrix} \tag{7}$$

The performance index consequently reduces to

$$J = \frac{1}{2}\int_0^{\infty} [w_{1a}x_1^2 + w_{1b}x_2^2 + w_3u^2]dt \quad . \tag{8}$$

so that each state is weighted independently.

If sinusoidal motion of the experiment is considered, so that

$$x(t) = B \sin \omega t$$

and $\ddot{x}(t) = \omega^2 x(t)$, the cost function can be expressed in terms of the acceleration and control as

$$J = \frac{1}{2}\int_0^{\infty} [(\frac{w_{1a}}{\omega^4} + \frac{w_{1b}}{\omega^2}) B^2\ddot{x}^2 + w_3u^2]\, dt \tag{9}$$

It is apparent that this performance index conveniently weights acceleration at low frequencies much more than at higher frequencies.

4. SOLUTION

Finding the optimal control to minimize Eqn. (4) is a variational problem of Lagrange, for which the initial steps of the solution are well–known (e.g., Elbert [4]). The variational approach is outlined below, following which the complications added by the nonhomogeneous term $\underline{f}(t)$ will be addressed. Current optimal controls texts either assume that $\underline{f}(t) \equiv 0$ (e.g., [4], p. 262) or require that it have a restricted range space (e.g., [6], p. 238). The solution that follows provides an analytical optimal without imposing such restrictions.

The argument of the cost function J from Eqn. (4) is augmented by the Lagrange multiplier $\underline{\lambda}$ times the system equation of motion Eqn. (3) where

$$\underline{\lambda} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} \tag{10}$$

The result $\bar{J}$ can be expressed as

$$\bar{J} = \int_0^{\infty} H \, dt \tag{11}$$

where the Hamiltonian H is

$$H = \frac{1}{2} (\underline{x}^T W_1 \underline{x} + w_3 u^2) + \underline{\lambda}^T (\dot{\underline{x}} - A\underline{x} - \underline{b}u - \underline{f}) \tag{12}$$

It is desired to obtain an optimal solution $u = u^*$ which minimizes $\bar{J}$.

The first variation of $\bar{J}(\underline{x}, u, \dot{\underline{x}})$ is

$$\delta\bar{J} = \int_0^{\infty} [\frac{\partial H}{\partial \underline{x}} \delta\underline{x} + \frac{\partial H}{\partial u} \delta u + \frac{\partial H}{\partial \dot{\underline{x}}} \delta\dot{\underline{x}}] \, dt$$

which is set equal to zero to minimize $\dot{J}$. However, integrating by parts,

$$\int_0^{\infty} \left(\frac{\partial H}{\partial \underline{\dot{x}}}\,\delta\underline{\dot{x}}\right) dt = -\int_0^{\infty} \underline{\dot{\lambda}}^T\,\delta\underline{x}\,dt$$

so that the above expression for $\delta\dot{J}$ becomes

$$\delta\dot{J} = \int_0^{\infty} \left[\left(\frac{\partial H}{\partial \underline{x}} - \underline{\dot{\lambda}}^T\right)\delta\underline{x} + \frac{\partial H}{\partial u}\,\delta u\right] dt = 0 \tag{13}$$

Both $\delta\underline{x}$ and δu are arbitrary variations, so $\delta\dot{J} = 0$ only if

$$\frac{\partial H}{\partial \underline{x}} = \underline{\dot{\lambda}}^T \tag{14a}$$

$$\frac{\partial H}{\partial u} = 0 \tag{14b}$$

The conditions given by Eqn. (5) still apply.

Solving Eqns. (14a) and (14b) yields

$$\underline{\dot{\lambda}} = W_1\,\underline{x} - A\,\underline{\lambda} \tag{15a}$$

$$u^* = \frac{1}{w_3}\,\underline{b}^T\,\underline{\lambda} \tag{15b}$$

Temporarily eliminating u^* produces the result

$$\begin{Bmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{Bmatrix} = \hat{A} \begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} + \begin{Bmatrix} \underline{f} \\ \underline{0} \end{Bmatrix} \tag{16}$$

where

$$\hat{A} = \left[\begin{array}{c|c} A & \frac{1}{w_3} \underline{b}\,\underline{b}^T \\ \hline W_1 & -A^T \end{array}\right]$$

If Eqn. (16) is now solved for $\underline{\lambda}$ in terms of $\underline{x}$ and $\underline{f}$, Eqn. (15b) will then furnish an expression for the optimal control u^*.

As noted before, optimal control texts generally treat the homogeneous problem (where $\underline{f}(t) \equiv 0$), but they do not provide an analytical solution to the nonhomogeneous system described by (5) and (16). Salukvadze has treated the nonhomogeneous problem [4,5], but his difficult treatment seems largely to have remained either uncomprehended or under–appreciated. This method is especially well–suited to low–frequency disturbance rejection, and has been applied below to the present problem.

The homogeneous solution to Eqn. (15), where $\underline{f} = \underline{0}$, is

$$\begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix}_h = e^{\hat{A}t} \begin{Bmatrix} \underline{x}_0 \\ \underline{\lambda}_0 \end{Bmatrix} \tag{17}$$

The four eigenvalues of $\hat{A}$ may be found to be, in ascending order of real parts.

$$\mu_1 = -\left[\frac{-\beta_1 + (\beta_1^{\,2} - 4\beta_2)^{1/2}}{2}\right]^{1/2} \tag{18a}$$

$$\mu_2 = -\left[\frac{-\beta_1 - (\beta_1^{\,2} - 4\beta_2)^{1/2}}{2}\right]^{1/2} \tag{18b}$$

$$\mu_3 = -\mu_1 \tag{18c}$$

$$\mu_4 = -\mu_2 \tag{18d}$$

where β_1 and β_2 are defined as follows:

$$\beta_1 = \frac{2k}{m} - \frac{c^2}{m^2} - \frac{\alpha \; w_{1b}}{m \; w_3} \tag{19a}$$

and

$$\beta_2 = \beta_1^{\,2} - 4\left[\frac{\alpha^2 \; w_{1a}}{m^2 w_3} + \frac{k}{m^2}\right] \tag{19b}$$

The eigenvectors of $\hat{A}$ corresponding to the respective eigenvalues μ_k may be chosen to be

$$\underline{p}_k = \left\{\begin{array}{c} 1 \\ \mu_k \\ \dfrac{\gamma_4}{\mu_k} + \dfrac{\gamma_1^{\,2}}{\gamma_3\mu_k} + \dfrac{\gamma_1(\gamma_2 + \mu_k)}{\gamma_3} \\ \dfrac{\gamma_1 + (\gamma_2 + \mu_k)\mu_k}{\gamma_3} \end{array}\right\} \tag{20a}$$

where γ_1, γ_2, γ_3, and γ_4 are defined below:

$$\gamma_1 = \frac{k}{m} \tag{20b}$$

$$\gamma_2 = \frac{c}{m} \tag{20c}$$

$$\gamma_3 = \frac{\alpha^2}{m^2 w_3} \tag{20d}$$

$$\gamma_4 = w_{1a} \tag{20e}$$

Using Eqns. (18) through (20) with (17) the solution to the homogeneous system is

$$\begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix}_h = \begin{Bmatrix} c_1 e^{\mu_1 t} p_{1_1} + c_2 e^{\mu_2 t} p_{2_1} + c_3 e^{-\mu_1 t} p_{3_1} + c_4 e^{-\mu_2 t} p_{4_1} \\ c_1 e^{\mu_1 t} p_{1_2} + c_2 e^{\mu_2 t} p_{2_2} + c_3 e^{-\mu_1 t} p_{3_2} + c_4 e^{-\mu_2 t} p_{4_2} \end{Bmatrix} \tag{21}$$

with $\underline{p}_k = \begin{Bmatrix} p_{k_1} \\ p_{k_2} \end{Bmatrix}$, $k = 1, \ldots, 4$ and where $c_1, \ldots, c_4$ are arbitrary constants.

Application of the variation of parameters method with terminal conditions (Eqns. 5b,c) leads to the general solution of the non–homogeneous system, with two constants of integration yet undetermined.

If the two constants of integration are eliminated by solving for $\underline{\lambda}$ in terms of $\underline{x}$ and $\underline{f}$, the general solutions for λ_1 and λ_2 become:

$$\lambda_1 = \xi_1 x_1 + \xi_2 x_2 + \xi_3 e^{-\mu_1 t} + \xi_4 e^{-\mu_2 t} \tag{22a}$$

$$\lambda_2 = \xi_5 x_1 + \xi_6 x_2 + \xi_7 e^{-\mu_1 t} + \xi_8 e^{-\mu_2 t} \tag{22b}$$

in which the ξ_i's are functions of the eigenvalues and eigenvectors of A, and of the disturbance $\underline{f}(t)$.

The Solution Form

Using the fact that

$$u^*(t) = \frac{1}{w_3}\underline{\lambda}^T\underline{b} \qquad \text{[cf. Eqn. (15b)]} \tag{23}$$

the optimal control is found to be

$$u^*(t) = \eta_1 x_1 + \eta_2 x_2 + \eta_3 e^{-\mu_1 t}\int e^{\mu_1 t} f_2(t)dt + \eta_4 e^{-\mu_2 t}\int e^{\mu_2 t} f_2(t)dt \tag{24a}$$

where

$$\eta_1 = \frac{-m}{\alpha}\left(\frac{k}{m} - \mu_1\mu_2\right) \tag{24b}$$

$$\eta_2 = \frac{-m}{\alpha}\left(\frac{c}{m} + \mu_1 + \mu_2\right) \tag{24c}$$

$$\eta_3 = \frac{m}{\alpha}\left(\frac{1}{\mu_1-\mu_2}\right)\left(\mu_1^2 + \frac{c}{m}\mu_1 + \frac{k}{m}\right) \tag{24d}$$

$$\eta_4 = -\frac{m}{\alpha}\left(\frac{1}{\mu_1-\mu_2}\right)\left(\mu_2^2 + \frac{c}{m}\mu_2 + \frac{k}{m}\right) \tag{24e}$$

(It should be noted that the feedback gains η_1 and η_2 are those which would result from applying standard LQR theory to the homogeneous system equation $\dot{\underline{x}} = A\underline{x} + b\underline{u}$). In Eqns. (24) μ_1, μ_2 are the eigenvalues of A with negative real parts, [see Eqns. (18a,b)] and

$$f_2(t) = \frac{k}{m} d + \frac{c}{m} \dot{d} \; . \tag{24f}$$

By repeated application of the method of integration by parts, the control may be re–expressed in terms of an infinite sum:

$$u^*(t) = \eta_1 x_1 + \eta_2 x_2 + \eta_3 \left[\sum_{r=0}^{\infty} \frac{(-1)^r \, f_2^{(r)}(t)}{\mu_1^{\,r+1}} \right] + \eta_4 \left[\sum_{r=0}^{\infty} \frac{(-1)^r \, f_2^{(r)}(t)}{\mu_2^{\,r+1}} \right] \tag{25}$$

Rewriting f_2 in terms of d and $\dot{d}$, the control function becomes

$$\begin{aligned} u^*(t) = {} & \eta_1 x(t) + \eta_2 \dot{x}(t) + \left[\frac{k}{m}\left(\frac{\eta_3}{\mu_1} + \frac{\eta_4}{\mu_2}\right)\right] d(t) \\ & + \sum_{i=1}^{r-1} \left[(-1)^{i-1} \frac{c}{m}\left(\frac{\eta_3}{\mu_1^{\,i}} + \frac{\eta_4}{\mu_2^{\,i}}\right) + (-1)^i \frac{k}{m}\left(\frac{\eta_3}{\mu_1^{\,i+1}} + \frac{\eta_4}{\mu_2^{\,i+1}}\right) \right] d^{(i)}(t) \\ & + \left[(-1)^{n-1} \frac{c}{m}\left(\frac{\eta_3}{\mu_1^{\,n}} + \frac{\eta_4}{\mu_2^{\,n}}\right)\right] d^{(n)}(t) + \text{higher order terms} \end{aligned} \tag{26}$$

This may be written in a more appealing form as

$$u^*(t) = c_p \, x(t) + c_v \, \dot{x}(t) + c_{d0} \, d(t) + c_{d1} \, \dot{d}(t) + \text{higher order terms} \tag{27}$$

in which the constant coefficients c_p, c_v, c_{d0}, and c_{d1} may be defined from Eqns. (24) and (26). Clearly, if the infinite sums converge rapidly enough, the optimal control can be approximated by

$$u^*(t) = c_p\, x(t) + c_v\, \dot{x}(t) + c_{d0}\, d(t) + c_{d1}\, \dot{d}(t) \tag{28}$$

For very low frequency disturbances the higher order terms in Eqn. (26) are negligibly small, and the control (Eqn. (28)) closely approximates the optimal. If, in fact, the second– and higher–order derivatives of d(t) are identically zero, the approximation is exact. It can be shown that for the critically damped closed loop system the eigenvalues are real and equal, and that the convergence is more rapid than for the overdamped system. Further, as the closed–loop system eigenvalues become more negative the convergence speed goes up as well.

5. CONTROL EVALUATION

Physical Realizability of the Control

The control, Eqn. (25), is physically realizable, if the states and sufficient derivatives of d(t) are accessible (or estimable by an observer), and if the higher order terms are negligible. It is not necessary that the eigenvalues be real, although the proof of this requires a more general linear–algebra or state–transition–matrix approach.

If values are assigned to the system parameters, associated controller gains can be evaluated. Suppose that m = 100 lbm, k = 0.3 lbf/ft, c = 0 lbf–sec/ft, and α = 10 lbf/Amp. With w_3 arbitrarily set at 1 and w_{1b} varied, associated integer values of w_{1a} can be found below which the eigenvalues μ_1 and μ_2 will always be real. Such values are tabulated in Table 1. Stated otherwise, the tabulated values of the weights w_{1a} and w_{1b} are those integer values (for the sake of simplicity) for which the closed loop system is closest to being critically damped without being underdamped. Corresponding controller feedback and feed–forward gains (for the first five derivatives) are also included.

The states x(t) and $\dot{x}(t)$ and the derivatives $d^{(0)}(t)$, $d^{(1)}(t)$ and $d^{(2)}(t)$ are clearly available for an earth–based system. However, in space, the only absolute measurements which can be directly available are $\ddot{x}(t)$ and $\ddot{d}(t)$, from which $\dot{x}(t)$, $\dot{d}(t)$ and x(t), d(t) are obtainable only by successive integration(s). Rearrangement of (28) into

$$u^*(t) = (c_p + c_{d0})x(t) + (c_v + c_{d1})\dot{x}(t) - c_{d0}[x(t) - d(t)] - c_{d1}[\dot{x}(t) - \dot{d}(t)] \tag{29}$$

or

$$u^*(t) = (c_p + c_{d0})d(t) + (c_v + c_{d1})\dot{d}(t) + c_p[x(t) - d(t)] + c_v[\dot{x}(t) - \dot{d}(t)] \tag{30}$$

obviates the need for one accelerometer, but one accelerometer plus two integrations remain necessary for either the platform or the experiment. Since $[x(t)-d(t)]$ (or one of its integrals) has not been weighted in the performance index J, experiment drift will be a problem that must be corrected either by another control loop or by a change of system states. The latter could be accomplished by incorporating an accelerometer attached to the experiment into the state equation. Alternatively, one could append an integrator to the plant, include the current $i(t)$ as a third state, and optimize the control di/dt. But for the sake of simplicity (i.e., fewer states) the former has been assumed (without development) in this paper.

The higher order terms of the control [Eqns. (25) and 26)] can be neglected, for low frequencies, if the eigenvalues μ_1 and μ_2 are of sufficient modulus. These eigenvalues, in turn, are under the control of the de igner, determined by his choice of weights w_{1a}, w_{1b}, and w_3. It is apparent from Eqn. (25) that $u^*(t)$ essentially reduces to two alternating power series. For a sinusoidal disturbance of frequency ω the series form of the control converges for $|\omega/\mu_i| < 1$ $(i = 1,2)$. It can be shown that each alternating power series converges like $\sum_{r=0}^{\infty} (-1)^r (\frac{\omega}{\mu})^{2r}$. With "low" frequency disturbances (i.e., small relative to system closed loop eigenvalues) a control formed by series truncation very closely approximates the optimal.

For example, suppose that the normalized frequencies $|\omega/\mu_i|$ for a sinusoidal disturbance are less than 1/5, and that only the feedforward control terms $c_{d0}d(t)$ and $c_{d1}\dot{d}(t)$ are included with the feedback terms. Even so, the feedforward portion of the truncated control, at any time t, will be a current that is still within 4% [i.e., $(1/5)^2$ of the feedforward portion of the actual optimal. If the normalized frequencies are below 1/10, this approximation error will be less than 1%. Table 1 shows that the gains c_{di} of higher order derivatives $d^{(i)}(t)$ [see Eqn. (26) for algebraic representations] are, in fact, quite small.

In some circumstances there may be design constraints which prevent the designer from selecting weights that will lead to sufficiently rapid convergence. However, since convergence occurs rapidly even for eigenvalues of relatively small modulus ($|\omega/\mu_i| < 1/3$), in a great many cases the designer will have much latitude in his choice of weights. For "low" frequency disturbances, in these cases, a control which includes only one or two feedforward terms will be "close" to the optimal. These frequencies will be well–attenuated.

Higher frequency disturbances will also be well–attenuated, provided the input–to–output transfer function(s) are at least strictly proper in the Laplace Transform variable s. This will not be the case for the present problem if more than three feedforward gains (c_{d0}, c_{d1}, c_{d2}) are included in the control. Practically, this means that only proportional and first– derivative feedforward [Eqn. (25) with r = 0,1 or Eqn. (26) with n = 2] should be added to the feedback control terms. As will be seen shortly, however, adding even the proportional feedforward term(s) can dramatically improve the disturbance rejection over that afforded by LQR feedback alone.

Transfer Function and Block Diagram

Neglecting the higher order terms, the transfer function between input and output accelerations or displacements is

$$\frac{s^2X(s)}{s^2D(s)} = \frac{X(s)}{D(s)} = \frac{(\frac{c}{\alpha} - c_{d1})s + (\frac{k}{\alpha} - c_{d0})}{(\frac{m}{\alpha})s^2 + (\frac{c}{\alpha} + c_v)s + (\frac{k}{\alpha} + c_p)} . \tag{31}$$

and a block diagram of the controlled system can be drawn as in Figure 2.

Control Stability, Stability Robustness, and General Robustness

Since the control feedback gains are the same as those obtained by solution of the standard Linear Quadratic Regulator (LQR) problem, the closed loop system is stable and enjoys the stability robustness characteristics guaranteed by the (LQR) approach to optimality, viz., a minimum of 60° phase margin, infinite positive gain margin, and 6 dB negative gain margin [6]. Additionally, numerical checks indicate that it enjoys substantial insensitivity, or general robustness to uncertainties in k, c, and m, as indicated by Table 2 and Figures 3 through 10. By comparing the Bode plots of Figures 3, 5, 7, and 9 (corresponding to controls using both LQR F/B and proportional F/F) with those of Figures 4, 6, 8, and 10, respectively (corresponding to controls using LQR F/B only), one can see that adding feed–forward substantially improves disturbance rejection at low frequencies. For example a comparison of Figures 3 with Figure 4 indicates that the optimal control method described above can lead to acceleration reductions of greater than four orders of magnitude for all frequencies. This reduction is more than two orders of magnitude below that afforded by LQR feedback alone at the lower frequencies, i.e., those most heavily weighted in the performance index.

The order of the reduction is eventually limited by control cost, of course, probably in terms either of actuator–related limitations (such as heat–removal or force–generation requirements) or of power limitations (especially in a space–station environment). The control also leads to displacement reductions of the same magnitude, limited in this case by actuator–stroke or spatial limitations. Providing a unit transmissibility for very low frequencies and weighting $(x-d)$ and/or $\int(x-d)$ in the performance index J would be steps toward addressing these latter limitations.

Computational Aspects

A significant amount of algebra was required to solve the two–state problem of this paper, and the labor involved increases dramatically with each additional state. However, such symbolic manipulators as MACSYMA may be used to ease the workload if a symbolic solution is desired. Further, well–known numerical methods exist (i.e., Potter's method [7] or Laub's method [8]) for solving the solution to the homogeneous system. These can readily provide the feedback gains in numerical form, even for problems with many states. It might be anticipated, then, that a numerical method also exists for finding the desired feed–forward gains. Such is the case, as will be shown in a later paper.

6. CONCLUSIONS

This paper has applied an existing method for obtaining an optimal control to the microgravity platform isolation problem, for which the disturbances to be rejected are low–frequency accelerations. The system was assumed to be representable in the form $\dot{\underline{x}} = A\ \underline{x} + \underline{b}\ u + \underline{f}$, with quadratic cost function $J = \frac{1}{2}\int_0^{\infty}(\underline{x}^T\ W_1\underline{x} + w_3u^2)dt$ and diagonal weighting matrix W_1. The resultant control law was found to be simple, stable, robust, and physically realizable. Further it was shown to have excellent acceleration– and displacement–attenuation characteristics, and to be frequency–weighted toward the low end of the acceleration spectrum.

The method is extendable to the case for which only relative positions and velocities, and absolute accelerations, are available; and can be applied so as to weight relative displacements in the performance index.

The approach as presented is algebraically intensive, but symbolic manipulators can be used to ease the algebraic labors. Further, since the method produces feedback gains identical to those obtained by the LQR approach to optimality, numerical computation of those gains is easily accomplished, even for large systems. The feed–forward gains can be found numerically with comparable ease.

7. ACKNOWLEDGEMENTS

The authors would like to recognize NASA for partial funding of this work, Eric Maslen at the University of Virginia for his many helpful control–related suggestions at problem points and regarding future work, and Dr. Gerald Brown of NASA Lewis for his insights into the microgravity isolation problem in an orbital environment.

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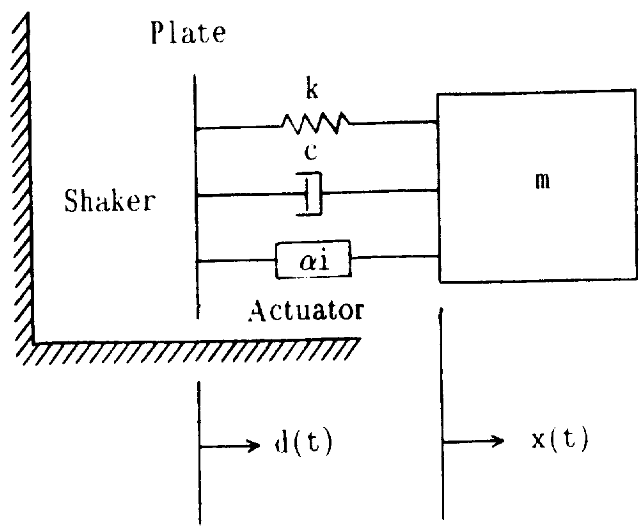


Figure 1. System Model

Table 1. Optimal F/F and F/B Gains for Selected State Variable and Control Weightings.

System Parameters:

m = 100 lbm k = 0.3 lbf/ft
c = 0.000622 lbf-sec/ft (ζ = 0.1%)
α = 10 lbf/amp

Weights			F/B Gains		F/F Gains					
w_{1a}	w_{1b}	w_3	C_p	C_v	C_{d0}	C_{d1}	C_{d2}	C_{d3}	C_{d4}	C_{d5}
2	1	1	1.3845	1.3637	0.0294	-0.0006	-0.0070	-0.0067	-0.0049	-0.0032
10	2	1	3.1324	1.9863	0.0297	-0.0001	-0.0030	-0.0019	-0.0009	-0.0004
23	3	.	4.7659	2.4413	0.0298	-0.0000	-0.0020	0.0010	-0.0004	-0.0001
41	4	1	6.3732	2.8210	0.0299	0.0000	-0.0015	-0.0007	-0.0002	-0.0001
64	5	1	7.9701	3.1514	0.0299	0.0000	-0.0012	-0.0005	-0.0001	-0.0000
92	6	1	9.5617	3.4552	0.0299	0.0000	-0.0010	-0.0004	-0.0001	-0.0000
126	7	1	11.1950	3.7354	0.0299	0.0000	-0.0008	-0.0003	-0.0001	-0.0000
165	8	1	12.8153	3.9949	0.0299	0.0000	-0.0007	-0.0002	-0.0001	-0.0000
209	9	1	14.4269	4.2380	0.0299	0.0000	-0.0006	-0.0002	-0.0000	-0.0000
258	10	1	16.0324	4.4674	0.0299	0.0000	-0.0006	-0.0002	-0.0000	-0.0000
581	15	1	24.0740	5.4729	0.0300	0.0001	-0.0004	-0.0001	-0.0000	-0.0000
1034	20	1	32.1259	6.3209	0.0300	0.0001	-0.0003	-0.0001	-0.0000	-0.0000
1617	25	1	40.1819	7.0680	0.0300	0.0001	-0.0002	-0.0000	-0.0000	-0.0000
2329	30	1	48.2297	7.7431	0.0300	0.0001	-0.0002	-0.0000	-0.0000	-0.0000
3171	35	1	56.2816	8.3640	0.0300	0.0001	-0.0002	0.0000	-0.0000	-0.0000
4443	40	1	64.3361	8.9420	0.0300	0.0001	0.0001	-0.0000	-0.0000	0.0000
9325	60	1	96.5360	10.9526	0.0300	0.0001	0.0001	-0.0000	0.0000	0.0000
16581	80	1	128.7372	12.6475	0.0300	0.0001	0.0001	-0.0000	-0.0000	-0.0000
25911	100	1	160.9389	14.1107	0.0300	0.0001	-0.0001	-0.0000	-0.0000	-0.0000

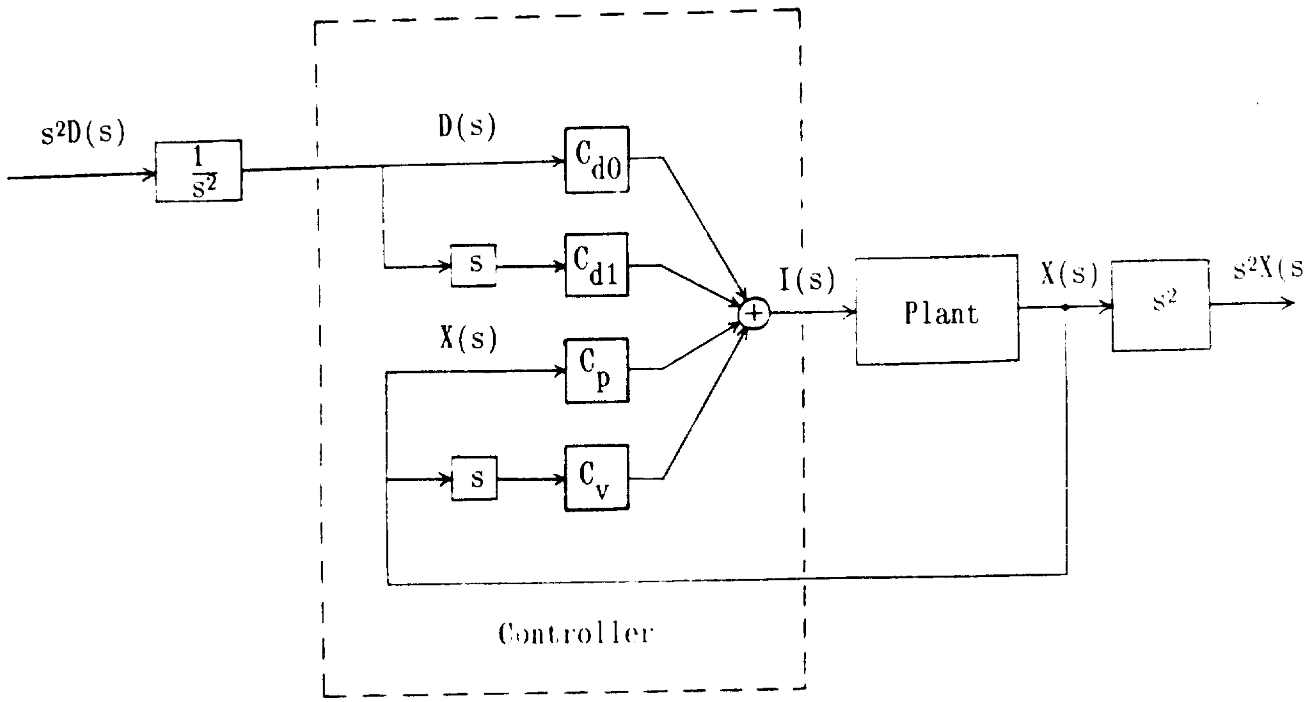


Figure 2. Block Diagram

Table 2. Closed loop transfer functions for system with design parameter values of k = 0.3, c = 0.000622, and m = 100; but with actual parameter values as shown. G1, G3, G5, and G7 include both **LQR F/B** and proportional F/F; G2, G4, G6, and G8 include **LQR** F/B alone. Weighting parameters used were w_{1a} = 258, w_{1b} = 10, w_3 = 1 (see Table 1).

System Parameters			Closed Loop Transfer Function $\frac{s^2X(s)}{s^2D(s)}$
k $(\frac{lbf}{ft})$	c $(\frac{lbf\text{-}sec}{ft})$	m (lbm)	
0.3	0.000622 (ζ=0.1%)	100	$G1(s) = \frac{0.0000622s + 0.0001}{0.31056s^2+4.4675s+16.0624}$
0.3	0.000622	100	$G2(s) = \frac{0.0000622s + 0.0300}{0.31056s^2+4.4675s+16.0624}$
0.45	0.000622	100	$G3(s) = \frac{0.0000622s + 0.0151}{0.31056s^2+4.4675s+16.0774}$
0.45	0.000622	100	$G4(s) = \frac{0.0000622s + 0.0450}{0.31056s^2+4.4675s+16.0774}$
0.3	0.00622	100	$G5(s) = \frac{0.000622s + 0.0001}{0.31056s^2+4.4680s+16.0624}$
0.3	0.00622	100	$G6(s) = \frac{0.000622s + 0.0300}{0.31056s^2+4.4680s+16.0624}$
0.45	0.00622	90	$G7(s) = \frac{0.000622s + 0.0151}{0.27950s^2+4.4680s+16.0774}$
0.45	0.00622	90	$G8(s) = \frac{0.000622s + 0.0450}{0.27950s^2+4.4680s+16.0774}$

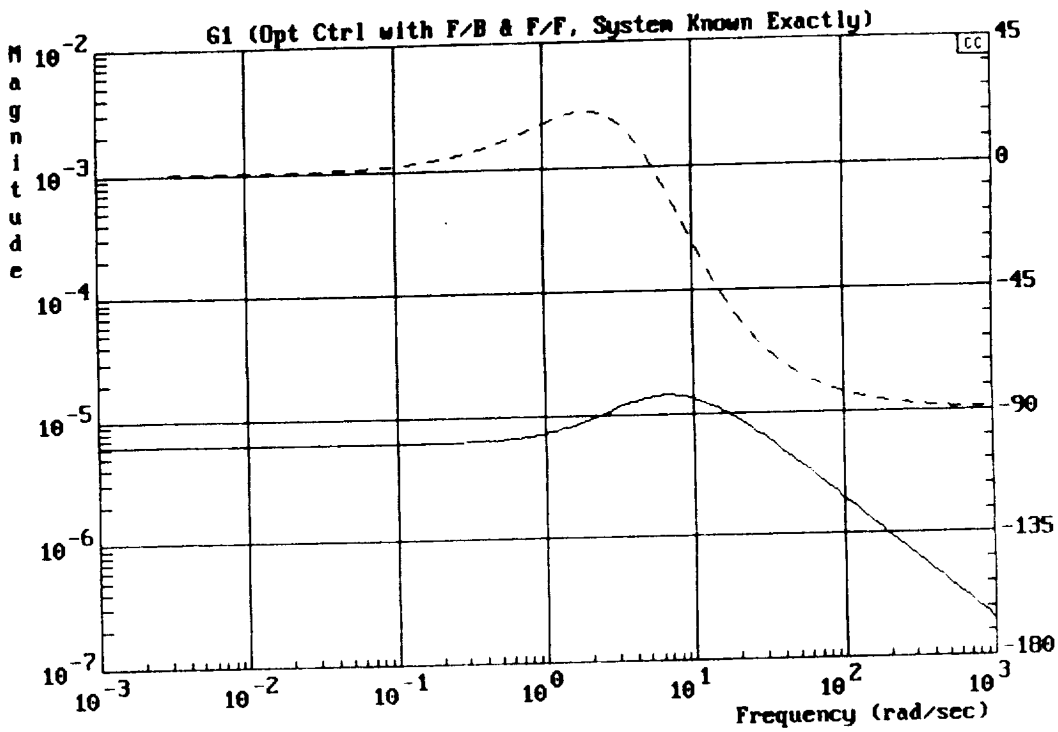

G1 (Opt Ctrl with F/B & F/F, System Known Exactly)
Magnitude
Phase
CC
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
45
0
-45
-90
-135
-180
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
Frequency (rad/sec)

Figure 3

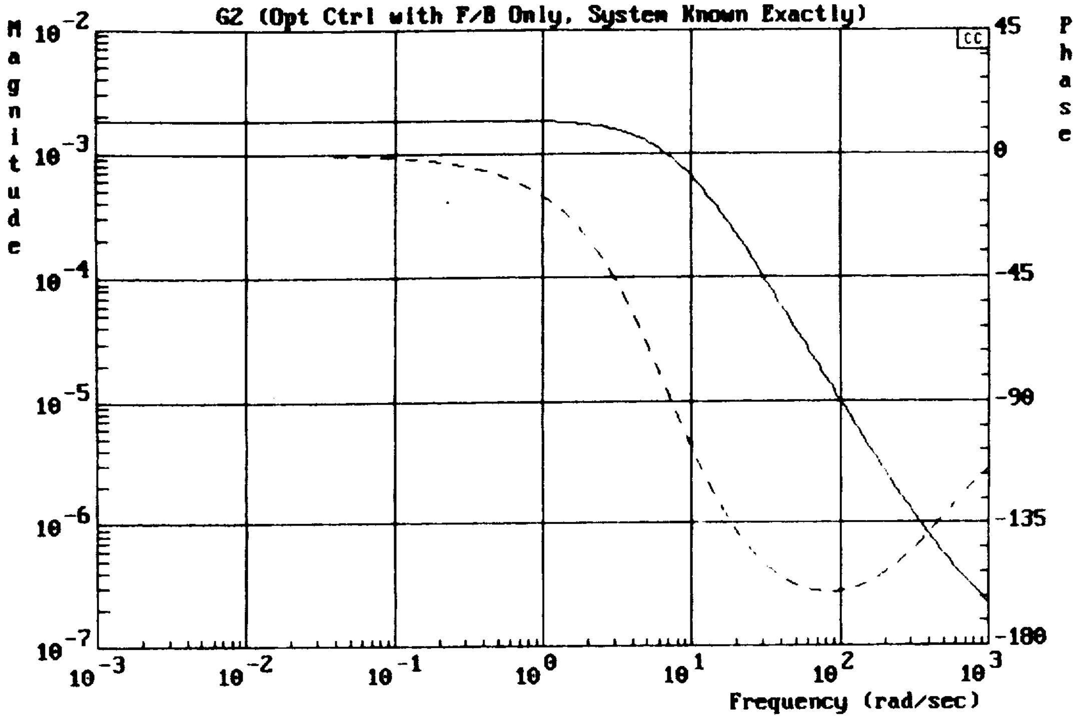
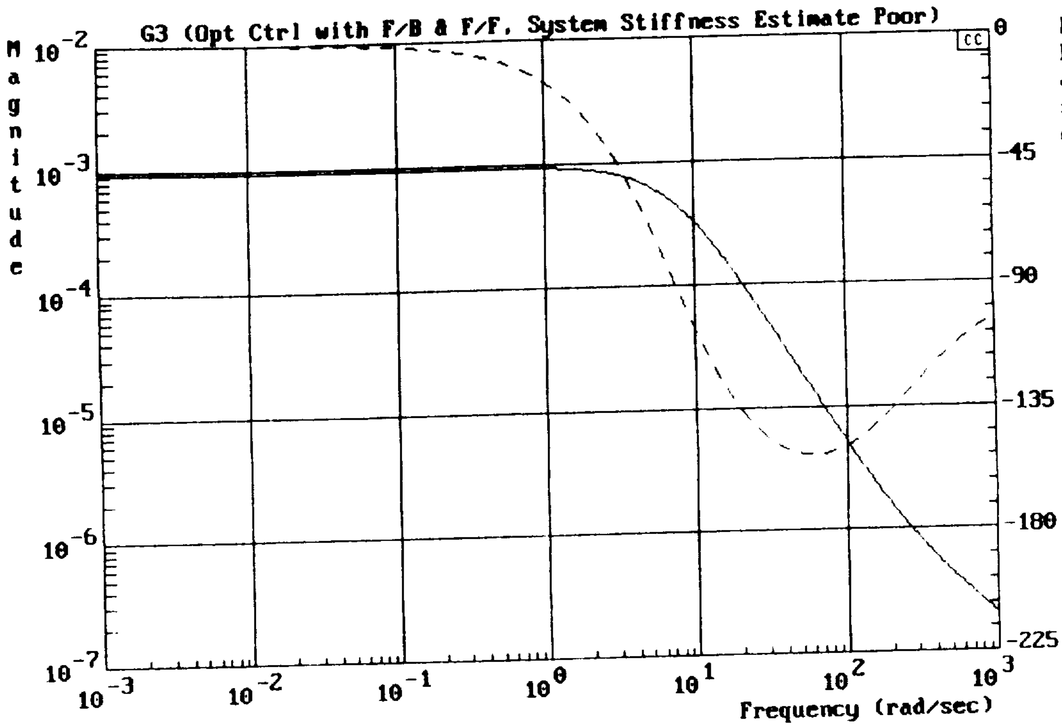
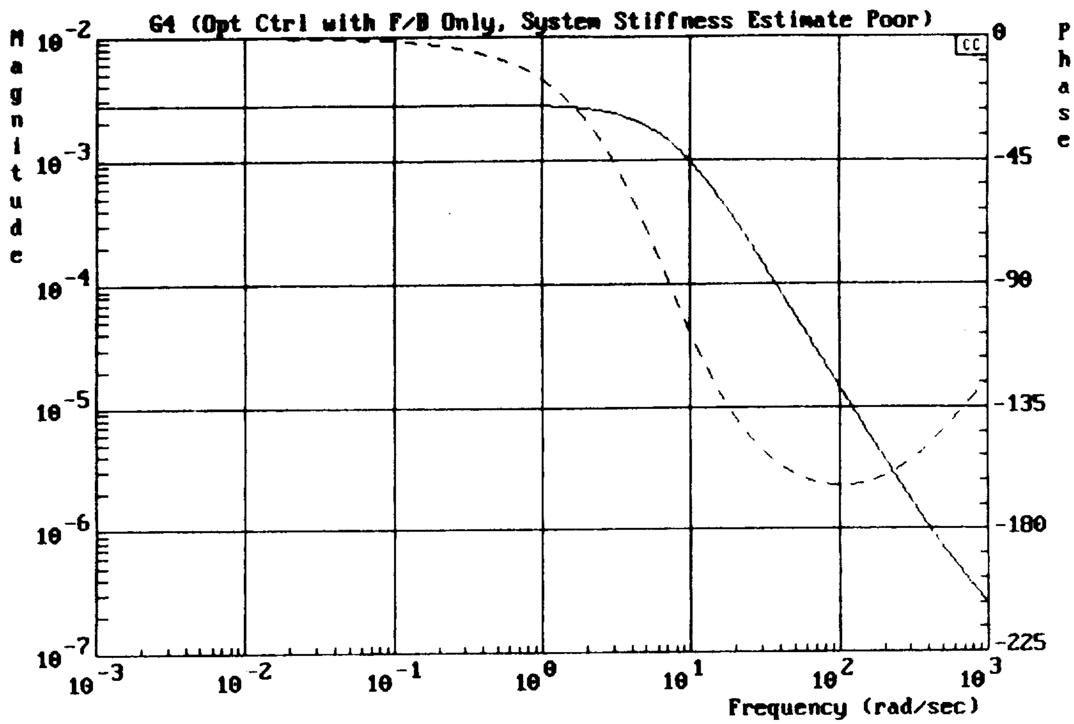

G2 (Opt Ctrl with F/B Only, System Known Exactly)
Magnitude
Phase
CC
Frequency (rad/sec)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
45
0
-45
-90
-135
-180
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

Figure 4



G3 (Opt Ctrl with F/B & F/F, System Stiffness Estimate Poor)
CC
Magnitude
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
Phase
0
-45
-90
-135
-180
-225
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Figure 5



G4 (Opt Ctrl with F/B Only, System Stiffness Estimate Poor)
Magnitude
Phase
CC
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
-45
-90
-135
-180
-225
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
Frequency (rad/sec)

Figure 6

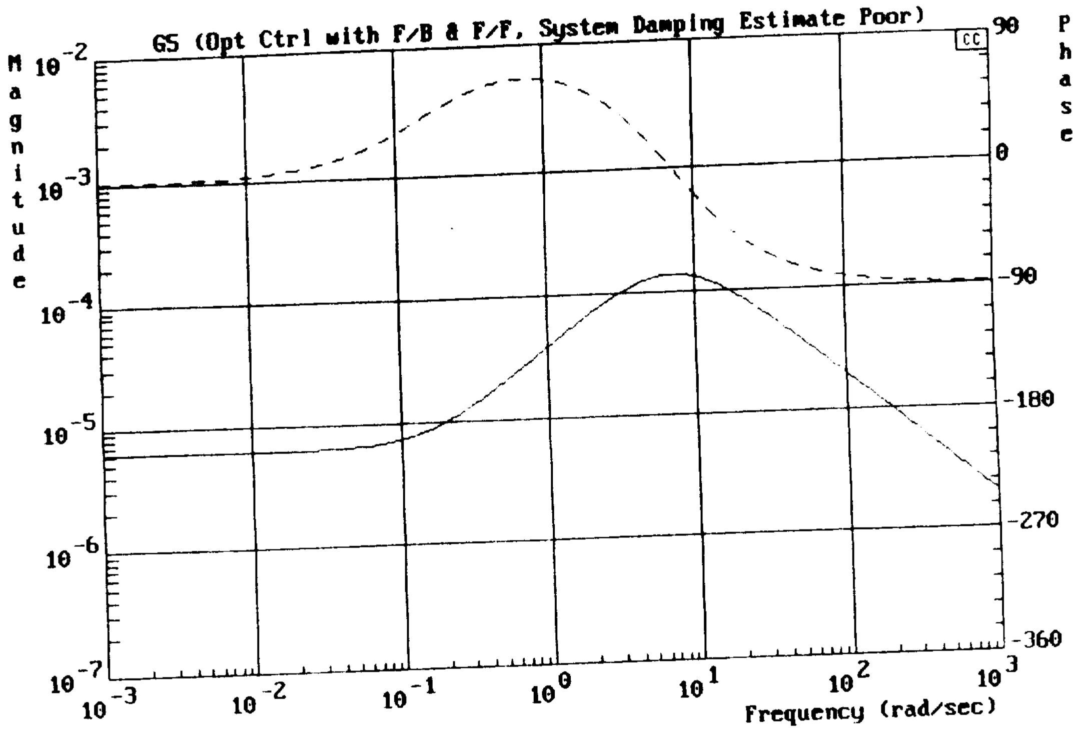

GS (Opt Ctrl with F/B & F/F, System Damping Estimate Poor)
Magnitude
Phase
Frequency (rad/sec)

Figure 7

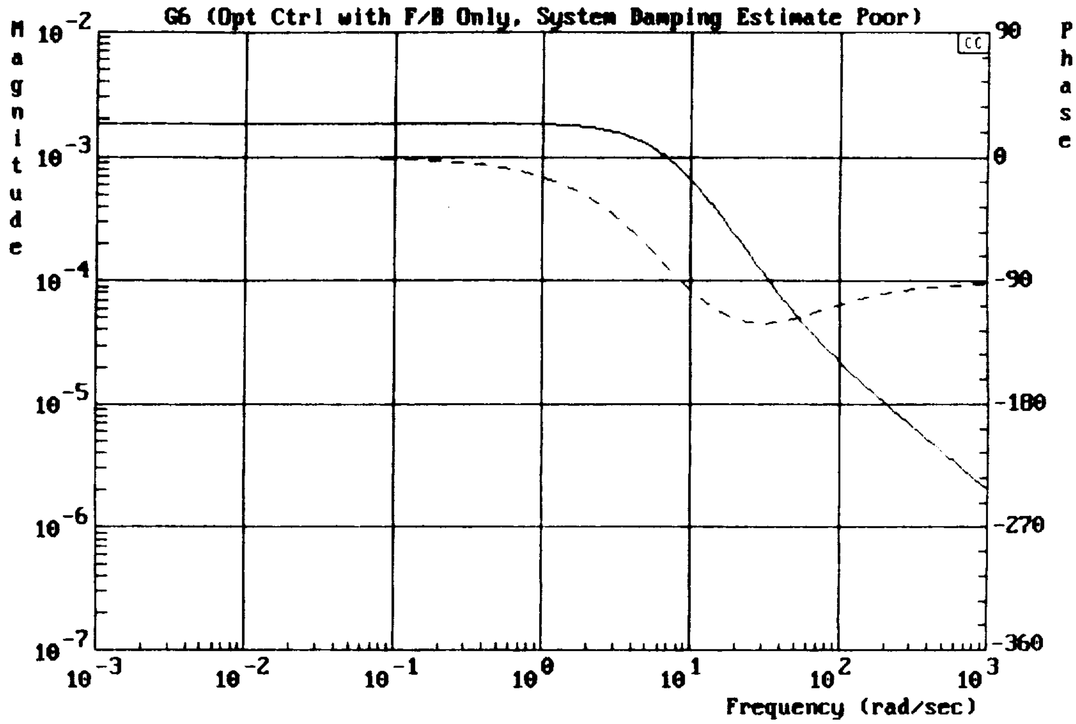

G6 (Opt Ctrl with F/B Only, System Damping Estimate Poor)
Magnitude
Phase
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
90
0
-90
-180
-270
-360
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Figure 8

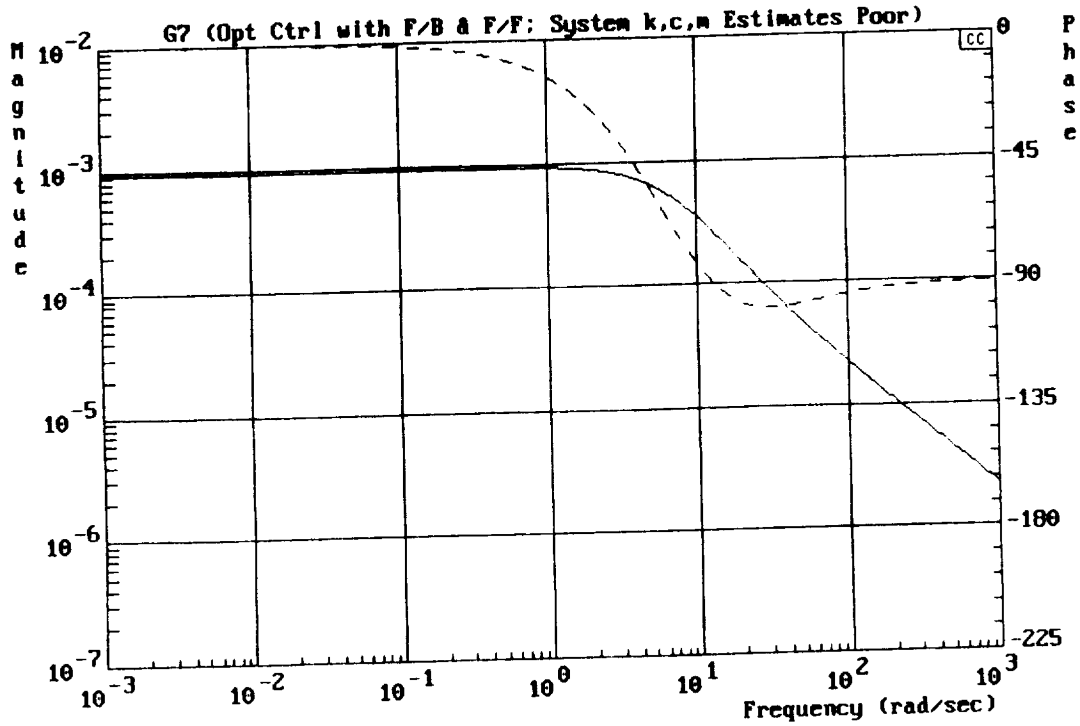

G7 (Opt Ctrl with F/B & F/F; System k,c,m Estimates Poor)
Magnitude
Phase
Frequency (rad/sec)
CC
0
-45
-90
-135
-180
-225

Figure 9

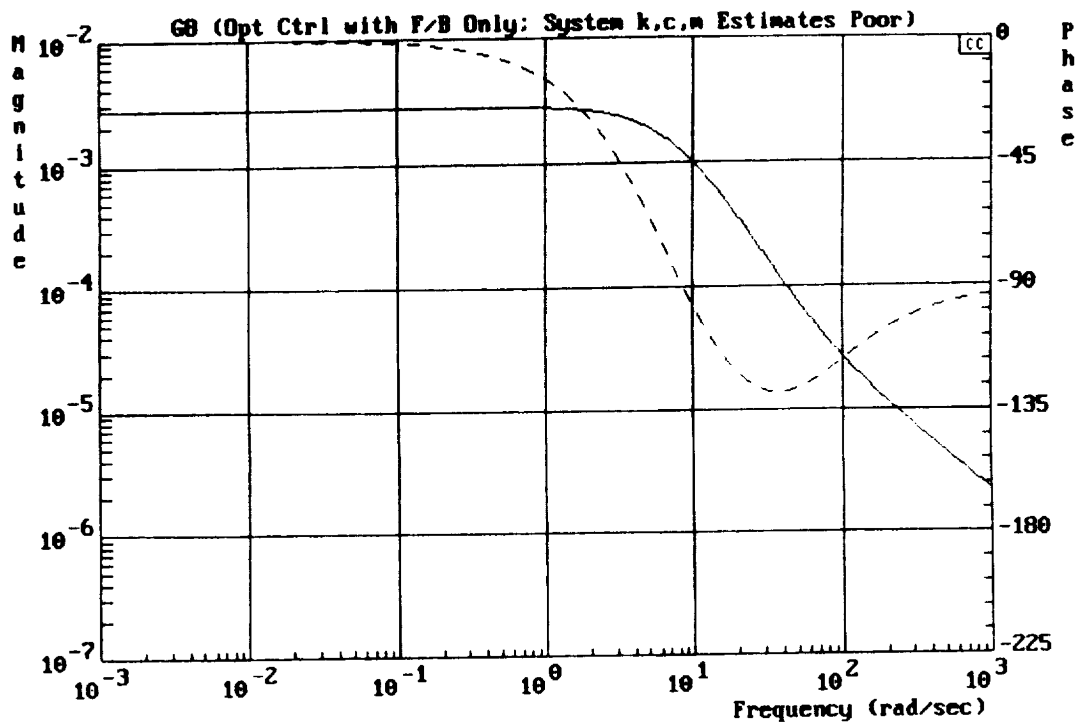
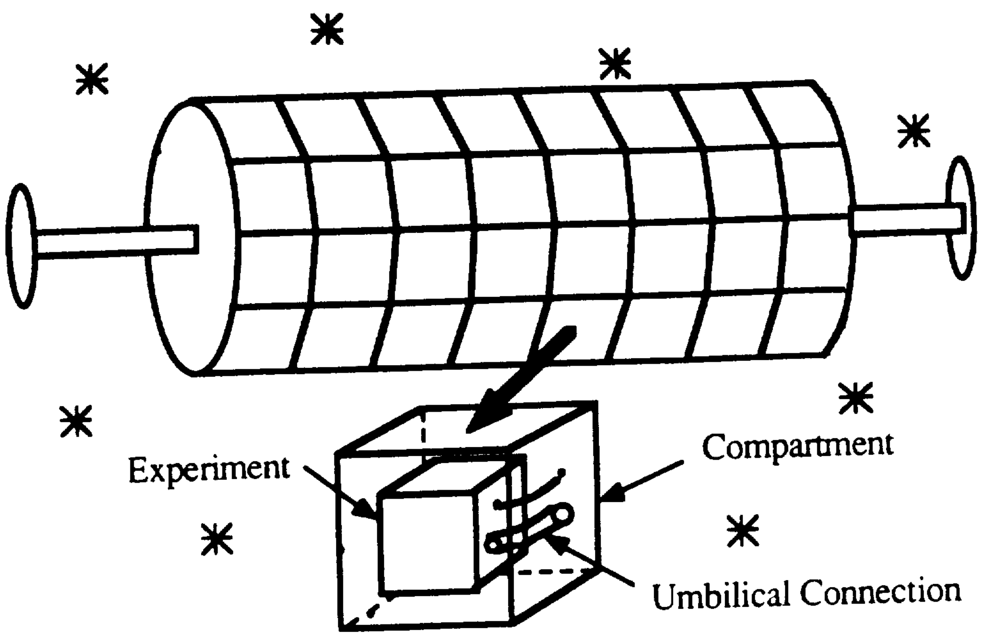

G8 (Opt Ctrl with F/B Only: System k,c,m Estimates Poor)
Magnitude
Phase
CC
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0
-45
-90
-135
-180
-225
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Figure 10



Scientific Experiment in Spacecraft

DISTURBANCE LEVELS

Quasi-Steady or "DC" Accelerations

Relative Gravity	Frequency (Hz)	Source
1E-7	0 to 1E-3	Aerodynamic Drag
1E-8	0 to 1E-3	Light Pressure
1E-7	0 to 1E-3	Gravity Gradient

Periodic Accelerations

Relative Gravity	Frequency (Hz)	Source
2E-2	9	Thruster Fire (orbital)
2E-3	5 to 20	Crew Motion
2E-4	17	Ku Band Antenna

Non-Periodic Accelerations

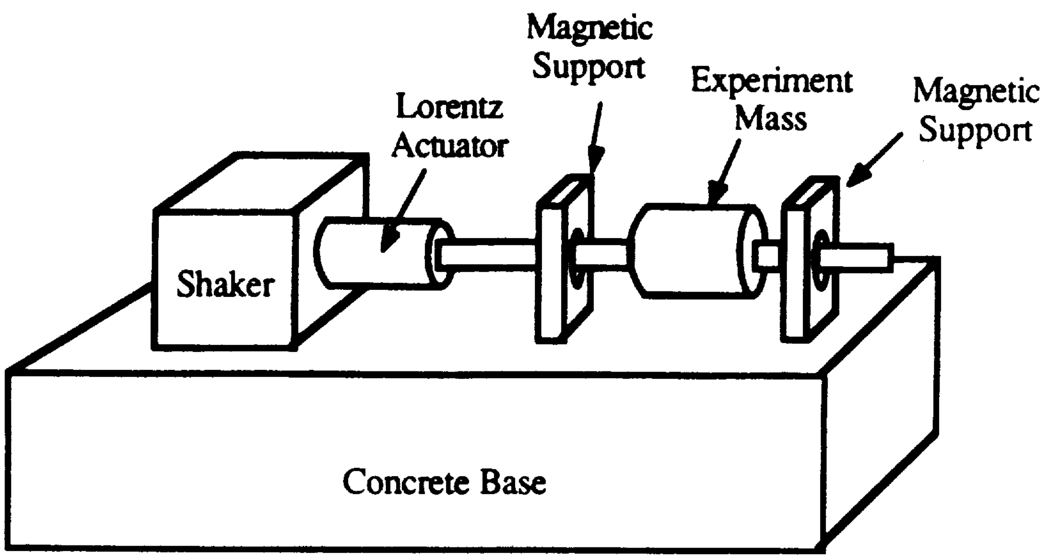
Relative Gravity	Frequency (Hz)	Source
1E-4	1	Thruster Fire (Attitudinal)
1E-4	1	Crew Push-Off

General Problem:

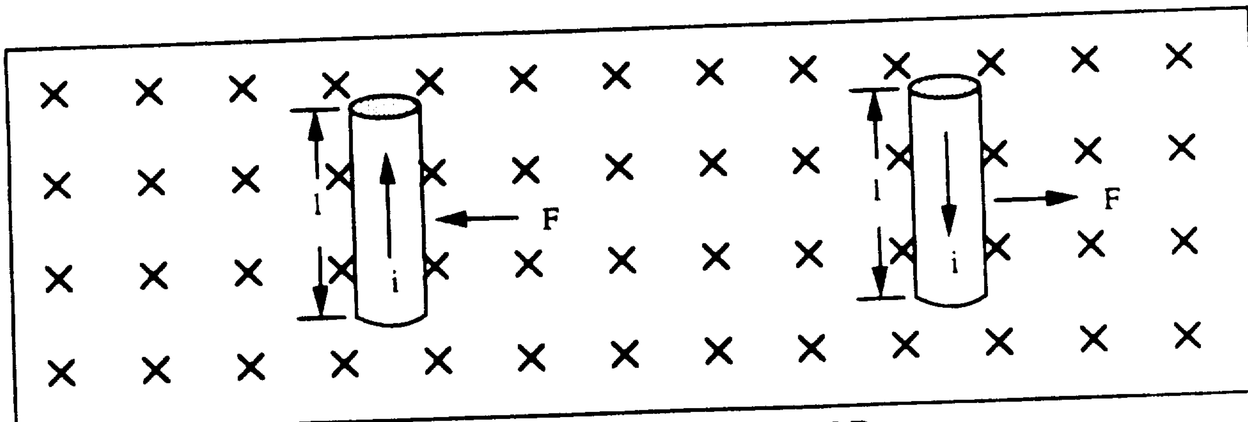
In the upcoming space station, planned by NASA for completion in the 1990's, minimize the low frequency accelerations transmitted from the space station to an experimental platform contained on (inside) the space station.

"minimize": reduce to $\sim 10^{-6}\ g/g_0$ if possible

"low frequency": 0.001 to 20 Hz



Control Law Validation Apparatus

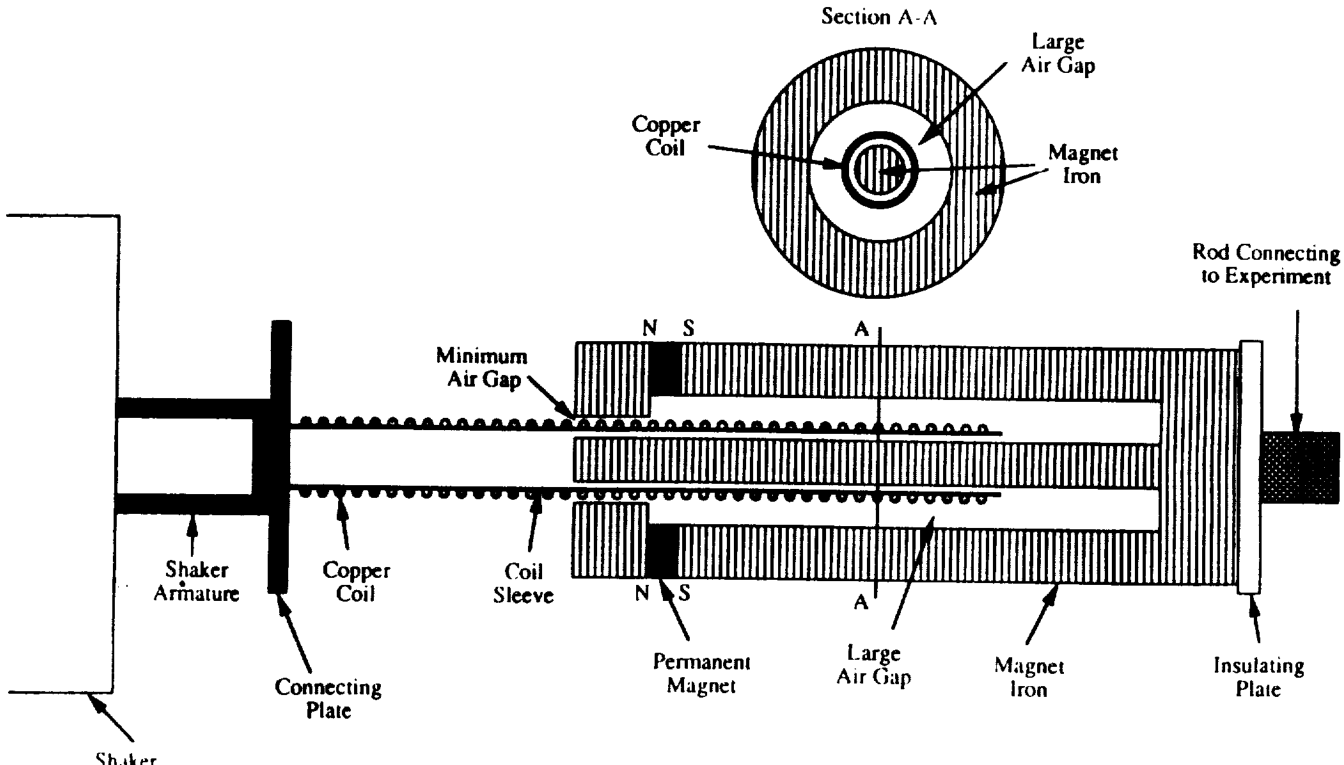


The Lorentz Equation: F = il X B

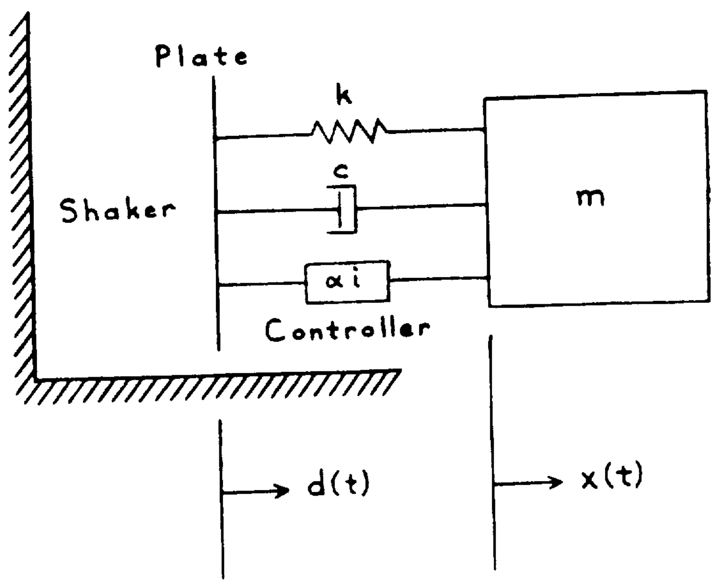
F = Force   l = length of wire   i = current

X Represents the tail feathers of a magnetic field **B** vector into the page

Lorentz Actuator Design



Cross-sectional View of Actuator



System Model

Objective:

Find the "best" $i(t)$,

to minimize $\ddot{x}(t)$.

Equation of Motion:

$$\ddot{x}(t) = \frac{-k}{m}\left[x(t) - d(t)\right] - \frac{c}{m}\left[\dot{x}(t) - \dot{d}(t)\right] - \frac{\alpha}{m}\, i(t)$$

State Equations:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{-\alpha}{m} \end{Bmatrix} i + \begin{Bmatrix} 0 \\ \frac{k}{m} d + \frac{c}{m} \dot{d} \end{Bmatrix}$$

OR

$$\underline{\dot{x}} = A \underline{x} + \underline{b} u + \underline{f}$$

where $\underline{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{bmatrix}$$

$$\underline{b} = \begin{Bmatrix} 0 \\ \frac{-\alpha}{m} \end{Bmatrix}$$

$$\underline{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{k}{m} d + \frac{c}{m} \dot{d} \end{Bmatrix}$$

Problem Statement:

Determine the control $\underline{u}(t)$ which minimizes the performance index

$$J = \frac{1}{2}\int_0^\infty \left(\underline{x}^T W_1 \underline{x} + \underline{u}^T W_3 \underline{u}\right) dt$$

for the system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} + \underline{f}$$

subject to the conditions

$$\underline{x}(0) = \underline{x}_0$$

$$\lim_{t \to \infty} \underline{x}(t) = \underline{0}$$

$$\lim_{t \to \infty} \underline{f}(t) = \underline{0}$$

Solution Method (Differential Equations Approach):

1. Augment the performance index J with the state equations using Lagrange multipliers.
2. Take the 1^{st} variation $\delta \bar{J}$ of the augmented performance index $\bar{J}$ and set it equal to zero:

$$\delta \bar{J} = \int_0^\infty \left[\frac{\partial H}{\partial \underline{x}} \delta \underline{x} + \frac{\partial H}{\partial \underline{u}} \delta \underline{u} + \frac{\partial H}{\partial \dot{\underline{x}}} \delta \dot{\underline{x}} \right] dt = 0$$

where $H = \frac{1}{2}\left(\underline{x}^T W_1 \underline{x} + \underline{u}^T W_3 \underline{u}\right) + \underline{\lambda}^T\left(\dot{\underline{x}} - A\underline{x} - B\underline{u} - \underline{f}\right)$

3. Integrate the third term of the integrand by parts, combine terms, and set coefficients of the arbitrary variations $\delta \underline{x}$ and $\delta \underline{u}$ equal to zero.

Result:
$$\dot{\underline{\lambda}} = W_1 \underline{x} - A^T \underline{\lambda}$$
$$\underline{u} = W_3^{-1} B^T \underline{\lambda}$$

4. Substitute for $\underline{u}$ in the state equations, to yield

$$\begin{Bmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{Bmatrix} = \begin{bmatrix} A & B W_3^{-1} B^T \\ W_1 & -A^T \end{bmatrix} \begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} + \begin{Bmatrix} \underline{f} \\ \underline{0} \end{Bmatrix}$$

The solution (i.e., the optimal control $\underline{u} = \underline{u}^*$) is now $\underline{u}^* = W_3^{-1} B^T \underline{\lambda}$ where $\underline{\lambda}$ is found by solving the system

$$\begin{Bmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{Bmatrix} = \begin{bmatrix} A & BW_3^{-1}B^T \\ W_1 & -A^T \end{bmatrix} \begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} + \begin{Bmatrix} \underline{f} \\ \underline{0} \end{Bmatrix}$$

subject to

$$\underline{x}(0) = \underline{x}_0$$

$$\lim_{t \to \infty} \underline{x}(t) = \underline{0}$$

$$\lim_{t \to \infty} \underline{f}(t) = \underline{0}$$

5. Find the solution of the homogeneous system, i.e., of $\begin{Bmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{Bmatrix} = \begin{bmatrix} A & BW_3^{-1}B^T \\ W_1 & -A^T \end{bmatrix} \begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix}$

6. Use the variation of parameters method to find the general solution of the nonhomogeneous system, i.e., of

$$\begin{Bmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{Bmatrix} = \begin{bmatrix} A & BW_3^{-1}B^T \\ W_1 & -A^T \end{bmatrix} \begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} + \begin{Bmatrix} \underline{f} \\ \underline{0} \end{Bmatrix}$$

7. Apply the terminal conditions on $\underline{x}(t)$, to conclude that n of the 2n arbitrary constants in the general solution are equal to zero.

8. Solve for $\underline{\lambda}(t)$ in terms of $\underline{x}(t)$ in a manner such that the remaining n arbitrary constants are eliminated.

9. Use the equation

$$\underline{u}^* = W_3^{-1} B^T \underline{\lambda}$$

to find $\underline{u}^*$ in terms of $\underline{x}$.

Result:

$$\underline{u}^*(t) = \left(W_3^{-1} B^T X_{21} X_{11}^{-1}\right) \underline{x} + \left(W_3^{-1} B^T X_{22}^{(-1)-1}\right) e^{-\Lambda t} \int e^{\Lambda t} X_{21}^{(-1)} \underline{f}(t)\, dt$$

where $\begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$ is the Jordan Canonical Form of the Hamiltonian matrix $\begin{bmatrix} A & B W_3^{-1} B^T \\ W_1 & -A^T \end{bmatrix}$,

where Λ contains only the negative eigenvalues of the Hamiltonian matrix, corresponding to the eigenvalues of the closed-loop system (assuming $\{A, B\}$ controllable),

where $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ is the eigenvector matrix which leads to the above J. C. F., where $X^{-1} = \begin{bmatrix} X_{11}^{(-1)} & X_{12}^{(-1)} \\ X_{21}^{(-1)} & X_{22}^{(-1)} \end{bmatrix}$,

and where integration of the indefinite integral requires constants of integration that are all identically zero.

10. Integrating repeatedly by parts and using the facts that

$$X_{22}^{(-1)T} \Lambda^T X_{22}^{(-1)-T} = X_{11} \Lambda X_{11}^{-1} = \bar{A} = A - B W_3^{-1} B^T P$$

where $P = -X_{21} X_{11}^{-1}$ [P is the solution to the well-known Algebraic Riccati Equation],

develop equivalent forms for $\underline{u}^*(t)$:

$$\underline{u}^*(t) = -W_3^{-1} B^T P \underline{x} + W_3^{-1} B^T X_{22}^{(-1)-1} e^{-\Lambda t} \int e^{\Lambda t} X_{21}^{(-1)} \underline{f}(t)\, dt$$

$$= -W_3^{-1} B^T P \underline{x} - W_3^{-1} B^T X_{22}^{(-1)-1} \sum_{r=0}^{\infty} (-\Lambda^{-1})^{r+1} X_{21}^{(-1)} \underline{f}^{(r)}$$

$$= -W_3^{-1} B^T P \underline{x} - W_3^{-1} B^T \sum_{r=0}^{\infty} \left(-\bar{A}^{-T}\right)^{r+1} P \underline{f}^{(r)}$$

$$= -W_3^{-1} B^T P \underline{x} - W_3^{-1} B^T X_{11}^{-T} \sum_{r=0}^{\infty} \left(-\Lambda^{-T}\right)^{r+1} X_{21}^T \underline{f}^{(r)}$$

Note: A state transition matrix approach yields

$$\underline{u}^*(t) = -W_3^{-1} B^T P \underline{x} - W_3^{-1} B^T X_{22}^{(-1)-1} e^{-\Lambda t} \int_t^{\infty} e^{\Lambda \tau} X_{21}^{(-1)} \underline{f}(\tau)\, d\tau$$

Solution:

$$\underline{u}^*(t) = -W_3^{-1}B^TP\,\underline{x} + W_3^{-1}B^TX_{22}^{(-1)-1}e^{-\Lambda t}\int e^{\Lambda t}X_{21}^{(-1)}\underline{f}(t)\,dt$$

$$= -W_3^{-1}B^TP\,\underline{x} - W_3^{-1}B^TX_{22}^{(-1)-1}e^{-\Lambda t}\int_t^{\infty} e^{\Lambda\tau}X_{21}^{(-1)}\underline{f}(\tau)\,d\tau$$

$$= -W_3^{-1}B^TP\,\underline{x} - W_3^{-1}B^TX_{22}^{(-1)-1}\sum_{r=0}^{\infty}(-\Lambda^{-1})^{r+1}X_{21}^{(-1)}\underline{f}^{(r)}$$

$$= -W_3^{-1}B^TP\,\underline{x} - W_3^{-1}B^T\sum_{r=0}^{\infty}(-\bar{A}^{-T})^{r+1}P\,\underline{f}^{(r)}$$

$$= -W_3^{-1}B^TP\,\underline{x} - W_3^{-1}B^TX_{11}^{-T}\sum_{r=0}^{\infty}(-\Lambda^{-1})^{r+1}X_{21}^{T}\underline{f}^{(r)}$$

Dropping higher order terms (for $r > 0$):

$$^{(0)}\tilde{\underline{u}}^*(t) = -W_3^{-1}B^TP\,\underline{x} + W_3^{-1}B^TX_{22}^{(-1)-1}\Lambda^{-1}X_{21}^{(-1)}\underline{f}$$

$$= -W_3^{-1}B^TP\,\underline{x} + W_3^{-1}B^TX_{22}^{(-1)-1}\Lambda^{-1}X_{22}^{(-1)}P\,\underline{f}$$

$$= -W_3^{-1}B^TP\,\underline{x} + W_3^{-1}B^TX_{11}^{-T}\Lambda^{-T}X_{11}^{T}P\,\underline{f}$$

$$= -W_3^{-1}B^TP\,\underline{x} + W_3^{-1}B^TX_{11}^{-T}\Lambda^{-T}X_{21}^{T}\underline{f}$$

$$\boxed{^{(0)}\tilde{\underline{u}}^*(t) = -W_3^{-1}B^TP\,\underline{x} + W_3^{-1}B^T\bar{A}^{-T}P\,\underline{f}}$$

where $\bar{A}^{-T} = -P(PA + W_1)^{-1} = (A - BW_3^{-1}B^TP)^{-T}$
$= (X_{11}\Lambda X_{11}^{-1})^{-T} = X_{22}^{(-1)-1}\Lambda^{-1}X_{22}^{(-1)}$

These are several forms for the control law.

Solution:

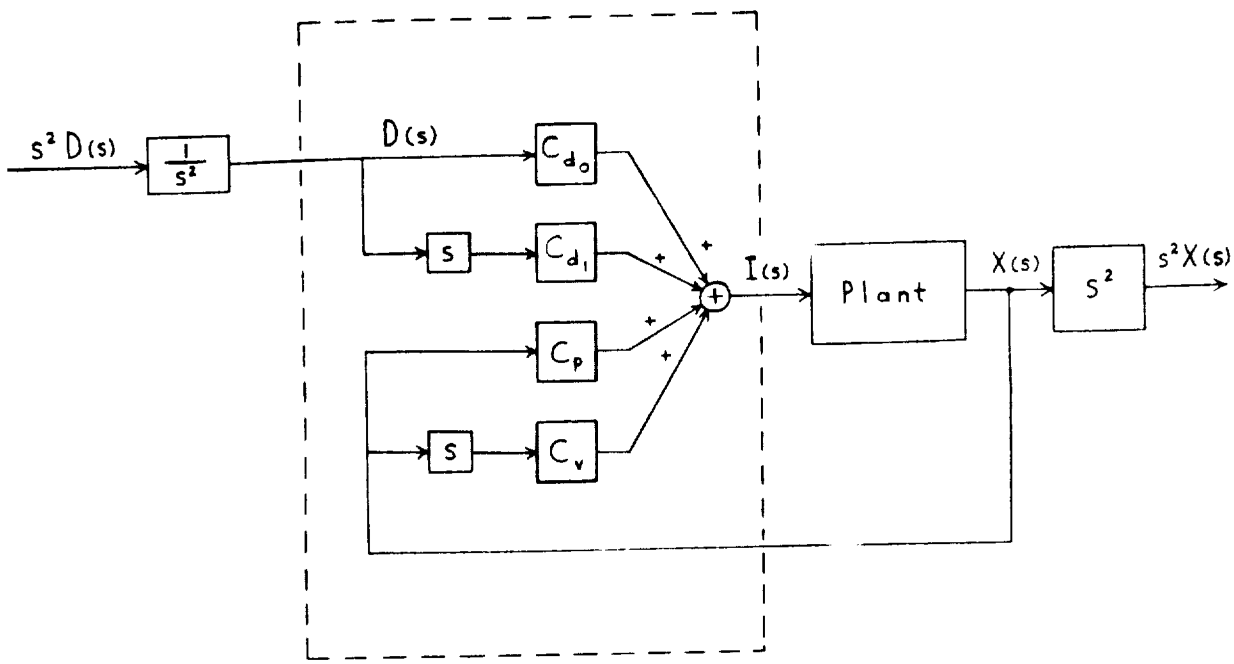
$$^{(o)}\,\underset{\sim}{\tilde{u}}^{*}(t) = \left(-W_3^{-1}B^{T}P\right)\underline{x}(t) + \left(W_3^{-1}B^{T}\bar{A}^{-T}P\right)\underline{f}(t)$$

where $\bar{A} = A - BW_3^{-1}B^{T}P$

Application to our specific problem:

$$i(t) = C_p\, x(t) + C_v\, \dot{x}(t) + C_{d_0}\, d(t) + C_{d_1}\, \dot{d}(t)$$

where C_p, C_v, C_{d_0}, and C_{d_1} are constant gains.



Block Diagram

Table 1 Optimal F/F and F/B Gains for Selected State Variable and Control Weightings.

System Parameters:

$m = 100$ lbm $\quad k = 0.3$ lbf/ft

$c = 0.000622$ lbf-sec/ft ($\zeta = 0.1\%$)

$\alpha = 10$ lbf/amp

Weights			F/B Gains		F/F Gains					
w_{1a}	w_{1b}	w_3	C_p	C_v	C_{d0}	C_{d1}	C_{d2}	C_{d3}	C_{d4}	C_{d5}
2	1	1	1.3845	1.3637	0.0294	-0.0006	-0.0070	-0.0067	-0.0049	-0.0032
10	2	1	3.1324	1.9863	0.0297	-0.0001	-0.0030	-0.0019	-0.0009	-0.0004
23	3	1	4.7659	2.4413	0.0298	-0.0000	-0.0020	-0.0010	-0.0004	-0.0001
41	4	1	6.3732	2.8210	0.0299	0.0000	-0.0015	-0.0007	-0.0002	-0.0001
64	5	1	7.9701	3.1544	0.0299	0.0000	-0.0012	-0.0005	-0.0001	-0.0000
92	6	1	9.5617	3.4552	0.0299	0.0000	-0.0010	-0.0004	-0.0001	-0.0000
126	7	1	11.1950	3.7354	0.0299	0.0000	-0.0008	-0.0003	-0.0001	-0.0000
165	8	1	12.8153	3.9949	0.0299	0.0000	-0.0007	-0.0002	-0.0001	-0.0000
209	9	1	14.42[illegible]	4.2380	0.0299	0.0000	-0.0006	-0.0002	-0.0000	-0.0000
258	10	1	16.0324	4.4674	0.0299	0.0000	-0.0006	-0.0002	-0.0000	-0.0000
581	15	1	24.0740	5.4729	0.0300	0.0001	-0.0004	-0.0001	-0.0000	-0.0000
1034	20	1	32.1259	6.3209	0.0300	0.0001	-0.0003	-0.0001	-0.0000	-0.0000
1617	25	1	40.1819	7.0680	0.0300	0.0001	-0.0002	-0.0000	-0.0000	-0.0000
2329	30	1	48.2297	7.7431	0.0300	0.0001	-0.0002	-0.0000	-0.0000	-0.0000
3171	35	1	56.2816	8.3640	0.0300	0.0001	-0.0002	-0.0000	-0.0000	-0.0000
4113	40	1	64.3361	8.9420	0.0300	0.0001	-0.0001	-0.0000	-0.0000	-0.0000
9325	60	1	96.5360	10.9526	0.0300	0.0001	-0.0001	-0.0000	-0.0000	-0.0000
16581	80	1	128.7372	12.6475	0.0300	0.0001	-0.0001	-0.0000	-0.0000	-0.0000
25911	100	1	160.9389	14.1407	0.0300	0.0001	-0.0001	-0.0000	-0.0000	-0.0000

C 2

Table 2. Closed loop transfer functions for system with design parameter values of k = 0.3, c = 0.000622, and m = 100; but with actual parameter values as shown. G1, G3, G5, and G7 include both LQR F/B and proportional F/F; G2, G4, G6, and G8 include LQR F/B alone. Weighting parameters used were w_{1a} = 258, w_{1b} = 10, w_3 = 1 (see Table 1).

System Parameters			Closed Loop Transfer Function $\frac{s^2X(s)}{s^2D(s)}$
k $(\frac{lbf}{ft})$	c $(\frac{lbf-sec}{ft})$	m (lbm)	
0.3	0.000622 (ζ=0.1%)	100	$G1(s) = \frac{0.0000622s + 0.0001}{0.31056s^2+4.4675s+16.0624}$
0.3	0.000622	100	$G2(s) = \frac{0.0000622s + 0.0300}{0.31056s^2+4.4675s+16.0624}$
0.45	0.000622	100	$G3(s) = \frac{0.0000622s + 0.0151}{0.31056s^2+4.4675s+16.0774}$
0.45	0.000622	100	$G4(s) = \frac{0.0000622s + 0.0450}{0.31056s^2+4.4675s+16.0774}$
0.3	0.00622	100	$G5(s) = \frac{0.000622s + 0.0001}{0.31056s^2+4.4680s+16.0624}$
0.3	0.00622	100	$G6(s) = \frac{0.000622s + 0.0300}{0.31056s^2+4.4680s+16.0624}$
0.45	0.00622	90	$G7(s) = \frac{0.000622s + 0.0151}{0.27950s^2+4.4680s+16.0774}$
0.45	0.00622	90	$G8(s) = \frac{0.000622s + 0.0450}{0.27950s^2+4.4680s+16.0774}$

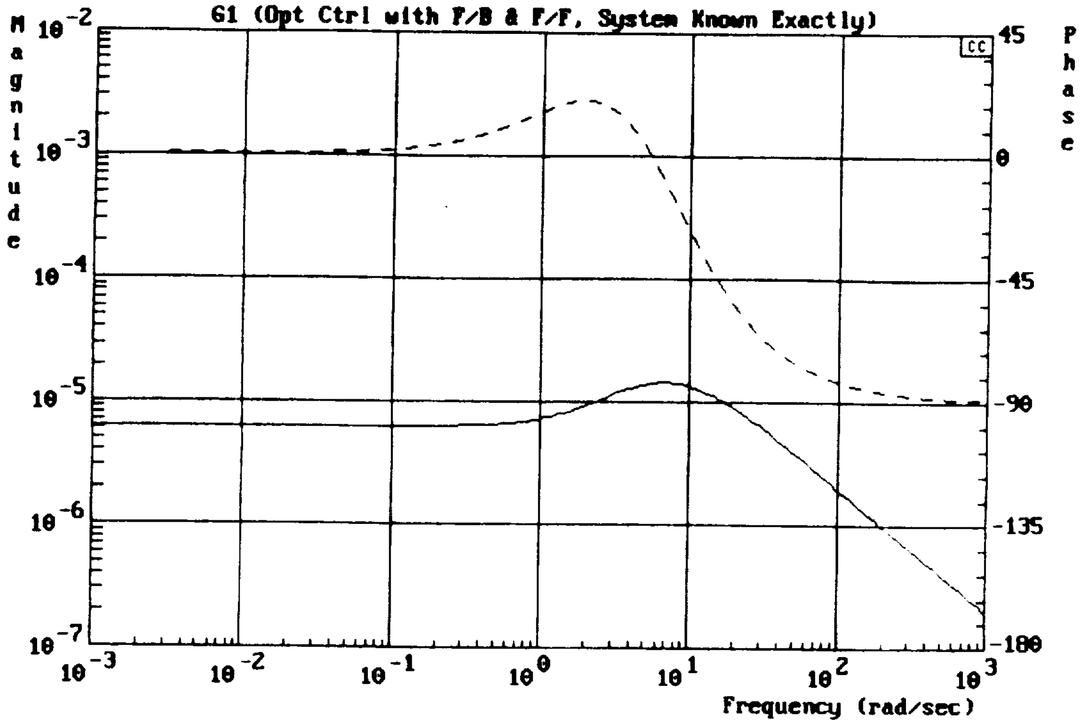

G1 (Opt Ctrl with F/B & F/F, System Known Exactly)
Magnitude
Phase
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
45
0
-45
-90
-135
-180
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
Frequency (rad/sec)
CC

Figure 3

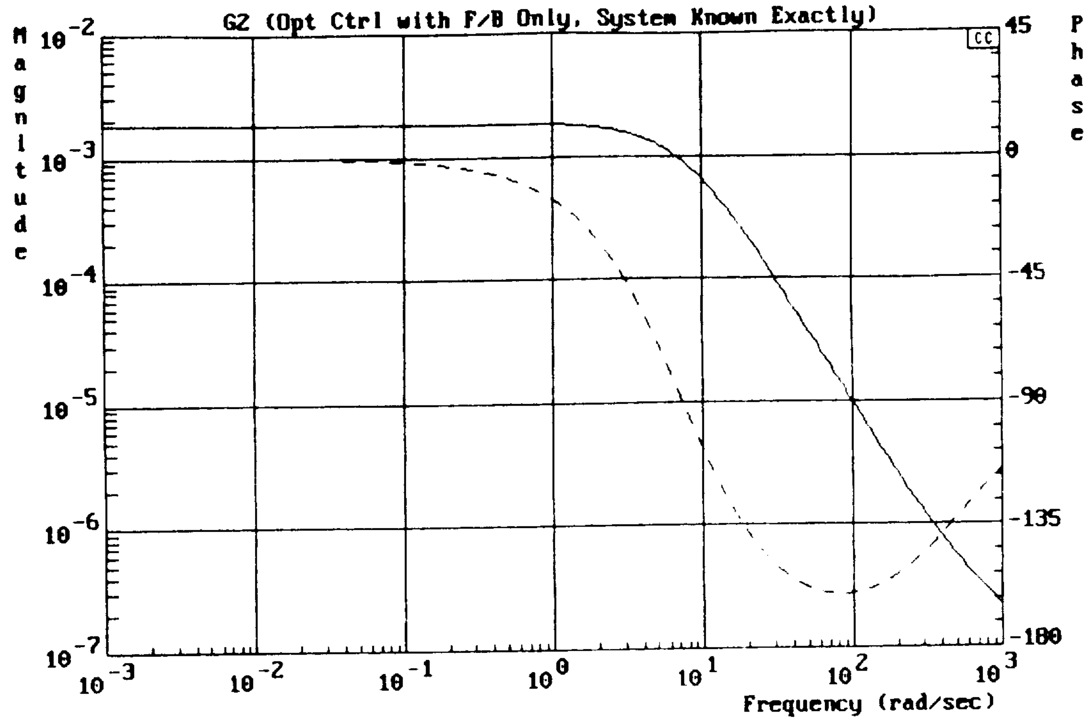

G2 (Opt Ctrl with F/B Only, System Known Exactly)
Magnitude
Phase
CC
Frequency (rad/sec)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
45
0
-45
-90
-135
-180

Figure 4

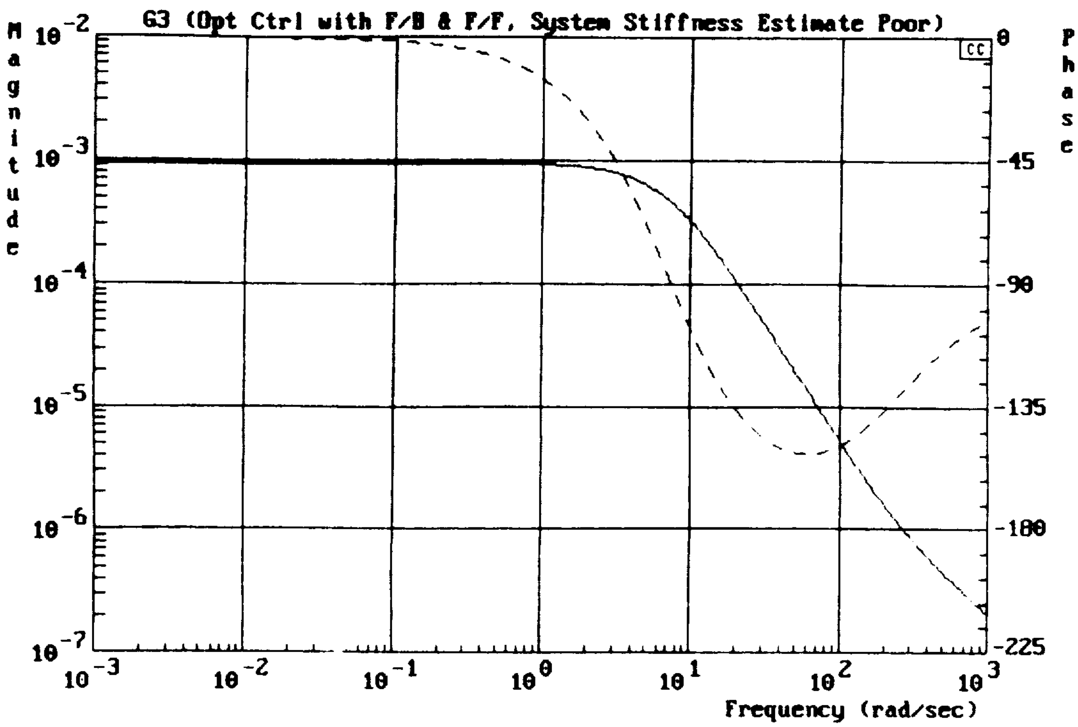

G3 (Opt Ctrl with F/B & F/F, System Stiffness Estimate Poor)
Magnitude
Phase
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
-45
-90
-135
-180
-225
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
Frequency (rad/sec)
CC

Figure 5

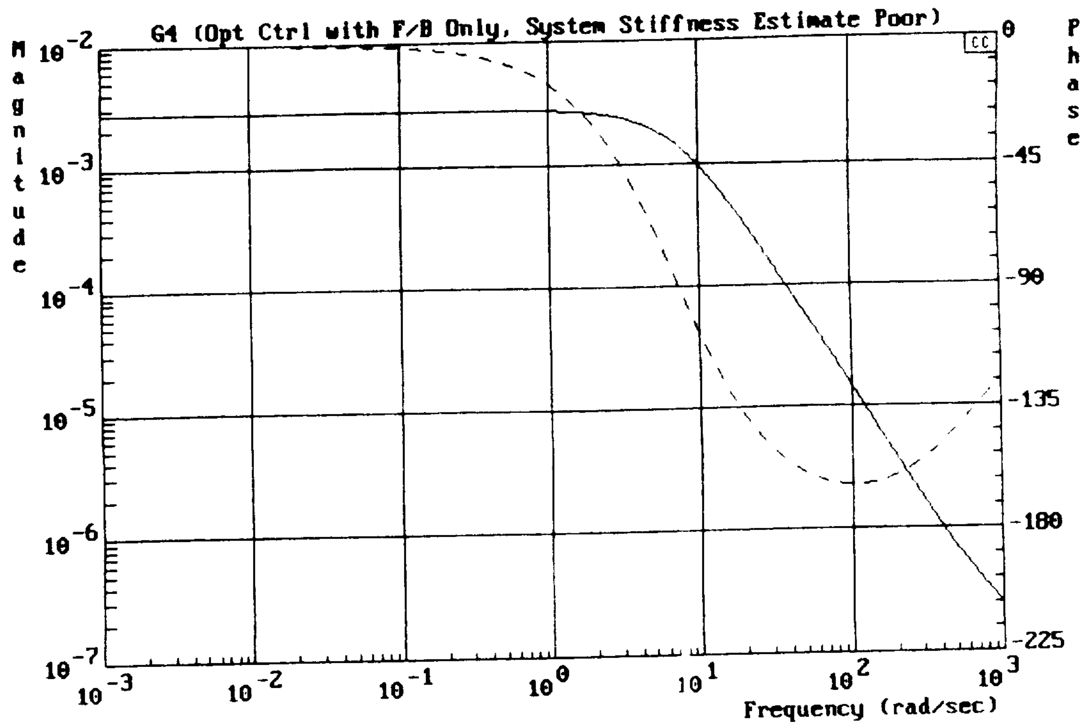

G4 (Opt Ctrl with F/B Only, System Stiffness Estimate Poor)
Magnitude
Phase
CC
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
-45
-90
-135
-180
-225
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
Frequency (rad/sec)

Figure 6

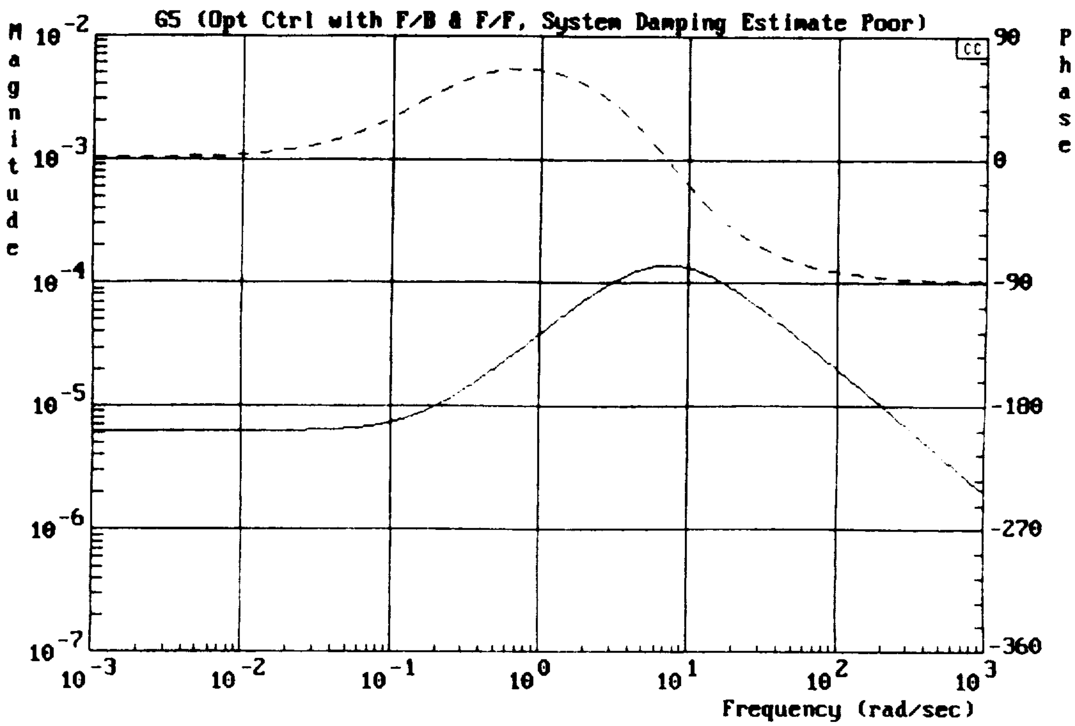

G5 (Opt Ctrl with F/B & F/F, System Damping Estimate Poor)
Magnitude
Phase
CC
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
90
0
-90
-180
-270
-360
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Figure 7

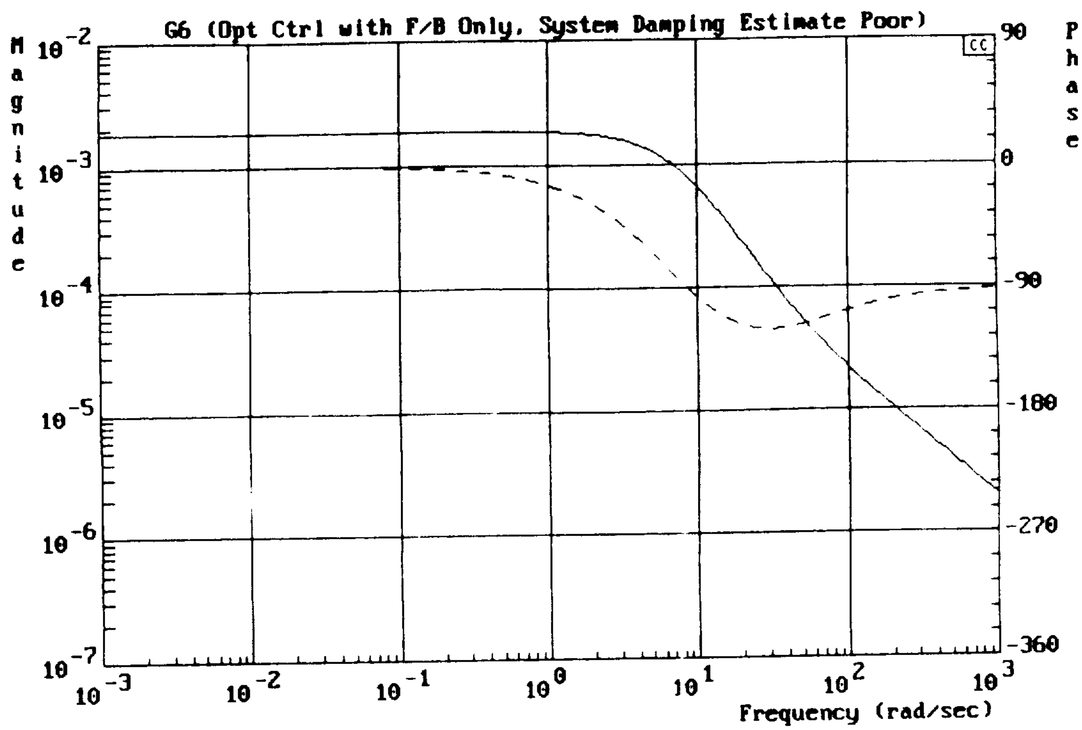

G6 (Opt Ctrl with F/B Only, System Damping Estimate Poor)
Magnitude
Phase
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
90
0
-90
-180
-270
-360
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Figure 8

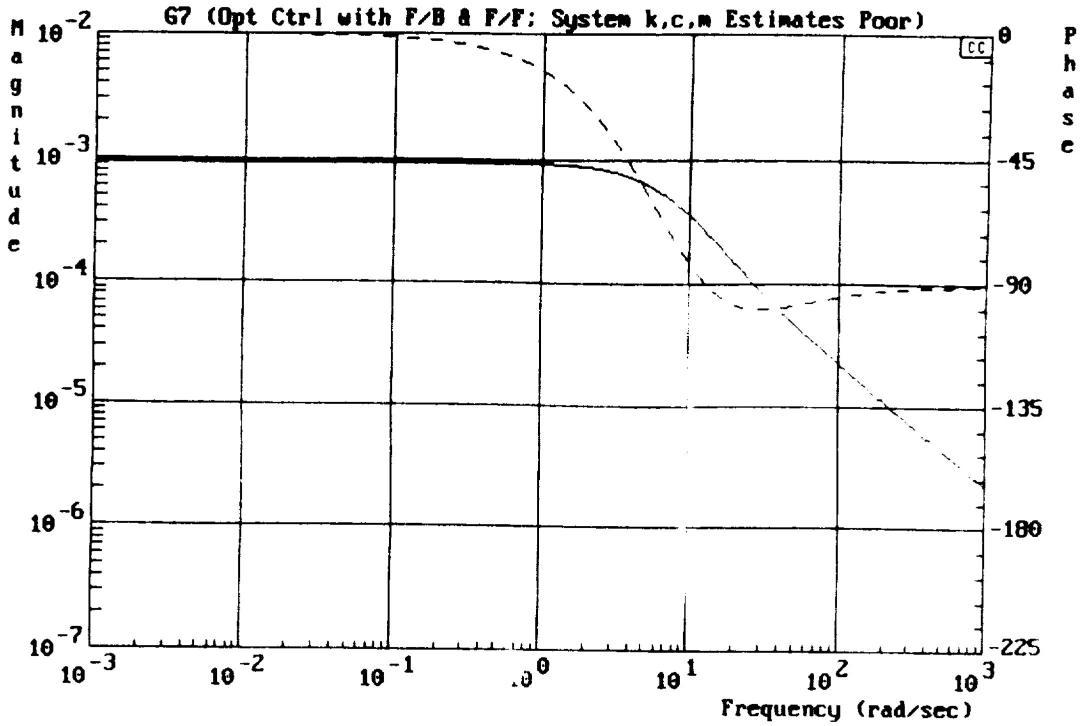

G7 (Opt Ctrl with F/B & F/F; System k,c,m Estimates Poor)
Magnitude
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
Phase
0
-45
-90
-135
-180
-225
10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3}
Frequency (rad/sec)
CC

Figure 9

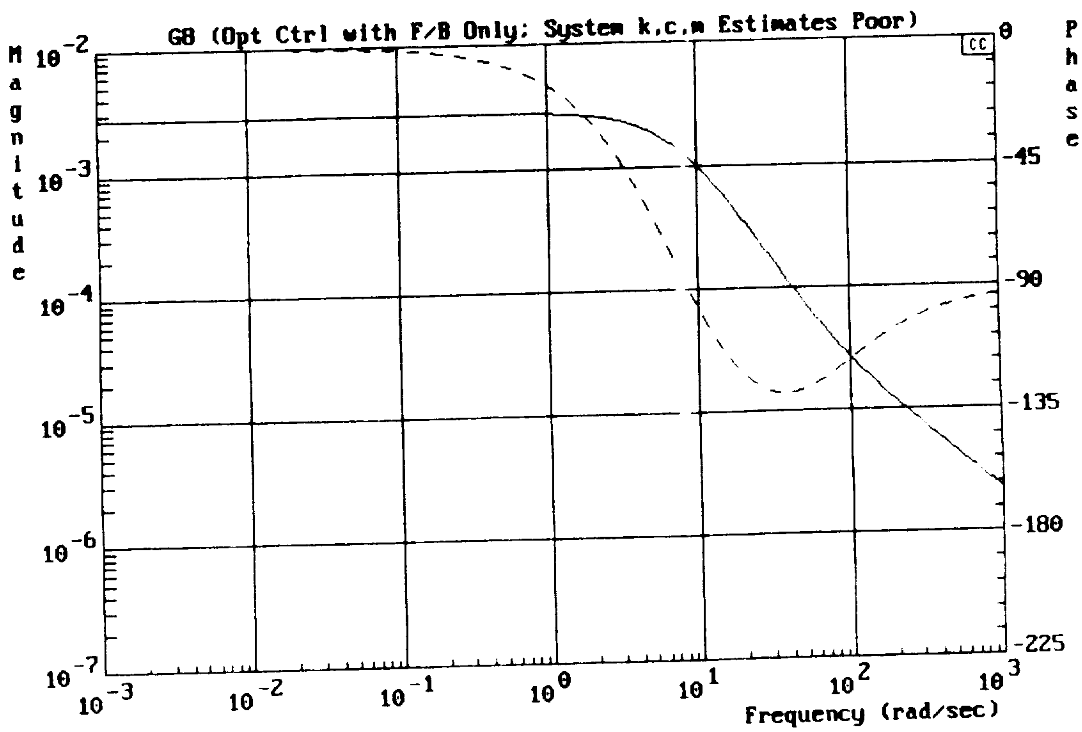

G8 (Opt Ctrl with F/B Only; System k,c,m Estimates Poor)
Magnitude
Phase
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
-45
-90
-135
-180
-225
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
Frequency (rad/sec)
CC

Figure 10

Conclusions:

1. An optimal control has been determined for the nonhomogeneous LQR problem.
2. An approximation to this optimal control has been found which uses constant feedback and feedforward gains.
3. The optimal control has the following advantages:
   a. The gains can be easily determined.
   b. The control is very robust (60° phase margin, infinite positive gain margin, 6 dB negative gain margin).
   c. The control is applicable to a wide range of problems.
   d. The control offers substantial improvement in disturbance rejection over that afforded by LQR feedback alone.
   e. The control can be easily implemented.