NASA Contractor Report 187531

ICASE Report No. 91–22

ICASE

PREDICTING CHAOS FOR INFINITE DIMENSIONAL DYNAMICAL SYSTEMS: THE KURAMOTO-SIVASHINSKY EQUATION, A CASE STUDY

Yiorgos S. Smyrlis

Demetrious T. Papageorgiou

Contract No. NAS1–18605

February 1991

Institute for Computer Applications in Science and Engineering

NASA Langley Research Center

Hampton, Virginia 23665–5225

Operated by the Universities Space Research Association

National Aeronautics and Space Administration

Langley Research Center

Hampton, Virginia 23665-5225

(NASA-CR-187531) PREDICTING CHAOS FOR INFINITE DIMENSIONAL DYNAMICAL SYSTEMS: THE KURAMOTO-SIVASHINSKY EQUATION, A CASE STUDY Final Report (ICASE) 15 p CSCL 20D

N91-21454

Unclas

G3/34 0003354

Predicting chaos for infinite dimensional dynamical systems: The Kuramoto-Sivashinsky equation, a case study.

Yiorgos S. Smyrlis
Department of Mathematics
University of California at Los Angeles
Los Angeles, CA 90024-1555

Demetrios T. Papageorgiou
Department of Mathematics
New Jersey Institute of Technology
Newark, NJ 07102

ABSTRACT

The results of extensive computations are presented in order to accurately characterize transitions to chaos for the Kuramoto-Sivashinsky equation. In particular we follow the oscillatory dynamics in a window that supports a complete sequence of period doubling bifurcations preceding chaos. As many as thirteen period doublings are followed and used to compute the Feigenbaum number for the cascade and so enable, for the first time, an accurate numerical evaluation of the theory of universal behavior of nonlinear systems, for an infinite dimensional dynamical system. Further more, the dynamics at the threshold of chaos exhibit a fractal behavior which is demonstrated and used to compute a universal scaling factor that enables the self-similar continuation of the solution into a chaotic regime.

Research was supported in part by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the authors were in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665. Additional support for the first author was provided by ONR Grant N-00014-86-K-0691 while he was at UCLA.

1. Introduction.

A central question in fluid dynamics that is attracting a considerable research effort, is the prediction of onset to turbulence. A general theory encompassing the Navier-Stokes equations of fluid motion, and consequently covering a large class of physical phenomena, is not available at present. As a result most contributions are focused on the analysis of model equations derived from the Navier-Stokes system by asymptotic methods, for example, or by finite-dimensional truncations [1]. In many cases this is a valid and useful approach, especially in the light of Feigenbaum's fascinating theory originally for one-dimensional nonlinear maps ([2]-[4]), which predicts universal nonlinear behavior and is believed to be applicable to many more complex nonlinear systems such as ordinary and partial differential equations. A brief review of Feigenbaum's theory for the quadratic map is in order here, but the interested reader should refer to the above mentioned articles (also [5]). The theory pertains to a large class of mappings of an interval onto itself, a representative member of which has

$$f(x) = 4\nu x(1-x) \quad , \quad \nu > 0 \quad , \quad x \in [0,1] \quad . \tag{1}$$

The flow map is represented by the repeated application of (1). This choice of (1) ensures that $x_1 = 0$ and $x_2 = 1-1/4\nu$ are always fixed points (i.e. solutions of $x = f(x)$) and that for any $0 < \nu < 1$ all iterates are contained in [0,1]. Now for $0 < \nu < 1/4$ the fixed point x_1 alone is in the range of the map, while for $1/4 < \nu < 1$ both fixed points x_1 and x_2 are in the range. Stability of a fixed point is determined by the condition $|f'(x)| < 1$, which tells us that for $0 < \nu < 1/4$ the fixed point x_1 is stable, whereas for $1/4 < \nu < 3/4$, x_1 is unstable and x_2 is stable (these results are easily obtainable by differentiation of (1)). Further more, at $\nu = 3/4$ we have $f'(x_2) = -1$ (i.e. x_2 is a *non-hyperbolic point.*) and so x_2 becomes unstable also for $\nu > 3/4$. To summarize, therefore, when $0 < \nu < 3/4$ repeated application of (1) starting from some initial position will yield convergence to one of the two fixed points and the flow is both deterministic and attractive to a global fixed point. Things become interesting when ν is increased and lies in the range $3/4 < \nu < 1$. For example, at values of ν just above 3/4 a cycle which contains two alternating fixed points is obtained. In other words, repeated application of (1) gives a sequence $x_{11}, x_{22}, x_{12}, x_{21}, \cdots$. Thus, a *period doubling* takes place and it is easy to find the values of x_{12} and x_{22} since they must be fixed points of the function $f(f(x))$, also written as $f^2(x)$. Further increase of ν makes these points unstable and a further split into a cycle containing 4 points heralds the second period doubling. Again the elements of this 4-cycle are computable by identification of the fixed points of $f^4(x)$. This process persists with period doublings *ad infinitum* towards an *accumulation point* beyond which chaos sets in. More precisely, if we define ν_n to be the value of the parameter where the n^{th} period doubling takes place, then the theory predicts that

$$\delta_n = \frac{\nu_{n+1} - \nu_n}{\nu_{n+2} - \nu_{n+1}} \quad , \quad \lim_{n \to \infty} \delta_n = \delta = 4.6692016... \quad . \tag{2}$$

The Feigenbaum number, δ, is universal for a variety of maps which have locally quadratic maxima. There is another important universal constant which arises from the theory of nonlinear maps. Without loss of generality, if we restrict our attention to the vicinity of the turning point $x = 1/2$ and follow a sequence of period doublings then it is found that the distance between neighboring elements of period doubled orbits (in the high-iteration limit) is reduced by a *universal* constant factor

$$\alpha = 2.502907875... \quad . \tag{3}$$

For a comprehensive review of nonlinear maps see [6]. This property can be used as a predictive tool to construct the behavior of higher-iterates and even chaotic motions beyond the accumulation point of the cascade (see [7], for an application in Rayleigh-Benard flow). In what follows we provide conclusive evidence that the universal behavior described above, also appears in the infinite-dimensional dynamical system provided by the Kuramoto-Sivashinsky equation, and both universal constants are computed with desirable accuracy. In achieving this, we obtain as many as 13 period doublings (the period increases from a value of for the first time as far as we know, conclusive evidence for a period doubling (classical) route to chaos for an infinite dimensional dynamical system.

The equation studied is the Kuramoto-Sivashinsky equation (KS), suitably written in the form

$$\begin{gathered} u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0 \quad , \\ (x,t) \in \mathbf{R}^1 \times \mathbf{R}^+ \quad , \\ u(x,0) = u_0(x) \quad , \quad u(x+2\pi,t) = u(x,t) \quad , \end{gathered} \tag{4}$$

where $\nu > 0$ is the *viscosity* of the system. This equation arises in a variety of problems such as concentration waves [8], flame propagation [9], free surface flows [10]. A generalized form, with (3) as a special case, has been derived by an asymptotic analysis of the Navier-Stokes in the context of two-phase flows in cylindrical geometries with applications in lubricated pipe-lining (for the efficient transport of crude oil) and oil recovery through porous media [11]. Much analytical and computational work has been completed in order to elucidate some of the highly complex and nonlinear behavior that

(4) is capable of producing. For many references see [12] and [13]. The former article concentrated on a fairly global characterization of solutions as ν varies, and in particular when it achieves fairly small values. Note that within the context of linear theory (for the periodic problem posed above), the number of unstable modes is equal to $[\nu^{-1/2}]$; as ν decreases, therefore, more unstable modes enter and our concern is with the long-time behavior of these under the action of nonlinearity.

2. Numerical solutions.

The results presented here, were obtained by numerical solution of the initial value problem (4) with the initial condition

$$u_0(x) = -\sin(x) \quad ,$$

for all values of ν. Inspection of (3) indicates that if the solution is an odd function of x initially, then it will remain so for all subsequent times. The advantage of such a choice is two-fold; first, the solution can be expanded as a Fourier sine-series and a spectral Galerkin approximation is readily applicable - secondly, existing analytical results are available that give global bounds for $u(x,t)$ (and higher derivatives) only in the odd-parity case [14], whereas analysis of the general case yields an estimate that the solution can grow at most exponentially [15]. The numerical scheme is described elsewhere as well as dynamics over a wide range of parameter values [12]. The truncation order of the Galerkin approximation depends on the value of ν, but a crude estimate that has proven practical, is the retention of a few frequencies more than the number of linearly unstable ones. This number is related to the dimension of the attractor for given ν but no conclusive remarks can be made at this point (note that the theoretical upper bound estimate for the Hausdorff dimension of the attractor yields a value proportional to $\nu^{-21/40}$ which is larger than $\nu^{-1/2}$ by a factor of $\nu^{-1/40}$, for details see [16] ; the multiplicative constant that comes with the bound, however, can be very large thus obstructing direct practical applications). A brief summary of the phase-space is now given, and documented in Table 1. The description we give starts at ν=1 and continues to ever-decreasing values. An initial bifurcation occurs when ν decreases below 1 giving unimodal fixed points, and as ν decreases further a series of Hopf bifurcations are encountered at ν=1/4,1/9,1/16,... which are seen as dimodal, trimodal, tetramodal etc. stationary solutions. Not all orbits are stable, however, and so are not necessarily realized. What is more important is that instability appears that gives the solutions an oscillatory nature. A time-periodic attractor is obtained for the first time at a value of $\nu \sim 0.06$; a period-doubling occurs in this window also, but as ν is decreased further the solutions are attracted to fully modal fixed points. A new time-periodic window with two period-doublings and a period-halving (in this order), is then found $\nu \in [.03729, .03962]$ to separate the fully modal fixed points with another laminar window supporting tetramodal fixed points. The next periodic window $\nu \in [.029969104, .034625896]$ has the added feature that it contains a complete sequence of period-doubling bifurcations which lead to chaos. We call this the third periodic window and describe and analyze data from it presently. Note that the first chaotic window as well as the periodic one that preceded it, are only the first in a series as ν is decreased. In fact at much smaller values of ν we found more periodic windows with a complete sequence of period-doubling bifurcations leading to a chaotic flow. The lengths, in parameter space, of such successive attractors shrink significantly, but nonetheless we were able to compute the Feigenbaum number for three successive time-periodic attractors that lead to chaos. In what follows we describe the first periodic attractor that contains such a sequence, in order to make an accurate comparison with Feigenbaum's theory as applied to a partial differential equation.

Computation of the Feigenbaum number.

Table 2 presents our evidence that supports Feigenbaum's universal theory for the Kuramoto-Sivashinsky equation. These results were generated by monitoring the time evolution of the *energy*, $E(t)$, of the solution (the L^2-norm) but any other positive definite quantity can be chosen also, for example the supremum of the solution. Each entry in Table 1 represents the beginning of successive sub-windows which support solutions that undergo period doublings. The sharp estimation of these boundaries is necessary if an accurate computation of the Feigenbaum number is desired. In all results provided here the boundaries were estimated with enough accuracy to yield the Feigenbaum number correct to the number of significant figures shown. The first column gives the value of ν where the window begins, the second column gives the period of the oscillation and the third column gives the ratio of successive sub-window lengths according to equation (2). A more visual form of these results can be seen from a sequence of energy phase planes which cover the first 5 period doublings. It is clear from the figures that the overall bounds of the solution, for example the maximum and minimum of E and $\dot{E}$, do not vary much beyond the second period doubling. Chaos sets in by the appearance of more turns in the phase plane, in other words by an index change of the curves (in fact the exact way in which the phase plane gains more turns before the appearance of chaos, is quantified in the next sub-section). Another qualitative aspect of the period doublings as seen through phase-plane sequences, is the index doubling of the curves at the beginning of successive sub-windows (the end of super-stable limit cycles) by the splitting of one branch into two; the two branches then drift apart very slowly as time evolves until a new sub-window is encountered and another index doubling takes place. The time period itself increases very slightly over the duration of a sub-window but period doublings are exact and spontaneous.

The universal limit of multiple period doublings.

Next we present a set of numerical results that exhibit very clearly the fractal nature of the period doubling bifurcations. The experiment we choose has a value of ν = .0299691035 and lies at the end of the third periodic window. The time period of the solution is 1798.2564595 units and is the result of a sequence of 12 period doublings (in Figure 1 we show only the first 5). Note that at a value of ν =.029969103484, i.e. a decrease of 1.6×10^{-11} chaotic solutions were observed. Consider, then, the energy phase plane at ν = .0299691035. This is not shown here but is similar to the last frame of Figure 1, except that it has more turns. We now look at that region of the phase plane which has $|\dot{E}(t)| \leq 0.01$ and whose horizontal axis is chosen so as to include initially all the *minima* of the energy time series (in other words all the branches on the left of the figure). This diagram is shown in Figure 2(a). Figure 2(b) is now obtained as follows : The minima on the right half of 2(a) are considered and the picture is enlarged and centered in exactly the same way that 2(a) was (this ensures that the position of the leftmost and rightmost branches relative to a given enlarged picture are congruent). We now take the left half of 2(b) and enlarge it as indicated above to produce 2(c). The right half of 2(c) is then enlarged to produce 2(d) and so on. The reason for alternating between right and left halves is that we obtain the correct regions that need to be enlarged in order to demonstrate self-similarity of two symmetrically placed regions of the phase plane. This process provides a picture of the dynamics in a continuously shrinking region of the phase plane and is the appropriate procedure to follow in searching for any kind of self-similarity or fractal properties of the flow. The pictures presented here involve scrutinization of the "central" part of the phase plane (in fact keeping track of the sequence of magnifications we see that the portion of 2(a) which is magnified all the way to the end, is in the vicinity of E = 17). Other parts of the curve could have been chosen also but the present choice provided the simplest algorithm. With the above construction we are in a position to exhibit the fractal nature of the dynamics. The pictures 2(c), 2(e), 2(g) and 2(i) are seen to be self-similar as also are 2(b), 2(d), 2(f), 2(h) and 2(j). Self-similarity is deduced by the identical geometrical arrangement of the various portions of the phase plane as the magnification proceeds. Also, figures which are side-by-side are reflections of each other about the vertical axis (this is to be expected by analogy with maps of quadratic nonlinearity). Next, consider the distance between the leftmost and rightmost elements of Figures 2(b)-2(j). If the ratio of successive distances is calculated as the magnification proceeds, the number found is 2.503 to within the accuracy of our measurements. This is the universal scaling factor α described in the Introduction (equation (3)), and so we have another instance of a complete confirmation of Feigenbaum's universal theory for the Kuramoto-Sivashinsky equation.

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Acknowledgements.

We would like to thank Professors P. D. Lax G.C. Papanicolaou and Stanley Osher for many useful discussions.

Overview of the *most* attracting manifolds	
Window range	**Description of the attractors**
$1 \leq \nu < \infty$	Constant states.
$.25 \leq \nu < 1$	Fully modal steady attractors.
$.0756 < \nu < .25$	Fully modal and dimodal steady attractors.
$.06697 \leq \nu \leq .0755$	Fully modal steady attractors.
$.05992 \leq \nu \leq .06695$	Trimodal steady attractors.
$.05516 \leq \nu \leq .05991$	Periodic attractors.
$.0396227 \leq \nu \leq .05515$	Fully modal steady attractors.
$.03729 \leq \nu \leq .0396226$	Periodic attractors.
$.0346259 \leq \nu \leq .03728$	Tetramodal steady attractors.
$.029969103484 \leq \nu \leq .0346258$	Periodic attractors. Complete period-doubling sequence.
$.02922 \leq \nu \leq .02969910348$	Chaotic oscillations.
$.02905 \leq \nu \leq .02921$	Periodic attractors.
$.02855 \leq \nu \leq .02904$	Chaotic oscillations.
$.02662 \leq \nu \leq .02854$	Periodic attractors.
$.02525 \leq \nu \leq .02661$	Chaotic oscillations.
$.02506 \leq \nu \leq .02524$	Periodic attractors.
$.0248607 \leq \nu \leq .02505$	Chaotic oscillations.
$.02445 \leq \nu \leq .0248606$	Periodic attractors. Complete period-doubling sequence.
$.0242861 \leq \nu \leq .02445$	Chaotic oscillations.
$.02367 \leq \nu \leq .02438608$	Periodic attractors. Complete period-doubling sequence.
$.0232 \leq \nu \leq .02386$	Chaotic oscillations.
$.0229 \leq \nu \leq .0231$	Periodic attractors.
$.0223 \leq \nu \leq .0228$	Chaotic oscillations.
$.022 \leq \nu \leq .0222$	Periodic attractors.
$? \leq \nu \leq .0219$	Chaotic oscillations.

TABLE 1

20-mode Galerkin Expansion			
Subwindow boundary	Length	Ratio	Period
.0346258	4.3083×10^{-3}		.44
.03031749	2.6825×10^{-4}		.88
.030049233	6.2786×10^{-5}	16.061	1.76
.029986446	1.3609×10^{-5}	4.2724	3.52
.0299728366	2.9330×10^{-6}	4.6136	7.03
.0299699036	6.288×10^{-7}	4.6399	14.05
.02996927484	1.3456×10^{-7}	4.6644	28.1
.02996914018	2.884×10^{-8}	4.6657	56.2
.02996911134	6.18×10^{-9}	4.667	112.4
.02996910516	1.32×10^{-9}	4.68	224.8
.029969103842	2.84×10^{-10}	4.65	449.6
.029969103558	6.0×10^{-11}	4.7	899.1
.029969103498	1.4×10^{-11}	4.	1798.2
.029969103484			∞

TABLE 2

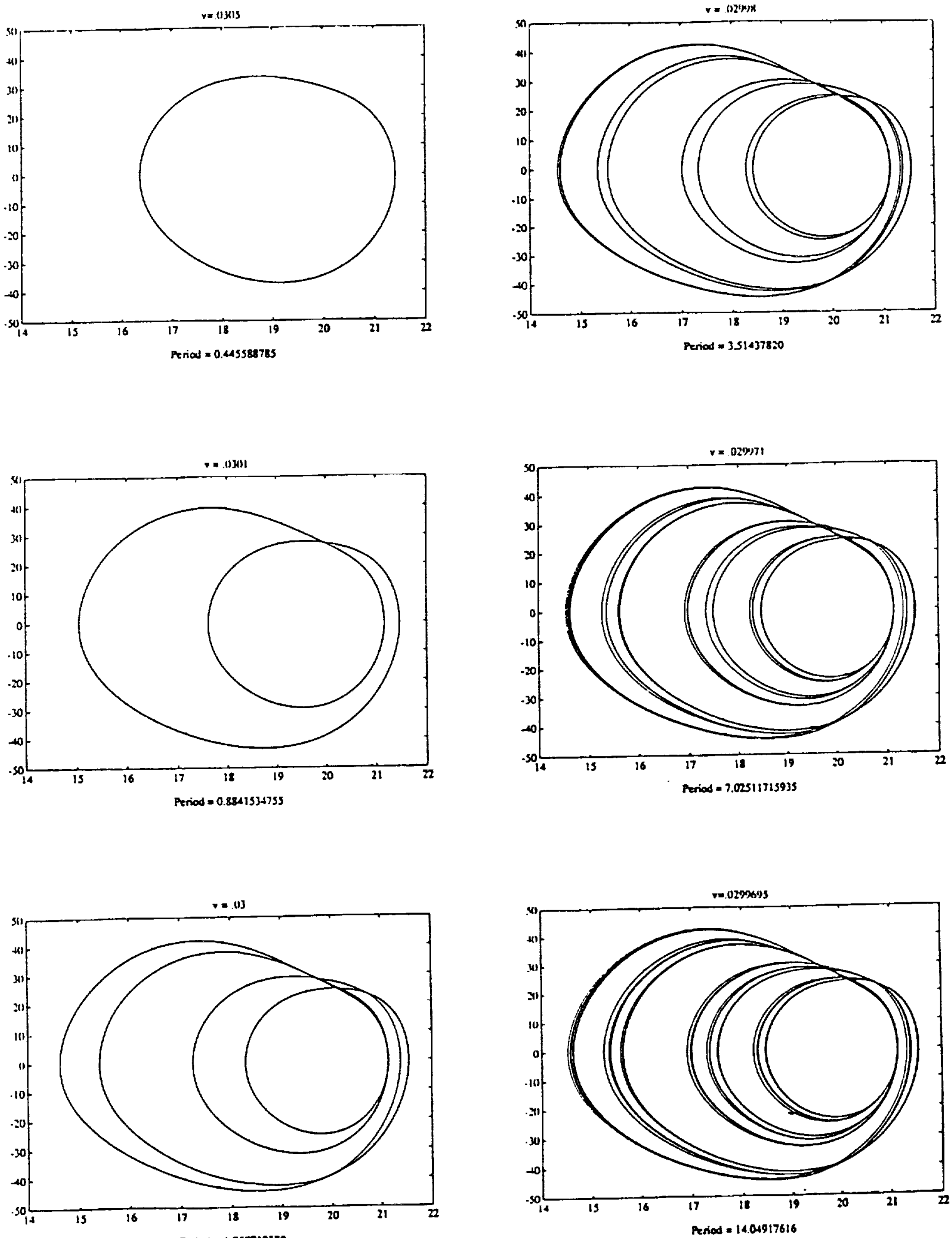

FIGURE 1 : The phase-plane showing the first five period-doublings. The values of v are given on the figure as well as the time periods.

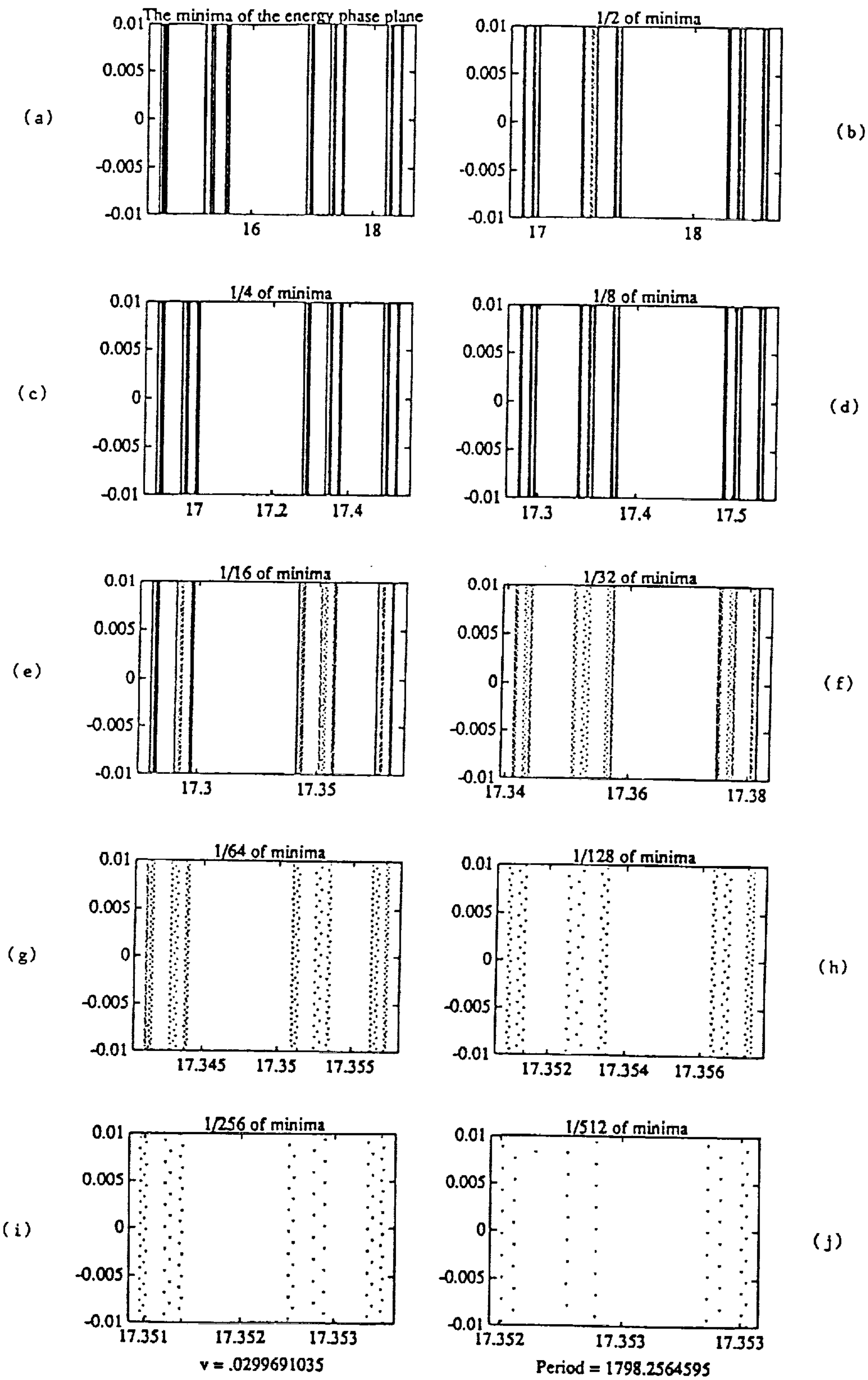

FIGURE 2 : 2(a)-2(j) - successive magnification of a portion of the phase-plane for the 2^{12}-cycle, showing the fractal characteristics of the attractor.

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National Aeronautics and Space Administration

Report Documentation Page

1. Report No. NASA CR-187531 ICASE Report No. 91-22	2. Government Accession No.	3. Recipient's Catalog No.

4. Title and Subtitle

PREDICTING CHAOS FOR INFINITE DIMENSIONAL DYNAMICAL SYSTEMS: THE KURAMOTO-SIVASHINSKY EQUATION, A CASE STUDY

5. Report Date

February 1991

6. Performing Organization Code

7. Author(s)

Yiorgos S. Smyrlis
Demetrious T. Papageorgiou

8. Performing Organization Report No.

91-22

10. Work Unit No.

505-90-52-01

9. Performing Organization Name and Address

Institute for Computer Applications in Science and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23665-5225

11. Contract or Grant No.

NAS1-18605

13. Type of Report and Period Covered

Contractor Report

12. Sponsoring Agency Name and Address

National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665-5225

14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor:
Michael F. Card

Submitted to Proceedings of the National Academy of Sciences

Final Report

16. Abstract

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17. Key Words (Suggested by Author(s))	18. Distribution Statement
chaos, infinite dimensional dynamical system, universal constants	34 - Fluid Mechanics and Heat Transfer Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	14	A03

NASA FORM 1626 OCT 86