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ABSTRACT

This paper is concerned with singularities in inviscid two dimensional finite amplitude water waves and inviscid Rayleigh-Taylor instability. For the deep water gravity waves of permanent form, through a combination of analytical and numerical methods, we present results describing the precise form, number and location of singularities in the unphysical domain as the wave height is increased. We then show how the information on the singularity can be used to calculate water waves numerically in a relatively efficient fashion. We also show that for two dimensional water waves in a finite depth channel, the nearest singularity in the unphysical region has the same form as for deep water waves. However, associated with such a singularity, there is a series of image singularities at increasing distances from the physical plane with possibly different behavior. Further, for the Rayleigh-Taylor problem of motion of fluid over vacuum, and for the unsteady water wave problem, we derive integro-differential equations valid in the unphysical region and show how these equations can give information on the nature of singularities for arbitrary initial conditions.

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I. INTRODUCTION

The study of singularities in the study of differential equations is quite old (for a review, see Ramani, Grammaticos & Bountis, 1989). Sometimes, a solution to a differential equation is completely characterized by the number of singularities and their location. A special example would be a differential equation in the infinite complex domain that admits only meromorphic functions for solutions. In that case, if we were to know the location of poles and the residues at the poles, we have a way of characterizing the solution completely when additional information is available about the behavior of solution at infinity. Some equations do not allow any singularity in the physical region of the flow; an example is the Laplace's equation. However, these equations do allow singularities in the appropriately continued functions in the unphysical domain. One relatively simple example is the steady periodic two dimensional water waves (gravity type). They were originally studied by Stokes (1849,1880) who developed perturbation expansions in powers of the wave height. Nekrasov (1921) proved the existence of solutions for sufficiently small wave height. Garabedian (1957) proved the existence and uniqueness of solutions with crests and troughs identical.

Actual computation of shapes have been done by Schwarz (1974), Longuet-Higgins (1975), Chen & Saffman (1980) and Zufiria (1987) among many others. Consider the conformal map from the standard domain such as a cut circle (shown in Fig. 1) to a semi-infinite strip shown in Fig. 2, such that A, B and D in Fig. 2 correspond to $\zeta = 1$, 0 and 1 respectively. The free boundary then corresponds to $|\zeta| = 1$. The conformal mapping function can be decomposed as:

$$z(\zeta) = 2\pi + i \ln \zeta + i f(\zeta) \tag{1.1}$$

where an assumed 2π periodicity in the x direction (see Fig.2) implies f is oblivious to the branch cut and therefore has a convergent power series representation:

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n \tag{1.2}$$

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The convergence of this power series is restricted by the nearest singularity from the origin in the ζ plane. If the interface is smooth (which earlier studies suggest it is for wave heights less than the limiting Stokes 120 degree cusped wave) then for large n, the coefficients $a_n \sim e^{-\alpha n}$ where $\alpha = ln \, |\zeta_0|$, ζ_0 being the nearest singularity of $z(\zeta)$ and hence of $f(\zeta)$ outside the unit circle. The characterization of such singularities in the unphysical domain is useful for at least three different reasons. First, if we are able to characterize the function $f(\zeta)$ completely in terms of all its singularity outside the unit circle (i.e. in the unphysical region in this case), then there is a chance of obtaining exact solutions for the problem. (While exact solutions are known for pure capillary waves (Crapper, 1971), no such solution is known to water waves of gravity type.) Even if this is not possible, the singularities of $z(\zeta)$ in the unphysical plane in the limit of their approach to the physical domain can be expected to be related to physical feature of the waves such as large variations of curvature near a crest. Finally, from a numerical standpoint, the knowledge of the singularity in the unphysical plane makes it easier to design a numerical algorithm that subtracts out the singularities, and reexpresses the remaining part of the analytic function in terms of a series like (1.2). The series for the remaining part will have a larger radius of convergence and therefore would require fewer terms in a truncated series representation for an accurate numerical calculation. While, the third reason is not as important for steady water waves where previous numerical work along traditional lines by Schwartz (1974), Longuet Higgins (1975), Chen & Saffman (1980) and Zufiria (1987) among others are able to describe reasonably steep water waves; the generalization of the above procedure for time dependent flows has the potential for far reaching consequences. Further, for idealized time dependent flows that are ill posed in the usual norms (such as the max norm), the study of singularities in the unphysical domain (see Orlenna & Caflisch, 1989 for Kelvin-Helmholtz instability and Siegel, 1990 for Rayleigh-Taylor instability) appears to be a natural way to understand the nature of the ill-posedness.

The Kelvin-Helmholtz interfacial evolution (i.e. the motion of the interface between two fluids moving with different velocities across the interface) is an example of such a flow where an initially analytic interface develops singularities in finite time as suggested by the asymptotic analysis of Moore (1979, 1985) and later supported by direct numerical computation by Shelley (1989). Caflisch & Orellana (1989) proved the existence of solutions in the function space of analytic norms up to a time consistent with Moore's formal asymptotic estimates for the formation of a singularity in the Kelvin-Helmholtz problem. In the formulation of Caflisch & Orellana (1989), the physical domain is the real line (unlike the example above for water waves we have just given), and they show that the occurrence of finite time interfacial singularities are due to singularities in the unphysical complex plane hitting the the real axis (i.e. physical domain) at some critical time. Following the earlier work of Moore, they showed how singularities can be formed in the complex plane by studying localized approximate equations and found the generic form of these singularities. Earlier, numerical evidence by Baker, Meiron & Orszag (1980) suggested that the finite time singularity also occurs in the Rayleigh-Taylor problem (the motion of the interface between fluids of differing densities) with non zero density ratio between the lighter and heavier fluids (non-unit Atwood ratio). This is supported by the analysis of Siegel (1990) based on an approximate localized equation.

These works suggest that the critical time of formation of interfacial singularities can be made arbitrarily small by perturbing initial data by any small amount in the physical domain if the analytic continuation of such a perturbation in the unphysical plane is appropriately large. This means that the Kelvin-Helmholtz and the Rayleigh-Taylor problem are ill-posed if we use any of the commonly used norms such as a Sobolev norm on the real line. However, for analytic norms that distinguish between behavior of functions in a complex strip surrounding the real physical domain, the problem is well-posed till the critical time of singularity formation on the interface. It is clear that despite the well posedness in terms of an appropriately defined norm, if the critical time value is

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finite the practical usefulness of the solutions to these equations is very restricted since the real observed phenomenon (Emmons, Chang & Watson, 1959) show no evidence of such singularities. It is clear that one must consider regularizing effects such as a finite thickness of the vortex layer (Baker & Shelley, 1989), or vortex blob (Bernard & Chorin, 1973, Krasny, 1986), to obtain well posed equations.

The study of singularities is an important first step necessary to understand within an analytical framework how small amounts of regularization affect the time evolution of the interface. For the unit Atwood ratio Rayleigh-Taylor problem, the work of Baker, Meiron & Orszag (1980) suggests that the critical time of singularity formation is infinity. This is supported by recent numerical work of Baker (1990) that shows that the critical time of singularity formation recedes to infinity as the Atwood ratio tends to unity. Thus, if this conjecture about critical times is correct, the singularity never appears in the physical domain in finite time though there is evidence to suggest that singularities appearing in the complex plane approach the physical domain arbitrary closely. One can then conjecture that for any given time T no matter how large, there should exist H so that for any $\epsilon > 0$, there exists δ so that if the initial conditions are close to each other within δ in an analytic norm with analyticity strip width H, the corresponding solutions do not differ from each other by more than ϵ on a Sobolev norm on the real axis upto time T. If this conjecture is true, one can choose T large enough so that formation of bubbles and spikes are observed making the unit Atwood idealized problem relevant to actual experiment. While the use of an analytic norm to make a problem well posed may be questioned since it should be enough to quantify the differences in initial conditions in the physical domain for an equation purporting to model a physical phenomenon, one can argue that neglected effects such as surface tension will prevent the sensitivity of the initial conditions to high wave number perturbation. Thus, it is conceivable that these neglected effects may in some way, not fully understood, filter out disturbances that are small in the physical plane but large in the unphysical plane, thereby justifying the use of an analytic

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norm in the theoretical study of the idealized equations.

Even if finite time singularities are apparently avoided in the Rayleigh-Taylor problem of fluid over vacuum, the continuing distortion causing large variation of the interface and the accompanying bubble and spike features observed in the experiment appears to suggest that the conformal mapping function $z(\zeta, t)$ from the unit circle to the physical domain as in Figures 1 and 2 will have singularities continually approaching the unit circle without actually reaching it. This is indirectly verified by noticing the actual asymptotic decay of the coefficients in (1.2) for large n in a numerical calculation. It is clear that any conventional numerical code such as that of Baker, Meiron & Orszag's (1980) will eventually fail because of limited resolution as the interface became highly convoluted unless the singularity in the unphysical plane is explicitly subtracted. Thus it appears that the singularity study in the unphysical plane is an important element in the process of construction of an efficient and accurate algorithm for highly convoluted interfaces.

The main purpose of this paper is to study the singularities in the unphysical plane by analytically continuing the equations into the unphysical plane for the water wave and Rayleigh-Taylor problems. An earlier version containing a basic skeleton of this paper was submitted to the Physics of Fluids (1988) but subsequently withdrawn. There we obtained the analytically continued equations for steady deep water waves (gravity type) and the unit Atwood number Rayleigh-Taylor problem. For water waves, we showed analytically that if we assumed a power law branch point behavior of the analytic function f (as defined in 1.1) for the leading order then the leading order behavior is of the squareroot type which was generally consistent with Schwartz's conclusion (1974) based on numerically implementing a Pade approximate method. However, a Pade approximate method is only very approximate and indeed erroneous conclusion on singularity nature from Schwartz's calculations could be drawn when the singularity is too close to the physical domain. Further the extraction of information from a Pade approximate method is practically feasible only for the nearest singularity and just to the leading order. Our result (1988)

was true regardless of the distance of the singularity once we assumed a certain general form. We subsequently found out about earlier work of Grant (1973) (this is referenced by Schwartz (1974) though that was mistakenly overlooked) that uses the same argument through in a slightly different formulation. However, in the earlier work of Schwartz and Grant (as with our earlier draft), there were several unanswered questions. The first is the relation of the number of singularities with the number of crests and troughs within one period of the wave.

Second is the actual location of the singularity as a function of height or speed of the water wave. Schwartz's result is only about the nearest singularity and that too is not accurate.

Third, the precise form of the singularity including higher order corrections to the squarerroot singularity were not addressed. Note that the higher order corrections to the leading order singularity at $\zeta = \zeta_0$ restricts the convergence of (1.2) just as much as the leading order. Finally, earlier work did not address the form of the singularity for water waves in a channel of finite depth.

These questions are answered in this paper. Using the rigorous work of Painleve (see Hille, 1979 and Ince, 1956), we establish that the only form of singularity in the finite ζ plane is of the square-root type such that near $\zeta = \zeta_0$ (the singularity point)

$$f(\zeta) = \sum_{n=0}^{\infty} d_n (\zeta - \zeta_0)^{n/2} \tag{1.3}$$

We then numerically calculate all possible ζ_0 and find that a water wave of one trough and one crest per period corresponds to a single singular point ζ_0 in the finite ζ plane lying outside the unit circle. We also numerically calculate the dependence of ζ_0 on the wave height for the class of symmetric waves that have identical troughs and crests (call it the Stokes waves). We then present an improved numerical method to calculate the Stokes deep water wave that incorporates the location of ζ_0. This method is not optimal since it does not actually make use of the complete representation (1.3); however, here we only present this as an example of developing concrete numerical algorithms,

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using singularity information. The problem of designing an optimal numerical algorithm is left for the future. While we could characterize the complete singularity structure of the analytic function f in the entire complex plane and find the leading order behavior of f at infinity; we were unable to find an exact solution because of the exceedingly complex behavior of f at ∞. We also discovered other singular points for f inside the unit circle (at $\zeta = \frac{1}{\zeta_0}$ and 0) in the unphysical Riemann sheet of the two sheeted Riemann surface centered at the branch point $\zeta = \zeta_0$ outside the unit circle. For steady periodic two dimensional water waves in a channel of finite depth, we also show that the form (1.3) is valid for the singularity closest to the physical plane for water waves in a channel of finite depth, where $f(\zeta)$ as defined in (1.1) is now related to the conformal map $z(\zeta)$ from an annular region with a cut (Fig. 3) into a period in the physical flow region (Fig. 4). We also show that corresponding to such a singularity ζ_0, there are an infinite set of image singularities further out in the unphysical domain. These induced singularities need not be of the form (1.3).

The earlier draft of this paper on the Rayleigh-Taylor problem (1988) did not contain any concrete results on the Rayleigh-Taylor problem other than the derivation of the governing integro-differential equations in the unphysical plane. Later, but independently, Orlenna & Caflisch (1989) (for the Kelvin-Helmholtz problem) and Siegel (1990) (for the Rayleigh-Taylor problem) showed how the analytic continuation of the equations to the unphysical domain together with a localized approximation that ignores global integral terms can result in a set of two first order hyperbolic partial differential equations in the unphysical domain whose solutions develop singularities. For the Kelvin-Helmholtz problem, these localized equations were equivalent to equations derived by Moore (1985) from a different perspective. While these studies have been useful in showing the formation of singularities far from the real axis (the physical domain in their formulation), they do not quite address the effect of the neglected integral terms, which are important when a singularity approaches the real physical domain. For the Rayleigh-Taylor equations, Siegel

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(1990) found an exact solution to his localized nonlinear hyperbolic set of partial differential equations describing the formation and motion of singularity in the unphysical plane and its approach to the physical domain. With initial conditions consistent with his exact solutions, he finds that as the Atwood ratio approaches unity (i.e the classical case), the singularity disappears even in the unphysical plane. It is unclear if this is true with the full equations and if it is a generic result for arbitrary initial conditions. Further, it is not clear in Siegel's work how to include the nonlocal integral terms in an analysis and numerical computation of singularity formation. These nonlocal terms, while negligible when the singularity is far from the physical domain need not be so when the singularity comes close to the physical domain. If we make the ansatz that the unit Atwood ratio Rayleigh-Taylor problem does not have a finite time singularity in the physical domain (as suggested by Baker, Meiron & Orszag's work), then it would seem reasonable that a highly deformed feature of the interface such as spikes corresponds to singularity approaching the real physical domain indefinitely without actually reaching it in finite time. If this is true, the Siegel (1990) localized equations may not be able to describe the most interesting stages of evolution: the formation and interaction of bubbles and spikes as observed in the experiments of Emmons, Chang & Watson (1959). If we use a conformal mapping representation as in (1.1) for the periodic 2-D Rayleigh-Taylor problem, the function $f(\zeta, t)$ can be represented in a power series as in (1.3). Numerical calculations to be reported elsewhere show that these coefficients appear to decay exponentially with n ; however the multiple of n in the exponent rapidly goes to zero for large times suggesting that for Atwood ratio unity, there is indeed a singularity in the unphysical domain that approaches the physical domain without actually entering it suggesting that Siegel's result for Atwood ratio unity is not generic. In terms of our exact integro-differential equations for the analytic function f and a companion function, we are able to show that in the unphysical domain, we obtain a set of two quasilinear first order partial differential equations when the integral terms involving f in the physical domain are considered known. There is only one set of

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characterestics in this problem. Through an appropriate linear combination, one of these equations is close to the inviscid Burger's equation which is known to develop fold singularity of the square-root type for general initial conditions. We find that such a behavior can be consistent with our equations though we do not address the question of other possible forms of singularities. We then suggest how a complete numerical calculation of the unphysical equations can shed light on the exact singularity structure in the unphysical plane leading the way for studying detailed bubble and spike structure for the classical Rayleigh-Taylor problem. Our calculations for early times appears to suggest that a spike corresponds to a singularity approaching the unit circle, i.e. the physical domain.

2. PERIODIC DEEP WATER WAVE

2.1 Mathematical formulation

We consider a two dimensional periodic deep water wave. We move to a frame of reference where the wave is stationary and the flow at $y = -\infty$ (Fig. 2) is a uniform flow to the right with speed c, the wave speed. Further, we shall assume for the present that the waves are symmetric, meaning that there exists some choice of the x origin so that the profiles have mirror symmetry about the y axis. For the smooth waves that we are considering, it is clear that such a choice of origin of x must coincide either with a crest or a trough. For symmetric waves with identical crests and troughs, without any loss of generality, we make the choice of origin coincide with a trough as shown in Fig. 2; otherwise, the origin will be chosen to be any such location (may not be a trough) about which there is symmetry. Our results on the form of the singularity hold equally for nonsymmetric waves (Zufiria, 1987) and we shall find out later how to extend the present analysis to a generally nonsymmetric wave. Without any loss of generality, the period is taken to be 2π and acceleration due to gravity g, acting in the negative y-direction is taken to be unity. In the complex velocity potential $W = \phi + i \psi$ plane, the flow domain within one period in the z-plane is a half strip as shown in Fig. 5. We consider the conformal map $z(\zeta)$ from the interior of a cut unit circle in the ζ plane (Fig. 1) into

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the interior of the semi-infinite strip in the physical plane (Fig. 2) such that the points A, B and D in Fig. 2 correspond to $\zeta = 1, 0$ and 1 on the two sides of the cut. Schwartz (1974) had earlier used the ζ plane for his direct numerical calculations. Note that the free boundary then corresponds to the circular part of the boundary. For waves with equal troughs and crests, our choice of origin of x and the symmetry of the flow imply that $\zeta = -1$ corresponds to the crest. We define the analytic function $f(\zeta)$ by writing

$$z = 2\pi + i(\ln\zeta + f) \qquad (2.1)$$

¿From the condition of wave symmetry,

$$Im\ f = 0 \qquad (2.2)$$

on the real ζ axis in the interval (-1, 1). Further from periodicity, it follows that f is analytic within the unit circle and does not "see" the branch cut. Thus it is possible to express f in a convergent power series representation:

$$f(\zeta) = \sum_0^\infty a_n\ \zeta^n \qquad (2.3)$$

¿From condition (2.2), it follows that all the a_n are real. We consider waves that have smooth profiles $z(\zeta)$ and hence $f(\zeta)$ is analytic on the boundary of the unit circle. This means that radius of convergence of the series in (2.3) is greater than unity. The mapping of the ζ plane into the flow domain in the complex velocity potential W -plane is given by

$$W(\zeta) = i\ c\ \ln\zeta + 2\pi c \qquad (2.4)$$

Note that equation (2.4) can be used to define ζ directly in terms of W and thus the singularity of f in the representation (2.1) does relate to the singularity of the actual flow $\frac{dW}{dz}$ in the unphysical domain. The dynamic condition (Bernoulli's equation) on the free-surface is

$$\frac{1}{2}(\nabla\phi)^2 + y = constant \qquad (2.5)$$

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¿From (2.1) and (2.4), it is clear that (2.5) implies that on $\zeta = e^{i\nu}$, ν real in the interval $(0, 2\pi)$,

$$Re\ f = -\frac{c^2}{2}\frac{1}{|1 + \zeta f_\zeta|^2} \tag{2.6}$$

Note that any additive constant appearing on the right hand side of (2.6) is eliminated by suitably redefining f. This corresponds to a choice of origin for y.

We now analytically continue equation (2.6) for any ζ not necessarily on the unit circle. We note for $\zeta = e^{i\nu}$, (2.6) can be written as

$$f(\zeta) + f(1/\zeta) = -\frac{c^2}{(1 + \zeta f_\zeta)(1 + \frac{1}{\zeta} f_\zeta(\frac{1}{\zeta}))} \tag{2.7}$$

¿From the principle of analytic continuation, (2.7) is valid for any ζ. This analytic continuation was previously realized by Grant (1973) in a differing but equivalent representation. Now, lets define

$$F(\zeta) = f(\zeta) + f(1/\zeta) \tag{2.8}$$

and

$$P(\zeta) = (1 + \frac{1}{\zeta}f'(1/\zeta)) \tag{2.9}$$

Then, (2.7) implies

$$\zeta\ F\ F_\zeta + P\ F = -\frac{\beta}{P} \tag{2.10}$$

where $\beta = c^2$. Introduce

$$F_1 = F^{-1} \tag{2.11}$$

Then equation (2.10) implies

$$F_1' = \frac{P}{\zeta}\ F_1^2 + \frac{\beta}{\zeta P}\ F_1^3 \tag{2.12}$$

Note that on or outside the unit circle, P is not known but must be analytic and nonzero for smooth water waves as otherwise the derivative of the conformal map z_ζ on or inside the unit circle at $\frac{1}{\zeta}$ would have to be zero. Thus the coefficients of F_1^2 and F_1^3 must be analytic everywhere outside the unit circle. If we think of P as given, (2.12) is a special

case of the nonlinear first order differential equations studied by Painleve (See Hille, 1976 & Ince, 1956). For solutions to (2.12), he rigorously established that the only form of movable singularity ζ_0 (i.e. a singularity not related to a singularity of the coefficients of F_1^2 and F_1^3 , but to the initial conditions on F) is such that in some neighborhood of ζ_0 ,

$$F_1(\zeta) = \frac{1}{(\zeta - \zeta_0)^{1/2}} \sum_{n=0}^{\infty} \tilde{b}_n \, (\zeta - \zeta_0)^{n/2} \tag{2.13}$$

It immediately follows that

$$F(\zeta) = \sum_{n=1}^{n=\infty} b_n \, (\zeta - \zeta_0)^{n/2} \tag{2.14}$$

Since $\frac{P}{\zeta}$ and $\frac{\beta}{P}$ are analytic everywhere outside the unit circle in the finite ζ plane, it follows that all singularities of F outside the unit circle are necessarily of this form. Further, when $|\zeta| > 1$, $f(1/\zeta)$ is clearly analytic and so (2.8) and (2.14) imply that

$$f(\zeta) = \sum_{n=0}^{n=\infty} d_n \, (\zeta - \zeta_0)^{n/2} \tag{2.15}$$

with

$$d_n = b_n \tag{2.16}$$

for odd n and for even n :

$$d_n = b_n - \frac{1}{\frac{n}{2}!} \frac{d^{n/2}}{d\zeta^{n/2}} f(1/\zeta)|_{\zeta = \zeta_0} \tag{2.17}$$

Note that for Stokes highest wave (120 degree cusp), these arguments do not go through because $P(\zeta)$ and $f(1/\zeta)$ are not analytic on the unit circle any more. Indeed, by directly substituting a singularity with a fractional power form into (2.7) near $\zeta = -1$ (crest), one can obtain the leading order behavior $(\zeta + 1)^{1/3}$ corresponding to the functional form of Stokes (see Lamb (1932)).

To study the behavior of F at infinity, we introduce

$$\zeta_1 = \frac{1}{\zeta} \tag{2.18}$$

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in which case (2.10) reduces to

$$-\zeta_1 F \frac{dF}{d\zeta_1} + P F = -\frac{\beta}{P} \tag{2.19}$$

It is clear from (2.3) and (2.9) that near $\zeta_1 = 0$, i.e $\zeta = \infty$,

$$P \sim 1 + a_1 \zeta_1 + 2a_2 {\zeta_1}^2 + .. \tag{2.20}$$

For leading order analysis, we substitute $P = 1$ into (2.19). Let us denote the corresponding solution F by F_0. The equation has an exact implicit solution which can be written as:

$$\zeta_1 = k \, e^{F_0} \, (\beta + F_0)^{-\beta} \tag{2.21}$$

where k is some constant. On inversion, this implies that as $\zeta \to \infty$,

$$F_0 \to -ln\zeta + \beta \, ln \, [\beta + \, ln \, (-ln\zeta)] + .. \tag{2.22}$$

Since $f(\frac{1}{\zeta})$ is analytic at $\zeta = \infty$, it follows that the leading order singularity of f at $\zeta = \infty$ is also given by (2.22). One can substitute (2.21) back into (2.20) to find a more accurate representation of P and then solve equation (2.19) to find a correction to F_0. It becomes clear that this process leads to very complex form of higher order corrections to the behavior (2.22) as $\zeta \to \infty$. Further from (2.10), it is clear that if F is zero at some finite ζ outside the unit circle, and F_ζ is necessarily singular at that point. This means that F cannot be analytic at a point in the finite ζ plane where $F = 0$. Thus the only kind of zeros of F are the singularity points ζ_0 around which (2.14) holds.

We end this section by noting how the analysis would change for nonsymmetric waves. First, if we take a semi-infinite strip as in Fig. 2, its image in the W plane will not be as shown in Fig. 5. since there is no reason to assume $Re \ W = constant$ at $x = 0$ and $x = 2\pi$. In this case, it is more convenient to take a semi-infinite strip in the W plane as shown in Fig. 5 and consider the corresponding physical region in the z plane formed by the conformal image $z(W)$. This will be some semi-infinite strip with curved

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boundaries. However, periodicity implies that $x(\phi + 2\pi c, \psi) = x(\phi, \psi) + 2\pi$ and that $y(\phi + 2\pi, \psi) = y(\phi, \psi)$. Thus if we use (2.4) to define ζ, it is clear that the image of the flow region in the ζ plane is the cut circle as shown in Fig. 1. Once again $z(\zeta)$ can be decomposed as in (2.1), with $f(\zeta)$ analytic everywhere within the unit circle, however the relation (2.2) is no longer valid. This means that the series coefficients a_n in (2.3) are not necessarily real. It is clear that that on $|\zeta| = 1$, $f^*(\zeta) \neq f(1/\zeta)$ and so the analytically continued equation (2.7) will not hold. However, if we define a function $g(\zeta)$ analytic everywhere outside the unit circle (including infinity) through the representation

$$g(\zeta) = \sum_{n=0}^{\infty} a_n^* \, \zeta^{-n} \tag{2.23}$$

then clearly on $|\zeta| = 1$, $g = f^*$, implying that (2.7) has to be replaced by

$$f(\zeta) + g(\zeta) = -\frac{c^2}{(1 + \zeta f_\zeta)(1 - \zeta g_\zeta)} \tag{2.24}$$

¿From (2.24), it is clear that (2.10) will hold provided we redefine $P(\zeta)$ as:

$$P(\zeta) = 1 - \zeta g_\zeta \tag{2.25}$$

The arguments about no fixed singularity in the finite ζ plane for the associated equation (2.12) would still hold provided $P(\zeta) \neq 0$ outside the unit circle. We now proceed to prove that this is indeed the case. Suppose, on the contrary, that there exists some $\zeta = \zeta_*$, $|\zeta_*| > 1$ at which $P(\zeta_*) = 0$. From the power series representation of g in (2.23), it follows that $1 + \sum_{n=0}^{\infty} n\, a_n^* \, \zeta_*^{-n} = 0$. On taking the complex conjugate, and relating it to $(1 + \zeta f_\zeta)$ through the series representation (2.3) for f, we find that this implies $0 = (1 + \zeta f_\zeta)|_{\zeta = 1/\zeta_*^*}$. Since $z_\zeta = \frac{i}{\zeta}(1 + \zeta f_\zeta)$, it follows that $z_\zeta(1/\zeta_*^*)$ is zero, which it can't be since $1/\zeta_*^*$ is inside the unit circle and the derivative of a conformal map must be nonzero at everypoint within the unit circle. This proves P is nonzero outside the unit circle. It is true on the boundary of the unit circle as well, if we assume that the water wave profile is smooth. Thus all the arguments leading to the representation (2.15) of f for the symmetric waves hold for nonsymmetric waves as well.

2.3 Numerical determination of the number of singular points ζ_0 and their locations

Here we make a concrete evaluation of the number of singularity points ζ_0 and their location. Earlier, Schwartz (1973) used Pade approximate methods to extract singularity information on the nearest leading order singularity. His results were generally consistent with Grant's (1973) analytical result though there was discrepancy when the singularity was too close to the unit circle. However, there was no information on how to obtain higher order corrections and how many singularities there were. Once again, since the chief interest here is illustrative, we will restrict our numerical calculations to Stokes waves, i.e. symmetric waves with identical crests and troughs. Without any loss of generality, we can then assume that one period of the wave contains only one crest and one trough.

First, for any given wave height h defined to be

$$h = \frac{1}{2} [\, Ref(-1) \, - \, Ref(1) \,] \tag{2.26}$$

we calculate wave speed c and a_0, a_1, ... a_{N-2} in an $(N-1)$ term truncated expansion of (2.3) and satisfy (2.6) at N uniformly spaced out points on the upper half unit semi-circular boundary including $+1$ and -1. This is done through Newton interation, a good initial guess for small wave height being the well known linear gravity wave solution. The solution was checked by increasing N and observing the consistency of the coefficients. The wave profile corresponding to the calculated a_n agreed with previous calculations of Stokes waves by Chen and Saffman (1980), providing a check of our numerical code. We note that once the a_n are known, the analytic functions $P(\zeta)$ and $f(1/\zeta)$ can be calculated at any point outside the unit circle. Note also that even when the precision of the numerically calculated a_n is not great, the calculation of $P(\zeta)$ and $f(1/\zeta)$ is relatively much more accurate for points outside the unit circle as $|\frac{1}{\zeta}| < 1$.

Given P, (2.10) is integrated numerically along any path outside the unit circle to determine F at each point on that path. The known value of $F(1) = 2f(1)$ calculated from the power series (2.3) provides the initial value. To calculate the total number of

possible ζ_0 and their numerical values, we followed a two pronged strategy. First we calculated

$$p = \frac{1}{2\pi i} \oint_C d\zeta \frac{F'}{F} = -\frac{1}{2\pi i} \oint_C d\zeta \left(\frac{\beta}{\zeta P F^2} + \frac{P}{\zeta F} \right) \qquad (2.27)$$

for various closed contours in the ζ plane outside the unit circle. The result was zero whenever the contour did not enclose any singularity ζ_0 of F as it must. When the answer was nonzero for a contour C, we deduced that there was a singularity ζ_0 inside C, in which case we went around once again to calculate

$$p_2 = \frac{1}{2\pi i} \oint_{C_2} d\zeta \frac{F'}{F} = -\frac{1}{2\pi i} \oint_{C_2} d\zeta \left(\frac{\beta}{\zeta P F^2} + \frac{P}{\zeta F} \right) \qquad (2.28)$$

where C_2 is a path that coincides with C but goes around twice. ¿From the singularity form in (2.14), it is clear that $p_2 = 1$ whenever there is only one singularity ζ_0 within the contour. Going around twice for a squareroot singularity ensures that we get back to the same Riemann sheet. Obviously, $p_2 = 0$ as is p when there are no singularities within C_2. However, if the contour C contains multiple singularities, $p_2 \neq 1$, and in general we cannot expect the answer to be an integer since going around twice on C_2 does not return us to the same Riemann sheet. However, for gravity waves with equal crests and troughs, the so called Stokes waves, we could only find one singularity ζ_0 outside the unit circle as evidenced by $p_2 = 1$ (within numerical precision) when a wavelength contained one trough and one crest. We consistently found the same value regardless of the size of the contour around $\zeta = \zeta_0$, when the contour was completely outside the unit circle. This suggests that even in the unphysical Riemann sheet of the two sheeted Riemann surface generated by the branch point at $\zeta = \zeta_0$, there are no other singularities outside the unit circle. By an iterative choice of contours, one can precisely locate the singularity; however, we found it much more efficient to calculate ζ_0 directly at the same time p_2 is calculated. We note that equation (2.14) can be locally inverted to give:

$$\zeta = \zeta_0 + \sum_{n=2}^{\infty} h_n F^n \qquad (2.29)$$

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and so

$$\frac{1}{2\pi i} \oint_{C_3} \frac{\zeta(F)}{F} \, dF = \zeta_0 \qquad (2.30)$$

where the closed path of integration C_3 is traversed in the positive sense in the F plane and corresponds to the image of C_2 in the F-plane. Using the differential equation (2.10), we get

$$\zeta_0 = -\frac{1}{2\pi i} \oint_{C_2} d\zeta \left(\frac{\beta}{PF^2} + \frac{P}{F} \right) \qquad (2.31)$$

where the contour of integration C_2 in the ζ plane goes around ζ_0 twice when it contains ζ_0 as evidenced by $p \neq 0$ and $p_2 = 1$. If the contour encloses no singularity the answer on the righthand side of (2.21) must be zero as each of P and F are analytic and can have no zeros. Besides calculation of p and p_2, this is further evidence that there are no singularities of F when the contour does not include ζ_0. To compute any of the d_n in the representation (2.15), we use (2.16) and (2.17) and the relation of F to its power series coefficient in (2.14) in powers of $(\zeta - \zeta_0)^{1/2}$ through contour integration on a closed path on the two sheeted Riemann surface around ζ_0. One finds:

$$d_n = \frac{1}{4\pi i} \oint_{C_2} d\zeta \, (\zeta - \zeta_0)^{-n/2 - 1} F(\zeta) \qquad (2.32)$$

for odd n and for even n,

$$d_n = \frac{1}{4\pi i} \oint_{C_2} d\zeta \, (\zeta - \zeta_0)^{-n/2 - 1} F(\zeta) - \frac{1}{\frac{n}{2}!} \frac{d^{n/2}}{d\zeta^{n/2}} f(1/\zeta)|_{\zeta = \zeta_0} \qquad (2.33)$$

Table 1 is the result of such numerical calculation for Stokes waves when there is only one trough and one crest in one wavelength. It shows that ζ_0 is on the negative real axis and approaches -1, i.e the crest, monotonically as the waveheight h is increased, at least in the range of calculation. It is clear that if there are M identical troughs and crests within one wavelength, there will be M possible ζ_0 at equal modulus and arguments equalling the arguments of M roots of -1. The bifurcation that Chen & Saffman (1980) observed of Stokes waves (class 1) to class 2 or class 3 waves respectively, where the two or three troughs and two or three crests are not identical would correspond

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(in the singularity picture) to a bifurcation where the two or three different ζ_0 are either no longer equidistant from the origin or are not equispaced in argument values or both. However, this has not been directly verified here.

2.4. Use of singularity information for an efficient numerical scheme

In the last section we obtained concrete information about the singularity by first computing the a_n by conventional methods. As mentioned before, the calculations of a_n becomes difficult when the singularity approaches the unit circle and the Stokes limiting wave is approached. An efficient numerical method should incorporate the singularity information so that a truncated basis representation does not need as many terms to describe the interface. One may be tempted to use the representation of $f(\zeta)$ in (2.15) as a basis representation and directly calculate the d_n by satisfying (2.6) on the unit circle. However, the convergence of this representation is restricted by a possible singularity at $\frac{1}{\zeta_0}$ inside the unit circle in the unphysical Riemann sheet of the two sheeted Riemann surface centered at the branch point $\zeta = \zeta_0$. We can expect singularities of F, and hence generally of f both at $\zeta = 1/\zeta_0$ and $\zeta = 0$ since $P(\zeta)$ is singular at those points. Since $f(1/\zeta)$ is also singular at that point, the chances are that the singularities will not cancel out in the unphysical Riemann sheet as they do in the physical sheet for $|\zeta| < 1$, and so $f(\zeta)$ is singular at these points inside the unit circle in the unphysical Riemann sheet associated with the squareroot singularity at ζ_0.

As a simple though less than optimal use of singularity information, we can use a fractional linear map

$$\zeta_3 = \frac{(\zeta + p)}{(1 + p\zeta)} \tag{2.34}$$

where

$$p = -\zeta_0 - \sqrt{\zeta_0^2 - 1} \tag{2.35}$$

and the squareroot is understood in the sense of the positive branch for positive arguments. Recalling that ζ_0 is negative and less than -1, it follows that $0 < p < 1$ and (2.34) is a conformal map of a circle into a circle. In the ζ_3 plane, $\zeta = \infty$ and $\zeta = \zeta_0$ are

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now mapped to $\zeta_3 = \pm\frac{1}{p}$ respectively and these are further out from the origin of the ζ_3 plane than ζ_0 was from the origin of the ζ plane. Thus if we represent

$$f = \sum_{n=0}^{\infty} \tilde{a}_n \, \zeta_3^n \qquad (2.36)$$

the coefficients $\tilde{a}_n$ will decay with n for large n faster than in the representation (2.3). Thus one can start first with the representation (2.3), compute a finite number of a_n without much precision, then compute ζ_0 as in the previous section. This gives at least a rough estimate of p and this can be used to introduce the fractional linear mapping ζ_3 and then continue a second time around, using the representation (2.36) for a finite truncated representation of f . A continuation process may be followed for systematic calculation of steeper and steeper waves. Unfortunately, the representation (2.36), while better than (2.3), still has the undesirable feature of $\frac{1}{p} \to 1$ as $\zeta_0 \to -1$, implying that for steep waves, the convergence of (2.36) will still be limited by a singularity approaching the ζ_3 unit circle.

Clearly, one needs a better algorithm that uses more of the singularity information in (2.14) than merely its location ζ_0 . This will be the material for further investigations in the future.

3. STEADY PERIODIC WATER WAVES IN A FINITE DEPTH CHANNEL

For symmetric periodic water waves of finite depth as shown in Fig. 4, the corresponding region in the complex potential W plane is shown in Fig. 6. Consider the additional mapping

$$\zeta = e^{\frac{(W - 2\pi c)}{i c}} \qquad (3.1)$$

This maps the physical flow region into the interior of the annular shaped region shown in Fig. 3 where there is a cut along the positive real axis. The correspondence of the boundary points to points in the ζ plane are marked in Fig. 3. It is clear that

$$W = i \, c \, ln \, \zeta + 2\pi c \qquad (3.2)$$

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as before for water waves of infinite depth. Note that the flat bottom of the water wave corresponds to the inner circle in the ζ plane of radius $\rho_0 = e^{-h}$ and the free boundary corresponds to $|\zeta| = 1$. Also note that h is defined in terms of the W plane image (see Fig. 6) and need not be the water depth in the physical plane. Now as before, we introduce an analytic function f through the representation

$$z(\zeta) = 2\pi + i\,[\,ln\,\zeta + f(\zeta)\,] \tag{3.3}$$

Then it is clear that the periodicity assumption implies that the branch cut in the ζ domain is invisible to the function $f(\zeta)$ and it possesses a convergent Laurent series expansion in the annular region in the ζ plane. Further, since $Im\,z = $ constant on $|\zeta| = \rho_0$, we obtain $Re\,f = $ constant on $|\zeta| = \rho_0$. Further, for a symmetric water wave, it is clear that on the segments in the domain coinciding with the real axis, $Im\,f = 0$. These imply that the Laurent series for f has the form:

$$f(\zeta) = a_0 + \sum_{n=1}^{\infty} a_n\,(\zeta^n - \rho_0^{2n}\,\zeta^{-n}) \tag{3.4}$$

If we define

$$g(\zeta) = \sum_{n=1}^{\infty} a_n\,\zeta^n \tag{3.5}$$

then $g(\zeta)$ is analytic within the unit circle and

$$f(\zeta) = a_0 + g(\zeta) - g(\rho_0^2/\zeta) \tag{3.6}$$

Further, with appropriate choice of origin of y the pressure condition is once again equivalent to

$$Re f = -\frac{c^2}{2}\,\frac{1}{|1 + \zeta\,f_\zeta|^2} \tag{3.7}$$

The analytical continuation of this equation is once again

$$\zeta\,F\,F_\zeta + P\,F = -\frac{\beta}{P} \tag{3.8}$$

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where

$$F(\zeta) = f(\zeta) + f(1/\zeta) \tag{3.9}$$

and

$$P = 1 + \frac{1}{\zeta} f'(1/\zeta) \tag{3.10}$$

There is now a difference between finite and infinite depth water waves. From (3.6), we have the analyticity of $P(\zeta)$ and $f(1/\zeta)$ guaranteed outside the unit circle only for $|\zeta|$ between 1 and $\frac{1}{\rho_0^2}$. There are two possibilities: (a) that the solution F to (3.8) in this region is free of singularities and (b) that it has a singularity in this region which must be of the type (2.14) as for deep water waves. In case (a), it immediately follows that $f(\zeta)$ does not have any singularity for $1 < |\zeta| < \frac{1}{\rho_0^2}$ implying that $g(\zeta)$ is free of singularity in that region as $g(\rho_0^2/\zeta)$ is analytic there. ¿From (3.6), this implies $f(\zeta)$ is analytic over the extended inner range $\rho_0^4 < |\zeta| < 1$ as well. This implies that $P(\zeta)$ and $f(1/\zeta)$ are analytic for $\rho_0^{-4} > |\zeta| > \rho_0^{-2}$. Thus, we are back to two cases (a) and (b) for a new region $\rho_0^{-4} > |\zeta| > \rho_0^{-2}$. It is clear that on repetition of the above argument that we can ensure that the nearest singularity to the physical domain outside the unit circle must have the same form (2.15) as for deep water waves. Once we find such a singularity, call it ζ_0 , it is clear that $g(\zeta)$ will have a singularity at that point. This implies that $g(\rho_0^2\zeta)$ will have a singularity at $\rho_0^{-2}\zeta_0$. This means that P and $f(1/\zeta)$ will have a singularity at that point. This singularity of the coefficients in (3.8), will generally induce a singularity of F and hence of f at that point. It is clear that there will be a sequence of such induced singularities at $\rho_0^{-2m} \zeta_0$, m ranging from 1 to ∞ . Similarly, inside the inner circle, there will be induced singularities at ρ_0^{2m}/ζ_0 , m ranging from 1 to ∞ . The form of these sequence of induced singularities is different from that of the deep water wave since Painleve's argument on movable singularities is not valid here. In Painleve's terminology, we have fixed singularities of F at such points. We do not address the question of what form the singularity must be at each of these points, except to note that at $\zeta = \rho_0^{-2}\zeta_0$, the nearest induced singularity outside the unit circle,

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one possible behavior is:

$$f(\zeta) \sim \text{constant} + \text{constant}(\zeta - \rho_0^{-2}\zeta_0)^{3/2} \qquad (3.11)$$

4. THE PERIODIC RAYLEIGH TAYLOR PROBLEM

In this case, the physical flow domain over a period is again given by Fig.2, where gravity ($g = 1$) is now assumed to act in the positive y direction. We assume that there is vacuum on top of the fluid and that there is no net motion at $y = -\infty$. As in the case of gravity waves, we decompose the conformal mapping function $z(\zeta, t)$ as in (2.1). The domain in the ζ plane is the interior of the cut unit circle in Fig. 3. The function f now depends on time t as well. We will assume that the initial conditions are symmetric, so that the interface remains symmetric about each crest and trough for later times. This assumption is only made for simplicity and generalizations for nonsymmetric disturbances can be made as for time dependent deep water wave (section 5). This means that on the real ζ -axis in the interval (-1,1), (2.2) holds for f and the complex velocity potential W satisfies

$$Im\ W = 0 \qquad (4.1)$$

But unlike the gravity wave case, there will be no singularity of W inside the unit circle. Also, as for unsteady gravity waves (section 5), the free surface is not a stream line. However, for all times, the unit circular boundary corresponds to the free boundary. The kinematic condition on the free boundary is that

$$\frac{D}{Dt}\ ln\ \rho(x,y,t) = 0 \qquad (4.2)$$

on $ln\ \rho(x,y,t) = 1$, where $\zeta = \rho\ e^{i\nu}$, with ν real. In this representation, $ln\ \rho$, ν and t can be thought of as three dependent variables depending on x , y and t . Switching the role of dependent and independent variables, the kinematic condition implies that

$$Re\ [\ \zeta\ W_\zeta - \zeta^*\ z_\zeta^*\ z_t] = 0 \qquad (4.3)$$

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Plugging in the representation for z from (2.1) on $|\zeta| = 1$, we find that (4.3) is equivalent to:

$$Re \left[\frac{f_t}{1 + \zeta f_\zeta} \right] = \frac{Re \, \zeta W_\zeta}{|1 + \zeta f_\zeta|^2} \tag{4.4}$$

on $\zeta = e^{i\nu}$ for ν in the interval $[0, \, 2\pi]$, The Bernoulli's equation on the free surface for this time dependent problem is

$$Re \left[W_t - \frac{\zeta W_\zeta f_t}{1 + \zeta f_\zeta} - f \right] = -\frac{1}{2} \frac{|W_\zeta|^2}{|1 + \zeta f_\zeta|^2} \tag{4.5}$$

on the unit circle $\zeta = e^{i\,\nu}$. The analytic continuation of this is:

$$W_t - \frac{\zeta \, W_\zeta \, f_t}{1 + \zeta \, f_\zeta} - f + \frac{W_\zeta \, W_\zeta(1/\zeta,t)}{(1 + \zeta \, f_\zeta)(1 + \frac{1}{\zeta} + f_\zeta(1/\zeta,t))} = -I_1 \tag{4.6}$$

where

$$I_1 = W_t(1/\zeta,t) - \frac{\frac{1}{\zeta}W_\zeta(\frac{1}{\zeta},t)f_t(\frac{1}{\zeta},t)}{(1 + \frac{1}{\zeta}f_\zeta(\frac{1}{\zeta},t))} - f(\frac{1}{\zeta},t)$$

and

$$\frac{f_t}{1 + \zeta \, f_\zeta} - \frac{\zeta \, W_\zeta + \frac{1}{\zeta} \, W_\zeta(1/\zeta,t)}{(1 + \zeta \, f_\zeta)(1 + \frac{1}{\zeta}f_\zeta(1/\zeta,t))} = I_2 \tag{4.7}$$

where

$$I_2 = -\frac{f_t(1/\zeta,t)}{1 + \frac{1}{\zeta}f_\zeta(\frac{1}{\zeta},t)}$$

We rewrite the two equations as

$$f_t - R_2 \, W_\zeta - R_3 \, f_\zeta = R_4 \tag{4.8}$$

and

$$W_t - \frac{\zeta W_\zeta}{1 + \zeta f_\zeta} (R_1 + R_4 + R_3 \, f_\zeta + R_2 \, W_\zeta) - f = -I_1 \tag{4.9}$$

where

$$R_1 = -\frac{\frac{1}{\zeta} \, W_\zeta(1/\zeta,t)}{1 + \frac{1}{\zeta} \, f_\zeta(1/\zeta,t)} \tag{4.10}$$

$$R_2 = \frac{\zeta}{1 + \frac{1}{\zeta} \, f_\zeta(1/\zeta,t)} \tag{4.11}$$

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$$R_3 = \zeta I_2 \tag{4.12}$$

$$R_4 = I_2 + \frac{\frac{1}{\zeta} W_\zeta(\frac{1}{\zeta}, t)}{1 + \frac{1}{\zeta} f_\zeta(1/\zeta, t)} \tag{4.13}$$

When ζ is outside the unit circle, each of the functions I_1, I_2 are regular since they only involve f and W and their derivatives inside the unit circle. The same is true for the functions $W(1/\zeta, t)$, $f(1/\zeta, t)$ and their derivatives. Thus it is clear that R_1, R_2, R_3 and R_4 are each regular outside the unit circle as well. Equations (4.6) and (4.7) are actually equivalent to a set of two coupled non-linear integro-differential equations for f and W. since each of I_1 and I_2 can be alternately expressed as integrals involving values of f_ζ and W_ζ on the unit circle by use of Poisson's integral formulae.

If these are considered known, then we get a set of 1st order fully nonlinear partial differential equations. On taking the ζ derivative of each of the equations, we get a system of quasi linear partial differential equations (P.D.E):

$$W_{\zeta_t} + Q_1 W_{\zeta\zeta} + Q_2 f_{\zeta\zeta} = T_1 \tag{4.14}$$

$$f_{\zeta_t} + Q_3 W_{\zeta\zeta} + Q_4 f_{\zeta\zeta} = T_2 \tag{4.15}$$

where in this case

$$Q_1 = -\frac{\zeta[R_1 + R_4 + R_3 f_\zeta + 2R_2 W_\zeta]}{(1 + \zeta f_\zeta)} \tag{4.16}$$

$$Q_2 = \frac{\zeta^2 W_\zeta[R_1 + R_4 + R_3 f_\zeta + R_2 W_\zeta]}{(1 + \zeta f_\zeta)^2} - \frac{\zeta W_\zeta R_3}{(1 + \zeta f_\zeta)} \tag{4.17}$$

$$Q_3 = -R_2 \tag{4.18}$$

$$Q_4 = -R_3 \tag{4.19}$$

$$T_1 = \frac{W_\zeta[R_1 + R_4 + R_3 f_\zeta + R_2 W_\zeta]}{(1 + \zeta f_\zeta)} - \frac{\zeta W_\zeta f_\zeta[R_1 + R_4 + R_3 f_\zeta + R_2 W_\zeta]}{(1 + \zeta f_\zeta)^2}$$
$$+ \frac{\zeta W_\zeta[R_{1_\zeta} + R_{4_\zeta} + R_{3_\zeta} f_\zeta + R_{2_\zeta} W_\zeta]}{(1 + \zeta f_\zeta)} + f_\zeta - I_{1_\zeta} \tag{4.20}$$

and

$$T_2 = R_{4_\zeta} + R_{2_\zeta} W_\zeta + R_{3_\zeta} f_\zeta \tag{4.21}$$

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On taking a linear combination of the two equations, we find that

$$\frac{d}{dt}W_\zeta + \lambda\frac{d}{dt}f_\zeta = T_1 + \lambda T_2 \qquad (4.22)$$

on

$$\frac{d}{dt}\zeta = Q_1 + \lambda Q_3 \qquad (4.23)$$

where

$$\lambda = \frac{Q_4 - Q_1 \pm \sqrt{(Q_1 - Q_4)^2 + 4Q_2Q_3}}{2Q_3} \qquad (4.24)$$

We denote the two roots above by λ_1 and λ_2 corresponding to the the plus and minus sign respectively. On substituting the expressions for each of Q_1 through Q_4 from (4.16) through (4.19), it is seen that

$$\lambda_1 = -\frac{\zeta W_\zeta}{(1 + \zeta f_\zeta)} - \frac{(\zeta R_1 + \zeta R_4 - R_3)}{R_2(1 + \zeta f_\zeta)} \qquad (4.25)$$

$$\lambda_2 = -\frac{\zeta W_\zeta}{(1 + \zeta f_\zeta)} \qquad (4.26)$$

On substituting the expressions for R_1 , R_4 and R_3 from (4.10), (4.13) and (4.12), we discover that the two roots λ_1 and λ_2 of (4.24) are coincident, implying the system of 1st order quasilinear P.D.E in (4.14) and (4.15) is parabolic. The slope of the characterestic is:

$$Q_1 + \lambda Q_3 = -R_3 - \frac{\zeta R_2 W_\zeta}{1 + \zeta f_\zeta} \qquad (4.27)$$

On multiplying (4.22) by $\frac{\zeta}{(1+\zeta f_\zeta)^2}$ and introducing the notation:

$$y_1 = \frac{\zeta W_\zeta}{1 + \zeta f_\zeta} \qquad (4.28)$$

$$y_2 = \frac{1}{1 + \zeta f_\zeta} \qquad (4.29)$$

then equation (4.22) is equivalent to:

$$y_{1_t} - (R_3 + R_2 y_1)y_{1_\zeta} = \zeta y_1 y_2 R_{1_\zeta} - (1 + \zeta I_{1_\zeta})y_2 + 1 \qquad (4.30)$$

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Note that the left hand side of (4.30) is independent of y_2 altogether and if we take $-R_3 - R_2 y_1$ rather than y_1 as our dependent variable, then we get a nonhomogeneous inviscid Burger's equation, when y_2 along with R_1, R_2, etc are considered known. However y_2 is an unknown and one needs a separate equation for the evolution of y_2. This can be found by multiplying (4.15) by $\frac{\zeta}{(1+\zeta I_\zeta)^3}$. We find

$$y_{2_t} - (R_3 + R_2 y_1) y_{2_\zeta} = -R_2 y_2 y_{1_\zeta} + \frac{R_3}{\zeta} y_2 - \frac{R_3}{\zeta} y_2^2 - \zeta R_{4_\zeta} y_2^2 - R_{2_\zeta} y_1 y_2 - R_{3_\zeta} y_2 + R_{3_\zeta} y_2^2$$

(4.31)

Note that the righthand side of (4.31) contains the differentiated term y_{1_ζ} which cannot be avoided as the system of equations (4.32) and (4.33) are a parabolic set of P.D.E.'s just like the original set (4.14) and (4.15). However, if instead of considering ζ and t as independent variables, if we consider ξ, the initial value of ζ on a characteristic, and t as our independent variables, then we obtain the following equations for the three variables, $y_1(\xi,t)$, $y_2(\xi,t)$ and $\zeta(\xi,t)$:

$$y_{1_t} = \zeta y_1 y_2 R_{1_\zeta} - (1 + \zeta I_{1_\zeta}) y_2 + 1 \tag{4.32}$$

$$y_{2_t} = -R_2 y_2 \frac{y_{1_\xi}}{\zeta_\xi} + \frac{R_3}{\zeta} y_2 - \frac{R_3}{\zeta} y_2^2 - \zeta R_{4_\zeta} y_2^2 - R_{2_\zeta} y_1 y_2 - R_{3_\zeta} y_2 + R_{3_\zeta} y_2^2 \tag{4.33}$$

and

$$\zeta_t = -(R_3 + R_2 y_1) \tag{4.34}$$

Since ζ_ξ appears in the denominator in (4.33), the above set of equations is not convenient for numerical integration. It is more convenient to define:

$$\tilde{y}_2 = \frac{y_2}{\zeta_\xi} \tag{4.35}$$

In that case, it is easy to show by using (4.33) and (4.34), (4.32) and (4.33) can be replaced by

$$y_{1_t} = \zeta y_1 \tilde{y}_2 \zeta_\xi R_{1_\zeta} - (1 + \zeta I_{1_\zeta}) \tilde{y}_2 \zeta_\xi + 1 \tag{4.36}$$

$$\tilde{y}_{2_t} = \frac{R_3}{\zeta} \tilde{y}_2 - \frac{R_3}{\zeta} \tilde{y}_2^2 \zeta_\xi - \zeta R_{4_\zeta} \zeta_\xi \tilde{y}_2^2 + R_{3_\zeta} \zeta_\xi \tilde{y}_2^2 \tag{4.37}$$

In principle, one can integrate this set of three equations (4.34), (4.36) and (4.37) numerically. Notice that the coefficients of each term multiplying the unknowns y_1, y_2 and ζ are analytic functions of ζ everywhere outside the unit circle. However, since this is a nonlinear set of differential equations, spontaneous singularity can be occur. If we assume that the solution to (4.34), (4.36) and (4.37) are locally smooth functions $y_1 = y_1(\xi, t)$, $\tilde{y}_2 = \tilde{y}_2(\xi, t)$, $\zeta = \zeta(\xi, t)$ of ξ, then in terms of the original independent variable ζ and t, we will get smooth solutions except where the inverse function theorem fails, i.e. where $\zeta_\xi = 0$. Generically, the zeros of an analytic function are simple. This implies that the inverse function $\xi = \xi(\zeta, t)$ will generically have squareroot singularites. This would then imply that y_1 and y_2 will have squareroot singularities of the type $(\zeta - \zeta_0(t))^{1/2}$. Also, if we directly substitute a representation near $\zeta = \zeta_0(t)$, the singularity point in the ζ plane corresponding to $\xi_0(t)$ where $\zeta_\xi = 0$ at time t, then an analytic dependence of y_1, $\tilde{y}_2$ and ζ on ξ at $\xi = \xi_0(t)$ would imply a local expansion of the form:

$$y_1(\zeta, t) = A_1(t) + A_2(t)(\zeta - \zeta_0(t))^{1/2} + .. \qquad (4.38)$$

and

$$y_2(\zeta, t) = B_1(t) + B_2(t)(\zeta - \zeta_0(t))^{1/2} + .. \qquad (4.39)$$

On substituting this form into (4.30) and (4.31) and equating the most singular terms, one finds

$$\frac{d}{dt}\zeta_0 + R_{3_0} + R_{2_0}A_1 = 0 \qquad (4.40)$$

and

$$B_2\frac{d}{dt}\zeta_0 + (R_{3_0} + R_{2_0}A_1)B_2 = R_2B_1A_2 \qquad (4.41)$$

where subscript 0 refers to the evaluation of those quantities at $\zeta = \zeta_0(t)$. It is clear that since R_2 is nonzero, the two equations are consistent for nonzero A_2 when $B_1 = 0$. Notice that $B_1 = 0$ follows from the assumption that $\tilde{y}_2$ is finite where $\zeta_\xi = 0$.

Thus, a square root singularity appears to be generic for arbitrary initial conditions. Whether (4.34), (4.36) and (4.37) allow singular solutions in the ξ, t variables has to be investigated through analytical and numerical means.

Thus, we are now at the stage of outlining a numerical procedure for the study of the exact form of a time moving singularity for the classical Rayleigh-Taylor problem. The first stage consists of expressing each of f and W in a truncated power series representation in ζ satisfying (4.4) and (4.5) on the unit circle to find the ordinary differential equations for the evolution of the power series coefficients. This can be done efficiently by use of fast Fourier transforms. Once these coefficients are calculated, we are in a position to calculate each of R_1, R_2 and other analytic functions for each ζ outside the unit circle for any time t. Equations (4.34), (4.36) and (4.37) then needs to be solved numerically to find $y_1(\xi,t)$, $\tilde{y}_2(\xi,t)$ and $\zeta(\xi,t)$ where ξ is the initial value of ζ. If these solutions are indeed smooth functions of ξ, then we track $\xi_0(t)$, where $\zeta_\xi = 0$. Then $\zeta_0(t) = \zeta(\xi_0(t),t)$ will be the location of the fold (i.e. generically squareroot singularity) in the ζ plane. This program needs to be carried out for a study of the evolution of each singularity in the unphysical plane. This information on the singularities can possibly be used to devise an efficient and accurate numerical method to study a continually deforming interface.

Now we comment on the relation of the observed spikes with these singularities. Once a singularity is formed, say for the variable y_1, it will propagate in the direction of the characteristics. To find the characterestics directions, we take

$$f(\zeta,0) = -\epsilon \zeta \tag{4.42}$$

$$W(\zeta,0) = -\epsilon \zeta \tag{4.43}$$

where $0 < \epsilon << 1$. Then for early times, inside the unit circle $|\zeta| < 1$, the linearized solution corresponds to:

$$f(\zeta,t) = -\epsilon\zeta e^t \tag{4.44}$$

$$W(\zeta, t) = -\epsilon \zeta e^t \tag{4.45}$$

Using this expression to calculate each of R_1 through R_4 outside the unit circle $|\zeta| > 1$ by using (4.10) through (4.13) and using (4.44) and (4.45), we find $-R_3 - R_2 y_1$ is real and positive for $|\zeta| > 1$. This implies that if any singularity forms on the negative real axis, it is swept towards $\zeta = -1$, the location of the initial crest (notice it will be a trough if gravity were pointing downwards). However, this is only a heuristic suggestion. Confirmation must await direct calculations.

5. THE TIME DEPENDENT DEEP WATER WAVE PROBLEM

We now consider the time dependent water wave problem. Once again we decompose $z(\zeta, t)$ as in (2.1):

$$z(\zeta, t) = 2\pi + i(\ln \zeta + f) \tag{5.1}$$

where $f = f(\zeta, t)$. However, unlike the the case of Rayleigh-Taylor problem, we also decompose

$$W(\zeta, t) = ic \ln \zeta + 2\pi c + ic\omega \tag{5.2}$$

where the log singularity at $\zeta = 0$ is due to the uniformly translating flow at $y = -\infty$ For the class of time dependent flow for which the flow at $y = -\infty$ is a constant flow, it seems natural to assume that each of $f(\zeta, t)$ and $\omega(\zeta, t)$ do not have a singularity at $\zeta = 0$ and therefore possess a convergent power series representation for $|\zeta| \leq 1$ given by

$$f(\zeta, t) = \sum_{n=0}^{\infty} a_n(t) \zeta^n \tag{5.2}$$

$$\omega(\zeta, t) = \sum_{n=0}^{\infty} b_n(t) \zeta^n \tag{5.3}$$

The time dependent Bernoulli equation and pressure condition implies:

$$\phi_t + y + \frac{1}{2} \left\{ \left| \frac{dW}{dz} \right|^2 \right\} = 0 \tag{5.4}$$

Translating this condition with (ζ, t) as independent variables rather than (z, t) implies:

$$Re\left[W_t - \frac{W_\zeta}{z_\zeta} z_t\right] + \frac{1}{2} \frac{|W_\zeta|^2}{|z_\zeta|^2} + Im\, z = 0 \tag{5.5}$$

Substitution of (5.1) and (5.2) into (5.4) implies

$$Re\left[ic\omega_t - i\,c\frac{(1+\zeta\omega_\zeta)}{(1+\zeta f_\zeta)}\,f_t\right] + \frac{c^2}{2}\,\left|\frac{(1+\zeta\omega_\zeta)}{(1+\zeta f_\zeta)}\right|^2 + Re\,f = 0 \qquad (5.6)$$

The kinematic condition (4.3) is valid and in this case becomes:

$$Re\left[ic\zeta\omega_\zeta - (1+\zeta^*\,f_\zeta^*)\,f_t\right] = 0 \qquad (5.7)$$

Equations (5.6) and (5.7), which are only satisfied on $|\zeta| = 1$ now have to be analytically continued outside the unit circle $|\zeta| > 1$. For that purpose, it is convenient to define analytic functions g and α related to f and ω respectively through the representation:

$$g(\zeta,t) = \sum_{n=0}^{\infty} a_n^*(t)\,\zeta^{-n} \qquad (5.8)$$

$$\alpha(\zeta,t) = \sum_{n=0}^{\infty} b_n^*(t)\,\zeta^{-n} \qquad (5.9)$$

Note that since f and ω are each analytic functions of ζ inside the unit circle, it follows from the representation (5.8) and (5.9) that g and α are analytic functions of ζ for $|\zeta| > 1$. We also note that on $|\zeta| = 1$, the complex conjugate $f^* = g$ and $[f_\zeta]^* = -\zeta^2\,g_\zeta$, and similar expressions connect α with ω. Equations (5.5) and (5.6) can then be written as:

$$\frac{ic}{2}\,\omega_t - \frac{ic}{2}\alpha_t - \frac{ic}{2}\,\frac{(1+\zeta\omega_\zeta)}{1+\zeta f_\zeta}f_t + \frac{ic}{2}\,\frac{(1-\zeta\alpha_\zeta)}{1-\zeta g_\zeta}\,g_t + \frac{c^2}{2}\,\frac{(1+\zeta\omega_\zeta)(1-\zeta\alpha_\zeta)}{(1+\zeta f_\zeta)(1-\zeta g_\zeta)} + \frac{1}{2}[f+g] = 0$$
$$(5.10)$$

$$\frac{ic}{2}\zeta\omega_\zeta + \frac{ic}{2}\zeta\alpha_\zeta - \frac{1}{2}\,(1-\zeta g_\zeta)f_t - \frac{1}{2}(1+\zeta f_\zeta)g_t = 0 \qquad (5.11)$$

Equations (5.10) and (5.11) provide the analytically continued equation off the unit circle. α and g can then be considered as known functions if we calculate ω and f through a power series representation on the unit circle. We can solve for ω and f everywhere outside the unit circle just by solving a system of parabolic quasi-linear p.d.e.s outside the

unit circle. These are the same as (4.30) and (4.31), except that the right hand side of (4.30) has to be replaced by:

$$\zeta y_1 y_2 R_{1_\zeta} \; - \; (-1 + \zeta I_{1_\zeta}) y_2 - 1$$

and I_1 is now given by

$$I_1 \; = \; -ic\alpha(\zeta, t) + \frac{ic(1 \; - \; \alpha_\zeta) g_t}{(1 \; - \; \zeta g_\zeta)} + g$$

Also in the formulae for R_1 , R_2 , R_3 , R_4 , I_2 , y_1 and y_2 in the last section, every occurrence of W_t has to be related by by $ic\omega_t$, ζW_ζ by $ic(1 + \zeta \omega_\zeta)$ $W_t(1/\zeta, t)$ by $-ic\alpha_t$, $\frac{1}{\zeta} W_\zeta(\frac{1}{\zeta}, t)$ by $-ic(1 - \zeta \alpha_\zeta)$, $f_t(1/\zeta, t)$ by g_t and $\frac{1}{\zeta} f_\zeta(\frac{1}{\zeta}, t)$ by $-\zeta g_\zeta$. The direct numerical study of (4.30) and (4.31) is likely to clarify questions on time dependent water waves including the question whether for some or generic initial conditions, the singularity formed in the unphysical ζ plane can actually hit the physical domain in finite time, however small the initial amplitude. This remains an important open question (P.G. Saffman, private communication).

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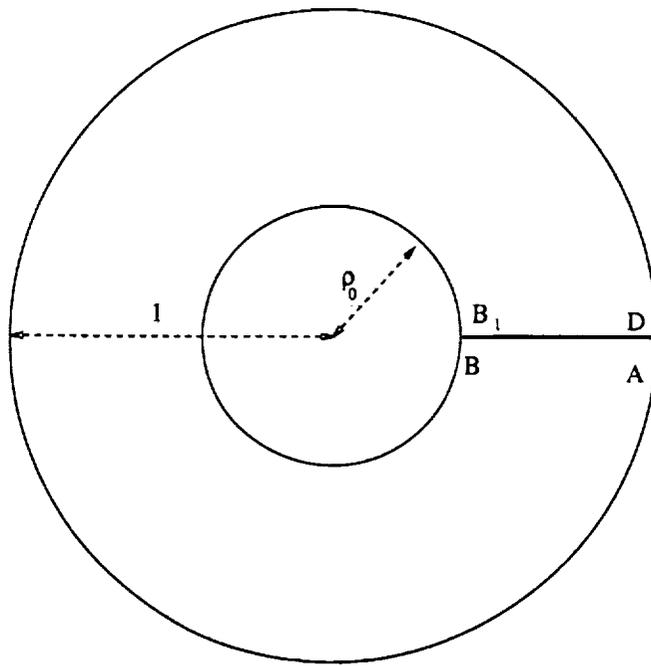
Table 1: Dependence of ζ_0 on wave height H

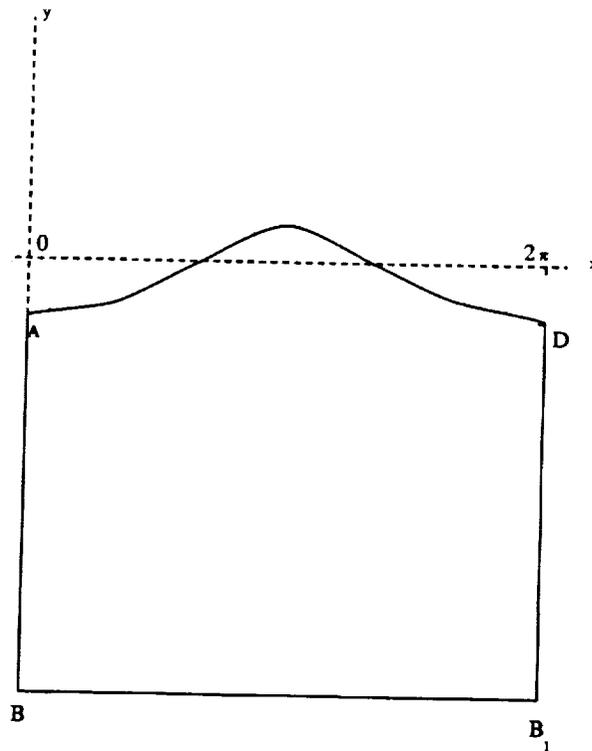| H	c	ζ_0
0.0199989	1.0002	−18.402
0.0998750	1.0050	−3.7166
0.1056818	1.0056	−3.5162
0.1262381	1.0080	−2.9563
0.1888844	1.0180	−2.0113
0.2431296	1.0220	−1.5960
0.3580657	1.0660	−1.1544
0.41015	1.0860	−1.0466
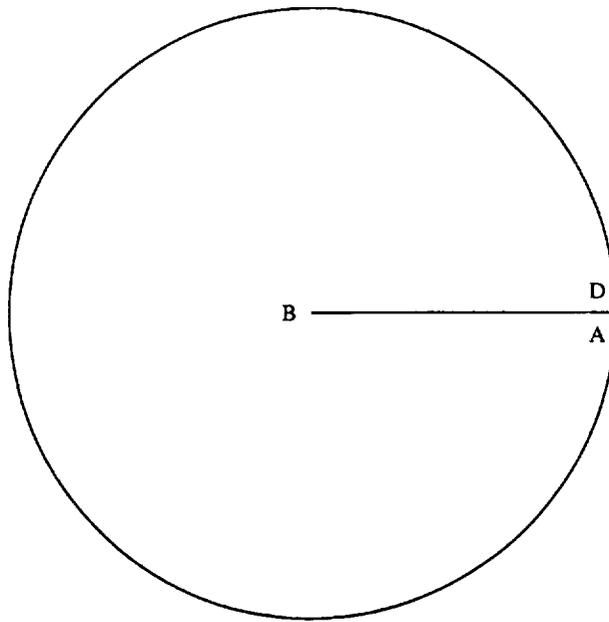
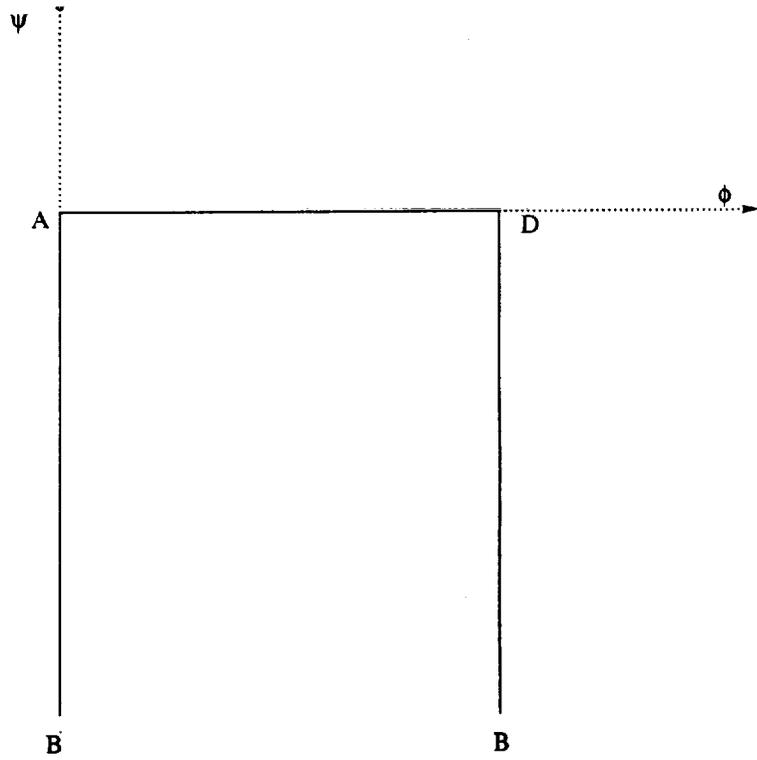1. Fig. 1: The complex ζ cut unit circle.
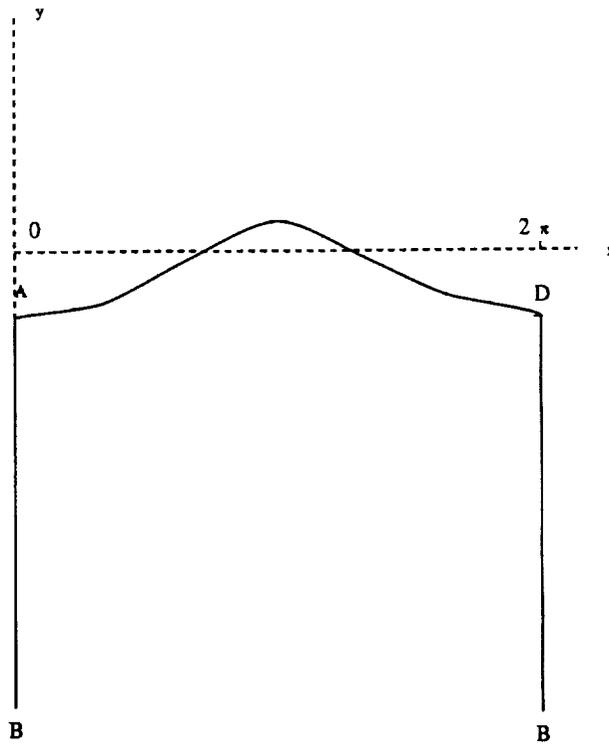
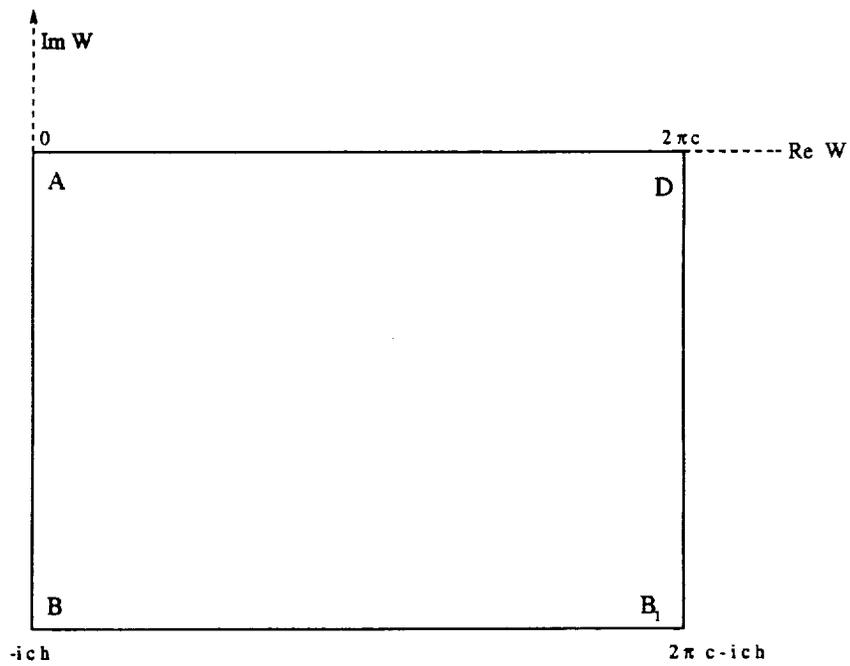2. Fig. 2: The physical $z = x + iy$ plane.

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3. Fig. 3: The ζ annular region with a cut corresponding to finite depth water waves.

4. Fig. 4: The physical $z = x + iy$ plane for finite depth water waves.

5. Fig. 5: $W = \phi + i\psi$ plane for steady deep water wave.

6. Fig. 6: W plane corresponding to finite depth steady water waves.

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Report Documentation Page

1. Report No. NASA CR-187530 ICASE Report No. 91-21	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle SINGULARITIES IN WATER WAVES AND RAYLEIGH-TAYLOR INSTABILITY		5. Report Date February 1991
		6. Performing Organization Code
7. Author(s) S. Tanveer		8. Performing Organization Report No. 91-21
		10. Work Unit No. 505-90-52-01
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		11. Contract or Grant No. NAS1-18605
		13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225		14. Sponsoring Agency Code

15. Supplementary Notes
Langley Technical Monitor: Submitted to Proc. Roy. Soc. Michael F. Card   LONDON A   Final Report

16. Abstract
This paper is concerned with singularities in inviscid two dimensional finite amplitude water waves and inviscid Rayleigh-Taylor instability. For the deep water gravity waves of permanent form, through a combination of analytical and numerical methods, we present results describing the precise form, number and location of singularities in the unphysical domain as the wave height is increased. We then show how the information on the singularity can be used to calculate water waves numerically in a relatively efficient fashion. We also show that for two dimensional water waves in a finite depth channel, the nearest singularity in the unphysical region has the same form as for deep water waves. However, associated with such a singularity, there is a series of image singularities at increasing distances from the physical plane with possibly different behavior. Further, for the Rayleigh-Taylor problem of motion of fluid over vacuum, and for the unsteady water wave problem, we derive integro-differential equations valid in the unphysical region and show how these equations can give information on the nature of singularities for arbitrary initial conditions.

17. Key Words (Suggested by Author(s)) water waves, Rayleigh-Taylor instability, singularities, free surface	18. Distribution Statement 34 - Fluid Mechanics and Heat Transfer Unclassified - Unlimited		
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of pages 39	22. Price A03